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Relay Sensor Placement in Wireless Sensor Networks

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Abstract

In this paper, we propose a novel idea of maintaining connectivity by introducing relay sensors in a *wireless sensor network*. We restrict our consideration to a very important class of wireless sensor networks such as biomedical sensor networks[13], in which the locations of the sensors are fixed and the placement can be pre-determined. We formulate our problem to the NP-hard network optimization problem named *Steiner Minimum Tree with Minimum number of Steiner Points (SMT-MSP)* [5] and present two approximate solutions. Meanwhile, we study the topology improvement in a wireless sensor network when relay sensors are introduced. In other words, we restrict transmission power of each sensor to a small value and use relay sensors to guarantee connectivity. The performance parameters under consideration are *P*, the total per node minimum power needed to maintain connectivity, and *D*, the maximum degree in the *minimum power topology* (maintained by *P*). Simulation study shows that with the introduction of relay sensors, we achieve better performance, especially for sparse topology.

Keywords: Wireless sensor networks, minimum power topology, relay sensor, SMT-MSP, power control

1 Introduction

Wireless Sensor Network (WSN) is an ad hoc multihop system containing sensors connected by wireless links. The possibility of wireless sensor networks is driven by the on-going improvement in sensor technology and VLSI [6]. WSNs have many possible applications, including environmental monitoring and biomedications. WSN is used to produce macro-scale effects from micro-devices through coordinated activities of many sensors, thus *connectivity* is an very important issue in WSN architecture design. On the other hand, wireless links are mainly determined by transmission powers of sensors, and higher transmission power produces rich connectivity.

However, "in the context of untethered nodes, the finite energy budget is a primary design constraint. Communications is a key energy consumer as the radio signal power in sensor networks drops off with r^4 [7] due to ground reflections from short antenna heights."(quoted from [3].) Here in this quote, r is the distance from the transmitter. This means to reach a slightly longer distance, the sensor needs to dispatch much higher transmission power. The second reason for the prohibitiveness of higher transmission power is the higher interference to on-going traffic. The higher the power a sensor transmits, the more the number of neighbors the sensor has, and the higher the negative influence the sensor has on the network throughput.

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The third reason is the lifetime of the network. Wireless sensors are battery powered. Either battery renewal is prohibited by economic considerations or it is impossible to renew a battery in a WSN. The lifetime of a WSN is mainly determined by the averaged battery life. In *biomedical sensor networks* [13], we have the forth reason: the heat dissipated by higher-power transmission may damage the surrounding tissue since the implantable devices are intended for long-term placement in the body.

Based on the above analysis, we conclude that a good WSN topology should be *uniform* and *regular* and the maximum degree (the number of neighbors with direct communication links) should be small. Of course the topology should be robust such that the removal of a few edges or nodes does not make the network disconnected. We are interested in the problem of *maintaining connectivity with minimum per node transmission power in wireless sensor networks*. The topology generated by minimum per node transmission power is called a *minimum power topology*. In this paper, we restrict our consideration to a very important class of wireless sensor networks such as biomedical sensor networks [13][14], in which *the locations of the sensors are fixed and the placement can be pre-determined*. In this kind of WSN, the global connectivity may not be guaranteed if transmission power is low and relay sensors need to be placed to maintain connectivity. We will study the following problem: *given a set of sensors in the plane, place minimum number of relay sensors to maintain global connectivity such that the transmission range of each sensor is at most R, where R is a constant.* This statement is formulated to the network optimization problem named *Steiner Minimum Tree with Minimum number of Steiner Points (SMT-MSP)* [5]:

- Given a set of terminals (denoted by V) in the plane and a constant R, find a Steiner tree τ spanning V with minimum number of Steiner points such that every edge in τ has length at most R.

In this description, "terminals" refer to "ordinary sensors" while "Steiner points" refer to "relay sensors". SMT-MSP is a generalized *Steiner Minimum Tree (SMT) problem*. A Steiner tree for terminal set V is a spanning tree over $V \cup S$, where S contains all points not in V, which are called *Steiner points*. A SMT is a Steiner tree with minimum total edge length. For a survey on SMT, we refer the readers to [2].

SMT-MSP is NP-hard [5]. Lin and Xue [5] also gave a ratio-5 approximation algorithm. In [1], Chen, Du *et al.* showed that the algorithm given by Lin and Xue [5] has performance ratio exactly 4, and they also presented a new $O(n^4)$-time approximation with performance ratio at most 3, where n is the number of given terminals. In this paper, we give a $O(n^3)$-time approximation with performance ratio at most 3, and a randomized approximation with performance ratio at most $\frac{5}{2}$. We also study the improvement on *maximum degree D* and *total consumed power P* to maintain global connectivity when relay sensors (Steiner points) are introduced in wireless sensor networks. In other words, we use P and D as performance parameters for topology control. We will answer the following questions:

1. How topology is improved when 1 or 2 relay sensors are introduced?

2. With a restricted low transmission power, how many relay sensors needed to maintain global connectivity? How topology is improved after these relay sensors are introduced?

Note that our starting point on *power efficient topology control* is quite different than those in literature. We maintain good topology by introducing relay sensors to keep transmission range small while most related research results focus on algorithm design to control the transmitted power dissipated by each sensor [9][4][10][12][15][17]. These techniques can be combined coherently. As mentioned earlier, we will study the improvement on the total transmitted power to maintain global connectivity with the introduction of relay sensors. Thus we need an algorithm to compute the total per node minimum power when connectivity is guaranteed. We will apply the optimal algorithm `CONNECT` provided by Ramanathan and Rosales-Hain in [10]. This algorithm contains two steps. First a minimum-cost spanning tree T is constructed where cost is the edge length. The transmitted power for each sensor is strong enough to reach the farthest neighbor in T. This step computes the minimum power p_{min} used to maintain connectivity but p_{min} is not *per node* *minimum*. The second step provides the optimization: for each sensor, decrease the transmission power until connectivity can not be observed! This algorithm guarantees per node minimum energy to maintain global connectivity.

This paper is organized as follows. Section 2 introduces the network model we will use. Section 3 studies the performance improvement when 1 or 2 relay sensors are introduced. Section 4 proposes a very simple algorithm to compute relay sensors when transmission range is restricted. We study the improvement on P and D by simulation when this algorithm is applied. Section 5 gives a randomized algorithm for the SMT-MSP problem. We conclude our paper in Section 6.

2 Network model

The given network contains n homogeneous sensors (nodes) located in a 2-dimensional plane. Each sensor is mounted an omni-directional antenna, which can transmit to all sensors in its coverage area. The topology is a graph $G = (V,E)$, where V is the set of n sensors (called nodes) and E is the set of transmission links (called edges) between pairs of nodes. Transmission links are well defined by transmission power. There is a link from sensor u to sensor v if and only if v is located in u's transmission range. u and v are called neighbors.

