ON QUASI-PRÜFER AND UMt DOMAINS

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ABSTRACT. In this note we show that an integral domain D of finite w-dimension is a quasi-Prüfer domain if and only if each overring of D is a w-Jaffard domain. Similar characterizations of quasi-Prüfer domains are given by replacing w-Jaffard domain by w-stably strong S-domain, and w-strong S-domain. We also give new characterizations of UMt domains.

1. INTRODUCTION

The quasi-Prüfer notion was introduced in [2] for rings (not necessarily domains). As in [9], we say that an integral domain D is a *quasi-Prüfer domain* if for each prime ideal P of D, if Q is a prime ideal of $D[X]$ with $Q \subseteq P[X]$, then $Q = (Q \cap D)[X]$. It is well known that an integral domain is a Prüfer domain if and only if it is integrally closed and quasi-Prüfer [11, Theorem 19.15]. There are several different equivalent condition for quasi-Prüfer domains (c.f. [9, 2, 3]).

On the other hand as a t-analogue, an integral domain D is called a *UMt domain* [12], if every upper to zero in $D[X]$ is a maximal t-ideal and has been studied by several authors (see [8], [6], and [18]). UMt domains are closely related to quasi-Prüfer domains in the sense that a domain D is a UMt domain if and only if D_P is a quasi-Prüfer domain for each t-prime ideal P of D [8, Theorem 1.5]. And the other relation is the characterization of quasi-Prüfer domains due to Fontana, Gabelli and Houston [8, Corollary 3.11]; a domain D is a quasi-Prüfer domain if and only if each overring of D is a UMt-domain.

In [16] we defined and studied the w-Jaffard domains and proved that all strong Mori domains (domains that satisfy the ACC on w-ideals) and all UMt domains of finite w-dimension, are w-Jaffard domains. In [17] we defined and studied a subclass of w-Jaffard domains, namely the w-stably strong S-domains and showed how this notion permit studies of UMt domains in the spirit of earlier works on quasi-Prüfer domains. The aim of this paper is to prove that, for a domain D with some condition on w-$\dim(D)$, the following statements are equivalent, which gives new descriptions of quasi-Prüfer domains; a result reminiscent of the well-known result of Ayache, Cahen and Echi [2] (see also [9, Theorem 6.7.8]).

(1) Each overring of D is a w-stably strong S-domain.
(2) Each overring of D is a w-strong S-domain.
(3) Each overring of D is a w-Jaffard domain.
(4) Each overring of D is a UMt domain.
(5) D is a quasi-Prüfer domain.

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Throughout, the letter D denotes an integral domain with quotient field K and $F(D)$ denotes the set of nonzero fractional ideals. Let $f(D)$ be the set of all nonzero finitely generated fractional ideals of D. Let $*$ be a star operation on the domain D. For every $A \in F(D)$, put $A^{*_f} := \bigcup F^*$, where the union is taken over all $F \in f(D)$ with $F \subseteq A$. It is easy to see that $*_f$ is a star operation on D. A star operation $*$ is called of *finite character* if $*_f = *$. We say that a nonzero ideal I of D is a $*$-*ideal* of D, if $I^* = I$; a $*$-*prime*, if I is a prime $*$-ideal of D. It has become standard to say that a star operation $*$ is *stable* if $(A \cap B)^* = A^* \cap B^*$ for all A, $B \in F(D)$. Given a star operation $*$ on an integral domain D it is possible to construct a star operation $\widetilde{*}$ which is stable and of finite character defined as follows: for each $A \in F(D)$,

$$A^{\widetilde{*}} := \{x \in K | xJ \subseteq A, \text{ for some } J \subseteq D, J \in f(D), J^* = D\}.$$

The $\widetilde{*}$-dimension of D is defined as follows:

$$\widetilde{*}\text{-}\dim(D) = \sup\{\text{ht}(P) \mid P \text{ is a } \widetilde{*}\text{-prime ideal of } D\}.$$

The most widely studied star operations on D have been the identity d, and v, $t := v_f$, and $w := \widetilde{v}$ operations, where $A^v := (A^{-1})^{-1}$, with $A^{-1} := (D : A) := \{x \in K | xA \subseteq D\}$.

Let D be a domain and T an overring of D. Let $*$ and $*'$ be star operations on D and T, respectively. One says that T is $(*, *')$-*linked to* D if $F^* = D \Rightarrow (FT)^{*'} = T$ for each nonzero finitely generated ideal F of D. As in [5] we say that T is t-linked to D if T is (t, t)-linked to D. As in [6] a domain D is called t-*linkative* if each overring of D is t-linked to D. As a matter of fact t-linkative domains are exactly the domains such that the identity operation coincides with the w-operation, that is *DW-domains* in the terminology of [15].

