RELATIVE CYCLOTOMIC SPECTRA AND TOPOLOGICAL CYCLIC HOMOLOGY VIA THE NORM

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Abstract. We describe a construction of the cyclotomic structure on topological Hochschild homology (THH) of a ring spectrum using the Hill-Hopkins-Ravenel multiplicative norm. Our analysis takes place entirely in the category of equivariant orthogonal spectra, avoiding use of the Bökstedt coherence machinery. As a consequence, we are able to define versions of topological cyclic homology (TC) relative to an arbitrary commutative ring spectrum A. We describe spectral sequences computing this relative theory ${}_ATR$ in terms of TR over the sphere spectrum and vice versa. Furthermore, our construction permits a straightforward definition of the Adams operations on TR and TC.

Contents

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1. Introduction

Over the last two decades, the calculational study of algebraic K-theory has been revolutionized by the development of trace methods. In analogy with the Chern character from topological K-theory to ordinary cohomology, there exist "trace maps" from algebraic K-theory to various more homological approximations, which also can be more computable. For a ring R, Dennis constructed a map $K(R) \to HH(R)$ that generalizes the trace of a matrix. Goodwillie lifted this trace map to negative cyclic homology

$$K(R) \longrightarrow HC^-(R) \longrightarrow HH(R)$$

and showed that rationally, this map can often be used to compute $K(R)$.

In his 1990 ICM address, Goodwillie conjectured that there should be a "brave new" version of this story involving "topological" analogues of cyclic and Hochschild homology (THH and TC), defined by changing the ground ring from $\mathbb{Z}$ to the sphere spectrum. Although the modern symmetric monoidal categories of spectra had not yet been invented, Bökstedt developed coherence machinery that enabled a definition of THH along these lines and constructed a "topological" Dennis trace map $K \to THH$ [Bokstedt]. Subsequently, Bökstedt-Hsiang-Madsen [BHM] defined TC and constructed the cyclotomic trace map

$$K \longrightarrow TC \longrightarrow THH$$

in the course of resolving the K-theory Novikov conjecture (for groups satisfying a mild finiteness hypothesis). Subsequently, seminal work of McCarthy [McCarthy] and Dundas [Dundas] showed that when working at a prime p, TC often captures a great deal of information about K-theory. Hesselholt and Madsen (inter alia, [HMannals]) then used TC to make extensive computations in K-theory including computational resolution of the Quillen-Lichtenbaum conjecture for certain fields.

The calculational power of trace methods depends on the ability to compute TC, which ultimately derives from the methods of equivariant stable homotopy theory, as TC is constructed from the S^1-action on THH. Specifically, Bökstedt's definition of THH closely resembles a cyclic bar construction, and as a consequence THH is an S^1-equivariant spectrum. In fact, $THH(R)$ has a very special equivariant structure: $THH(R)$ is a *cyclotomic* spectrum, which is an S^1-equivariant spectrum equipped with additional data that models the structure of the free loop space ΛX.

The cyclic bar construction can be formed in any symmetric monoidal category $(A, \boxtimes, 1)$; we will let $N^{\mathrm{cyc}}_{\boxtimes}$ denote the resulting simplicial (or cyclic) object. Recall that in the category of spaces, for a group-like monoid M, there is a natural map

$$|N^{\mathrm{cyc}}_{\times} M| \longrightarrow \mathrm{Map}(S^1, BM) = \Lambda BM$$

(where $|\cdot|$ denotes geometric realization) that is a weak equivalence on fixed points for any finite subgroup $C_n \subset S^1$. Moreover, for each such C_n, the free loop space is equipped with equivalences (in fact homeomorphisms)

$$(\Lambda BM)^{C_n} \cong \Lambda BM$$

of S^1-spaces, where $(\Lambda BM)^{C_n}$ is regarded as an S^1-space (rather than an S^1/C_n-space) via pullback along the nth root isomorphism

$$\rho_n \colon S^1 \cong S^1/C_n.$$

In analogy, a cyclotomic spectrum is an S^1-equivariant spectrum equipped with compatible equivalences of S^1-spectra

$$t_n\colon \rho_n^* L\Phi^{C_n} X \longrightarrow X,$$

where $L\Phi^{C_n}$ denotes the (left derived) "geometric" fixed points functor.

The construction of the cyclotomic structure on THH has classically been one of the more subtle and mysterious parts of the construction of TC. In a modern symmetric monoidal category of spectra (e.g., symmetric spectra or EKMM S-modules), one can simply define $THH(R)$ as

$$THH(R) = |N_\wedge^{\mathrm{cyc}} R|,$$

but the resulting equivariant spectrum did not have the correct homotopy type. Only Bökstedt's original construction of THH seemed to produce the cyclotomic structure.

Although this situation has not impeded the calculational applications, reliance on the Bökstedt construction has limited progress in certain directions. For one thing, it does not seem to be possible to use the Bökstedt construction to define TC relative to a ground ring that is not the sphere spectrum S. Moreover, the details of the Bökstedt construction make it difficult to understand the equivariance (and therefore relevance to TC) of various additional algebraic structures that arise on THH, notably the Adams operations and the coalgebra structures.

The purpose of this paper is to introduce a new approach to the construction of the cyclotomic structure on THH using an interpretation of THH in terms of the Hill-Hopkins-Ravenel multiplicative norm. Our point of departure is the observation that the construction of the cyclotomic structure on $THH(R)$ ultimately boils down to having good models of the smash powers

$$R^{\wedge n} = \underbrace{R \wedge R \wedge \ldots \wedge R}_{n}$$

of a spectrum R as a C_n-equivariant spectrum such that there is a suitably compatible collection of diagonal equivalences

$$R \longrightarrow \Phi^{C_n} R^{\wedge n}.$$

The recent solution of the Kervaire invariant one problem involved the detailed analysis of a multiplicative norm construction in equivariant stable homotopy theory that has precisely this behavior. Although Hill-Hopkins-Ravenel studied the norm construction N_H^G for a finite group G and subgroup H, using the cyclic bar construction one can extend this construction to a norm $N_e^{S^1}$ on associative ring orthogonal spectra; such a construction first appeared in the thesis of Martin Stolz [37].

For the following definition, we need to introduce some notation. Let $\mathscr{I}\mathcal{S}$ denote the category of orthogonal spectra and let $S^1\mathscr{I}\mathcal{S}^U$ denote the category of orthogonal S^1-spectra indexed on the complete universe U. Finally, let $\mathscr{I}\mathcal{S}[\mathbb{T}]$ and $S^1\mathscr{I}\mathcal{S}^U[\mathbb{T}]$ denote the categories of associative ring orthogonal spectra and associative ring orthogonal S^1-spectra, respectively.

defn:Tnorm

Definition 1.1. Define the functor

$$N_e^{S^1}\colon \mathscr{I}\mathcal{S}[\mathbb{T}] \longrightarrow S^1\mathscr{I}\mathcal{S}^U$$

to be the composite functor

$$R \mapsto \mathcal{I}^U_{\mathbb{R}^\infty} |N_\wedge^{\mathrm{cyc}} R|,$$

with $|N^{\mathrm{cyc}}_{\wedge} R|$ regarded as an orthogonal S^1-spectrum indexed on the standard trivial universe $\mathbb{R}^\infty$. Here $\mathcal{I}^U_{\mathbb{R}^\infty}$ denotes the change of universe functor (see Definition 2.6).

Since both the cyclic bar construction and the change of universe functor preserve commutative ring orthogonal spectra, the norm above also preserves commutative ring orthogonal spectra. In the following proposition, let $\mathscr{I}\mathcal{S}[\mathbb{P}]$ and $S^1\mathscr{I}\mathcal{S}^U[\mathbb{P}]$ denote the categories of commutative ring orthogonal spectra and commutative ring orthogonal S^1-spectra, respectively.

Proposition 1.2. *$N_e^{S^1}$ restricts to a functor*

$$N_e^{S^1} \colon \mathscr{I}\mathcal{S}[\mathbb{P}] \longrightarrow S^1\mathscr{I}\mathcal{S}^U[\mathbb{P}]$$

that is the left adjoint to the forgetful functor from commutative ring orthogonal S^1-spectra to commutative ring orthogonal spectra.

Since the forgetful functor from commutative ring orthogonal S^1-spectra to commutative ring orthogonal spectra is the composite of the change of universe functor $\mathcal{I}_U^{\mathbb{R}^\infty}$ and the functor that forgets equivariance, the previous proposition therefore identifies $N_e^{S^1} \colon \mathscr{I}\mathcal{S}[\mathbb{P}] \to S^1\mathscr{I}\mathcal{S}^U[\mathbb{P}]$ as the composite functor

$$R \mapsto \mathcal{I}^U_{\mathbb{R}^\infty}(R \otimes S^1).$$

Here $\otimes$ denotes the tensor of a commutative ring orthogonal spectrum with an unbased space, and we regard $R \otimes S^1$ as a functor from commutative ring orthogonal spectra to commutative ring orthogonal spectra with an action of S^1.

The deep aspect of the Hill-Hopkins-Ravenel treatment of the norm functor is their analysis of the left derived functors of the norm. As part of this analysis they show that the norm N_H^G preserves certain weak equivalences. For our norm $N_e^{S^1}$ into $S^1\mathscr{I}\mathcal{S}^U$, we work with the homotopy theory defined by the *$\mathcal{F}$-equivalences* of orthogonal S^1-spectra, where an $\mathcal{F}$-equivalence is a map that induces an isomorphism on all the homotopy groups at the fixed point spectra for the finite subgroups of S^1.

prop:derive **Proposition 1.3.** *Assume that $\bar{R}$ is a cofibrant associative ring orthogonal spectrum and R is either a cofibrant associative ring orthogonal spectrum or a cofibrant commutative ring orthogonal spectrum. If $\bar{R} \to R$ is a weak equivalence, then $N_e^{S^1}(\bar{R}) \to N_e^{S^1}(R)$ is an $\mathcal{F}$-equivalence in $S^1\mathscr{I}\mathcal{S}^U$.*

As a consequence we obtain the following additional observation about the adjunction in the commutative case.

prop:dercom **Proposition 1.4.** *The left derived functor of*

$$N_e^{S^1} \colon \mathscr{I}\mathcal{S}[\mathbb{P}] \longrightarrow S^1\mathscr{I}\mathcal{S}^U[\mathbb{P}]$$

(for the $\mathcal{F}$-equivalences on the codomain) exists and is left adjoint to the right derived forgetful functor.

Our first main theorem is that when R is a cofibrant associative ring orthogonal spectrum, $N_e^{S^1}R$ is a cyclotomic spectrum. To be precise, we use the point-set model of cyclotomic spectra from [7], which provides a definition entirely in terms of the category of S^1-equivariant orthogonal spectra.

thm:maincyc **Theorem 1.5.** *Let R be a cofibrant associative or cofibrant commutative ring orthogonal spectrum. Then $N_e^{S^1}R$ has a natural structure of a cyclotomic spectrum.*

Proposition 1.4, which describes $N_e^{S^1}$ as the homotopical left adjoint to the forgetful functor, suggests a generalization of our construction of THH that takes ring orthogonal C_n-spectra as input. For commutative ring orthogonal C_n-spectra, we can define $N_{C_n}^{S^1}$ as the left adjoint to the forgetful functor. However, to extend to the non-commutative case, we need an explicit construction. We give such a construction in Section 8 in terms of a cyclic bar construction, which we denote as $N_\wedge^{\mathrm{cyc},C_n}R$. Its geometric realization $|N^{\mathrm{cyc},C_n}R|$ has an S^1-action, and promoting it to the complete universe, we obtain a genuine S^1-equivariant orthogonal spectrum that we denote as $N_{C_n}^{S^1}R$. The following proposition is a consistency check.

Proposition 1.6. *Let R be a commutative ring orthogonal C_n-spectrum. Then $N_{C_n}^{S^1}R$ is isomorphic to the left adjoint of the forgetful functor from commutative ring orthogonal S^1-spectra to commutative ring orthogonal C_n-spectra.*

Again, we can describe the left adjoint in terms of a tensor

$$N_{C_n}^{S^1} = \mathcal{I}^U_{\mathbb{R}^\infty}(R \otimes_{C_n} S^1),$$

where the relative tensor $R\otimes_{C_n} S^1$ may be explicitly constructed as the coequalizer

$$(i^*R)\otimes C_n \otimes S^1 \rightrightarrows (i^*R)\otimes S^1$$

of the canonical action of C_n on S^1 and the action map $(i^*R)\otimes C_n \to i^*R$, where i^* denotes the change-of-group functor to the trivial group. Choosing an appropriately subdivided model of the circle produces the isomorphism between the two descriptions.

As above, by cofibrantly replacing R we can compute the left-derived functor of $N_{C_n}^{S^1}$, and in this case $N_{C_n}^{S^1}R$ is a p-cyclotomic spectrum (see Definition 3.1) provided either n is prime to p or R is "C_n-cyclotomic" (q.v. Definition 8.7 below). This leads to the obvious definition of $TC_{C_n}R$. This C_n-relative THH (and the associated constructions of TR and TC) is expected to be both interesting and comparatively easy to compute for some of the equivariant spectra that arise in Hill-Hopkins-Ravenel, in particular the real cobordism spectrum $MU_{\mathbb{R}}$.

We can also consider another kind of relative construction, namely in the situation where R is an algebra over an arbitrary commutative ring orthogonal spectrum A. One of the principal advantages of Definition 1.1 is that it can be easily extended to this relative setting; the equivariant indexed product can be carried out in any symmetric monoidal category, and the homotopical analysis (after change of universe) extends to A-modules.

Definition 1.7. Let A be a cofibrant commutative ring orthogonal spectrum, and denote by $\mathscr{I}\mathcal{S}_A[\mathbb{T}]$ the category of A-algebras. We define the A-relative norm functor

$${}_AN_e^{S^1} : \mathscr{I}\mathcal{S}_A[\mathbb{T}] \longrightarrow S^1\mathscr{I}\mathcal{S}^U_{\mathcal{I}^U_{\mathbb{R}^\infty}A}$$

by

$$R \mapsto \mathcal{I}^U_{\mathbb{R}^\infty} N^{cy}_{\wedge_A} R.$$

Here, in the construction $\mathcal{I}^U_{\mathbb{R}^\infty}A$, we regard A as a commutative ring orthogonal S^1-spectrum (on the universe $\mathbb{R}^\infty$) with trivial S^1-action. Then $\mathcal{I}^U_{\mathbb{R}^\infty}A$ is a commutative ring orthogonal S^1-spectrum (on the universe U) and $S^1\mathscr{I}\mathcal{S}^U_{\mathcal{I}^U_{\mathbb{R}^\infty}A}$ denotes the category of $\mathcal{I}^U_{\mathbb{R}^\infty}A$-modules in $S^1\mathscr{I}\mathcal{S}^U$.

We write ${}_ATHH(R)$ for the underlying non-equivariant spectrum of ${}_AN_e^{S^1}R$; this spectrum was denoted $thh^A(R)$ in [EKMM, IX.2.1]. When R is a commutative A-algebra, ${}_AN_e^{S^1}R$ is naturally a commutative $\mathcal{I}^U_{\mathbb{R}^\infty}A$-algebra. The functor

$$ {}_AN_e^{S^1} : \mathscr{I}\mathcal{S}_A[\mathbb{P}] \longrightarrow S^1\mathscr{I}\mathcal{S}^U_{\mathcal{I}^U_{\mathbb{R}^\infty}A}[\mathbb{P}] $$

is again left adjoint to the forgetful functor. An argument analogous to the proof of Theorem 1.5 then establishes the following theorem.

Theorem 1.8. *Let R be a cofibrant associative A-algebra or cofibrant commutative A-algebra. Then ${}_AN_e^{S^1}R$ is a cyclotomic spectrum with structure map a map of A-modules. Moreover, the natural map $N_e^{S^1}R \to {}_AN_e^{S^1}R$ induced by the lax symmetric monoidal natural transformation $\wedge \to \wedge_A$ is a cyclotomic map.*

As a consequence, we can define the A-relative topological cyclic homology ${}_ATC(R)$ (as the usual homotopy limit over the Frobenius and restriction maps) and the cyclotomic map $N_e^{S^1}R \to {}_AN_e^{S^1}R$ induces a map $TC(R) \to {}_ATC(R)$. The relative topological cyclic homology is therefore the target for an A-relative cyclotomic trace $K(R) \to {}_ATC(R)$, factoring though the usual cyclotomic trace $K(R) \to TC(R)$. Experts will recognize that one can also give a direct construction of the relative cyclotomic trace induced by the inclusion of objects in a spectral category enriched in orthogonal spectra (e.g., see [BMtw]).