We assume that there is a maximum power P_{max} at which sensors can transmit. If all sensors transmit with P_{max}, the topology is *complete*, which contains all possible bidirectional links. We denote this topology by $G_C = (V,E_C)$. If each sensor transmits with different power, the topology will contain unidirectional links. Denote this topology by $G_P = (V,E_p)$. Note that the complete topology G_C may not be a complete graph. A complete graph contains an edge for every pair of nodes. Also note that G_P is a subgraph of G_C, which means $E_P \subseteq E_C$, if we consider any bidirectional link in E_C as two independent unidirectional links.

A transmission between sensors u and v takes power $p(u,v) \leq P_{max}$, which is a linear function of $d(u,v)^\alpha$ [11], where $\alpha > 2$ is the path-loss exponent of outdoor radio propagation with a typical value of 4 and $d(u,v)$ is the Euclidean distance between sensors u and v. With P_{max}, sensor u can reach sensor v if $d(u,v) \leq R_{max}$, where R_{max} is the maximum transmission range. Other power consumption sources include signal reception and process. A reception at any sensor takes constant power [11]. The processing power consumed by CPU, buffer and other electronics is negligible. With this analysis, the power consumption model for sensor u to relay message to sensor v is $p(u,v) = k \cdot (d(u,v))^4 + c$, where k is an appropriate constant related to path-loss. Since in this study we are considering the problem of controlling topology with transmission power, we will ignore the reception and computation powers. For simplicity, in all the simulation studies, we apply $p(u,v) = (d(u,v))^4$ as the power consumption model for the transmission from sensor u to v. This will not affect the results too much since we are considering the improvement (%), not the absolute value, on the amount of needed power to maintain the topology.

3 Performance improvement when one or two relay sensors are introduced

Given a complete topology $G_C = (V,E_C)$ with maximum degree d^C_{max}, what can we achieve when one relay sensor or two relay sensors are introduced? In other words, with the introduction of one or two relay sensors, how much improvement on the amount of total battery power needed to maintain global connectivity? Here we assume that G is *strongly connected*. A strongly connected graph has a path between any pair of nodes.

We run algorithm `CONNECT` over G_C to compute $G_P = (V,E_P)$, the topology maintained by per node minimum power. By placing a node (relay sensor) r in the middle of the longest edge in E_P, we get another topology G_S, which contains one more node. Note that we add the relay sensor to the original topology G_C to get G_S. Now we run algorithm `CONNECT` over G_S to compute G_{P_S}. To place the second relay sensor, we treat G_S as G_C and repeat the previous procedure.

The power improvement is defined to be $(P_{G_P} - P_{G_{P_S}})/P_{G_P}$, where P_G is the minimum total power needed to maintain topology G. As mentioned in Section 1, we also consider the improvement on the maximum degree. The maximum degree improvement is defined to be $D_{G_P} - D_{G_{P_S}})/D_{G_P}$, where D_G is the maximum degree of topology G.

Let's first look at a simple example. The given topology G_C is shown in Figure 1(a); After the operation of algorithm `CONNECT`, the per node minimum power topology G_P is shown in Figure 1(b); We introduce a relay sensor in the middle of edge $(0,2)$ to get topology G_S (1(c)) and run `CONNECT` to compute G_{P_S} (1(d)). The power improvement is 35.55%. Note that from Figure 1, even though there is no improvement for maximum degree, but the number of sensors with maximum degree is in fact decreased.

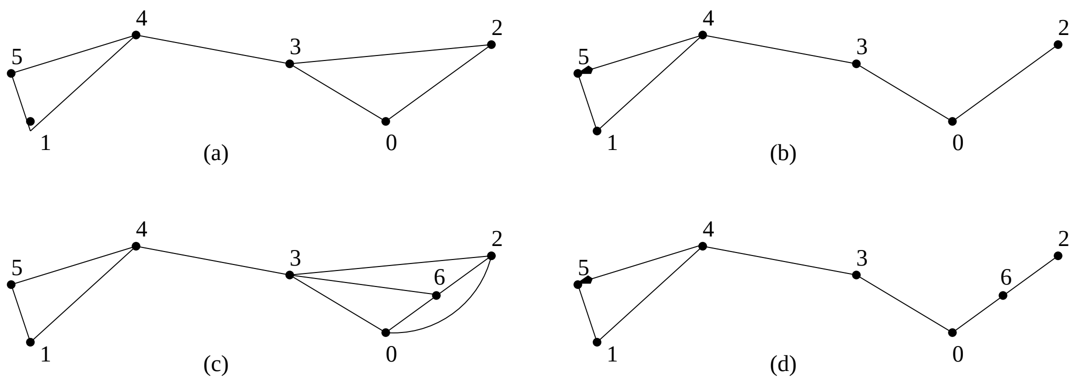

Figure 1: *(a) The given topology* $G_C = (V, E_C)$, *where* $V = \{0(75,22), 1(40,22), 2(88,13), 3(65,16), 4(51,12), 5(37,17)\}$ *and transmission range* $R = 25$. *Here* $i(x,y)$ *represents the coordinates for sensor i. (b) Topology* G_P*, which is a subgraph of* G_C*. Note that there is one uni-directional link from sensor 4 to sensor 5. (c) Topology* G_S*. Host 6 is the inserted relay sensor. (d) Topology* G_{P_S}*.*

We study the performance improvement with 1 and 2 relay sensors on random topology by simulation. We assume there are N sensors distributed randomly in a 100×100 rectangular region. Transmission range R_{max} is chosen to be 50 units. The averaged results for 1000 runs on the selected N values are reported in Figures 2 and 3.

Note that Figures 2 and 3 illustrate very similar curves. We have the following observations:

- The total consumed power for maintaining connectivity is decreased when 1 or 2 relay sensors are introduced. In the simulation range ($N = 10$ to 100), the total consumed power is decreased by 13.1% to 39.8% when one relay sensor is introduced, by 20.4% to 60.6% when two relay sensors are introduced. This improvement favors sparse graph, which means that the sparser the graph, the higher the improvement on the total consumed power.

- The maximum degree for maintaining connectivity is also decreased when very few relay sensors are introduced. In the simulation range ($N = 10$ to 100), the maximum degree is decreased by 4.8% to 9.4% when one relay sensor is introduced, by 7.2% to 16.1% when two relay sensors are introduced. For sparser graphs, the improvement is relatively higher.