If $F \subseteq K$ are fields, then $\text{tr.deg.}_F(K)$ stands for the *transcendence degree* of K over F. If P is a prime ideal of the domain D, then we set $\mathbb{K}(P) := D_P/PD_P$.

2. w-Jaffard domains

First we recall a special case of a general construction for semistar operations (see [16]). Let D be an integral domain with quotient field K, let X, Y be two indeterminates over D and $*$ be a star operation on D. Set $D_1 := D[X]$, $K_1 := K(X)$ and take the following subset of $\text{Spec}(D_1)$:

$$\Theta_1^* := \{Q_1 \in \text{Spec}(D_1) |\ Q_1 \cap D = (0) \text{ or } (Q_1 \cap D)^{*_f} \subsetneq D\}.$$

Set $\mathfrak{S}_1^* := D_1[Y] \backslash (\bigcup\{Q_1[Y] | Q_1 \in \Theta_1^*\})$ and:

$$E^{\circlearrowleft_{\mathfrak{S}_1^*}} := E[Y]_{\mathfrak{S}_1^*} \cap K_1, \text{ for all } E \in F(D_1).$$

It is proved in [16, Theorem 2.1] that the mapping $*[X] := \circlearrowleft_{\mathfrak{S}_1^*}: F(D_1) \to F(D_1)$, $E \mapsto E^{*[X]}$ is a stable star operation of finite character on $D[X]$, i.e., $\widetilde{*[X]} = *[X]$. It is also proved that $\widetilde{*}[X] = *_f[X] = *[X]$, $d_D[X] = d_{D[X]}$. If $X_1, \cdots, X_r$ are indeterminates over D, for $r \geq 2$, we let

$$*[X_1, \cdots, X_r] := (*[X_1, \cdots, X_{r-1}])[X_r].$$

For an integer r, put $*[r]$ to denote $*[X_1, \cdots, X_r]$ and $D[r]$ to denote $D[X_1, \cdots, X_r]$.

Let $*$ be a star operation on D. A valuation overring V of D is called a $*$-*valuation overring of* D provided that $F^* \subseteq FV$, for each $F \in f(D)$. Following [16], the $*$-*valuative dimension* of D is defined as:

$$*\text{-}\dim_v(D) := \sup\{\dim(V) | V \text{ is } *\text{-valuation overring of } D\}.$$

It is shown in [16, Theorem 4.5] that

$$\widetilde{*}\text{-}\dim_v(D) = \sup\{w\text{-}\dim(R) | R \text{ is a } (*, t)\text{-linked over } D\}.$$

It is observed in [16] that we have always the inequality $\widetilde{*}$-dim$(D) \leq \widetilde{*}$-dim$_v(D)$. We say that D is a $*$-*Jaffard domain*, if $*\text{-}\dim(D) = *\text{-}\dim_v(D) < \infty$. When $* = d$ the identity operation then d-Jaffard domain coincides with the classical Jaffard domain (cf. [1]). It is proved in [16], that D is a $\widetilde{*}$-Jaffard domain if and only if

$$*[X_1, \cdots, X_n]\text{-}\dim(D[X_1, \cdots, X_n]) = \widetilde{*}\text{-}\dim(D) + n,$$

for each positive integer n. In [19] we gave examples to show that the two classes of w-Jaffard and Jaffard domains are incomparable by constructing a w-Jaffard domain which is not Jaffard and a Jaffard domain which is not w-Jaffard.

We are now prepared to state and prove the first main result of this paper.

Theorem 2.1. *Let D be an integral domain of finite w-dimension. Then the following statements are equivalent:*

(1) *Each overring of D is a w-Jaffard domain.*
(2) *D is a quasi-Prüfer domain.*

Proof. (1) $\Rightarrow$ (2) Let Q be a prime ideal of an overring T of D, and set $\mathfrak{q} := Q \cap D$. Let $\tau : T_Q \to \mathbb{K}(Q)$ be the canonical surjection and let $\iota : \mathbb{K}(\mathfrak{q}) \to \mathbb{K}(Q)$ be the canonical embedding. Consider the following pullback diagram:

$$\begin{array}{ccc} D(Q) := \tau^{-1}(\mathbb{K}(\mathfrak{q})) = D_{\mathfrak{q}} + QT_Q & \longrightarrow & \mathbb{K}(\mathfrak{q}) \\ \downarrow & & \downarrow \\ T_Q & \xrightarrow{\ \tau\ } & \mathbb{K}(Q). \end{array}$$