Theorem 1.9. *Let R be a cofibrant associative A-algebra or a cofibrant commutative A-algebra. There is an A-relative cyclotomic trace map $K(R) \to {}_ATC(R)$ making the following diagram commute*

$$\begin{array}{ccccc} K(R) & \longrightarrow & TC(R) & \longrightarrow & THH(R) \\ & \searrow & \downarrow & & \downarrow \\ & & {}_ATC(R) & \longrightarrow & {}_ATHH(R). \end{array}$$

Using the identification $N_e^{S^1}A \cong \mathcal{I}^U_{\mathbb{R}^\infty}(A \otimes S^1)$ in the commutative context, the map $S^1 \to *$ induces a map of equivariant commutative ring orthogonal spectra $N_e^{S^1}A \to \mathcal{I}^U_{\mathbb{R}^\infty}A$. Just as in the non-equivariant case, we can identify ${}_AN_e^{S^1}(R)$ as extension of scalars along this map.

Proposition 1.10. *There is a natural isomorphism*

$$ {}_AN_e^{S^1}(R) \cong N_e^{S^1}(R) \wedge_{N_e^{S^1}A} \mathcal{I}^U_{\mathbb{R}^\infty}A. $$

When R is a cofibrant associative A-algebra or cofibrant commutative A-algebra, this induces a natural isomorphism in the stable category

$$ {}_AN_e^{S^1}(R) \cong N_e^{S^1}(R) \wedge^{\mathbf{L}}_{N_e^{S^1}A} \mathcal{I}^U_{\mathbb{R}^\infty}A. $$

The equivariant homotopy groups $\pi_*^{C_n}(N_e^{S^1}R)$ are the TR-groups $TR^n_*(R)$ and so $\pi_*^{C_n}({}_AN_e^{S^1}R)$ are by definition the relative TR-groups ${}_ATR^n_*(R)$. The Künneth spectral sequence of [LewisMandell2] can be combined with the previous theorem to compute the relative TR-groups from the absolute TR-groups and Mackey functor $\underline{\mathrm{Tor}}$. More often we expect to use the relative theory to compute the absolute theory. Non-equivariantly, the isomorphism

eq:changebasering

$$ THH(R) \wedge A \cong {}_ATHH(R \wedge A) \tag{1.11} $$

gives rise to a Künneth spectral sequence

$$\mathrm{Tor}^{A_*(R\wedge_S R^{\mathrm{op}})}_{*,*}(A_*(R), A_*(R)) \quad \Longrightarrow \quad A_*(THH(R)).$$

An Adams spectral sequence can then in practice be used to compute the homotopy groups of $THH(R)$. For formal reasons, the isomorphism (1.11) still holds equivariantly, but now we have three different versions of the non-equivariant Künneth spectral sequence (none of which have quite as elegant an E^2-term) which we use in conjunction with equation (1.11). We discuss these in Section 9.

A further application of our model of THH and TC is a construction when R is commutative of Adams operations on $N_e^{S^1}R$ and ${}_AN_e^{S^1}R$ that are compatible with the cyclotomic structure. McCarthy explained how Adams operations can be constructed on any cyclic object that when viewed as a functor from the cyclic category, factors through the category of finite sets (and all maps). As a consequence, it is possible to construct Adams operations on THH of a commutative monoid object in any symmetric monoidal category of spectra. The advantage of our formulation is that we can verify the equivariance of these operations and in particular show they descend to TC and ${}_ATC$.

Theorem 1.12. *Let A be a commutative ring orthogonal spectrum and R a commutative A-algebra. There are "Adams" operations $\psi^r\colon {}_AN_e^{S^1}R \to {}_AN_e^{S^1}R$. When r is prime to p, the operation ψ^r commutes with the restriction and Frobenius maps on the p-cyclotomic spectrum ${}_ATHH(R)$ and so induces a corresponding operation on ${}_ATR(R)$ and ${}_ATC(R)$.*

We have organized the paper to contain a brief review with references to much of the background needed here. Section 2 is entirely review of [30] and [22, App. B], and Section 3 is in part a review of [7, §4]. In addition, the main results in Sections 4 and 5 overlap significantly with [37], although our treatment is very different: we rely on [22] to study the absolute S^1-norm whereas [37] directly analyzes the construction by using a somewhat different model structure and focuses on the case of commutative ring orthogonal spectra.

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2. Background on equivariant stable homotopy theory

In this section, we briefly review necessary details about the category of orthogonal G-spectra and the geometric fixed point and norm functors. Our primary sources for this material are the monographs of Mandell and May [30] and the appendix to Hill-Hopkins-Ravenel [22]. See also [7, §2] for a review of some of these details. We begin with two subsections discussing the point-set theory followed by two subsections on homotopy theory and derived functors.

2.1. The point-set theory of equivariant orthogonal spectra.

Let G be a compact Lie group. We denote by $G\mathcal{T}$ the category of based G-spaces and G-maps. The smash product of G-spaces makes this a closed symmetric monoidal category, with function object $F(X,Y)$ the based space of (non-equivariant) maps from X to Y with the conjugation G-action. In particular, $G\mathcal{T}$ is enriched over G-spaces. We will denote by U a fixed universe of G-representations [30, §II.1.1], by which we

mean a countable dimensional vector space with linear G-action and G-fixed inner product that contains $\mathbb{R}^\infty$, is the sum of finite dimensional G-representations, and that has the property that any G-representation that occurs in U occurs infinitely often. Let $\mathcal{V}^G(U)$ denote the set of finite-dimensional G-vector subspaces of U, regarded as G-inner product spaces with the inherited inner product. Except in this section, we always assume that G is a *complete* G-universe, meaning that all finite dimensional irreducible G-represntations are reprented as subspaces. For V, W in $\mathcal{V}^G(U)$, denote by $\mathscr{I}_G(V,W)$ the space of (non-equivariant) isometric isomorphisms $V \to W$, regarded as a G-space via conjugation. Let $\mathscr{I}_G^U$ be the category enriched in G-spaces with $\mathcal{V}^G(U)$ as its objects and $\mathscr{I}_G(V,W)$ as its morphism G-spaces; we write just $\mathscr{I}_G$ when U is understood.

Definition 2.1 ([MM, II.2.6])**.** An orthogonal G-spectrum is a G-equivariant continuous functor $X\colon \mathscr{I}_G \to G\mathcal{T}$ equipped with a structure map

$$\sigma_{V,W}\colon X(V) \wedge S^W \longrightarrow X(V \oplus W)$$

that is a natural transformation of enriched functors $\mathscr{I}_G \times \mathscr{I}_G \to G\mathcal{T}$ and that is associative and unital in the obvious sense. A map of orthogonal G-spectra $X \to X'$ is an equivariant natural transformation that commutes with the structure map.

We denote the category of orthogonal G-spectra by $G\mathscr{I}\mathcal{S}$. When necessary to specify the universe U, we include it in the notation as $G\mathscr{I}\mathcal{S}^U$.

The category of orthogonal G-spectra is enriched over based G-spaces, where the G-space of maps consists of all natural transformations (not just the equivariant ones). Tensors and cotensors are computed levelwise. The category of orthogonal G-spectra is a closed symmetric monoidal category with unit the equivariant sphere spectrum S_G (with $S_G(V) = S^V$).

For technical reasons, it is often convenient to give an equivalent formulation of orthogonal G-spectra as diagram spaces. Following [MM, §II.4], we consider the category $\mathscr{J}_G$ which has the same objects as $\mathscr{I}_G$ but morphisms from V to W given by the Thom space of the complement bundle of linear isometries from V to W.

prop:orthodesc

Proposition 2.2 ([MM, II.4.3])**.** *The category $G\mathscr{I}\mathcal{S}$ of orthogonal G-spectra is equivalent to the category of $\mathscr{J}_G$-spaces, i.e., the continuous equivariant functors from $\mathscr{J}_G$ to $\mathcal{T}_G$. The symmetric monoidal structure is given by the Day convolution.*

This description provides simple formulas for suspension spectra and desuspension spectra in orthogonal G-spectra.

Definition 2.3 ([MM, II.4.6])**.** For any finite-dimensional G-inner product space V we have the shift desuspension spectrum functor

$$F_V\colon G\mathcal{T} \longrightarrow G\mathscr{I}\mathcal{S}$$

defined by

$$(F_V A)(W) = \mathscr{J}_G(V,W) \wedge A.$$

This is the left adjoint to the evaluation functor which evaluates an orthogonal G-spectrum at V.

Remark 2.4. In [HHR], the desuspension spectrum $F_V S^0$ is denoted as S^{-V} and $F_0 A$ is denoted as $\Sigma^\infty A$ in a nod to the classical notation. (They write $S^{-V} \wedge A$ for $F_V A \cong F_V S^0 \wedge A$.)

Since the category $G\mathscr{I}\mathcal{S}$ is symmetric monoidal under the smash product, we have categories of associative and commutative monoids, i.e., algebras over the monads $\mathbb{T}$ and $\mathbb{P}$ that create associative and commutative monoids in symmetric monoidal categories (e.g., see [EKMM, §II.4] for a discussion).

notn:rings **Notation 2.5.** Let $G\mathscr{I}\mathcal{S}[\mathbb{T}]$ and $G\mathscr{I}\mathcal{S}[\mathbb{P}]$ denote the categories of associative and commutative ring orthogonal G-spectra.

For a fixed object A in $G\mathscr{I}\mathcal{S}[\mathbb{P}]$, there is an associated symmetric monoidal category $G\mathscr{I}\mathcal{S}_A$ of A-modules in orthogonal G-spectra, with product the A-relative smash product $\wedge_A$. As in Notation 2.5, there are categories $G\mathscr{I}\mathcal{S}_A[\mathbb{T}]$ of A-algebras, and $G\mathscr{I}\mathcal{S}_A[\mathbb{P}]$ of commutative A-algebras [MM, III.7.6].

We now turn to the description of various useful functors on orthogonal G-spectra. We begin by reviewing the change of universe functors. In contrast to the classical framework of "coordinate-free" equivariant spectra [LMS], orthogonal G-spectra disentangle the point-set and homotopical roles of the universe U. A first manifestation of this occurs in the behavior of the point-set "change of universe" functors.

def:changeuniverse **Definition 2.6** ([MM, V.1.2])**.** For any pair of universes U and U', the point-set change of universe functors

$$\mathcal{I}_U^{U'} \colon G\mathscr{I}\mathcal{S}^U \longrightarrow G\mathscr{I}\mathcal{S}^{U'}$$

are defined by $\mathcal{I}_U^{U'} X(V) = \mathscr{J}(\mathbb{R}^n, V) \wedge_{O(n)} X(\mathbb{R}^n)$ for V in $\mathcal{V}^G(U')$, where $n = \dim V$.

These functors are strongly symmetric monoidal equivalences of categories:

Proposition 2.7 ([MM, V.1.1,V.1.5])**.** *Given universes U, U', U'',*

1. *$\mathcal{I}_U^U$ is naturally isomorphic to the identity.*
2. *$\mathcal{I}_{U'}^{U''} \circ \mathcal{I}_U^{U'}$ is naturally isomorphic to $\mathcal{I}_U^{U''}$.*
3. *$\mathcal{I}_U^{U'}$ is strong symmetric monoidal.*

We are particularly interested in the change of universe functors associated to the universes U and U^G. The latter of these universes is isomorphic to the standard trivial universe $\mathbb{R}^\infty$. Note that the category of orthogonal G-spectra on $\mathbb{R}^\infty$ is just the category of orthogonal spectra with G-actions.

Given a subgroup $H \subset G$, we can regard a G-space $X(V)$ as an H-space $\iota_H^* X(V)$. The space-level construction gives rise to a spectrum-level change-of-group functor.

def:changegroup **Definition 2.8** ([MM, V.2.1])**.** For a subgroup $H \subset G$, define the functor

$$\iota_H^* \colon G\mathscr{I}\mathcal{S}^U \longrightarrow H\mathscr{I}\mathcal{S}^{\iota_H^* U}$$

by

$$(\iota_H^* X)(V) = \mathscr{J}_H(\mathbb{R}^n, V) \wedge_{O(n)} \iota_H^*(X(\mathbb{R}^n))$$

for V in $\mathcal{V}^H(\iota_H^* U)$, where $n = \dim(V)$.

As observed in [MM, V.2.1, V.1.10], for V in $\mathcal{V}^G(U)$,

$$(\iota_H^* X)(\iota_H^* V) \cong \iota_H^*(X(V)).$$

In contrast to the category of G-spaces, there are two reasonable constructions of fixed-point functors: the "categorical" fixed points, which are based on the description of fixed points as G-equivariant maps out of G/H, and the "geometric" fixed

points, which commute with suspension and the smash product (on the homotopy category level). Again, the description of orthogonal G-spectra as $\mathscr{I}_G$-spaces in Proposition 2.2 provides the easiest way to construct the categorical and geometric fixed point functors [MM, §V].

For any normal $H \lhd G$, let $\mathscr{I}_G^H(U,V)$ denote the G/H-space of H-fixed points of $\mathscr{I}_G(U,V)$. Given any orthogonal spectrum X, the collection of fixed points $\{X(V)^H\}$ forms a $\mathscr{I}_G^H$-space. We can turn this collection into a $\mathscr{I}_{G/H}$-space in two ways. There is a functor $q\colon \mathscr{I}_{G/H} \to \mathscr{I}_G^H$ induced by the pullback of H-trivial G-representations to G/H-representations along the quotient map $G \to G/H$.

Definition 2.9 ([MM, §V.3])**.** For H a normal subgroup of G, the categorical fixed point functor

$$(-)^H\colon G\mathscr{I}\mathcal{S}^U \longrightarrow (G/H)\mathscr{I}\mathcal{S}^{U^H}$$

is computed as the pullback of the $\mathscr{I}_{G/H}$-space $\{X(V)^H\}$ along q.

On the other hand, there is an equivariant continuous functor $\phi\colon \mathscr{I}_G^H \to \mathscr{I}_{G/H}$ induced by taking a G-representation V to the G/H-representation V^H.

Definition 2.10 ([MM, §V.4])**.** For H a normal subgroup of G, the geometric fixed point functor

$$\Phi^H(-)\colon G\mathscr{I}\mathcal{S}^U \longrightarrow (G/H)\mathscr{I}\mathcal{S}^{U^H}$$

is constructed by taking the left Kan extension of the $\mathscr{I}_G^H$-space $\{X(V)^H\}$ along ϕ.

Both fixed-point functors are lax symmetric monoidal [MM, V.3.8, V.4.7] and so descend to categories of associative and commutative ring orthogonal G-spectra.

prop:fixsymmon

Proposition 2.11. *Let $H \subset G$ be a normal subgroup. Let X and Y be orthogonal G-spectra. There are natural maps*

$$\Phi^H X \wedge \Phi^H Y \longrightarrow \Phi^H(X \wedge Y) \qquad \text{and} \qquad X^H \wedge Y^H \longrightarrow (X \wedge Y)^H$$

that exhibit Φ^H and $(-)^H$ as lax symmetric monoidal functors.

Therefore, there are induced functors

$$\Phi^H, (-)^H\colon G\mathscr{I}\mathcal{S}[\mathbb{T}] \longrightarrow (G/H)\mathscr{I}\mathcal{S}[\mathbb{T}]$$

and

$$\Phi^H, (-)^H\colon G\mathscr{I}\mathcal{S}[\mathbb{P}] \longrightarrow (G/H)\mathscr{I}\mathcal{S}[\mathbb{P}].$$

For a commutative ring orthogonal G-spectrum A, a corollary of Proposition 2.11 is that the fixed-point functors interact well with the category of A-modules.

Corollary 2.12. *Let A be a commutative ring orthogonal spectrum. The fixed-point functors restrict to functors*

$$\Phi^H\colon G\mathscr{I}\mathcal{S}_A \longrightarrow (G/H)\mathscr{I}\mathcal{S}_{\Phi^H A}$$

and

$$(-)^H\colon G\mathscr{I}\mathcal{S}_A \longrightarrow (G/H)\mathscr{I}\mathcal{S}_{A^H}.$$

Remark 2.13. We can extend these constructions to subgroups $H \subset G$ that are not normal by considering the normalizer NH and quotient $WH = G/NH$. However, since we do not need this generality herein, we do not discuss it further.

2.2. **The point-set theory of the norm.** Central to our work is the realization by Hill, Hopkins, and Ravenel [HHR] that a tractable model for the "correct" equivariant homotopy type of a smash power can be formed as a point-set construction using the point-set change of universe functors. It is "correct" insofar as there is a diagonal map which induces an equivalence onto the geometric fixed points (see Section 2.3 below). They refer to this construction as the norm after the norm map of Greenlees-May [19], which in turn is named for the norm map of Evens in group cohomology [16, Chapter 6].

The point of departure for the construction of the norm is the use of the change-of-universe equivalences to regard orthogonal G-spectra on any universe U as G-objects in orthogonal spectra. (Good explicit discussions of the interrelationship can be found in [MM, §V.1] and [schwegui, 2.7].) We now give a point-set description of the norm following [schwegui] and [bohmann]; these descriptions are equivalent to the description of [HHR, §A] by the work of [bohmann].