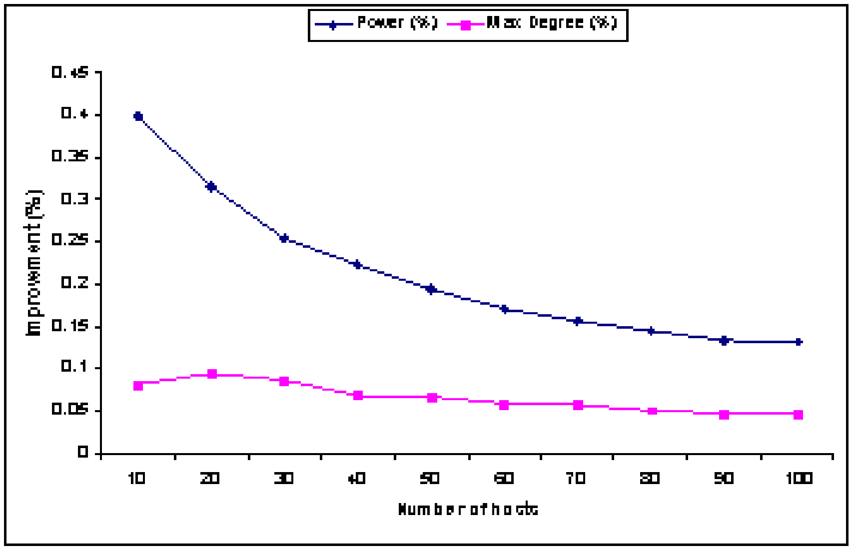

Figure 2: *The improvement on total consumed power and maximum degree when 1 relay sensor is introduced.*

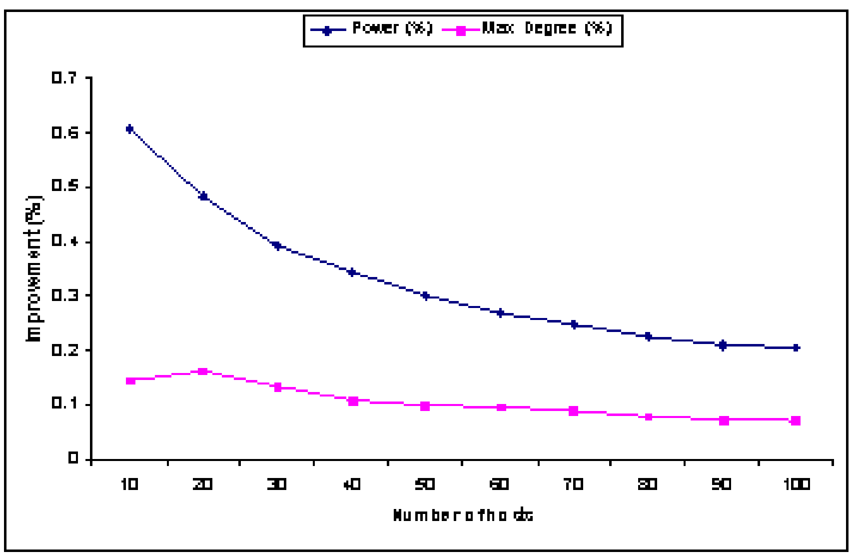

Figure 3: *The improvement on total consumed power and maximum degree when 2 relay sensors are introduced.*

- There is a bump in the curve representing the improvement of maximum degree, which corresponds to $N = 20$, a moderately sparse graph.

4 Performance study when transmission range is bounded

In this section, we will study the improvement on P and D when transmission range is restricted to a small value and the connectivity is ensured by the introduction of relay sensors. We first provide a greedy algorithm to compute relay sensors in Subsection 4.1. We will use graph-theoretic terminology in this subsection to state the algorithm. That is, we use "terminal" to represent an "ordinary sensor" and "Steiner point" to represent a "relay sensor". In other words, we are proposing an algorithm for STP-MSP. In Subsection 4.2, we study the performance of this algorithm on P and D by simulation.

4.1 A ratio 3 algorithm for STP-MSP

Given a set P of n terminals in the Euclidean plane, and a positive constant R, we want to find a Steiner tree with minimum number of Steiner points such that each edge in the tree has length at most R. In [1], Chen *et al.* presented an $O(n^4)$-time approximation with performance ratio at most 3. With a slightly modification, we may reduce the running time to $O(n^3)$.

Algorithm A: 3-approximation Algorithm for STP-MSP

Input A set P of n terminals, a positive constant R.
Output A Steiner tree T_A in which each edge has length at most R.

0. Sort all $\frac{n(n-1)}{2}$ possible edges between the n terminals of P in length increasing order $e_1, e_2, \ldots, e_{\frac{n(n_1)}{2}}$, and set $T_A = \emptyset$;
1. **for** every e_i such that $|e_i| \leq R$ **do**
   if e_i connects two different connected components of T_A
   then put e_i into T_A;
2. **for** each subset of three terminals a, b, c respectively in three connected components of T_A **do**
   if there exists a points s within distance R from a, b and c
   then put the 3-star, consisting of three edges sa, sb, sc into T_A;
3. **for** every e_i **do**
   if e_i connects two different connected components of T_A
   then put e_i into T_A.

Figure 4: The ratio-3 algorithms.

Our algorithm is given in Figure 4. Since we construct 3-stars in Step 2, the algorithm runs in $O(n^3)$ time. Now we analyze this algorithm theoretically.

For a given set P of terminals, a *minimum spanning tree* is a tree interconnecting the terminals in P with edge between terminals. For a given constant R, a *steinerized minimum spanning tree* is a tree obtained from a minimum spanning tree by inserting $\lceil \frac{|ab|}{R} \rceil - 1$ Steiner points to break each edge ab into small pieces of length at most R.

Let T be a Steiner tree and e be a line segment. $C(T)$ and $C(e)$ denote the numbers of Steiner points in T and e, respectively. $|e|$ denotes the length of e.

Lemma 4.1 *[1] Every steinerized minimum spanning tree has the minimum number of Steiner points among steinerized spanning trees.*

Lemma 4.2 *[5] There exists a shortest optimal Steiner tree T^* for $STP-MSP$ such that every vertex in T^* has degree at most five.*

Lemma 4.3 *[1] Let T^* be a shortest optimal tree for STP-MSP such that every Steiner point has degree at most five. Let T_j be a full component of T^*. Then the following hold:*

(1) The steinerized minimum spanning tree on terminals in T_j has at most $3 \cdot C(T_j) + 1$ Steiner points.

(2) If T_j contains a Steiner point of degree at most four, then the steinerized minimum spanning tree on terminals in T_j has at most $3 \cdot C(T_j)$ Steiner points.

(3) If the steinerized minimum spanning tree on terminals in T_j has an edge (of length at most R) between two terminals, then it contains at most $3 \cdot C(T_j)$ Steiner points.

From (3) of Lemma 4.3, we know that if the number of Steiner points contained in a steinerized spanning tree on terminals in a full component T_j reaches the upper bound $3 \cdot C(T_j) + 1$, then any two terminals are not connected directly by a single edge of length at most R, i.e., there must be a Steiner point between them.