Since T_Q is quasilocal and $\mathbb{K}(\mathfrak{q})$ is a DW-domain, then $D(Q)$ is a DW-domain by [15, Theorem 3.1(2)]. Thus the w-operation coincides with the identity operation d for $D(Q)$. Since by the hypothesis $D(Q)$ is a w-Jaffard domain we actually have $D(Q)$ is a Jaffard domain. On the other hand by [1, Proposition 2.5(a)] we have

$$\dim_v(D(Q)) = \dim_v(T_Q) + \operatorname{tr.deg}_{\mathbb{K}(\mathfrak{q})}(\mathbb{K}(Q)).$$

In particular $\operatorname{tr.deg}_{\mathbb{K}(\mathfrak{q})}(\mathbb{K}(Q))$ and $\dim_v(T_Q)$ are finite numbers. Note that by [7, Proposition 2.1(5)] we have $\dim(D(Q)) = \dim(T_Q)$ and since $\dim_v(D(Q)) = \dim(D(Q))$, we obtain that

$$\dim(T_Q) = \dim_v(T_Q) + \operatorname{tr.deg}_{\mathbb{K}(\mathfrak{q})}(\mathbb{K}(Q)).$$

Since $\dim(T_Q) \leq \dim_v(T_Q)$, then $\operatorname{tr.deg}_{\mathbb{K}(\mathfrak{q})}(\mathbb{K}(Q)) = 0$. Consequently D is a residually algebraic domain, and hence is a quasi-Prüfer domain by [3, Corollary 2.8].

(2) $\Rightarrow$ (1) Let T be an overring of D. We claim that T is of finite w-dimension. Since D is a quasi-Prüfer domain, [6, Theorem 2.4] implies that D is a t-linkative and UMt domain. Thus in particular T is a t-linked overring of D. Then

$$\begin{aligned} w\text{-}\dim(T) &\leq \sup\{w\text{-}\dim(R) | R \text{ is } t\text{-linked over } D\} \\ &= w\text{-}\dim_v(D) = w\text{-}\dim(D) < \infty, \end{aligned}$$

where the first equality is by [16, Theorem 4.5]. Finally by [17, Corollary 2.6], every UMt domain of finite w-dimension is a w-Jaffard domain to deduce that T is a w-Jaffard domain. $\square$

As an immediate corollary we have:

Corollary 2.2. *Let D be an integral domain of finite w-dimension. Then the following statements are equivalent:*

(1) *Each t-linked overring of D is a w-Jaffard domain.*
(2) *D is a UMt domain.*

Proof. (1) $\Rightarrow$ (2) Let P be a t-prime ideal of D, and T be an overring of D_P. Thus $T = T_{D\setminus P}$ is a t-linked overring of D by [5, Proposition 2.9]. Therefore T is a w-Jaffard domain by the hypothesis. Consequently D_P is a quasi-Prüfer domain by Theorem 2.1. Then D is a UMt domain by [8, Theorem 1.5].

(2) $\Rightarrow$ (1) Let T be a t-linked overring of D. Then as the proof of Theorem 2.1 we have

$$\begin{aligned} w\text{-}\dim(T) \leq & \sup\{w\text{-}\dim(R) | R \text{ is } t\text{-linked over } D\} \\ = & w\text{-}\dim_v(D) = w\text{-}\dim(D) < \infty. \end{aligned}$$

By [17, Corollary 2.6] we get that T is a w-Jaffard domain. □

3. w-STABLY STRONG S-DOMAINS

Let $*$ be a star operation on D. Following [17] the domain D is called a *$*$-strong S-domain*, if each pair of adjacent $*$-prime ideals $P_1 \subset P_2$ of D, extend to a pair of adjacent $*[X]$-prime ideals $P_1[X] \subset P_2[X]$, of $D[X]$. If for each $n \geq 1$, the polynomial ring $D[n]$ is a $*[n]$-strong S-domain, then D is said to be an *$*$-stably strong S-domain*. It is observed in [17] that a domain D is $*$-strong S-domain (resp. $*$-stably strong S-domain) if and only if D_P is strong S-domain (resp. stably strong S-domain) for each $*$-prime ideal P of D. Thus a strong S-domain (resp. stably strong S-domain) D is $*$-strong S-domain (resp. $*$-stably strong S-domain) for each star operation $*$ on D. However, the converse is not true in general; i.e., for some star operation $*$, the domain D might be $*$-strong S-domain (resp. $*$-stably strong S-domain), but D is not strong S-domain (resp. stably strong S-domain). In [14, Example 4.17] Malik and Mott gave an example of a UMt domain (in fact a Krull domain) which is not strong S-domain. But a UMt domain is a w-stably strong S-domain (and hence w-strong S-domain as well) by [17, Corollary 2.6].