For the construction of the norm, it is convenient to use BG to denote the category with one object, whose monoid of endomorphisms is the finite group G. The category $(\mathscr{I}\mathcal{S})^{BG}$ of functors from BG to the category $\mathscr{I}\mathcal{S}$ of (non-equivariant) orthogonal spectra indexed on the universe $\mathbb{R}^\infty$ is isomorphic to the category $G\mathscr{I}\mathcal{S}^{\mathbb{R}^\infty}$ of orthogonal G-spectra indexed on the universe $\mathbb{R}^\infty$. We can then use the change of universe functor $\mathcal{I}^U_{\mathbb{R}^\infty}$ to give an equivalence of $(\mathscr{I}\mathcal{S})^{BG}$ with the category $G\mathscr{I}\mathcal{S}^U$ of orthogonal G-spectra indexed on U.

def:indexedsmash **Definition 2.14.** Let G be a finite group and $H \subset G$ be a finite index subgroup with index n. Fix an ordered set of coset representatives $(g_1, \dots, g_n)$, and let $\alpha\colon G \to \Sigma_n \wr H$ be the homomorphism

$$\alpha(g) = (\sigma, h_1, \dots, h_n)$$

defined by the relation $gg_i = g_{\sigma(i)}h_i$. The indexed smash-power functor

$$\wedge^G_H\colon (\mathscr{I}\mathcal{S})^{\mathcal{B}H} \longrightarrow (\mathscr{I}\mathcal{S})^{\mathcal{B}G}$$

is defined as the composite

$$(\mathscr{I}\mathcal{S})^{\mathcal{B}H} \xrightarrow{\wedge^n} (\mathscr{I}\mathcal{S})^{B(\Sigma_n \wr H)} \xrightarrow{\alpha^*} (\mathscr{I}\mathcal{S})^{\mathcal{B}G}.$$

The norm functor

$$N^G_H\colon H\mathscr{I}\mathcal{S}^U \longrightarrow G\mathscr{I}\mathcal{S}^{U'}$$

is defined to be the composite

$$X \mapsto \mathcal{I}^{U'}_{\mathbb{R}^\infty} \wedge^G_H \mathcal{I}^{\mathbb{R}^\infty}_U X.$$

This definition depends on the choice of coset representatives; however, any other choice gives a canonically naturally isomorphic functor (the isomorphism induced by permuting factors and multiplying each factor by the appropriate element of H). As observed in [HHR, ?], in fact it is possible to give a description of the norm which is independent of any choices and is determined instead by the universal property of the left Kan extension. Alternatively, Schwede [schwegui, 9.3] gives another way of avoiding the choice above, using the set $\langle G : H\rangle$ of all choices of ordered sets of coset representatives; $\langle G : H\rangle$ is a free transitive $\Sigma_n \wr H$-set and the inclusion of $(g_1, \dots, g_n)$ in $\langle G : H\rangle$ induces an isomorphism

$$\wedge^G_H X \cong \langle G : H\rangle \wedge_{\Sigma_n \wr H} X^{(n)}$$

(where (n) indicates the nth smash power). In our work, G will be the cyclic group $C_{nr} < S^1$ and $H = C_r$ (usually for $r = 1$), and we have the obvious choice of coset representatives $g_k = e^{2\pi(k-1)/nr}$, letting us take advantage of the explicit formulas. In the case $r = 1$, we have the following.

prop:pointsetnorm

Proposition 2.15. *Let G be a finite group and U a complete G-universe. The norm functor*

$$N_e^G \colon \mathscr{I}\mathcal{S} \longrightarrow G\mathscr{I}\mathcal{S}^U$$

is given by the composite

$$X \mapsto \mathcal{I}^U_{\mathbb{R}^\infty} X^{\wedge G},$$

where X^G denotes the smash power indexed on the set G.

When dealing with commutative ring orthogonal G-spectra, the norm has a particularly attractive formal description [HHR, 22, ?], which is a consequence of the fact that the norm is a symmetric monoidal functor.

thm:normadj

Theorem 2.16. *Let G be a finite group and let H be a subgroup of G. The norm restricts to the left adjoint in the adjunction*

$$N_H^G \colon H\mathscr{I}\mathcal{S}[\mathbb{P}] \leftrightarrows G\mathscr{I}\mathcal{S}[\mathbb{P}] \colon \iota_H^*,$$

where ι_H^* denotes the change of group functor along $H < G$.

The relationship of the norm with the geometric fixed point functor is encoded in the diagonal map [HHR, 22, ?].

def:diag

Definition 2.17. Let G be a finite group, $H \subset G$ a subgroup, and $K \lhd G$ a normal subgroup. Let X be an orthogonal H-spectrum. There is a natural diagonal map of orthogonal G/K-spectra

$$\Delta \colon N^{G/K}_{HK/K} \Phi^{K\cap H} X \longrightarrow \Phi^K N_H^G X.$$

For any fixed commutative ring orthogonal spectrum A, the indexed smash-power construction of Definition 2.14 can be carried out in the symmetric monoidal category $\mathscr{I}\mathcal{S}_A$. Denote the A-relative indexed smash-power by $(\wedge_A)_e^G$. To make sense of this, just as in the absolute case, we have to use the change of universe functors: For X an A-module, we understand $(\wedge_A)_e^G X$ to be

$$(\wedge_A)_e^G X := \alpha^* X^{(n)},$$

where the nth smash power is over A and α^* is as in Definition 2.14. This is an A-module (in $G\mathscr{I}\mathcal{S}^{\mathbb{R}^\infty}$). We then have the following definition of the A-relative norm functor:

Definition 2.18. Let A be a commutative ring orthogonal spectrum. Write A_G for the commutative ring orthogonal G-spectrum $\mathcal{I}^U_{\mathbb{R}^\infty} A$ obtained by regarding A as an object of $(\mathscr{I}\mathcal{S})^{BG}$ and applying the change of universe functor. The A-relative norm functor

$${}_A N_e^G \colon \mathscr{I}\mathcal{S}_A \longrightarrow G\mathscr{I}\mathcal{S}^U_{A_G}$$

is defined to be the composite

$$X \mapsto \mathcal{I}^U_{\mathbb{R}^\infty}((\wedge_A)_e^G X).$$

There is an A-relative diagonal map constructed in exactly the same way as the absolute diagonal map. For the following proposition, note that $A \cong \Phi^G A_G$.

prop:reldiag **Proposition 2.19.** *Let G be a finite group. Let A be a commutative ring orthogonal spectrum and let X be an A-module. There is a natural diagonal map of A-modules*

$$\Delta_A\colon X \longrightarrow \Phi^G {}_A N_e^G X.$$

The diagonal map for the norm shares the following extended naturality property of other diagonal maps. Let $z \in G$ be an element in the center of G. Then multiplication by z is a natural automorphism on objects of $G\mathscr{I}\mathcal{S}^{\mathbb{R}^\infty}$ or on objects of $G\mathscr{I}\mathcal{S}_A^{\mathbb{R}^\infty}$, and so induces a natural automorphism $\mathcal{I}^U_{\mathbb{R}^\infty} z$ of $N_H^G X$ for any X in $H\mathscr{I}\mathcal{S}^{\iota_H^* U}$ or of ${}_A N_H^G X$ for any X in $H\mathscr{I}\mathcal{S}_A^{\iota_H^* U}$. We use the following observation in Sections 4 and 6.

sec:THH sec:relTHH

prop:diagdiag **Proposition 2.20.** *Let z be an element of the center of G and let $H < G$. Then for any orthogonal H-spectrum X indexed on $\iota_H^* U$, the following diagram commutes in $G\mathscr{I}\mathcal{S}^U$:*

$$\begin{array}{ccc} & & \Phi^G N_H^G X \\ & \overset{\Delta}{\nearrow} & \big\downarrow {\scriptstyle \mathcal{I}^U_{\mathbb{R}^\infty} z} \\ \Phi^H X & & \\ & \underset{\Delta}{\searrow} & \\ & & \Phi^G N_H^G X. \end{array}$$

For a commutative ring orthogonal spectrum A and an A-module X, the following diagram commutes in $G\mathscr{I}\mathcal{S}^U$:

$$\begin{array}{ccc} & & \Phi^G {}_A N_e^G X \\ & \overset{\Delta}{\nearrow} & \big\downarrow {\scriptstyle \mathcal{I}^U_{\mathbb{R}^\infty} z} \\ X & & \\ & \underset{\Delta}{\searrow} & \\ & & \Phi^G {}_A N_e^G X. \end{array}$$

Proof. The diagonal map $X \to \Phi^G N_H^G X$ is induced by the space-level diagonal map

$$(X(V))^H \longrightarrow (X(V)^{\wedge n})^G \longrightarrow (X^{\wedge n}(G \otimes_H V))^G$$

for which the analogous diagram is clear. □

sec:horev

2.3. **Homotopy theory of orthogonal spectra.** We now review the homotopy theory of orthogonal G-spectra with a focus on discussing the derived functors associated to the point-set constructions of the preceding section. We begin by reviewing the various model structures on orthogonal G-spectra. All of these model structures are ultimately derived from the standard model structure on $G\mathcal{T}$ (the category of based G-spaces), which we begin by reviewing.

Following the notational conventions of [MM, 30], we start with the sets of maps

$$I = \{(G/H \times S^{n-1})_+ \longrightarrow (G/H \times D^n)_+\}$$

and

$$J = \{(G/H \times D^n)_+ \longrightarrow (G/H \times (D^n \times I))_+\},$$

where $n \geq 0$ and H varies over the closed subgroups of G. Recall that there is a compactly generated model structure on the category $G\mathcal{T}$ in which I and J are the generating cofibrations and generating acyclic cofibrations (e.g., [MM, 30, III.1.8]). The weak equivalences and fibrations are the maps $X \to Y$ such that $X^H \to Y^H$ is a weak equivalence or fibration for each closed $H \subset G$. Transporting this structure levelwise in $\mathcal{V}^G(U)$, we get the level model structure in orthogonal G-spectra.

Proposition 2.21 ([MM, III.2.4])**.** *Fix a G-universe U. There is a compactly generated model structure on $G\mathscr{I}\mathcal{S}^U$ in which the weak equivalences and fibrations are the maps $X \to Y$ such that each map $X(V) \to Y(V)$ is a weak equivalence or fibration of G-spaces. The sets of generating cofibrations and acyclic cofibrations are given by $I^U_G = \{F_V i \mid i \in I\}$ and $J^U_G = \{F_V j \mid j \in J\}$, where V varies over $\mathcal{V}^G(U)$.*

The level model structure is primarily scaffolding to construct the stable model structures. In order to specify the weak equivalences in the stable model structures, we need to define equivariant homotopy groups.

Definition 2.22. Fix a G-universe U. The homotopy groups of an orthogonal G-spectrum X are defined for a subgroup $H \subset G$ and an integer q as

$$\pi^H_q(X) = \begin{cases} \operatorname*{colim}_{V \in \mathcal{V}^G(U)} \pi_q((\Omega^V X(V))^H) & q \geq 0 \\ \operatorname*{colim}_{\substack{\mathbb{R}^{-q} \subset V \\ V \in \mathcal{V}^G(U)}} \pi_0((\Omega^{V-\mathbb{R}^{-q}} X(V))^H) & q < 0, \end{cases}$$

(see [MM, §III.3.2]).

These are the homotopy groups of the underlying G-prespectrum associated to X (via the forgetful functor from orthogonal G-spectra to prespectra). We define the stable equivalences to be the maps $X \to Y$ that induce isomorphisms for all homotopy groups.

Proposition 2.23 ([MM, 4.2])**.** *Fix a G-universe U. The standard stable model structure on $G\mathscr{I}\mathcal{S}^U$ is the compactly generated symmetric monoidal model structure with the cofibrations given by the level cofibrations, the weak equivalences the stable equivalences, and the fibrations determined by the right lifting property. The generating cofibrations are given by I^U_G as above, and the generating acyclic cofibrations K are the union of J^U_G and certain additional maps described in* [MM, 4.3].

We will also use a variant of the standard stable model structure that can be more convenient when working with the derived functors of the norm. We refer to this as the complete stable model structure. See [HHR, ?] for a comprehensive discussion of this model structure, and [Ullman, §A] for a brief review. In order to describe this, denote by I^U_H and J^U_H the generating cofibrations for the stable model structure on orthogonal H-spectra indexed on the universe $\iota^*_H U$.

thm:normstable

Theorem 2.24 ([HHR, ?.?])**.** *Fix a G-universe U. There is a compactly generated symmetric monoidal model structure on $G\mathscr{I}\mathcal{S}$ with generating cofibrations and acyclic cofibrations the sets $\{G_+ \wedge_H i \mid i \in I^{\iota^*_H U}_H,\ H < G\}$ and $\{G_+ \wedge_H \mid j \in J^{\iota^*_H U}_H,\ H < G\}$ respectively. The weak equivalences are the stable equivalences, and the fibrations are determined by the right lifting property.*

Both the standard stable model structure and the complete model structure of Theorem 2.24 can be lifted to model structures on the category $G\mathscr{I}\mathcal{S}^U[\mathbb{T}]$ of associative monoids in orthogonal G-spectra.

Theorem 2.25 ([MM, 7.6.(iv)],[HHR, ?.?])**.** *Fix a G-universe U. There are compactly generated model structures on $G\mathscr{I}\mathcal{S}^U[\mathbb{T}]$ in which the weak equivalences are the stable equivalences of underlying orthogonal G-spectra indexed on U and the fibrations are respectively*

(1) *the maps which are stable fibrations of underlying orthogonal G-spectra indexed on U, or*
(2) *the maps which are complete stable fibrations of underlying orthogonal G-spectra indexed on U.*

In each case, the cofibrations are determined by the left-lifting property.

To obtain model structures on commutative ring orthogonal spectra, we also need the "positive" variants of the standard stable and complete stable model structures. We define the positive model structures in terms of generating cofibrations I^+ and J^+ that are defined analogously with I and J except that we restrict to representations V that contain a nonzero trivial representation.

Theorem 2.26 ([HHR, ?.?])**.** *Fix a G-universe U. There are compactly generated model structures on $G\mathscr{I}\mathcal{S}^U[\mathbb{P}]$ in which the weak equivalences are the stable equivalences of the underlying orthogonal G-spectra and fibrations are respectively*

(1) *the maps which are positive stable fibrations of underlying orthogonal G-spectra indexed on U, or*
(2) *the maps which are positive complete stable fibrations of underlying orthogonal G-spectra indexed on U.*

In each case, the cofibrations are determined by the left-lifting property.

For a fixed object A in $G\mathscr{I}\mathcal{S}^U[\mathbb{P}]$, there are also lifted model structures on the categories $G\mathscr{I}\mathcal{S}^U_A$ of A-modules, $G\mathscr{I}\mathcal{S}^U_A[\mathbb{T}]$ of A-algebras, and $G\mathscr{I}\mathcal{S}^U_A[\mathbb{P}]$ of commutative A-algebras in both the stable and complete stable model structures ([MM, III.7.6] and [HHR,]). (There are also lifted model structures on the category $G\mathscr{I}\mathcal{S}^U_A$ of A-modules when A is an object of $G\mathscr{I}\mathcal{S}^U_A[\mathbb{T}]$, but we will not need these.)

Theorem 2.27 ([HHR, ?.?])**.** *Fix a G-universe U. Let A be a commutative ring orthogonal G-spectrum indexed on U. There are compactly generated model structures on the categories $G\mathscr{I}\mathcal{S}^U_A$ and $G\mathscr{I}\mathcal{S}^U_A[\mathbb{T}]$ in which the fibrations and weak equivalences are created by the forgetful functors to the stable and complete stable model structures on $G\mathscr{I}\mathcal{S}^U$. There are compactly generated model structures on $G\mathscr{I}\mathcal{S}^U_A[\mathbb{P}]$ in which the fibrations and weak equivalences are created by the forgetful functors to the positive stable and positive complete stable model structures on $G\mathscr{I}\mathcal{S}^U_A$.*

Finally, when dealing with cyclotomic spectra, we need to use variants of these model structures where the stable equivalences are determined by a family of subgroups of G. Recall from [MM, IV.6.1] the definition of a family: a family $\mathcal{F}$ is a collection of closed subgroups of G that is closed under taking closed subgroups (and conjugation). We say a map $X \to Y$ is an $\mathcal{F}$-equivalence if it induces an isomorphism on homotopy groups π_*^H for all H in $\mathcal{F}$. All of the model structures described above have analogues with respect to the $\mathcal{F}$-equivalences (e.g., see [MM, IV.6.5]), which are built from sets I and J where the cells $F_V(G/H \times S^{n-1})_+ \to F_V(G/H \times D^n)_+$ and $F_V(G/H \times D^n)_+ \to F_V(G/H \times D^n \times I)_+$ are restricted to $H \in \mathcal{F}$. We record the situation in the following omnibus theorem.

thm:fmodel

Theorem 2.28. *There are stable and complete stable compactly generated model structures on the categories $G\mathscr{I}\mathcal{S}^U$, $G\mathscr{I}\mathcal{S}^U[\mathbb{T}]$, and $G\mathscr{I}\mathcal{S}^U[\mathbb{P}]$ where the weak equivalences are the $\mathcal{F}$-equivalences.*

Let A be a commutative ring orthogonal G-spectrum. There are stable and complete stable compactly generated model structures on the categories $G\mathscr{I}\mathcal{S}^U_A$, $G\mathscr{I}\mathcal{S}^U_A[\mathbb{T}]$, and $G\mathscr{I}\mathcal{S}^U_A[\mathbb{P}]$ where the weak equivalences are the $\mathcal{F}$-equivalences.