Theorem 4.4 *Let T^* be an optimal tree for STP-MSP and T_A an approximation produced by Algorithm A. Then $C(T_A) \leq 3C(T^*)$.*

Proof. Let T^S be a steinerized minimum spanning tree on all terminals, and let k be the number of 3-stars produced by Step 2 of **Algorithm A**. Then

$$C(T_A) \leq C(T^S) - k.$$

By Lemma 4.2, we assume that each Steiner point of T^* has degree at most five. Assume that T^* has h full components $T_1, T_2, \ldots, T_h$. For $i = 1, 2$, let $T^{(i)}$ be the components produced by Step i of **Algorithm A**. We construct a steinerized spanning tree T as follows: Initially, set $T := T^{(1)}$, then for each full component T_j ($1 \leq j \leq h$), add to T the steinerized minimum spanning tree H_j on terminals of T_j, if the resulted tree has a cycle, then destroy the cycle by deleting some edges of H_j. Without loss of generality, suppose that $T_1, T_2, \ldots, T_g$ ($g \leq h$) are the full components in T^* such that every Steiner point has degree five and $T^{(1)} \cup T_j$ has no cycle. Combining Lemma 4.1 and Lemma 4.3 with the fact that for destroying a cycle from $T \cup H_j$, a Steiner point must be removed unless H_j contains an edge between two terminals, we have

$$C(T^S) \leq C(T) \leq 3C(T^*) + g,$$

i.e.,

$$C(T_A) \leq 3C(T^*) + g - k.$$

Suppose that $T^{(1)}$ has p components. Then, $T^{(2)}$ has $p - 2k$ components $C_1, C_2, \ldots, C_{p-2k}$. Now we construct another graph H on all terminals as follows: Initially put all edges of $T^{(1)}$ into H, then consider every T_j ($1 \leq j \leq g$). If T_j has a unique Steiner point(this Steiner point connects five terminals which must lie in at most two $C_i' s$), then among the five terminals there are three pairs (edges) of terminals, each pair (edge) lies in the same C_i. We add the three edges into H. If T_j has at least two Steiner points, then there are two Steiner points each connecting four terminals, and we can also find three pairs (edges) of terminals such that each pair (edge) lies in the same C_i. Thus, we can add the three edges into H. It is clear that H has at most $p - 3g$ components. Since each components of H is contained by a C_i, we have $p - 2k \leq p - 3g$, then $g - k \leq \frac{3g}{2} - k \leq 0$. This ends the proof. ■

4.2 Performance study by simulation

In this subsection, we study topology improvement by simulation when transmission power is restricted to a small value. For any given topology and transmission range R, we first run Algorithm A to make the graph connected. Then we run `CONNECT` to compute the per node minimum power needed to maintain connectivity. We alsoe compute the maximum degree in the minimum power topology.

As in Section 3, we assume there are N sensors distributed randomly in a 100×100 rectangular region. Transmission range R_{max} is chosen to be 50 units. The transmission range is restricted to 10, 20, 30, and 40 units. Again we run each scenario 1000 times. The averaged results are reported in Figures 5-8, which illustrate similar results as in Section 3 when 1 or 2 relay sensors are introduced. Note that we also combine the improvement on P and D for all transmission ranges in Figures 10-11.

We also report the average number of relay sensors introduced for each simulation scenario in figure 9. Note that with low transmission power, we need more relay sensors to maintain connectivity. Also note that when the transmission range is enlarged from $R = 10$ to $R = 20$, the number of relay sensors is increased drastically. But $R = 10$ gives the best topology improvement as shown in Figures 10-11. This tells us that we can add many relay sensors to make a sparse topology "dense" to achieve better saving on transmission power.

Based on the analysis on Figures 9-11, we obtain the following obervations:

- Making a sparse topology "dense" can achieve better power usage.
- For each topology, there exists some crossover transmission range or a span of transmission range such that much more relay sensors are needed if we want to decrease transmission power. For example, for $N = 20$, we need about 5 relay sensors if $R = 20$. If we force $R = 10$, we need 23 relay sensors to maintain the topology while if $R = 30$, we only need 1 or 2 relay sensors. In this case, the crossover transmission range is between $R = 10$ and $R = 20$.
- For an already dense topology, the maximum degree may be increased when many relay sensors are introduced. For example, when $N = 80$ and $R = 10$, the addition of 20 steiner points make the maximum degree increase by 2%.

5 2.5-approximation of STP-MSP

In this section, we give a randomized algorithm of ratio-$\frac{5}{2}$ for the STP-MSP problem. The following are some useful terminologies and Lemmas.

A *full component* of a Steiner tree is a subtree in which each terminal is a leaf and each internal node is a Steiner point. A Steiner tree for n terminals is a *k-restricted Steiner tree* if each full component spans at most k terminals.

A path $q_1q_2\ldots,q_m$ in a tree T is called a *convex* path if for every $i = 1,2,\ldots,m-3$, q_iq_{i+2} intersects $q_{i+1}q_{i+3}$. An angle of degree more than 120° is called a *big* angle. An angle of degree less than or equal to 120° is called a *small* angle.

Lemma 5.1 *[1] Let $q_1q_2\ldots,q_m$ be a convex path. Suppose there are b big angles among $m-2$ angles $\angle q_1q_2q_3, \angle q_2q_3q_4, \ldots, \angle q_{m-2}q_{m-1}q_m$. Then, $|q_1q_m| \leq (b+2)R$.*

Note that if there is no small angle, $|q_1q_m| \leq (b+1)R$. Thus, this lemma is useful only when there are many small angles in the convex path.

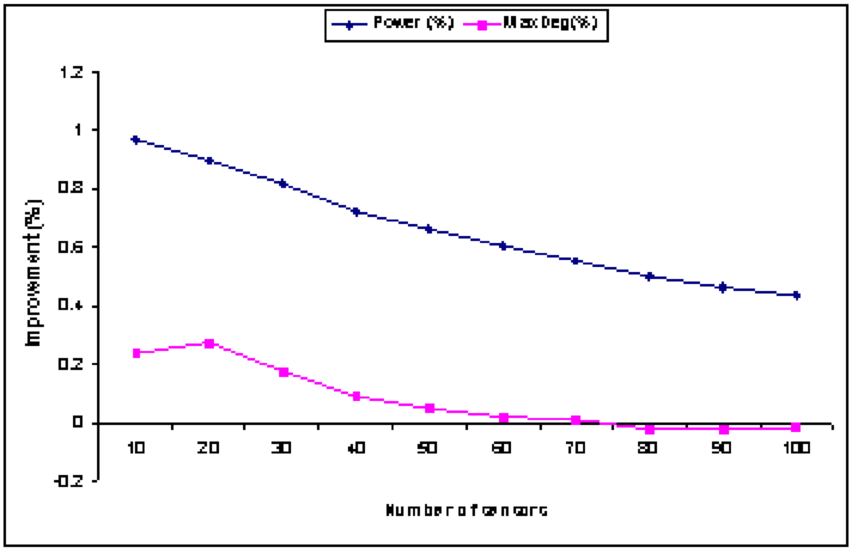

Figure 5: *The improvement on total consumed power and maximum degree when* $R = 10$.