We observe [17, Corollary 2.3] that a finite w-dimensional w-stably strong S-domain is a w-Jaffard domain.

We are now prepared to state and prove the second main result of this paper.

Theorem 3.1. *Let D be an integral domain of finite w-valuative dimension. Then the following statements are equivalent:*

(1) *Each overring of D is a w-stably strong S-domain.*
(2) *Each overring of D is a w-strong S-domain.*
(3) *Each overring of D is a UMt domain.*
(4) *D is a quasi-Prüfer domain.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial, and (3) $\Rightarrow$ (1) holds by [17, Corollary 2.6].

(2) ⇒ (4) Let Q be a prime ideal of an overring T of D and set $\mathfrak{q} := Q \cap D$. As in the proof of Theorem 2.1 we have the following pullback diagram:

$$\begin{array}{ccc} D(Q) & \longrightarrow & \mathbb{K}(\mathfrak{q}) \\ \downarrow & & \downarrow \\ T_Q & \xrightarrow{\tau} & \mathbb{K}(Q). \end{array}$$

Since T_Q is quasilocal and $\mathbb{K}(\mathfrak{q})$ is a DW-domain, then $D(Q)$ is a DW-domain by [15, Theorem 3.1(2)]. Thus the w-operation coincides with the identity operation d for $D(Q)$. Since by the hypothesis $D(Q)$ is a w-strong S-domain, we actually have $D(Q)$ is a strong S-domain. Next we claim that $D(Q)$ is of finite dimension. Indeed since $D(Q)$ is a DW-domain it is in fact a t-linked overring of D. Then

$$\begin{aligned} \dim(D(Q)) =& w\text{-}\dim(D(Q)) \\ \leq& \sup\{w\text{-}\dim(R) | R \text{ is } t\text{-linked over } D\} \\ =& w\text{-}\dim_v(D) < \infty, \end{aligned}$$

where the second equality is by [16, Theorem 4.5]. On the other hand by [1, Proposition 2.7] we have the inequality belove

$$\begin{aligned} 1 + \dim(T_Q) + \min\{\operatorname{tr.deg.}_{\mathbb{K}(\mathfrak{q})}(\mathbb{K}(Q)), 1\} \leq& \dim(D(Q)[X]) \\ =& \dim(D(Q)) + 1 \\ =& \dim(T_Q) + 1. \end{aligned}$$

The first equality holds since $D(Q)$ is strong S-domain and [13, Theorem 39], and the second one holds by [7, Proposition 2.1(5)]. Thus $\operatorname{tr.deg.}_{\mathbb{K}(\mathfrak{q})}(\mathbb{K}(Q)) = 0$. Consequently D is a residually algebraic domain and hence is a quasi-Prüfer domain by [3, Corollary 2.8].

(4) ⇒ (3) Suppose that D is a quasi-Prüfer domain and let T be an overring of D. Thus T is also a quasi-Prüfer domain. Therefore T is a UMt domain by [6, Theorem 2.4]. □

As an immediate corollary we have:

Corollary 3.2. *Let D be an integral domain of finite w-valuative dimension. Then the following statements are equivalent:*

(1) *Each t-linked overring of D is a w-stably strong S-domain.*
(2) *Each t-linked overring of D is a w-strong S-domain.*
(3) *Each t-linked overring of D is a UMt domain.*
(4) *D is a UMt domain.*

Proof. The implication (1) ⇒ (2) is trivial.

For (2) ⇒ (4) let P be a t-prime ideal of D, and T be an overring of D_P. Thus $T = T_{D \setminus P}$ is a t-linked overring of D by [5, Proposition 2.9]. Therefore T is a w-strong S-domain by the hypothesis. Consequently D_P is a quasi-Prüfer domain by Theorem 3.1. Then D is a UMt domain by [8, Theorem 1.5].

(4) ⇒ (3) Suppose T is a t-linked overring of D. Then T is a UMt domain by [18, Theorem 3.1].

(3) ⇒ (1) Is true by [17, Corollary 2.6]. □

Note that the equivalence (3) $\Leftrightarrow$ (4) in Theorem 3.1 (resp. Corollary 3.2) is well known [8, Corollary 3.11] (resp. [4, Theorem 2.6]), but our proof is completely different.

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