We are most interested in case of $G = S^1$ and $\mathcal{F}_{\mathrm{Fin}}$ the family of finite subgroups of S^1 and the family $\mathcal{F}_p$ of the p-subgroups $\{C_{p^n}\}$ of S^1 for a fixed prime p.

2.4. **Derived functors of fixed points and the norm.** We now discuss the use of the model structures described in the previous section to construct the derived functors of the categorical fixed points, the geometric fixed points, and the norm functors. We begin with the categorical fixed point functor. Since this is a right adjoint, we have right-derived functors computed using fibrant replacement (in any of our available stable model structures):

thm:catder **Theorem 2.29.** *Let $H \subset G$ be a normal subgroup. Then the categorical fixed-point functor $(-)^H \colon G\mathscr{I}\mathcal{S}^U \to (G/H)\mathscr{I}\mathcal{S}^{U^H}$ is a Quillen right adjoint; in particular, it preserves fibrations and weak equivalences between fibrant objects in the stable and complete stable model structures (and their positive variants) on $G\mathscr{I}\mathcal{S}^U$.*

As the fibrant objects in the model structures on associative and commutative ring orthogonal spectra are fibrant in the underlying model structures on orthogonal G-spectra, we can derive the categorical fixed points by fibrant replacement in any of the settings in which we work.

In contrast, the geometric fixed point functor admits a Quillen left derived functor (see [MM, V.4.5] and [HHR, ?]).

thm:geoder **Theorem 2.30.** *Let H be a normal subgroup of G. The functor $\Phi^H(-)$ preserves cofibrations and weak equivalences between cofibrant objects in the stable and complete stable model structures (and their positive variants) on $G\mathscr{I}\mathcal{S}^U$.*

Since the cofibrant objects in the lifted model structures on $G\mathscr{I}\mathcal{S}[\mathbb{T}]$ are cofibrant when regarded as objects in $G\mathscr{I}\mathcal{S}$, an immediate corollary of Theorem thm:geoder 2.30 is that we can derive Φ^H by cofibrant replacement when working with associative ring orthogonal G-spectra. In contrast, the underlying orthogonal G-spectra associated to cofibrant objects in $G\mathscr{I}\mathcal{S}[\mathbb{P}]$ in either of the model structures we study are essentially never cofibrant and a separate argument is needed in that case in our work below. The first part of the following theorem is [HHR, ?]; the case of A-modules is similar and discussed in Section sec:amodcyc 7.

thm:derivenorm **Theorem 2.31.** *The norm $N_H^G(-)$ preserves weak equivalences between cofibrant objects in any of the various stable model structures on $H\mathscr{I}\mathcal{S}$ and $H\mathscr{I}\mathcal{S}[\mathbb{P}]$. Let A be a commutative ring orthogonal spectrum. Then the A-relative norm ${}_AN_e^G(-)$ preserves weak equivalences between cofibrant objects in $\mathscr{I}\mathcal{S}_A$ and $\mathscr{I}\mathcal{S}_A[\mathbb{P}]$.*

The utility of the complete model structure is the following homotopical version of Theorem thm:mapHHRadj 2.16 [HHR, 22, ?].

Theorem 2.32. *Let H be a subgroup of G. The adjunction*

$$N_H^G \colon H\mathscr{I}\mathcal{S}[\mathbb{P}] \leftrightarrows G\mathscr{I}\mathcal{S}[\mathbb{P}] \colon \iota_H^*$$

is a Quillen adjunction for the positive complete stable structures.

Finally, we have the following result about the derived version of the diagonal map [HHR, 22, ?]. We note the strength of the conclusion: the diagonal map is an isomorphism on cofibrant objects, not just a weak equivalence. The case of A-modules is similar and discussed in Section sec:amodcyc 7.

thm:normiso **Theorem 2.33.** *Let H be a normal subgroup of G. The diagonal map*

$$\Delta\colon \Phi^H X \longrightarrow \Phi^G N_H^G X$$

is an isomorphism of orthogonal spectra (and in particular a weak equivalence) when X is cofibrant in any of the stable model structures on $G\mathscr{I}\mathcal{S}$, when X is a cofibrant object in $G\mathscr{I}\mathcal{S}[\mathbb{T}]$, or when X is a cofibrant object in $G\mathscr{I}\mathcal{S}[\mathbb{P}]$.

Let A be a commutative ring orthogonal spectrum. The A-relative diagonal map of orthogonal spectra

$$\Delta_A\colon X \longrightarrow \Phi^{G}{}_A N_e^G X,$$

is an isomorphism (and in particular a weak equivalence) of orthogonal spectra when X is cofibrant in $\mathscr{I}\mathcal{S}_A$ or in $\mathscr{I}\mathcal{S}_A[\mathbb{P}]$.

3. Cyclotomic spectra and TC

sec:cyclosec

In this section, we review the details of the category of p-cyclotomic spectra and the construction of topological cyclic homology (TC). The diagonal maps that naturally arise in the context of the norm go in the opposite direction to the usual structure maps, and so we also explain how to construct TC from these "op"-cyclotomic spectra. In the following, fix a prime p and a complete S^1 universe U.

sec:cyclo

3.1. **Background on p-cyclotomic spectra.** In this section, we briefly review the point-set description of p-cyclotomic spectra from [BM7, §4]; we refer the reader to that paper for more detail discussion.

def:pcyclotomic **Definition 3.1** ([BM7, 4.5])**.** A *p-cyclotomic spectrum* X consists of an orthogonal S^1-spectrum X together with a map of orthogonal S^1-spectra

$$t_p\colon \rho_p^*\Phi^{C_p} X \longrightarrow X,$$

such that the induced map on the derived functor $\rho_p^* L\Phi^{C_p} X \to X$ is an $\mathcal{F}_p$-equivalence. Here ρ_p denotes the p-th root isomorphism $S^1 \to S^1/C_p$. A morphism of p-cyclotomic spectra consists of a map of orthogonal S^1-spectra $X \to Y$ such that the diagram

$$\begin{array}{ccc} \rho_p^*\Phi^{C_p} X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \rho_p^*\Phi^{C_p} Y & \longrightarrow & Y \end{array}$$

commutes.

Remark 3.2. A *cyclotomic spectrum* is an orthogonal spectrum with p-cyclotomic structures for all primes p satisfying certain compatibility relations; see [BM7, 4.7–8] for details.

Following [BM7, 5.4–5], we have the following weak equivalences for p-cyclotomic spectra.

Definition 3.3. A map of p-cyclotomic spectra is a *weak equivalence* when it is a weak equivalence of the underlying (non-equivariant) orthogonal spectra.

prop:wkeqivcyc **Proposition 3.4** ([BM7, 5.5])**.** *A map of p-cyclotomic spectra is a weak equivalence if and only if it is an $\mathcal{F}_p$-equivalence of the underlying orthogonal S^1-spectra.*

3.2. **Constructing TR and TC from a cyclotomic spectrum.** In this section, we give a very rapid review of the definition of TR and TC in terms of the point-set category of cyclotomic spectra described above. The interested reader is referred to the excellent treatment in Madsen's CDM notes [29] for more details on the construction in terms of the classical (homotopical) definition of a cyclotomic spectrum.

For a p-cyclotomic spectrum X, the collection $\{X^{C_{p^n}}\}$ of (point-set) categorical fixed points is equipped with functors

$$F, R\colon X^{C_{p^n}} \longrightarrow X^{C_{p^{n-1}}}$$

for all n defined as follows. The Frobenius maps F are simply the obvious inclusion of fixed-point maps, and the restriction maps R are constructed as the composite

$$X^{C_{p^n}} \cong (\rho_p^* X^{C_p})^{C_{p^{n-1}}} \longrightarrow (\rho_p^* \Phi^{C_p} X)^{C_{p^{n-1}}} \xrightarrow{(t_p)^{C_{p-1}}} X^{C_{p^{n-1}}}.$$

The Frobenius and restriction maps satisfy the identity $F \circ R = R \circ F$. When X is fibrant in the $\mathcal{F}_p$-model structure (of Theorem 2.28), we then define

$$TR(X) = \operatorname{holim}_R X^{C_{p^n}} \qquad \text{and} \qquad TC(X) = \operatorname{holim}_{R,F} X^{C_{p^n}}.$$

In general we define TR and TC using a fibrant replacement that preserves the p-cyclotomic structure; such a functor is provided by the main theorems of [7, §5], which construct a model structure on p-cyclotomic spectra where the fibrations are the fibrations of the underlying orthogonal S^1-spectra in the $\mathcal{F}_p$-model structure. Alternatively, an explicit construction of a fibrant replacement functor on orthogonal spectra that preserves cyclotomic structures is given in [5, 4.6–7].

Proposition 3.5 (cf. [7, 1.4])**.** *A weak equivalence $X \to Y$ of p-cyclotomic spectra induces weak equivalences $TR(X_f) \to TR(Y_f)$ and $TC(X_f) \to TC(Y_f)$ of orthogonal spectra, where $(-)_f$ denotes any fibrant replacement functor in p-cyclotomic spectra.*

Remark 3.6. We do not yet have an abstract homotopy theory for multiplicative objects in cyclotomic spectra, and the explicit fibrant replacement functor $Q^{\mathcal{I}}$ of [5, 4.6] is lax monoidal but not lax symmetric monoidal. As a consequence, at present we do not know how to convert a p-cyclotomic spectrum which is also a commutative ring orthogonal S^1-spectrum into a cyclotomic spectrum that is a fibrant commutative ring orthogonal S^1-spectrum.

3.3. **Op-cyclotomic spectra.** For our construction of THH based on the norm (in the next section), the diagonal map $X \to \Phi^G N_e^G X$ is in the opposite direction of the cyclotomic structure map needed in the definition of a cyclotomic structure. In the case when X is cofibrant (or a cofibrant ring or cofibrant commutative ring orthogonal spectrum), the diagonal map is an isomorphism and so presents no difficulty; in the case when X is just of the homotopy type of a cofibrant orthogonal spectrum, the fact that the structure map goes the wrong way necessitates some technical maneuvering in order to construct TR and TC.

Definition 3.7. An op-p-cyclotomic spectrum X consists of an orthogonal S^1-spectrum X together with a map of orthogonal S^1-spectra

$$\gamma\colon X \longrightarrow \rho_p^* \Phi^{C_p} X$$

that is a $\mathcal{F}_p$-equivalence. A map of op-p-cyclotomic spectra is a map of orthogonal S^1-spectra that commutes with the structure map. A map of op-p-cyclotomic spectra is a weak equivalence when it is an $\mathcal{F}_p$-equivalence of the underlying orthogonal S^1-spectra.

Note that the definition above uses a condition on the point-set geometric fixed point functor rather than the derived geometric fixed point functor. In practice, we should restrict to those op-p-cyclotomic spectra X where the canonical map in the S^1-equivariant stable category $\rho_p^* L\Phi^{C_p} X \to \rho_p^* \Phi^{C_p} X$ is an $\mathcal{F}_p$-equivalence; the full subcategory of such op-p-cyclotomic spectra is closed under weak equivalence. In this subcategory, a map is a weak equivalence if and only if it is a weak equivalence of the underlying (non-equivariant) orthogonal spectra.

Rather than study the category of op-p-cyclotomic spectra in detail, we simply explain an approach to constructing TR and TC from this data. In what follows, let $(-)_f$ denote a fibrant replacement functor in the $\mathcal{F}_p$-model structure on orthogonal S^1-spectra; to be clear, we assume the given natural transformation $X \to X_f$ is always an acyclic cofibration. Then for a op-p-cyclotomic spectrum X, we get a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\gamma} & \rho_p^*\Phi^{C_p}X & & \\ \simeq\downarrow & \overset{\simeq}{\searrow} & \simeq\downarrow & & \\ X_f & \xrightarrow[\gamma_f]{} & (\rho_p^*\Phi^{C_p}X)_f & \xrightarrow[\simeq]{} & (\rho_p^*\Phi^{C_p}(X_f))_f \end{array}$$

where the right vertical map is an acyclic cofibration because ρ_p^* and Φ^{C_p} preserve acyclic cofibrations, and the bottom right horizontal map is a weak equivalence for the same reason. In place of the restriction map R, we have a zigzag

$$R\colon (X_f)^{C_{p^n}} \longrightarrow ((\rho_p^*\Phi^{C_p}(X_f))_f)^{C_{p^{n-1}}} \xleftarrow{\simeq} (X_f)^{C_{p^{n-1}}}$$

constructed as the following composite

$$\begin{array}{ccccc} (X_f)^{C_{p^n}} & \xrightarrow{\cong} & (\rho_p^*(X_f)^{C_p})^{C_{p^{n-1}}} & \xrightarrow{\simeq} & ((\rho_p^*(X_f)^{C_p})_f)^{C_{p^{n-1}}} \\ & & & \swarrow & \\ ((\rho_p^*\Phi^{C_p}(X_f))_f)^{C_{p^{n-1}}} & \xleftarrow[\simeq]{} & ((\rho_p^*\Phi^{C_p}X)_f)^{C_{p^{n-1}}} & \xleftarrow[\simeq]{} & (X_f)^{C_{p^{n-1}}}. \end{array}$$

We can use this as an analogue of TR.

defn:zTR **Definition 3.8.** Define ${}^{op}TR$ as the homotopy limit of the diagram

$$\cdots \longleftarrow (X_f)^{C_{p^n}} \longrightarrow ((\rho_p^*\Phi^{C_p}(X_f))_f)^{C_{p^{n-1}}} \longleftarrow (X_f)^{C_{p^{n-1}}} \longrightarrow \cdots$$
$$\cdots \longleftarrow (X_f)^{C_p} \longrightarrow (\rho_p^*\Phi^{C_p}(X_f))_f \longleftarrow X_f$$

The zigzags R are compatible with the inclusion maps

$$F\colon (X_f)^{C_{p^n}} \longrightarrow (X_f)^{C_{p^{n-1}}}$$

in the sense that the following diagram commutes

$$\begin{array}{ccccc} (X_f)^{C_{p^{n+1}}} & \longrightarrow & ((\rho_p^*\Phi^{C_p}(X_f))_f)^{C_{p^n}} & \longleftarrow & (X_f)^{C_{p^n}} \\ & \searrow^{F} & \downarrow{\scriptstyle F} & & \searrow^{F} \\ & & (X_f)^{C_{p^n}} & \longrightarrow & ((\rho_p^*\Phi^{C_p}(X_f))_f)^{C_{p^{n-1}}} \longleftarrow (X_f)^{C_{p^{n-1}}}. \end{array}$$

We can therefore form an analogue of TC.

defn:zTC **Definition 3.9.** Define ${}^{op}TC$ by taking the homotopy limit over the diagram

$$\cdots \longleftarrow ((X)_f)^{C_{p^n}} \longleftarrow \cdots \longrightarrow ((X)_f)^{C_{p^{n-1}}} \longrightarrow \cdots$$

where the top parts are the R zigzags and the bottom the F maps.

This has the expected homotopy invariance property.

Proposition 3.10. *Let $X \to Y$ be a weak equivalence of op-p-cyclotomic spectra. The induced maps ${}^{op}TR(X) \to {}^{op}TR(Y)$ and ${}^{op}TC(X) \to {}^{op}TC(Y)$ are weak equivalences.*

Although we have nothing to say in general about the relationship between p-cyclotomic spectra and op-p-cyclotomic spectra or between ${}^{op}TC$ and TC, in the case when X has compatible p-cyclotomic and op-p-cyclotomic structures, we have the following comparison result. This in particular applies when X has the homotopy type of a cofibrant orthogonal spectrum, as we explain in Section 6. sec:cyclostruc

prop:zconsist **Proposition 3.11.** *Let X be an op-p-cyclotomic spectrum and a p-cyclotomic spectrum and assume that the composite of the two structure maps*

$$\rho_p^* \Phi^{C_p} X \longrightarrow X \longrightarrow \rho_p^* \Phi^{C_p} X$$

is homotopic to the identity. Then there is a zig-zag of weak equivalences connecting $TR(X)$ and ${}^{op}TR(X)$ and a zig-zag of weak equivalences connecting $TC(X)$ and ${}^{op}TC(X)$.