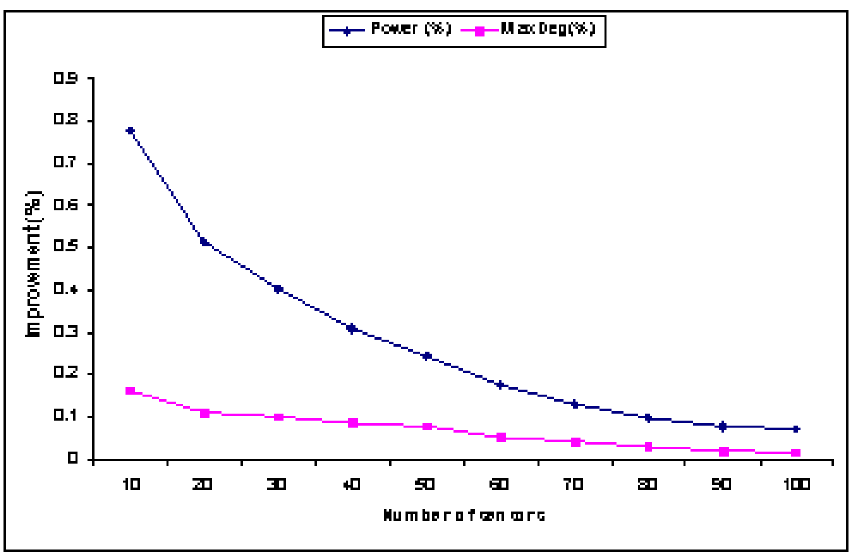

Figure 6: *The improvement on total consumed power and maximum degree when* $R = 20$.

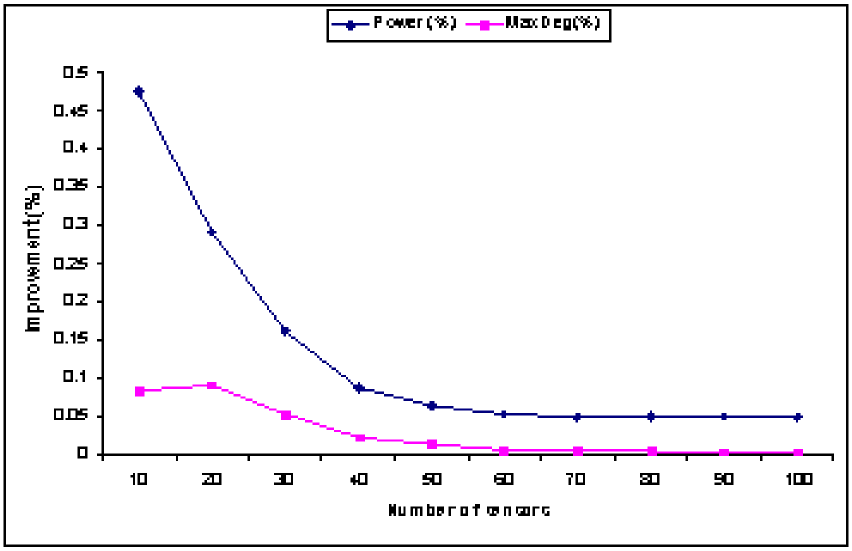

Figure 7: *The improvement on total consumed power and maximum degree when* $R = 30$.

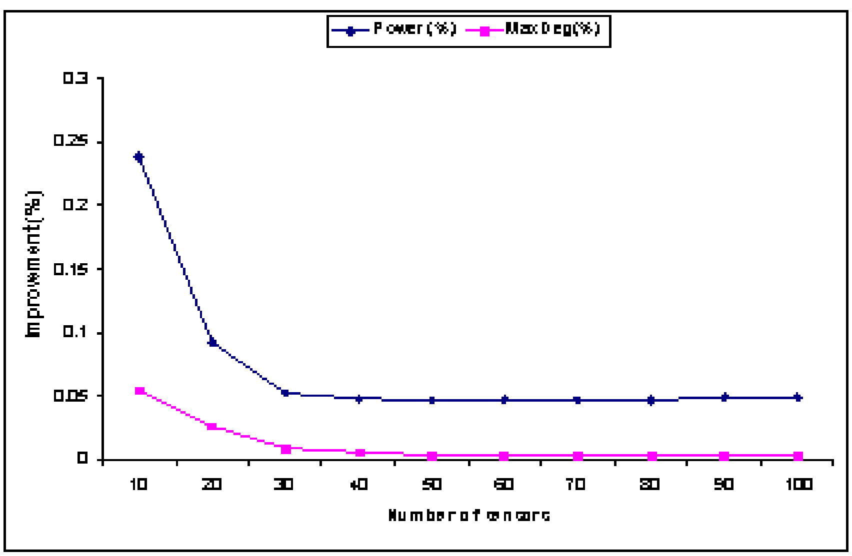

Figure 8: *The improvement on total consumed power and maximum degree when* $R = 40$.