Proof. In the case of the comparison of $TR(X)$ and ${}^{op}TR(X)$, it suffices to show that the homotopy limits of diagrams of fibrant objects of the form

eq:comptr1 (3.12) $$\ldots \longleftarrow Y_n \xrightarrow{f_n} Y'_n \xleftarrow{g_n^{-1}} Y_{n-1} \longrightarrow \ldots$$

and

eq:comptr2 (3.13) $$\ldots \longrightarrow Y_n \xrightarrow{f_n} Y'_n \xrightarrow{g_n} Y_{n-1} \longrightarrow \ldots$$

are equivalent, where $g_n^{-1} \circ g_n$ is homotopic to the identity. This kind of rectification argument is standard, although we are not sure of a place in the literature where the precise fact we need is spelled out. We argue as follows. Choosing a homotopy H from the identity to $g_n^{-1} \circ g_n$, we get a strictly commuting diagram of the form

$$\begin{array}{ccccccccc}
Y_n & \xrightarrow{f_n} & Y'_n & \xrightarrow{\mathrm{id}} & Y'_n & \xleftarrow{\mathrm{id}} & Y'_n & \xleftarrow{g_n^{-1}} & Y_{n-1} \\
\uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle H} & & \uparrow{\scriptstyle g_n^{-1}\circ g_n} & & \uparrow{\scriptstyle \mathrm{id}} \\
Y_n & \xrightarrow{f_n} & Y'_n & \xrightarrow{\mathrm{id}\times\{0\}} & Y'_n \times I & \xleftarrow{\mathrm{id}\times\{1\}} & Y'_n & \xrightarrow{g_n} & Y_{n-1} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
Y_n & \xrightarrow{f_n} & Y'_n & \xrightarrow{\mathrm{id}} & Y'_n & \xrightarrow{\mathrm{id}} & Y_{n'} & \xrightarrow{g_n} & Y_{n-1}.
\end{array}$$

Note that all the vertical maps are weak equivalences, and therefore the induced maps between the homotopy limits of the rows are both weak equivalences. The homotopy limit of the top row is weakly equivalent to the homotopy limit of (3.12) eq:comptr1

and the homotopy limit of the bottom row is weakly equivalent to the homotopy limit of (3.13). This completes the comparison of $TR(X)$ and ${}^{op}TR(X)$; the argument for comparing $TC(X)$ and ${}^{op}TC(X)$ is analogous using "ladders" in place of rows. □

4. The construction and homotopy theory of the S^1-norm

In this section, we construct the norm from the trivial group to S^1 and study its basic point-set and homotopy properties. In the next section, we prove that under mild hypotheses it gives a model for THH with the correct equivariant homotopy type. Unlike norms for finite groups, the S^1-norm does not apply to arbitrary orthogonal spectra; instead we need an associative ring structure. In the case when R is commutative, we identify the S^1-norm as the left adjoint of the forgetful functor from orthogonal S^1-spectra indexed on a complete universe to (non-equivariant) orthogonal spectra.

Throughout this section, we fix a complete S^1-universe U. As in the definition of the norm for finite groups, the (point-set) equivalence of categories $\mathcal{I}^U_{\mathbb{R}^\infty}$ discussed in Section 2.1 will play a key technical role.

For an orthogonal ring spectrum R, let $N^{\mathrm{cyc}}_\wedge R$ denote the cyclic bar construction with respect to the smash product; i.e., the cyclic object in orthogonal spectra with k-simplices

$$[k] \longrightarrow \underbrace{R \wedge R \wedge \ldots \wedge R}_{k+1}$$

and the usual cyclic structure maps induced from the ring structure on R.

Lemma 4.1. *Let R be an object in $\mathscr{I}\mathcal{S}[\mathbb{T}]$. Then the geometric realization of the cyclic bar construction $|N^{\mathrm{cyc}}_\wedge R|$ is naturally an object in $S^1\mathscr{I}\mathcal{S}^{\mathbb{R}^\infty}$.*

Proof. It is well known that the geometric realization of a cyclic space has a natural S^1-action [23, 3.1]. Since geometric realization of an orthogonal spectrum is computed levelwise, it follows by continuous naturality that the geometric realization of a cyclic object in orthogonal spectra has an S^1-action. As noted in Section 2.1, the category $S^1\mathscr{I}\mathcal{S}^{\mathbb{R}^\infty}$ of orthogonal S^1-spectra indexed on $\mathbb{R}^\infty$ is isomorphic to the category of orthogonal spectra with S^1-actions. □

Using the point-set change of universe functors we can regard this as indexed on the complete universe U. The following definition repeats Definition 1.1 from the introduction.

Definition 4.2. Let R be a ring orthogonal spectrum. Define the functor

$$N^{S^1}_e : \mathscr{I}\mathcal{S}[\mathbb{T}] \longrightarrow S^1\mathscr{I}\mathcal{S}^U$$

to be the composite functor

$$N^{S^1}_e R = \mathcal{I}^U_{\mathbb{R}^\infty} |N^{\mathrm{cyc}}_\wedge R|.$$

When R is a commutative ring orthogonal spectrum, the usual tensor homeomorphism [15, IX.3.3]

$$|N^{\mathrm{cyc}}_\wedge R| \cong R \otimes S^1$$

yields the following characterization:

prop:commuT **Proposition 4.3.** *The restriction of $N_e^{S^1}$ to $\mathscr{I}\mathcal{S}[\mathbb{P}]$ lifts to a functor*

$$N_e^{S^1}\colon \mathscr{I}\mathcal{S}[\mathbb{P}] \longrightarrow S^1\mathscr{I}\mathcal{S}^U[\mathbb{P}]$$

that is left adjoint to the forgetful functor

$$\iota^*\colon S^1\mathscr{I}\mathcal{S}^U[\mathbb{P}] \longrightarrow \mathscr{I}\mathcal{S}[\mathbb{P}].$$

Proof. To obtain the refinement of $N_e^{S^1}$ to a functor $\mathscr{I}\mathcal{S}[\mathbb{P}] \to S^1\mathscr{I}\mathcal{S}^U[\mathbb{P}]$, it suffices to construct a refinement of $|N_\wedge^{\mathrm{cyc}}|$ to a functor

$$|N_\wedge^{\mathrm{cyc}}|\colon \mathscr{I}\mathcal{S}[\mathbb{P}] \longrightarrow S^1\mathscr{I}\mathcal{S}^{\mathbb{R}^\infty}[\mathbb{P}].$$

We obtain this immediately from the strong symmetric monoidal isomorphism

$$|X_\bullet| \wedge |Y_\bullet| \cong |X_\bullet \wedge Y_\bullet|$$

for simplicial objects $X_\bullet$,$Y_\bullet$ in orthogonal spectra and the easy observation that the map is S^1-equivariant for cyclic objects. Indeed, using the isomorphism

$$\mathbb{P}|X_\bullet| \cong |\mathbb{P}X_\bullet|,$$

we can identify $|N_\wedge^{\mathrm{cyc}}\mathbb{P}X|$ as $\mathbb{P}(X \wedge S^1_+)$. Now using the canonical reflexive coequalizer

$$\mathbb{P}\mathbb{P}R \rightrightarrows \mathbb{P}R \longrightarrow R$$

we can identify $|N_\wedge^{\mathrm{cyc}}R|$ as the reflexive coequalizer

$$\mathbb{P}\mathbb{P}(R \wedge S^1_+) \rightrightarrows \mathbb{P}(R \wedge S^1_+) \longrightarrow R \otimes S^1,$$

constructing the tensor of R with the unbased space S^1 in the category of commutative ring orthogonal spectra. A formal argument now identifies this as the left adjoint to the forgetful functor

$$\iota^*\colon S^1\mathscr{I}\mathcal{S}^{\mathbb{R}^\infty}[\mathbb{P}] \longrightarrow \mathscr{I}\mathcal{S}[\mathbb{P}]$$

and it follows that $N_e^{S^1}$ is the left adjoint to the forgetful functor indicated in the statement. □

We now turn to the question of understanding the derived functors of $N_e^{S^1}$. Recall that when dealing with cyclic sets, the S^1-fixed points do not usually carry homotopically meaningful information. As a consequence, we will work with the model structure on $S^1\mathscr{I}\mathcal{S}^U$ provided by Theorem 2.28 with weak equivalences the $\mathcal{F}_{\mathrm{Fin}}$-equivalences, i.e., the maps which are isomorphisms on the homotopy groups of the (categorical or geometric) fixed point spectra for the finite subgroups of S^1 (irrespective of what happens on the fixed points for S^1). We will now write $S^1\mathscr{I}\mathcal{S}^U_{\mathcal{F}_{\mathrm{Fin}}}$ for $S^1\mathscr{I}\mathcal{S}^U$ to emphasize that we are using the $\mathcal{F}_{\mathrm{Fin}}$-equivalences. We use analogous notation for the categories of ring orthogonal S^1-spectra and commutative ring orthogonal S^1-spectra.

thm:fmodel

We now show that $N_e^{S^1}$ admits (left) derived functors when regarded as landing in $S^1\mathscr{I}\mathcal{S}^U_{\mathcal{F}_{\mathrm{Fin}}}$ and (in the commutative case) $S^1\mathscr{I}\mathcal{S}^U_{\mathcal{F}_{\mathrm{Fin}}}[\mathbb{P}]$. Our analysis requires the following observation about the point-set description of the C_n-action on $N_e^{S^1}(R)$, which follows from inspection of the definitions.

lem:pointset **Lemma 4.4.** *Let R be a ring orthogonal spectrum or commutative ring orthogonal spectrum. Fix a cyclic subgroup $C_n \subset S^1$ and let $\tilde{U} = \iota_{C_n}^* U$, a complete C_n-universe. There is an isomorphism of orthogonal C_n-spectra indexed on $\tilde{U}$*

$$\iota_{C_n}^* N_e^{S^1} R \cong \mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}}(\iota_{C_n}^* |N_\wedge^{\mathrm{cyc}} R|),$$

where $\iota_{C_n}^*$ is as in Definition 2.8. def:changegroup

Using the results reviewed in Section 2.3 about sec:horev deriving the norm, we now analyze the $\mathcal{F}_{\mathrm{Fin}}$-homotopy type of $N_e^{S^1} R$. In the discussion that follows, for computing the left derived functor of geometric fixed points, we can use cofibrancy conditions in the positive complete stable model structure (a fortiori, in the complete, the stable, and the positive stable model structures).

thm:Tder **Theorem 4.5.** *Let $R \to R'$ be a weak equivalence of cofibrant orthogonal ring spectra. Then the induced map $N_e^{S^1} R \to N_e^{S^1} R'$ is an $\mathcal{F}_{\mathrm{Fin}}$-equivalence.*

Proof. Fix a cyclic subgroup $C_n \subset S^1$ and consider the derived geometric fixed points, which we can compute as $L\Phi^{C_n} N_e^{S^1} R \simeq L\Phi^{C_n} \iota_{C_n}^* N_e^{S^1} R$. By Lemma 4.4 lem:pointset we can understand the C_n-action on $N_e^{S^1} R$ in terms of the C_n-action on $|N_\wedge^{\mathrm{cyc}} R|$. Since $|N_\wedge^{\mathrm{cyc}} R|$ is the geometric realization of a cyclic spectrum, the C_n action can be computed in terms of the edgewise subdivision of the cyclic spectrum $N_\wedge^{\mathrm{cyc}} R$ [BHM 11, §1]. Specifically, the nth edgewise subdivision $\mathrm{sd}_n N_\wedge^{\mathrm{cyc}} R$ is a simplicial orthogonal spectrum with a simplicial C_n-action such that there is a natural isomorphism of orthogonal S^1-spectra

$$|\,\mathrm{sd}_n N_\wedge^{\mathrm{cyc}} R| \cong |N_\wedge^{\mathrm{cyc}} R|,$$

where the S^1-action on the left extends the C_n-action induced from the simplicial structure. Thus, by Lemma 4.4, lem:pointset we have $L\Phi^{C_n} N_e^{S^1} R \simeq L\Phi^{C_n} \mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} |\,\mathrm{sd}_n N_\wedge^{\mathrm{cyc}} R|$.

Under the hypotheses,

$$\mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} |\,\mathrm{sd}_n N_\wedge^{\mathrm{cyc}} R| \cong |\mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} (\mathrm{sd}_n N_\wedge^{\mathrm{cyc}} R)|$$

is cofibrant in $C_n\mathscr{I}\mathcal{S}^{\tilde{U}}$ and so we can compute $L\Phi^{C_n}$ using the point-set geometric fixed point functor Φ^{C_n}, which commutes with geometric realization. By Proposition 2.15, prop:pointsetnorm

$$N_e^{C_n} X \cong \mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} X^{\wedge n},$$

and so we can identify the kth level of the simplicial object in terms of the C_n-norm,

$$\mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} (\mathrm{sd}_n N_\wedge^{\mathrm{cyc}} R)_k \cong (N_e^{C_n} R)^{\wedge(k+1)}.$$

(In the next section, we will observe that the diagonal maps fit together into a simplicial map.) Since $R \to R'$ is a weak equivalence of cofibrant objects, this isomorphism and Theorem 2.31 thm:derivenorm imply that

$$\Phi^{C_n} \mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} (\mathrm{sd}_n N_\wedge^{\mathrm{cyc}} R)_k \longrightarrow \Phi^{C_n} \mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} (\mathrm{sd}_n N_\wedge^{\mathrm{cyc}} R')_k$$

is a weak equivalence of orthogonal spectra for each k, i.e., a simplicial-level weak equivalence of simplicial orthogonal spectra. To conclude a weak equivalence on geometric realization, we just need to see that at each V in $\mathcal{V}(\mathbb{R}^\infty)$, the simplicial spaces are "proper", i.e., each degeneracy map is an h-cofibration of spaces. This again follows from the hypothesis that R and R' are cofibrant, as the maps $R^{\wedge(k)} \to R^{\wedge(k+1)}$ induced by the inclusion of the unit are cofibrations and in particular are spacewise h-cofibrations and give spacewise h-cofibrations on $\Phi^{C_n} \mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}}$. □

In the commutative case, we have the following derived functor result.

prop:Tcommquillen

Proposition 4.6. *Regarded as a functor on commutative ring orthogonal spectra, the functor $N_e^{S^1}$ is a left Quillen functor with respect to the positive complete model structure on $\mathscr{I}\mathcal{S}[\mathbb{P}]$ and the $\mathcal{F}_{\mathrm{Fin}}$-model structure on $S^1\mathscr{I}\mathcal{S}^U[\mathbb{P}]$.*

Proof. The forgetful functor preserves fibrations and acyclic fibrations. □

The following proposition compares the derived functor on commutative ring orthogonal spectra with the derived functor on ring orthogonal spectra.

prop:Tdercompder

Proposition 4.7. *Let $R \to R'$ be a weak equivalence of ring orthogonal spectra where R is cofibrant and R' is a cofibrant commutative ring orthogonal spectrum. Then the induced map $N_e^{S^1}R \to N_e^{S^1}R'$ is an $\mathcal{F}_{\mathrm{Fin}}$-equivalence.*

Proof. By [HHR, ?.?], the point-set geometric fixed point functor Φ^{C_n} computes the derived geometric fixed point functor when applied to the norm of a cofibrant commutative ring orthogonal spectrum. The argument of Theorem thm:Tder 4.5 now generalizes to the case in the statement. □

5. The cyclotomic structure on $N_e^{S^1}R$

sec:cyclostruc

In this section, we show that the S^1-norm $N_e^{S^1}R$ has the correct equivariant homotopy type, in the sense that $N_e^{S^1}R$ is a cyclotomic spectrum in orthogonal S^1-spectra.

The proof of Theorem thm:Tder 4.5 yields a description of the orthogonal C_n-spectrum $\iota_{C_n}^* N_e^{S^1}(R)$ as the geometric realization of a simplicial orthogonal C_n-spectrum having k-simplices given by norms

$$(N_e^{C_n}R)^{\wedge(k+1)} \cong \mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}}(R^{\wedge n(k+1)}),$$

where C_n acts by block permutation on $R^{\wedge n(k+1)}$ and $\tilde{U} = \iota_H^* U$ (for U a complete S^1-universe). The faces are also given blockwise, with d_i for $0 \le i \le k-1$ the induced map on norms of the multiplication of the $(i+1)$st and $(i+2)$nd factors of R,

$$N_e^{C_n}(R^{\wedge(k+1)}) \longrightarrow N_e^{C_n}(R^{\wedge k}).$$

The face map d_k is a bit more complicated and uses both an internal cyclic permutation inside the last $N_e^{C_n}R$ factor (as in Proposition prop:diagdiag 2.20) and a permutation of the $k+1$-factors of $(N_e^{C_n}R)^{\wedge(k+1)}$ together with the multiplication d_0. Writing $g = e^{2\pi i/n}$ for the canonical generator of $C_n < S^1$ and α for the natural cyclic permutation on $X^{\wedge(k+1)}$, then d_k is the composite

$$(N_e^{C_n}R)^{\wedge(k+1)} \xrightarrow{\mathrm{id}^{\wedge k} \wedge \mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} g} (N_e^{C_n}R)^{\wedge(k+1)} \xrightarrow{\alpha} (N_e^{C_n}R)^{\wedge(k+1)} \xrightarrow{d_0} (N_e^{C_n}R)^{\wedge k}.$$

In fact, we have the following concise description of the C_n-action on $N_e^{S^1}$-bimodule terms. We obtain a $(N_e^{C_n}R, N_e^{C_n}R)$-bimodule ${}^g N_e^{C_n}R$ by using the standard right action, but twisting the left action using $\mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} g$ (for $g = e^{2\pi i/n}$, the canonical generator of $C_n < S^1$). In the following statement, we use the cyclic bar construction with coefficients in a bimodule, q.v. [BHM, §2].

Theorem 5.1. *Let R be a ring orthogonal spectrum. For any $C_n \subset S^1$, there is an isomorphism of orthogonal C_n-spectra*

$$\iota_{C_n}^* N_e^{S^1}(R) \cong N_\wedge^{\mathrm{cyc}}(N_e^{C_n}R, {}^g N_e^{C_n}R),$$

where the cyclic bar construction is taken in the symmetric monoidal category $C_n\mathscr{I}\mathcal{S}^{\tilde{U}}$.

Next we assemble the diagonal maps into a map $N_e^{S^1}R \to \rho_n^*\Phi^{C_n}N_e^{S^1}R$ of orthogonal S^1-spectra The following lemma (which is just a specialization of Proposition 2.20) provides the basic compatibility we need.

lem:multcompat

Lemma 5.2. *Let R be a cofibrant orthogonal spectrum, let $H < S^1$ be a finite subgroup, and let $h \in H$. Then the diagram*

$$\begin{array}{ccc} & & \Phi^H N_e^H R \\ & \overset{\Delta}{\nearrow} & \big\downarrow {\scriptstyle \mathcal{I}^{\tilde{U}}_{\mathbb{R}^\infty} h} \\ R & & \\ & \underset{\Delta}{\searrow} & \\ & & \Phi^H N_e^H R \end{array}$$

commutes.

We now prove the main theorem about the diagonal map cyclotomic structure; in light of our assembled work, the proof is straightforward.

thm:cyclodiag

Theorem 5.3. *Let R be a ring orthogonal spectrum. The diagonal maps*

$$\Delta_n \colon R^{\wedge(k+1)} \longrightarrow \Phi^{C_n} N_e^{C_n} R^{\wedge(k+1)}$$

assemble into natural maps of S^1-equivariant spectra

$$\tau_n \colon N_e^{S^1} R \longrightarrow \rho_n^* \Phi^{C_n} \mathcal{I}^{\tilde{U}}_{\mathbb{R}^\infty} |\operatorname{sd}_n N_\wedge^{\mathrm{cyc}} R|.$$

If R is cofibrant or cofibrant as a commutative ring orthogonal spectrum, then these maps are isomorphisms.

Proof. Lemma 5.2 implies that the diagonal maps Δ_n assemble into natural maps of cyclic spectra, and the properties of the edgewise subdivision [BHM, 1.11] imply that on realization we have maps of orthogonal S^1-spectra. Under cofibrancy hypotheses, Theorem 2.33 now implies that the realization of the diagonal maps is an isomorphism. □

When R is cofibrant, the preceding theorem immediately provides the cyclotomic structure on $N_e^{S^1}R$. If R only has the homotopy type of a cofibrant object, application of Proposition 3.11 allows us to functorially work with ${}^{op}TR$ and ${}^{op}TC$ as models of TR and TC.

sec:relTHH

6. A description of relative THH as the relative S^1-norm

In this section, we extend the work of Section 4 to the setting of A-algebras for a commutative orthogonal ring spectrum A. The category of A-modules is a symmetric monoidal category with respect to $\wedge_A$, the smash product over A. As explained in [HHR, ?.?], the construction of the indexed smash product can be carried out in the symmetric monoidal category of A-modules. Our construction of relative cyclotomic THH will use the associated A-relative norm.

We will write A_G to denote the commutative ring orthogonal G-spectrum obtained by regarding A as having trivial G-action; i.e., $A_G = \mathcal{I}^U_{\mathbb{R}^\infty} A$. (This is a commutative ring orthogonal G-spectrum since $\mathcal{I}^U_{\mathbb{R}^\infty}$ is a symmetric monoidal functor.) For example, if A is the sphere spectrum then A_G is the G-equivariant sphere spectrum. For an R-algebra A, let $N^{\mathrm{cyc}}_{\wedge_A} R$ denote the cyclic bar construction with

respect to the smash product over A. The same proof as Lemma 4.1 implies the following.

Lemma 6.1. *Let R be an object in $\mathscr{I}\mathcal{S}_A[\mathbb{T}]$. Then the geometric realization of the cyclic bar construction $|N^{\mathrm{cyc}}_{\wedge_A} R|$ is naturally an object in $S^1\mathscr{I}\mathcal{S}_A^{\mathbb{R}^\infty}$.*

Using the point-set change of universe functors we can turn this into an orthogonal S^1-spectrum indexed on the complete universe U.

Definition 6.2. Let R be an orthogonal ring spectrum. Define the functor

$$ {}_AN_e^{S^1} : \mathscr{I}\mathcal{S}_A[\mathbb{T}] \longrightarrow S^1\mathscr{I}\mathcal{S}^U_{A_{S^1}} $$

as the composite

$$ {}_AN_e^{S^1} R = \mathcal{I}^U_{\mathbb{R}^\infty} |N^{\mathrm{cyc}}_{\wedge_A} R|. $$

The argument for Proposition 4.3 also proves the following relative version.

prop:relcommuT

Proposition 6.3. *The restriction of ${}_AN_e^{S^1}$ to commutative A-algebras lifts to a functor*

$$ {}_AN_e^{S^1} : \mathscr{I}\mathcal{S}_A[\mathbb{P}] \longrightarrow S^1\mathscr{I}\mathcal{S}^U_{A_{S^1}}[\mathbb{P}] $$

that is left adjoint to the forgetful functor

$$ \iota^* \colon S^1\mathscr{I}\mathcal{S}^U_{A_{S^1}}[\mathbb{P}] \longrightarrow \mathscr{I}\mathcal{S}_A[\mathbb{P}]. $$

We now make a non-equivariant observation about relative THH (ignoring the group action temporarily) that informs our description of the equivariant structure. Similar theorems have appeared previously in the literature, e.g., [33, §5].

lem:nonequirel

Lemma 6.4. *Let R be an A-algebra in orthogonal spectra. Then there is an isomorphism*

$$ {}_STHH(R) \wedge_{{}_STHH(A)} A \cong {}_ATHH(R). $$

Proof. Commuting the smash product with geometric realization reduces the lemma to verifying the formula

$$ (R \wedge R \wedge \ldots \wedge R) \wedge_{A\wedge A\wedge\ldots\wedge A} R \cong R \wedge_A R \wedge_A \ldots \wedge_A R, $$

which is a straightforward calculation. □

We now generalize Lemma 6.4 to take advantage of the equivariant structure.

lem:basechange

Proposition 6.5. *Let G be a finite group. Let A be a ring orthogonal spectrum and M an A-module. The A-relative norm is obtained by base-change from the usual norm,*

$$ {}_AN_e^G M \cong N_e^G M \wedge_{N_e^G A} A_G. $$

Proof. Since M is an A-module, we know that $N_e^G M$ is an $N_e^G A$-module (in the category $G\mathscr{I}\mathcal{S}^U$), using the fact that the norm is a symmetric monoidal [HHR, ?.?] functor. The right hand side is the extension of scalars along the canonical map $N_e^G A \to A_G$ obtained as the adjoint of the natural (non-equivariant) map $A \to A_G$. Again because the norm is monoidal, we obtain a canonical map from $N_e^G M \wedge_{N_e^G A} A_G$ to ${}_AN_e^G M$; this map is an isomorphism because it is clearly an isomorphism after forgetting the equivariance. □

Extending this to S^1, if R is an A-algebra, then we have the following characterizations of relative THH as an S^1-spectrum that follows by essentially the same argument.

prop:extscal **Proposition 6.6.** *Let R be an A-algebra in orthogonal spectra. Then we have an isomorphism*

$$ {}_AN_e^{S^1}R \cong N_e^{S^1}R \wedge_{N_e^{S^1}A} A_{S^1} $$

We now turn to the homotopical analysis of ${}_AN_e^{S^1}$. The proof of Theorem 4.5 (and the analogous statement to Lemma 4.4) give rise to the following result.

thm:relTder **Theorem 6.7.** *Let $R \to R'$ be a weak equivalence of cofibrant A-algebras. Then the induced map ${}_AN_e^{S^1}R \to {}_AN_e^{S^1}R'$ is an $\mathcal{F}_{\mathrm{Fin}}$-equivalence.*

Similarly, we can extend the homotopical statement of Proposition 4.6 to the relative setting.

prop:relTcommquillen **Proposition 6.8.** *Regarded as a functor on commutative A-algebras, the functor $N_e^{S^1}$ is a left Quillen functor with respect to the positive complete model structure on $\mathscr{I}\mathcal{S}_A[\mathbb{P}]$ and the $\mathcal{F}_{\mathrm{Fin}}$-model structure on $S^1\mathscr{I}\mathcal{S}^U_{A_{S^1}}[\mathbb{P}]$.*

prop:relTdercompder **Proposition 6.9.** *Let $R \to R'$ be a weak equivalence of A-algebras where R is cofibrant and R' is a cofibrant commutative A-algebra. Then the induced map $N_e^{S^1}R \to N_e^{S^1}R'$ is an $\mathcal{F}_{\mathrm{Fin}}$-equivalence.*

7. The cyclotomic structure on ${}_AN_e^{S^1}R$

sec:amodcyc

The main application of the perspective of THH as the S^1-norm is the existence of a cyclotomic structure on ${}_AN_e^{S^1}R$, which we now construct in this section. In contrast to the absolute situation when $A = S$, the A-relative S^1-norm ${}_AN_e^{S^1}R$ and the associated constructions of ${}_ATR$ and ${}_ATC$ are novel and could not have been done using the Bökstedt construction of THH.

We begin by explaining what we mean by a cyclotomic spectrum in A_G-modules for A a commutative ring orthogonal spectrum. The geometric fixed point functor Φ^G is lax monoidal, and therefore gives rise to a functor

$$ \Phi^G \colon G\mathscr{I}\mathcal{S}^U_{A_G} \longrightarrow \mathscr{I}\mathcal{S}^{U^G}_{\Phi^G A_G}. $$

In the case of a finite subgroup $C_n < S^1$, for an A_{S^1}-module X, we have that $\Phi^{C_n}X$ is an orthogonal S^1/C_n-spectrum and a module over A_{S^1/C_n}. Pulling back along the nth root isomorphism $\rho_n \colon S^1 \to S^1/C_n$ again gives rise to an orthogonal S^1-spectrum $\rho_n^*\Phi^{C_n}X$ that is a module over $A_{S^1} \cong \rho_n^*A_{S^1/C_n}$.

defn:relcyclo **Definition 7.1.** A p-cyclotomic spectrum relative to A consists of an A_{S^1}-module X together with a map of A_{S^1}-modules

$$ t_p \colon \rho_p^*\Phi^{C_p}X \longrightarrow X, $$

that induces an $\mathcal{F}_p$-equivalence in the homotopy category of A_{S^1}-modules

$$ \rho_p^*L\Phi^{C_p}X \longrightarrow X. $$

As in the absolute case, the diagonal map

$$\Delta_n\colon R^{\wedge_A(k+1)} \longrightarrow \Phi^{C_n}\, {}_AN_e^{C_n} R^{\wedge_A(k+1)}$$

is an isomorphism both for cofibrant A-algebras and for cofibrant commutative A-algebras. Following the development in the absolute setting, we can then establish (using the same argument as for Theorem 5.3) the cyclotomic structure on ${}_AN_e^{S^1}R$.

thm:relcyclodiag

Theorem 7.2. *Let R be an A-algebra. The diagonal maps*

$$\Delta_n\colon R^{\wedge_A(k+1)} \longrightarrow \Phi^{C_n}\, {}_AN_e^{C_n} R^{\wedge_A(k+1)}$$

assemble into natural maps of A_{S^1}-modules

$$\tau_n\colon {}_AN_e^{S^1} R \longrightarrow \rho_n^*\Phi^{C_n}\mathcal{I}^{\tilde{U}}_{\mathbb{R}^\infty}|\operatorname{sd}_n N^{\mathrm{cyc}}_{\wedge_A} R|,$$

where $\tilde{U} = \iota_{C_n}^* U$. If R is cofibrant or cofibrant as a commutative A-algebra, then these maps are isomorphisms.

When R is cofibrant or a cofibrant commutative A-algebra, these maps give a cyclotomic structure on ${}_AN_e^{S^1}R$. If R only has the homotopy type of a cofibrant object, we can form the relative analogues of Definitions 3.8 and 3.9, which we denote ${}^{op}_ATR$ and ${}^{op}_ATC$.

Suppose that we are given a map of commutative ring orthogonal spectra $\phi\colon A \to A'$, and an A'-algebra R. Pullback along ϕ allows us to regard R as an A-algebra, and this gives rise to an induced map on relative THH, TR, and TC.

Proposition 7.3. *Let R be a (commutative) A-algebra and $\phi\colon A \to A'$ a map of commutative ring orthogonal spectra. Then we have a map*

$${}_AN_e^{S^1} R \longrightarrow {}_{A'}N_e^{S^1} R$$

of cyclotomic spectra that gives rise to maps ${}_ATR(R) \to {}_{A'}TR(R)$, ${}_ATC(R) \to {}_{A'}TC(R)$, ${}^{op}_ATR(R) \to {}^{op}_{A'}TR(R)$, and ${}^{op}_ATC(R) \to {}^{op}_{A'}TC(R)$.

Proof. The natural map $R \wedge_A R \to R \wedge_{A'} R$ gives rise to a map of orthogonal S^1-spectra ${}_AN_e^{S^1}R \to {}_{A'}N_e^{S^1}R$. Since the relative diagonal map is functorial in ϕ, it follows that this is a map of cyclotomic spectra. The remaining statements now follow from the functoriality of all of the constructions involved in defining $TR_{(-)}$, $TC_{(-)}$, ${}^{op}TR_{(-)}$, and ${}^{op}TC_{(-)}$. □

8. THH of C_n-equivariant ring spectra

sec:cncyclo

For G a finite group and $H < G$ a subgroup, the norm N_H^G provides a functor from orthogonal H-spectra to orthogonal G-spectra. In this section, we generalize this construction to a relative norm $N_{C_n}^{S^1}$, which we view as a "C_n-relative THH". We begin with an explicit construction in terms of a cyclic bar construction, which generalizes the simplicial object studied in Section 5 on the edgewise subdivision of the cyclic bar construction.

defn:ncntos1

Definition 8.1. Let R be an associative ring orthogonal C_n-spectrum indexed on the trivial universe $\mathbb{R}^\infty$. Let $N_\wedge^{\mathrm{cyc},C_n}R$ denote the simplicial object that in degree q is $R^{\wedge(q+1)}$, has degeneracy s_i (for $0 \le i \le q$) induced by the inclusion of the unit in the i-th factor, has face maps d_i for $0 \le i < q$ induced by multiplication of the ith and $(i+1)$th factors. The last face map d_q is given as follows. Let α_q be the automorphism of $R^{\wedge(q+1)}$ that cyclically permutes the factors putting the

last factor in the zeroth position and then acts on that factor by the generator $g = e^{2\pi i/n}$ of C_n. The last face map is $d_q = d_0 \circ \alpha_q$.

The previous definition constructs a simplicial object but not a cyclic object. Nevertheless it does have extra structure, of the same sort found on the edgewise subdivision of a cyclic object. The operator α_q in simplicial degree q is the generator of a $C_{n(q+1)}$-action (the action obtained by regarding $R^{\wedge(q+1)}$ as an indexed smash product for $C_n < C_{n(q+1)}$). The faces, degeneracies, and operators t satisfy the following relations in addition to the usual simplicial relations:

$$\begin{aligned} \alpha_q^{n(q+1)} &= \mathrm{id} \\ d_0\alpha_q &= d_q \\ d_i\alpha_q &= \alpha_{q-1}d_{i-1} \text{ for } 1 \le i \le q \\ s_i\alpha_q &= \alpha_{q+1}s_{i-1} \text{ for } 1 \le i \le q \\ s_0\alpha_q &= \alpha_{q+1}^2 s_q \end{aligned}$$

This defines a Λ_n^{op}-object in the notation of [BHM, II, 1.5]. As explained in [BHM, II, 1.6–8], the geometric realization has an S^1-action extending the C_n-action.

defn:Cnnorm **Definition 8.2.** Let R be an associative ring orthogonal C_n-spectrum indexed on the universe $\tilde{U} = \iota_{C_n}^* U$. The relative norm $N_{C_n}^{S^1} R$ is defined as the composite functor

$$N_{C_n}^{S^1} R = \mathcal{I}_{\mathbb{R}^\infty}^U N_\wedge^{\mathrm{cyc},C_n}(\mathcal{I}_{\tilde{U}}^{\mathbb{R}^\infty} R)$$

When R is a commutative ring orthogonal C_n-spectrum, we have the following analogue of Proposition 4.3.

prop:commT **Proposition 8.3.** *The restriction of $N_{C_n}^{S^1}$ to $C_n\mathscr{I}\mathcal{S}[\mathbb{P}]$ lifts to a functor*

$$N_{C_n}^{S^1} \colon C_n\mathscr{I}\mathcal{S}[\mathbb{P}] \longrightarrow S^1\mathscr{I}\mathcal{S}[\mathbb{P}]$$

that is left adjoint to the forgetful functor

$$\iota^* \colon S^1\mathscr{I}\mathcal{S}[\mathbb{P}] \longrightarrow C_n\mathscr{I}\mathcal{S}[\mathbb{P}].$$

We now describe the homotopical properties of the relative norm. The following analogue of Theorem 4.5 has the same proof.

thm:Ider **Theorem 8.4.** *Let $R \to R'$ be a weak equivalence of cofibrant associative ring orthogonal C_n-spectra. Then $N_{C_n}^{S^1} R \to N_{C_n}^{S^1} R'$ is a $\mathcal{F}_{\mathrm{Fin}}$-equivalence.*

In the commutative case, we have the following analogue of Proposition 4.6 (also using the identical proof).

prop:Tcommquillen **Theorem 8.5.** *Regarded as a functor on commutative ring orthogonal C_n-spectra, the functor $N_{C_n}^{S^1}$ is a left Quillen functor with respect to the positive complete model structure on $C_n\mathscr{I}\mathcal{S}[\mathbb{P}]$ and the $\mathcal{F}_{\mathrm{Fin}}$-model structure on $S^1\mathscr{I}\mathcal{S}^U[\mathbb{P}]$.*

We now turn to the question of the cyclotomic structure.