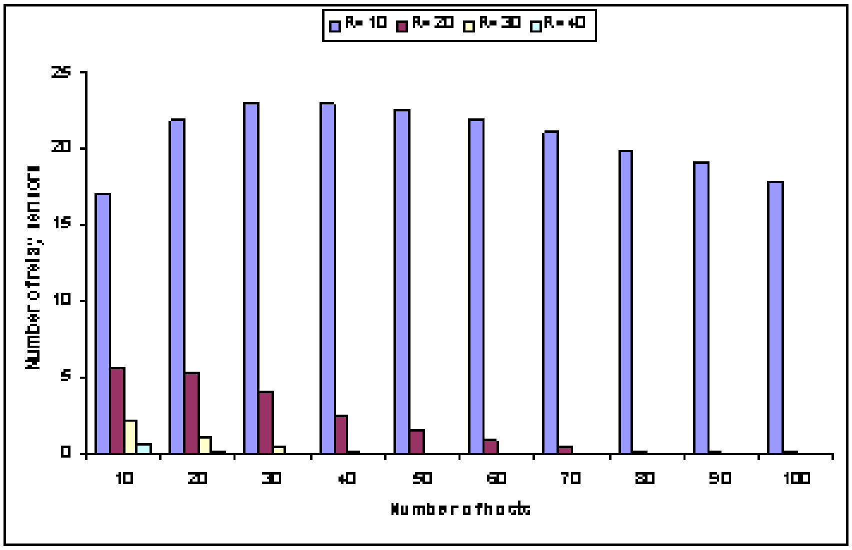

Figure 9: *The number of relay sensors in each simulation scenario.*

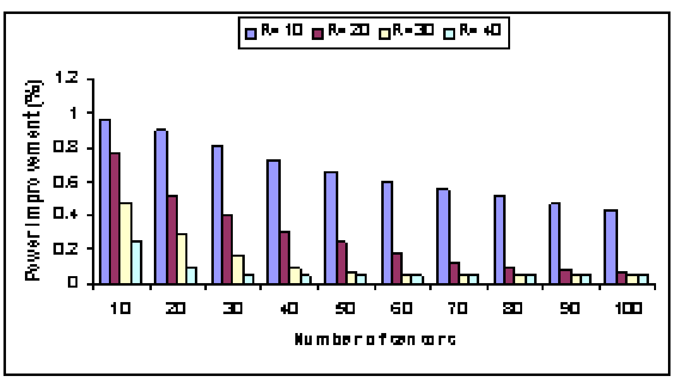

Figure 10: *The improvement on total consumed power.*

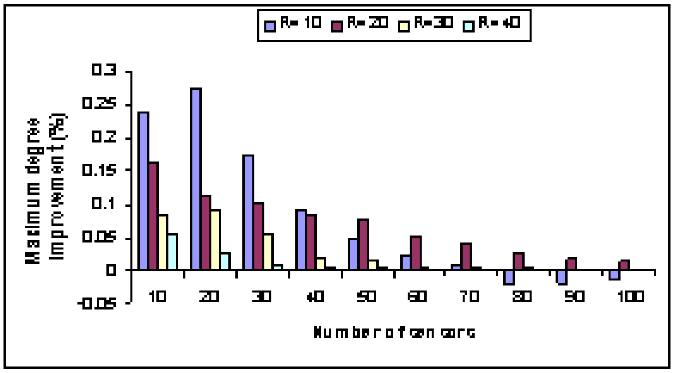

Figure 11: *The improvement on maximum degree .*

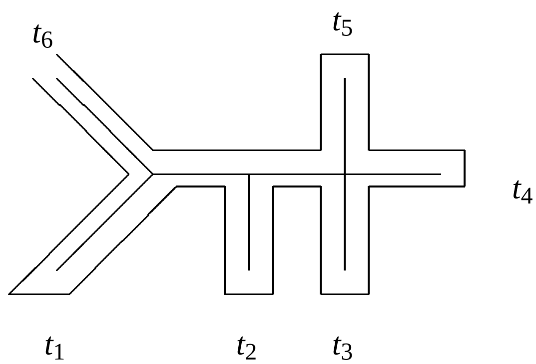

Figure 12: The tour F that visits all the terminals

Lemma 5.2 *[1] In a shortest optimal tree T for $STP-MSP$, there are at most two big angles at a point of degree three, there is at most one big angle at a point of degree four, and there is no big angle with degree five.*

Let T^* be a shortest optimal tree for $STP-MSP$ on n terminals which is a full Steiner tree. Let s_i denote the number of Steiner points of degree i in T^*.

Lemma 5.3 *[1]* $3s_5+2s_4+s_3=n-2$.

Theorem 5.4 *[8] There exists a randomized algorithm for the minimum spanning tree problem in* 3*-hypergraphs running in poly*(n, w_{max}) *time with probability at least* 0.5*, where n is the number of nodes in the hypergraph and w_{max} is the largest weight of edges in the hypergraph.*

Lemma 5.5 *Consider a clockwise tour F of T^* that visits the n terminals in the order $t_1, t_2, \ldots, t_n, t_1$. (See Figure 12.) Then,*

(i) the tour F has exactly n convex paths $P_1, P_2, \ldots, P_n$ such that P_i connects two terminals t_i and t_{i+1} ($t_{n+1}=t_1$);

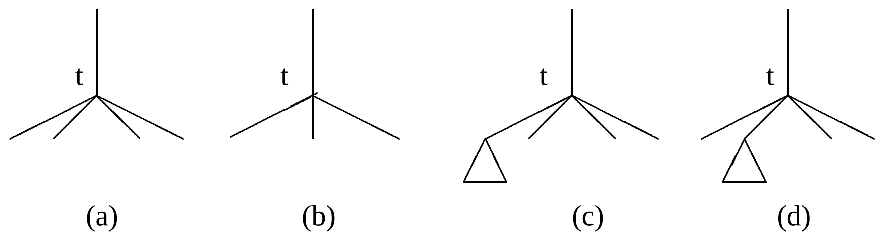

Figure 13: Four possible cases of type (1) good point

(ii) *each angle at a Steiner point appears in those n convex paths exactly once.*

(iii) *connect the two ends of P_i by an edge e_i and then steinerized e_i, $i = 1, 2, \ldots, n$, the total number of Steiner points in any $n-1$ e_i's $\bar{C} = \sum_{i=1}^{n-1} C(e_i)$ is upper bounded as follows:*

$$\begin{aligned} \bar{C} &\leq s_4 + 2s_3 + 2s_2 + n - C(e_n) \\ &= 3(s_5 + s_4 + s_3) + 2s_2 + 2 - C(e_n) \end{aligned} \quad (1)$$

We denote by T_F the tree consisting of n terminals and $(n-1)$ edges $e_1, e_2, \ldots, e_{n-1}$.

Proof. (i) and (ii) are very easy to see from the structure of T^*. Now, we prove (iii). Consider the tour F. By Lemma 5.1, if there are a_i big angles in P_i, then there are at most $a_i + 1$ Steiner points on e_i, and so the total number of Steiner points in F is at most n plus the number of big angles in T^*. By Lemma 5.2, there are at most $2s_2 + 2s_3 + s_4$ big angles in T^*. From Lemma 5.3, we know that (iii) is valid. ■

Let T^* be a shortest optimal tree for STP-MSP which is a full Steiner tree on n terminals. Without loss of generality, we assume that T^* has Steiner points of degree at least three. Selecting an arbitrary Steiner point of degree at least three as the root of T^*, we get a rooted tree. A *good point t* in T^* is a Steiner point that is adjacent to some terminals and satisfies one of the following:

(i) (type (1)) t has three or more terminals as children;

(ii) (type (2)) t has two terminals as children and the degree of t is 4;

(iii) (type (3)) t is a point of degree 3.

Note that a good point is of degree at least 3. A *bad* point is a Steiner point of degree at least 3 in T^* that is not a good point.

Theorem 5.6 *There is a 3-restricted Steiner tree such that each edge has length at most R and the Steiner points is at most $\frac{5}{2}$ times the optimum.*

Proof. Let F and T_F be defined in Lemma 5.5. From Lemma 5.5, we know that in T_F (1) each degree 2 Steiner point is used at most twice, and (2) each Steiner point of degree at least 3 is used at most three times.

We modify T_F into a 3-restricted tree T such that each of the good point of type (1) or type (3) in T^* is used at most twice, at least half of the type (2) good points is used at most twice, and each of the rest of Steiner points of degree at least 3 in T^* is used at most three times.

Our modification is as follows:

(i) Let t be a type (1) good point of degree d. We have to consider the four cases as shown in Figure 13.

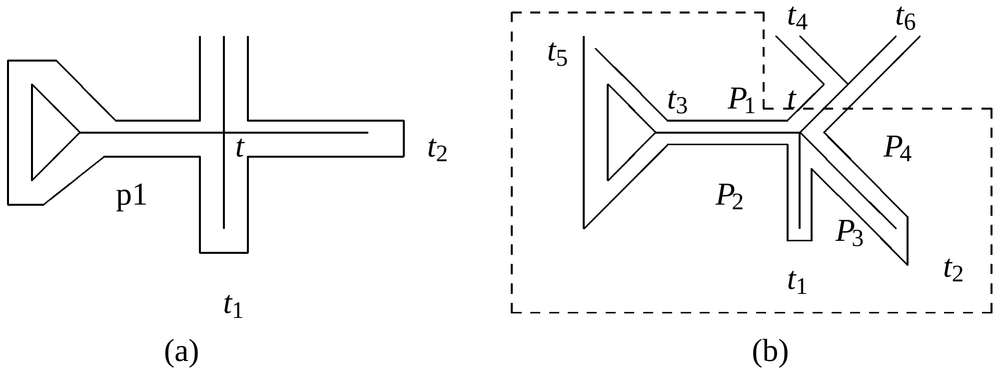

Figure 14: Two cases of type (2) good point. (a) Case 1. (b) Case 2.

There are d convex paths in F that go through t. We can use a 3-star connecting the three terminals to replace two of the d convex paths in the tour F. It is easy to verify that the number of times that t is used in the new 3-restricted T is reduced by 1, i.e., t is used at most twice (instead of three times).

(ii) Let t be a type (2) good point. Thus, t is of degree four. If there is no big angle at t, by Lemmas 5.1 and 5.2, the total number of big angles in T^* is reduced by 1, and t is used at most twice. So, we can assume that there is a big angle at t.

Let P_1 be the convex path in the tour F having the big angle at t. Let t_1 and t_2 be the two terminals adjacent to t. Two cases arise.

Case 1. t_1 or t_2, say, t_1, is in the convex path P_1 connecting t_1 and another terminal t_3 (see Figure 14 (a)). If $t_3 = t_2$, then the conex path in T_F connecting t_1 and t_2 has no small angle. Thus, the length $|t_1t_2| \leq (b+1)R$, i.e., the upper bound $(b+2)R$ in Lemma 5.1 is not tight. Therefore, the number of times that t is used in T_F is at most 2 (not 3). In this case, we do not have to modify the tree. If $t_3 \neq t_2$, we connect t_2 to the convex path P_1 at point t. This forms a 3-star with t as the center connecting three terminals t_1, t_2 and t_3. We then use the three line segments tt_1, tt_2 and tt_3 (not the 3 convex paths) to form the three edges of the 3-star. Let t_i and t_j be two points. We use t_it_j to represent the line segment connecting t_i and t_j. From Lemma 5.1, the big angle at t ensures that

$$C(tt_1) + C(tt_3) = C(tt_3).$$

Moreover, $C(tt_2) = b$ (t is not counted and there is no small angle in tt_2), where b is the number of big angles in edge tt_2 in T_F. Note that , $C(t_1t_2)$ is estimated as at least $b+1$, in Lemma 5.5. Thus, the number of times that t is used in T is reduced from at most 3 to at most 2.

Case 2. Neither t_1 nor t_2 is in the convex path that has the big angle. Let t_3 be the other child of t other than t_1 and t_2. See Figure 14 (b). In this case, t_3 is either the leftmost child or the rightmost child of t. Without loss of generality, we assume that t_3 is the leftmost child, and let P_1 be the convex path in F which connects two terminals t_4 and t_5 and contains the big angle at t(see Figure 14 (b)). Note that all the descendent terminals of t (in the dashed circle in Figure 14 (b)) are connected with paths P_1, P_2, P_3 and P_4. Thus, we can shorten P_1 to obtain P_1' by cutting off the part from t to t_5. By doing this, we get a 3-star with t as the center connecting t_1, t_2 and t_4. Since P_1 has a big angle at t, from Lemma 5.1, we know that the number of times that t is used in T is at most 2 (not 3).

In above discussion, we just consider the case where there is only one type (2) good point in the convex path P_1. Now, consider the case where there is more than one type (2) good point in P_1. See Figure 15 (a). Note that T_F is a tree having $n-1$ edges (corresponding to convex paths) connecting the n terminals. Let e be the edge in T_F which corresponds to P_1. Deleting the edge e forms two components (inside a dashed box),

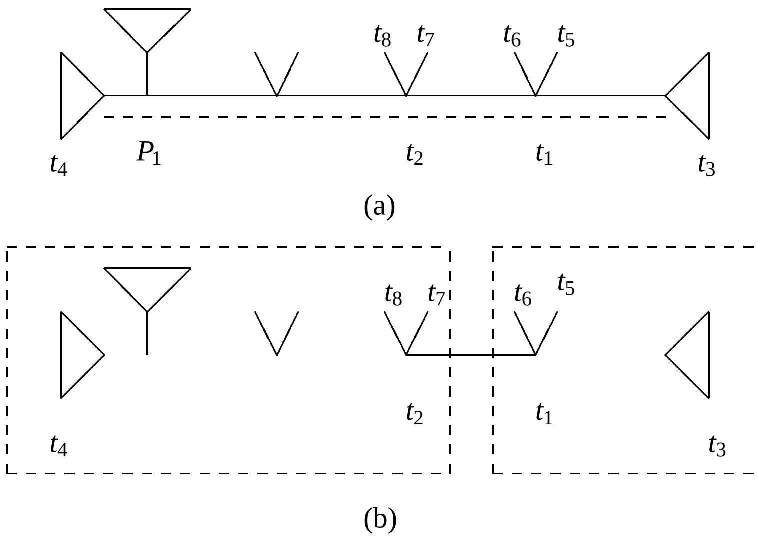

Figure 15: (a) The original T^*. The dashed line stands for P_1. (b) After deleting P_1, we have two components in the dashed boxes. We use an edge connecting t_1 and t_2 to form a tree again.

each contains an end of e (t_3 or t_4). (see Figure 15 (b)). The terminals in each of the two components are connected by the rest of the $(n-2)$ edges (possibly replaced by some 3-stars in the modification process).