Theorem 8.6. *Let R be a cofibrant associative ring orthogonal C_n-spectrum or a cofibrant commutative ring orthogonal C_n-spectrum. If p is prime to n, then $N_{C_n}^{S^1} R$ has the natural structure of a p-cyclotomic spectrum.*

Proof. As in the proof of Theorem 5.3, we can identify $\iota^*_{C_{pn}} N^{S^1}_{C_n} R$ as the geometric realization of a simplicial orthogonal C_{pn}-spectrum of the form

$$N^{C_{pn}}_{C_n}(R^{(\bullet+1)}).$$

Since p is prime to n, we have a diagonal map $R^{\wedge(q+1)} \to \Phi^{C_p} N^{C_{pn}}_{C_n} R^{\wedge(q+1)}$, which again commutes with the simplicial structure and induces a diagonal map

$$\tau_p \colon N^{S^1}_{C_n} R \longrightarrow \rho_p^* \Phi^{C_p} N^{S^1}_{C_n} R.$$

Under the hypothesis that R is cofibrant as an orthogonal C_n-spectrum or cofibrant as a commutative ring orthogonal C_n-spectrum [22, ?.?] shows that the diagonal map $R^{\wedge(q+1)} \to \Phi^{C_p} N^{C_{pn}}_{C_n} R^{\wedge(q+1)}$ is an isomorphism, and it follows that τ_p is an isomorphism. The inverse gives the p-cyclotomic structure map. □

As usual, we can construct $TR_{C_n} R$ and $TC_{C_n} R$ from the cyclotomic structure on $N^{S^1}_{C_n} R$. And as before, when R only has the homotopy type of a cofibrant object, application of Proposition 3.11 allows us to work with ${}^{op}TR_{C_n}$ and ${}^{op}TC_{C_n}$.

When p divides n, the diagonal map is of the form

$$N^{S^1}_{C_{n/p}} \Phi^{C_p} R \longrightarrow \Phi^{C_p} N^{S^1}_{C_n}(R)$$

and is an isomorphism when R is cofibrant as an orthogonal C_n-spectrum or as a commutative ring orthogonal C_n-spectrum. In this case, we can get a p-cyclotomic structure map if we have one on R of the following form:

defn:cncyclo

Definition 8.7. For $p|n$, a C_n p-cyclotomic spectrum consists of an orthogonal C_n-spectrum X together with a map of orthogonal C_n-spectra

$$t \colon N^{C_n}_{C_{n/p}} \Phi^{C_p} X \longrightarrow X$$

that induces a weak equivalence from the derived composite functors.

Proposition 8.8. *Assume $p|n$ and let R be a associative ring orthogonal C_n-spectrum with a C_n p-cyclotomic structure such that the structure map t is a ring map. Then $N^{S^1}_{C_n} R$ has the natural structure of a p-cyclotomic spectrum.*

At present, we do not know if the previous proposition is interesting. For any (non-equivariant) ring orthogonal spectrum R', $R = N^{C_n}_e R'$ satisfies the hypothesis of the previous proposition, and $N^{S^1}_{C_n} R \cong N^{S^1}_e R'$. We know of no other examples.

9. First examples of ${}_ATC$

sec:computations

In this section, we begin to study the computational aspects of ${}_ATR$ and ${}_ATC$. We discuss relative analogues of standard classical computations of TR and TC. In the following section we construct a number of spectral sequences to compute the relative TR. We intend to return to more detailed calculations in future work.

We begin by computing ${}_ATHH(A)$, ${}_ATR(A)$, and ${}_ATC(A)$. Non-equivariantly, the isomorphism $A \wedge_A A \cong A$ implies that ${}_ATHH(A) \cong A$. The equivariant structure on ${}_ATHH(A)$ arises from the description

$${}_ATHH(A) \cong A \wedge S_{S^1}.$$

That is, the cyclotomic structure is entirely induced from the cyclotomic structure on the (equivariant) sphere spectrum. In order to use this observation to compute ${}_ATR(A)$ and ${}_ATC(A)$, we need the following lemma.

Lemma 9.1. *Let Z be an orthogonal spectrum (regarded as an orthogonal G-spectrum with trivial action) and X an orthogonal G-spectrum on a universe U. Then for any closed subgroup $H \subset G$ the natural composite map in the stable category*

$$Z \wedge X^H \longrightarrow Z^H \wedge X^H \longrightarrow (Z \wedge X)^H$$

is an isomorphism.

Proof. It suffices to consider the case when Z is a finite complex. Then we have an isomorphism in the equivariant stable category $Z \wedge X \simeq F(DZ, X)$ where DZ denotes the dual and F the mapping spectrum functor (cotensor of the equivariant stable category enriched over the stable category). Then we have

$$(Z \wedge X)^H \simeq F(DZ, X)^H \simeq F(DZ, X^H) \simeq Z \wedge X^H$$

and it is straightforward to identify this weak equivalence with the map above. □

Coupled with the identification of ${}_ATHH(A)$ above, the arguments in [BHM, §5] allow the computation of ${}_ATC(A)$. (See [MadsenTraces, 4.4.9] for the version of [BHM, 5.15] needed here for the last equivalence below.)

Proposition 9.2. *There are p-equivalences*

$${}_ATR(A) \simeq \lim(A \wedge TR^n(S)) \simeq_p A \vee \prod(A \wedge BC_{p^n})$$
$${}_ATC(A) \simeq \lim(A \wedge TC^n(S)) \simeq_p A \wedge TC(S).$$

Indeed, the arguments in [BHM] are stated in terms of "spherical group rings" $S[\Gamma] = \Sigma^\infty \Gamma_+$ where Γ is a group-like topological monoid. For relative TC, we should consider the group A-algebra $A[\Gamma] = A \wedge S[\Gamma]$. The argument for [BHM, 5.17] extends to the relative context (see also [MadsenTraces, 4.4.11] for the version of this argument we need) and we obtain the following theorem.

Theorem 9.3. *Let Γ be a group-like monoid. There is a p-equivalence*

$${}_ATC(A[\Gamma]) \simeq_p A \wedge TC(S[\Gamma]).$$

The spherical group ring is an example of a Thom spectrum for the trivial classifying map. More generally, the computation of the THH of Thom spectra [BThom, BCS] extends to the context of the relative THH of the generalized Thom spectra over $B\,\mathrm{GL}_1\,A$ [Bunits1, Bunits2]. We will study this issue in future work.

sec:spectralsequences

10. Spectral sequences for ${}_ATR$

In this section we present four spectral sequences for computing ${}_ATR$. In each case we actually have two spectral sequences, one graded over the integers and a second graded over $RO(S^1)$. We follow the modern convention of denoting an integral grading with $*$ and an $RO(S^1)$-grading with $\star$. Although the two look formally similar, they are very different computationally, for reasons explained in the introduction to [LewisMandell2]: the Tor terms are computed using very different notions of projective module. Specifically, for V a non-trivial representation $\underline{\pi}_*^{(-)}(\Sigma^V R)$ cannot be expected to be projective as a $\underline{\pi}_*^{(-)}R$ Mackey functor module; however, $\underline{\pi}_\star^{(-)}(\Sigma^V R)$ is of course projective as a $\underline{\pi}_\star^{(-)}R$ Mackey functor module, being just a shift of the free module $\underline{\pi}_\star^{(-)}R$.

10.1. **The absolute to relative spectral sequence.** The equivariant homotopy groups $\pi_*^{C_n}(N_e^{S^1}R)$ are the TR-groups $TR^n_*(R)$ and so $\pi_*^{C_n}({}_AN_e^{S^1}R)$ are by definition the relative TR-groups ${}_ATR^n_*(R)$.

Notation 10.1. Let

$$TR^{(-)}_*(R) = \underline{\pi}^{(-)}_*(N_e^{S^1}(R)) \qquad TR^{(-)}_\star(R) = \underline{\pi}^{(-)}_\star(N_e^{S^1}(R))$$
$${}_ATR^{(-)}_*(R) = \underline{\pi}^{(-)}_*({}_AN_e^{S^1}(R)) \qquad {}_ATR^{(-)}_\star(R) = \underline{\pi}^{(-)}_\star({}_AN_e^{S^1}(R))$$

Using the isomorphism of Proposition 6.6 prop:extscal

$${}_AN_e^{S^1}(R) \cong N_e^{S^1}(R) \wedge_{N_e^{S^1}A} A_{S^1},$$

we can apply the Künneth spectral sequences of [24] LewisMandell2 to compute the relative TR-groups from the absolute TR-groups and Mackey functor $\underline{\mathrm{Tor}}$. Technically, to apply [24] LewisMandell2 and for ease of statement, we restrict to a finite subgroup $H < S^1$.

Theorem 10.2. *Let A be a cofibrant commutative orthogonal spectrum and let R be a cofibrant associative A-algebra or cofibrant commutative A-algebra. For each finite subgroup $H < S^1$, there is a natural strongly convergent spectral sequence of H-Mackey functors*

$$\underline{\mathrm{Tor}}^{TR^{(-)}_*(A)}_{*,*}(TR^{(-)}_*(R), \underline{\pi}^{(-)}_*(A_H)) \quad \Longrightarrow \quad {}_ATR^{(-)}_*(R),$$

compatible with restriction among finite subgroups of S^1.

Compatibility with restriction among finite subgroups of S^1 refers to the fact that for $H < K$, the restriction of the K-Mackey functor $\underline{\mathrm{Tor}}$ to an H-Mackey functor is canonically isomorphic to the H-Mackey functor $\underline{\mathrm{Tor}}$ and the corresponding isomorphism on E^∞-terms induces the same filtration on $\underline{\pi}_*$. (Free K-Mackey functor modules restrict to free H-Mackey functor modules essentially because finite K-sets restrict to finite H-sets.)

We also have corresponding Kunneth spectral sequences graded on $RO(H)$ for $H < S^1$ or $RO(S^1)$. We choose to state our results in terms of the $RO(S^1)$-grading because this makes the behavior of the restriction among subgroups easier to describe; in the following theorem, $\star$ denotes the $RO(S^1)$-grading.

Theorem 10.3. *Let A be a cofibrant commutative orthogonal spectrum and let R be a cofibrant associative A-algebra or cofibrant commutative A-algebra. For each finite subgroup $H < S^1$, there is a natural strongly convergent spectral sequence of H-Mackey functors*

$$\underline{\mathrm{Tor}}^{TR^{(-)}_\star(A)}_{*,\star}(TR^{(-)}_\star(R), \underline{\pi}^{(-)}_\star(A_H)) \quad \Longrightarrow \quad {}_ATR^{(-)}_\star(R),$$

compatible with restriction among finite subgroups of S^1.

10.2. **The simplicial filtration spectral sequence.** The spectral sequence of the preceding subsection essentially gives a computation of the relative theory in terms of absolute theory. More often we expect to use the relative theory to compute the absolute theory. Non-equivariantly, the isomorphism

eq:changebaseringtxt

$$THH(R) \wedge A \cong {}_ATHH(R \wedge A) \tag{10.4}$$

gives rise to a Künneth spectral sequence

$$\mathrm{Tor}^{A_*(R\wedge_S R^{\mathrm{op}})}_{*,*}(A_*(R), A_*(R)) \quad \Longrightarrow \quad A_*(THH(R)).$$

An Adams spectral sequence can then in practice be used to compute the homotopy groups of $THH(R)$. For formal reasons, the isomorphism (10.4) still holds equivariantly, but now we have three different versions of the non-equivariant Künneth spectral sequence (none of which have quite as elegant an E^2-term) which we use in conjunction with equation (10.4).

The first equivariant spectral sequence generalizes the Künneth spectral sequence in the special case when π_*A is a field. Non-equivariantly, it derives from the simplicial filtration of the cyclic bar construction; equivariantly, we restrict to a finite subgroup $C_n < S^1$ and look at the simplicial filtration on the nth edgewise subdivision (described in the proof of Theorem 4.5).

Theorem 10.5. *Let A be a cofibrant commutative orthogonal spectrum and let R be a cofibrant associative A-algebra or cofibrant commutative A-algebra. Let H be a finite subgroup of S^1*

(1) *There is a natural spectral sequence strongly converging to the integer graded H-Mackey functor ${}_ATR^{(-)}_*(R)$ with E^1-term*
$$\underline{E}^1_{s,t} = \underline{\pi}_t({}_AN^H_e(R^{\wedge(s+1)})).$$
(2) *There is a natural spectral sequence strongly converging to the $RO(S^1)-$ graded H-Mackey functor ${}_ATR^{(-)}_\star(R)$ with E^1-term*
$$\underline{E}^1_{s,\tau} = \underline{\pi}_\tau({}_AN^H_e(R^{\wedge(s+1)})).$$

The E^2-terms of both spectral sequences are compatible with restriction among finite subgroups of S^1.

To see the compatibility with restriction among subgroups, we note that for $H = C_{mn}$, the E^2-term $(E^2_{*,\tau})^{C_m}$ is the homology of the simplicial object

$$\mathrm{sd}_n\, \pi^{C_m}_\star((N^{C_m}_e A)^{\wedge(\bullet+1)}).$$

For $H < K$, the subdivision operators then induce an isomorphism on E_2-terms.

In general, we do not know how to describe the E^2-term of these spectral sequences. One can formulate box-flatness hypotheses that would permit the identification of the E^2-term as a kind of Mackey functor Hochschild homology [11]; however, such hypotheses will rarely hold in practice. On the other hand, when $A = H\mathbb{F}$ for $\mathbb{F}$ a field, for formal reasons, the E^1-term is a purely algebraic functor of the graded vector space π_*R. We conjecture that the E^2-term is a functor of the graded $\mathbb{F}$-algebra π_*R.

10.3. **The cyclic filtration spectral sequence.** We have a second spectral sequence arising from the filtration on cyclic objects constructed by Fiedorowicz and Gajda [17]. Although they work in the context of spaces, their arguments generalize to provide an $\mathcal{F}_{\mathrm{Fin}}$-equivalence

$$|EX_\bullet| \longrightarrow |X_\bullet|$$

for cyclic orthogonal spectra, where E is the evident orthogonal spectrum generalization of the construction in their Definition 1,

$$EX_\bullet = \int_{[m]\in\Lambda_{\mathrm{face}}} X_m \wedge \Lambda(\bullet, [m])_+.$$

The proof of their Proposition 1 (which in fact only gives an $\mathcal{F}_{\mathrm{Fin}}$-equivalence for spaces) also applies in the orthogonal spectrum context, substituting geometric fixed points for fixed points, to prove the $\mathcal{F}_{\mathrm{Fin}}$-equivalence for orthogonal spectra. Change of universe $\mathcal{I}^U_{\mathbb{R}^\infty}$ commutes with geometric realization, and we use the coend filtration of $EX_\bullet$ for $X_\bullet = N^{\mathrm{cyc}}_{\wedge_A} R$ to obtain the following Fiedorowicz-Gajda cyclic filtration spectral sequences.

Theorem 10.6. *Let A be a cofibrant commutative orthogonal spectrum and let R be a cofibrant associative A-algebra or cofibrant commutative A-algebra. Let H be a finite subgroup of S^1.*

(1) *There is a natural spectral sequence of integer graded H-Mackey functors strongly converging to ${}_ATR^{(-)}_*(R)$ with E^1-term*

$$\underline{E}^1_{s,t} = \underline{\pi}_t(\mathcal{I}^U_{\mathbb{R}^\infty}(S^1_+ \wedge_{C_{s+1}} R^{\wedge(s+1)})).$$

(2) *There is a natural spectral sequence of $RO(S^1)$-graded H-Mackey functors strongly converging to ${}_ATR^{(-)}_\star(R)$ with E^1-term*

$$\underline{E}^1_{s,\tau} = \underline{\pi}_\tau(\mathcal{I}^U_{\mathbb{R}^\infty}(S^1_+ \wedge_{C_{s+1}} R^{\wedge(s+1)})).$$

The E^1-terms are compatible with restriction among finite subgroups of S^1.

10.4. **The relative cyclic bar construction spectral sequence.** The third spectral sequence directly involves Mackey functor $\underline{\mathrm{Tor}}$. For an A-algebra R, let ${}^g_A N^{C_n}_e R$ denote the $({}_AN^{C_n}_e R, {}_AN^{C_n}_e R)$-bimodule obtained by twisting the left action of ${}_AN^{C_n}_e R$ on ${}_AN^{C_n}_e R$ by the generator $g = e^{2\pi i/n}$ of C_n. We can identify the C_n-homotopy type of ${}_AN^{S^1}_e R$ in terms of this bimodule,

$${}_AN^{S^1}_e R \cong \mathcal{I}^U_{\tilde{U}} N^{\mathrm{cyc}}_{\wedge_A}({}_AN^{C_n}_e R, {}^g_A N^{C_n}_e R),$$

where the cyclic bar construction on the right is taken in the symmetric monoidal category of A-modules in orthogonal C_n-spectra and $\tilde{U} = \iota^*_{C_n} U$ denotes U viewed as a complete C_n-universe. A consequence of this description is that the main theorem of [24] constructing the equivariant Kunneth spectral sequence applies:

Theorem 10.7. *Let A be a cofibrant commutative orthogonal spectrum and let R be a cofibrant associative A-algebra or cofibrant commutative A-algebra. Fix $n > 0$.*

(1) *There is a natural strongly convergent spectral sequence of integer graded C_n-Mackey functors*

$$E^2_{*,*} = \underline{\mathrm{Tor}}^{N^{C_n}_e(R\wedge_A R^{\mathrm{op}})}_{*,*}(\underline{\pi}_{*\,A}N^{C_n}_e R, \underline{\pi}_{*\,A}{}^g N^{C_n}_e R) \Longrightarrow {}_ATR^{(-)}_*(R).$$

(2) *There is a natural strongly convergent spectral sequence of $RO(S^1)$-graded C_n-Mackey functors*

$$E^2_{*,\star} = \underline{\mathrm{Tor}}^{N^{C_n}_e(R\wedge_A R^{\mathrm{op}})}_{*,\star}(\underline{\pi}_{\star\,A}N^{C_n}_e R, \underline{\pi}_{\star\,A}{}^g N^{C_n}_e R) \Longrightarrow {}_ATR^{(-)}_\star(R).$$

We see no reason why the E^2-terms for the spectral sequences of the previous theorem should be compatible under restriction among finite subgroups of S^1.

11. Adams operations

sec:adams

In this section, we study the circle power operations on $THH(R)$ for a commutative ring R and on ${}_ATHH(R)$ for a commutative A-algebra R. Such operations were first defined on Hochschild homology by Loday [26] and Gerstenhaber-Schack [18] and explained by McCarthy [32] in terms of covering maps of the circle and extended to THH by [34]. Following [12, 4.5.3], we refer to these as *Adams operations* and denote as ψ^r (though in older literature [27, 4.5.16], the Adams operations differ by a factor of the operation number r). Specifically, we study how the operations interact with the equivariance, and we show that when r is prime to p, ψ^r descends to an operation on $TR(R)$, $TC(R)$, and in the commutative A-algebra context to an operation on ${}_ATR(R)$ and ${}_ATC(R)$, cf. [12, §7]. We study the effect of ψ^r on ${}_ATR_0(R)$ and ${}_ATC_0(R)$, where it is shown to be the identity when R is connective.

We recall the construction of McCarthy's Adams operations, which ultimately derives from the identification of $N^{\mathrm{cyc}}_{\wedge_A}R$ as the tensor $R\otimes S^1$ in the category of commutative A-algebras. Using the standard model for the circle as the geometric realization of a simplicial set $S^1_\bullet$ (with one 0-simplex and one non-degenerate 1-simplex), the tensor identification is just observing that $N^{\mathrm{cyc}}_{\wedge_A}R$ is the simplicial object obtained by taking $S^1_\bullet$ coproduct factors of R in simplicial degree $\bullet$,

$$N^{\mathrm{cyc}}_{\wedge_A}R = R\otimes S^1_\bullet.$$

The operation ψ^r is induced by the r-fold covering map

$$q_r\colon S^1\longrightarrow S^1,\qquad e^{i\theta}\mapsto e^{ri\theta}.$$

after tensoring with R.

def:Adams

Definition 11.1. Let A be a commutative ring orthogonal spectrum and R a commutative A-algebra. For $r\neq 0$, the Adams operation

$$\psi^r\colon {}_ATHH(R)\longrightarrow {}_ATHH(R)$$

is the map of (non-equivariant) commutative A-algebras obtained as the tensor of R with the quotient map $q_r\colon S^1\to S^1$.

We will study the equivariance of ψ^r using the C_n-action that arises on the edgewise subdivision sd_n of a cyclic set. To make this section more self-contained, we again recall from [11, §1] how this works. There is a natural homeomorphism

$$\delta_n\colon |\mathrm{sd}_n X|\longrightarrow |X|$$

for the n-fold edgewise subdivision of a simplicial space or simplicial orthogonal spectrum and the canonical isomorphism of simplicial objects $\mathrm{sd}_r\,\mathrm{sd}_s X\to \mathrm{sd}_{rs}X$, which together make the following diagram commute [11, 1.12]:

eq:subdiv-compat

$$\begin{array}{ccc} |\mathrm{sd}_r\,\mathrm{sd}_s X| & \longrightarrow & |\mathrm{sd}_{rs}X| \\ \downarrow{\scriptstyle \delta_r} & & \downarrow{\scriptstyle \delta_{rs}} \\ |\mathrm{sd}_s X| & \xrightarrow[\delta_s]{} & |X|. \end{array} \tag{11.2}$$

When X has a cyclic structure, $\mathrm{sd}_n X$ comes with a natural C_n-equivariant structure which on the geometric realization is the restriction to C_n of the natural S^1-action; moreover, in the diagram above, the left hand isomorphism is C_s-equivariant [11, 1.7–8].

We have a simplicial model of ψ^r by McCarthy's observation that q_r is the geometric realization of a quotient map of simplicial sets $\mathrm{sd}_r\, S^1_\bullet \to S^1_\bullet$. By naturality, diagram (11.2) is compatible with this quotient map.

thm:Adamsequiv

Proposition 11.3. *Let A be a commutative ring orthogonal spectrum and R a commutative A-algebra. For $r \neq 0$ and n relatively prime to r, the restriction of q_r is the multiplication by r isomorphism $C_n \to C_n$ and the Adams operations ψ^r is a map of commutative ring orthogonal C_n-spectra*

$$\psi^r \colon \iota^*_{C_n A} N_e^{S^1} R \longrightarrow q_r^* \iota^*_{C_n A} N_e^{S^1} R.$$

Moreover, for s relatively prime to n, the formula

$$(q_r)^*(\psi^s) \circ \psi^r = \psi^{rs} \colon \iota^*_{C_n A} N_e^{S^1} R \longrightarrow q_{rs}^* \iota^*_{C_n A} N_e^{S^1} R.$$

holds.

Proof. As above, the r-fold covering map defining the Adams operations becomes a C_n-equivariant map

$$\mathrm{sd}_n(\mathrm{sd}_r\, S^1) \longrightarrow (q_r \mid_{C_n})^*(\mathrm{sd}_n\, S^1).$$

Tensoring levelwise and applying $\mathcal{I}^{\tilde{U}}_{\mathbb{R}^\infty}$, we obtain a map of simplicial commutative A-algebras

$$\mathcal{I}^{\tilde{U}}_{\mathbb{R}^\infty}(R \otimes (\mathrm{sd}_n\, \mathrm{sd}_r\, S^1)) \longrightarrow q_r^* \mathcal{I}^{\tilde{U}}_{\mathbb{R}^\infty}(R \otimes \mathrm{sd}_n\, S^1).$$

The result now follows from diagram (11.2) and its compatibility with the covering projections q_r. □

In the case when $p \not| r$, the previous proposition shows that in particular the operation ψ^r should pass to categorical C_{p^n}-fixed points (in the derived category of A). Taking fibrant replacements, we get a map (of non-equivariant A-modules)

$$\psi^r \colon ({}_A N_e^{S^1} R)_f^{C_{p^n}} \longrightarrow ({}_A N_e^{S^1} R)_f^{C_{p^n}}$$

making the diagram

$$\begin{array}{ccc} ({}_A N_e^{S^1} R)_f^{C_{p^{n+1}}} & \xrightarrow{\psi^r} & ({}_A N_e^{S^1} R)_f^{C_{p^{n+1}}} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ ({}_A N_e^{S^1} R)_f^{C_{p^n}} & \xrightarrow[\psi^r]{} & ({}_A N_e^{S^1} R)_f^{C_{p^n}} \end{array}$$

commute, where F is the natural inclusion of fixed-points. Passing to the homotopy limit, we get an Adams operation ψ^r on ${}_A TR(R)$.

thm:AdamsTR

Theorem 11.4. *Let A be a commutative ring orthogonal spectrum and R a commutative A-algebra. For $p \not| r$, the Adams operation ψ^r induces a map*

$$\psi^r \colon {}_A TR(R) \longrightarrow {}_A TR(R)$$

natural in the derived category of A.

We next argue that for $p \not| r$, the Adams operation ψ^r descends to ${}_A TC(R)$.

thm:AdamsTC **Theorem 11.5.** *Let A be a commutative ring orthogonal spectrum and R a commutative A-algebra. For $p \not| r$, the Adams operation ψ^r induces a map*

$$\psi^r \colon {}_ATC(R) \longrightarrow {}_ATC(R)$$

natural in the derived category of A.

Proof. It suffices to consider the case when R is cofibrant as a commutative A-algebra and to show that ψ^r commutes with the cyclotomic structure map t_p, or equivalently, its inverse isomorphism

$$\tau_p \colon {}_AN_e^{S^1} \longrightarrow \rho_p^* \Phi^{C_p} \mathcal{I}^{\tilde{U}}_{\mathbb{R}^\infty} |\operatorname{sd}_p N^{\mathrm{cyc}}_{\wedge_A} R|.$$

This is clear from the naturality of (eq:subdiv-compat 11.2). □

Finally, we provide the following computation for the action of the Adams operations on ${}_ATR_0$ and ${}_ATC_0$.

thm:Adamspi0 **Theorem 11.6.** *Let A be a commutative ring orthogonal spectrum and R a commutative A-algebra. Assume that R is connective. Then for $p \not| r$, the Adams operation ψ^r acts by the identity on ${}_ATR_0(R)$.*

Proof. Under the hypothesis of connectivity,

$${}_ATR_0(R) = \pi_0 {}_ATR(R) \cong \pi_0 TR(R) \cong \pi_0 TR(H\pi_0 R)$$

and so it suffices to consider the case when $A = S$ and $R = H\pi_0 R$. Writing $R_0 = \pi_0 R$, by [20, Addendum 2.3] (HMfinitealgs), we have a canonical isomorphism of $TR_0(R)$ with the p-typical Witt ring $W(R_0)$ and canonical isomorphisms of $\pi_0^{C_{p^n}} TR(R)$ with $W_{n+1}(R_0)$, the p-typical Witt vectors of length $n+1$. Letting R_0 vary over all commutative rings, ψ^r then restricts to a natural transformation ψ^r_{n+1} of rings $W_{n+1}(-) \to W_{n+1}(-)$. We complete the proof by arguing that this natural transformation is the identity.

Since W_{n+1} is representable, it suffices to prove that ψ^r_{n+1} is the identity when R_0 is the representing object $\mathbb{Z}[x_0, \dots, x_n]$, or, since this is torsion free, when $R_0 = \mathbb{Q}[x_-, \dots, x_n]$. A fortiori, it suffices to prove ψ^r_{n+1} is the identity when R_0 is a $\mathbb{Q}$-algebra. Since for a $\mathbb{Q}$-algebra $W_{n+1}(R_0)$ is isomorphic as a ring to the cartesian product of $n+1$ copies of R_0 via the ghost coordinates, the only possible natural ring endomorphisms of W_{n+1} are the maps that permute the factors. Since ψ^r commutes with the restriction map R on $TR(R)$, and on the ghost coordinates the restriction map induces the projection onto the first n factors, it follows by induction that ψ^r_n is the identity. □

Since a connective commutative A-algebra R admits a canonical map of commutative A-algebras $R \to H\pi_0 R$, we obtain the following corollary of the previous theorem and its proof.

cor:AdamsTCpi0 **Corollary 11.7.** *Let A be a commutative ring orthogonal spectrum and R a commutative A-algebra. Assume that R is connective and that $p \not| r$. Then ${}_ATC_0(R)$ has the Frobenius invariants of $W(\pi_0 R)$ as a quotient and the action of ψ^r descends to the identity map on this quotient.*

12. Madsen's remarks

In his CDM notes [29, p. 218], Madsen describes the restriction map, and notes that the inverse is not as readily accessible even in the algebraic setting since "$\Delta(r) = r \otimes \cdots \otimes r$ is not linear". In our framework, we naturally get the inverse to the cyclotomic structure map, rather than the cyclotomic structure map itself. At first blush, this poses a curious contradiction. The answer arises from the transfer: $v \mapsto v^{\otimes p}$ is linear modulo the ideal generated by the transfer, and this is exactly the ideal killed by Φ^H.

The observation that the ideal killed by Φ^H coincides with the ideal generated by the transfer is essentially formal from the definition: $\Phi^H(X) = (X \wedge \widetilde{E\mathcal{P}})^H$ is a composite of the categorical fixed points with the localization killing cells of the form S^1/K for K a proper subgroup of H. Computationally, this means that all transfers from proper subgroups of H are killed.

The observation that the algebraic diagonal map is linear modulo the transfer is more interesting and relies on quite elementary algebra (undoubtedly familiar to the readers with experience with TNR-functors). Consider the C_p-module

$$N_e^{C_p}\big(\mathbb{Z}\{x,y\}\big) = \big(\mathbb{Z}\{x,y\}\big)^{\otimes p},$$

where $\mathbb{Z}\{x,y\}$ is the free abelian group on the set $\{x,y\}$. Inside is the element $(x+y)^{\otimes p}$, which is obviously in the fixed points of the C_p-action. Madsen's remark is essentially that $(x+y)^{\otimes p}$ is not $x^{\otimes p} + y^{\otimes p}$. We can expand $(x+y)^{\otimes p}$ using a non-commutative version of the binomial theorem as follows. Observing that the full symmetric group Σ_p acts on the tensor power (and the C_p action is just the obvious restriction), if we group all terms with i tensor factors of x and $p-i$ tensor factors of y, then we see that the symmetric group permutes these and a subgroup conjugate to $\Sigma_i \times \Sigma_{p-i}$ stabilizes each element. We therefore see that the sum of all of such terms for a fixed i can be expressed as the transfer

$$\mathrm{Tr}^{\Sigma_p}_{\Sigma_i \times \Sigma_{p-i}} x^{\otimes i} \otimes y^{\otimes (p-i)}.$$

Letting i vary and summing the terms (and then restricting back to C_p) shows that

$$(x+y)^{\otimes p} = x^{\otimes p} + y^{\otimes p} + \mathrm{Res}^{\Sigma_p}_{C_p}\big(\sum_{i=1}^{p-1} \mathrm{Tr}^{\Sigma_p}_{\Sigma_i \times \Sigma_{p-i}} x^{\otimes i} y^{\otimes (p-i)}\big).$$

All of the terms involving transfers are in the ideal generated by transfers by definition, and so we conclude that the p^{th} power map is linear modulo these.

The story is actually a bit more complex, as we illustrate with an algebraic example. In algebra, the fixed points of G-module play the role of the geometric fixed points in equivariant stable homotopy theory.

Let $p = 2$, and let $R = \mathbb{Z}[x]$. Then the two-fold tensor power, C_2-equivariantly, is

$$\mathbb{Z}[C_2 \cdot x] = \mathbb{Z}[x, gx].$$

The transfer ideal is generated by 2 and $x + gx$, and modulo 2 and $x + gx$, the map $x \mapsto x \cdot gx$ induces the canonical surjection

$$\mathbb{Z}[x] \longrightarrow \mathbb{Z}[x \cdot gx].$$

In this example, the map from R to the fixed points of $R^{\otimes 2}$ is not an isomorphism; we can interpret the failure to be an isomorphism as a failure to correctly interpret

the transfer of the element 1. In particular, restricting to the submodule generated by 1 we implicitly computed

$$N_e^{C_2}\mathbb{Z} = \mathbb{Z},$$

endowed with the trivial action. To better model the genuine equivariant story, we must use a richer algebraic norm (the Tambara functor valued norm). For our purposes, this is the left-adjoint to the forgetful functor from Tambara functors to commutative rings.

An explicit construction of the left adjoint is $\underline{\pi}_0 N_e^G(HR)$, reflecting the fact that for this richer norm, the algebraic model much better approximates the stable story. For $G = C_2$ and for $R = \mathbb{Z}[x]$, the fixed points are the ring

$$\mathbb{Z}[t, y, x \cdot gx]/(t^2 - 2t, ty - 2y),$$

with the elements t and y the transfers of 1 and x respectively. Thus, modulo the image of the transfer, this ring is simply $\mathbb{Z}[x \cdot gx]$, and the norm map $x \mapsto x \cdot gx$ is an isomorphism.

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