Let t_1 and t_2 be the two type (2) good point in P_1 that are the leftmost and rightmost in the two components, respectively. See Figure 15 (b).

We replace e by a segment (see the line in Figure 15 (b) connecting the two boxes) connecting t_1 and t_2 directly. Thus, we get a tree again. This makes all type (2) good point in P_1 other than t_1 and t_2 appear in T at most twice (not three times). Moreover, we can form a 3-star with t_1 as the center connecting t_5, t_6 and t_7, or with t_2 as the center connecting t_5, t_7 and t_8. By doing this, one of t_1 and t_2 appears in T at most twice instead of three times.

Thus we can conclude that at least half of the type (2) good points are used in T at most twice instead of three times.

(iii) t has at least one child, say, t_1, and the degree of t is 3 (see Figure 16). If there is at most one big angle at t, then in Lemma 5.5, t is overestimated, i.e., T is used in the tour F at most twice (not three times). So, we assume that there are two big angles at t. Thus, at least one of the two convex paths P_2 and P_3, say, P_3, (See Figure 16.) has a big angle at t. We then can shorten the edge e corresponding to P_3 in T_F by cutting off the part from t to t_1 and form a 3-star with t as center connecting t_1, t_2 and t_3 (the other end of P_3). By doing this, we save at least one Steiner point, and thus the number of times that t is used in T is at most 2 (not 3).

Note that, in above modification, we merge P_2 and P_3 into a 3-star and save one Steiner point by taking the advantage of a big angle at t. Each convex path can only be used to form a 3-star once. Otherwise, we get an i-star for $i > 3$. Thus, we have to make sure that each type (3) good point t can match a unique convex path that has a big angle at t. This can be done since each type (3) good point has degree 3 and there would be two big angles at t. (If there is only one or zero big angle at t, then t is used only once or twice in T_F. Thus we do not have to do any modification.)

Consider the case that in the convex path P_1 there are many type (2) good points and type (3) good points. Using the same argument as in (ii) demonstrated in Figure 15, we can replace the edge corresponding to P_1 in T_F by the segment connecting t_1 and t_2 as in Figure 15. Thus, every type (2) and type (3) good point other

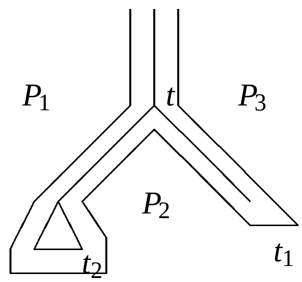

Figure 16: The type (3) Steiner point.

than t_1 and t_2 is used at most twice (not three times). In this case, t_1 and t_2 competes the edge t_1t_2 to form a 3-star. We assign t_1t_2 to either t_1 or t_2 using the following strategy: If a type (2) good point competes t_1t_2 with a type (3) good point, say, t, we always let the type (2) good point have t_1t_2 since the type (3) good point has another big angle at it, but the type (2) good point does not. If next time t competes with another type (2) good point or type (3) good point, we let t win. Thus, every type (3) good point appears at most twice in T and at least half of the type (2) good points appear in T at most twice.

Now, we can make sure that in T each good point of type (1) and (3) appears at most twice, and at least half of the type (2) good point appear twice. From Lemma 5.5, we have

$$C(T) \leq 2g_1 + 2.5g_2 + 3b + 2s_2 + 2 - C(e_n),$$

where g_1 is the number of good points of type (1) and (3), g_2 is the number of type (2) good points, and b is the number of bad points.

We can delete a convex path P_n from tour F to form T_F. We always delete the convex path such that the corresponding edge e_n is the longest.

In the following, we show that $b < g_1$. Let T' be the tree obtained from T^* by deleting all terminals. Obviously, each leave in T' is a good point. Therefore, the number of bad points is the number of points of degree at least 3 in T'. Thus, $b < g_1$.

If $C(e_n) \geq 2$, then we have

$$C(T) \leq 2g_1 + 2.5g_2 + 3b + 2s_2. \tag{2}$$

Therefore,

$$C(T) \leq 2g_1 + 2.5g_2 + 2.5b + 0.5b + 2s_2 \leq 2.5g_1 + 2.5g_2 + 2.5b + 2s_2 \leq 2.5C(T^*).$$

If $C(e_n) = 1$, then there is no big angle in T^*. Thus, there is no degree-3 point in T^*. Suppose that there are degree-4 points in T^*. Since there is not big angle in T^*, the number of times that the degree-4 Steiner points is overestimated. Thus, (2) still holds.

Now, we only have to consider the case where each Steiner point in T^* has degree 5. In this case, each point in T' is either of degree 1 or degree at least 3. Thus, the root of T' is either of degree 1 or degree at least 3. In this case, it is easy to see that the number of leaves of T' is at least two more than the number of points of degree at least 3, i.e., $g_1 \geq b + 2$. Therefore,

$$C(T) \leq 2g_1 + 2.5g_2 + 2.5b + 0.5b + 2s_2 + 1 \leq 2.5g_1 + 2.5g_2 + 2.5b + 2s_2 \leq 2.5C(T^*).$$

■

$\frac{5}{2}$-approximation Algorithm for STP-MSP

Input A set P of n terminals in the Euclidean plane, a positive constant R

Output A 3-restricted Steiner tree T in which each edge has length at most R.

1. Construct a weighted hypergraph $H_3(V,F,w)$.
2. Call the randomized algorithm in [8] to compute a minimum spanning tree T for $H_3(V,F,w)$;
3. Replace every edge f of the minimum spanning tree T' on $H_3(V,F,w)$ with a Steiner tree with $w(f)$ Steiner points such that the maximum length of each edge in the tree is at most R and output the obtained tree.

Figure 17: The complete algorithm.

Now, we focus on the computation of an optimal 3-restricted tree.

Let $H_3(V,F,W)$ be a weighted 3-hypergraph, where $V=P$, $F=\{(a,b)|a\in V \text{ and } b\in V\}\cup\{(a,b,c)|a\in V, b\in V \text{ and } c\in V\}$, and for each edge $e\in F$, $w(e)$ is the smallest number of Steiner points to form an optimal solution of the STP-MSP problem on the terminals in e.

Given three points a,b and c on the on the Euclidean plane, let s be the Steiner point which minimizes $(|sa|+|sb|+|sc|)$, and let k be the number of Steiner points in an optimum solution T of $STP-MSP$ on $\{a,b,c\}$ with constant R.

Lemma 5.7

$$\lceil\frac{|sa|}{R}\rceil+\lceil\frac{|sb|}{R}\rceil+\lceil\frac{|sc|}{R}\rceil-2\geq k\geq\lfloor\frac{|sa|}{R}\rfloor+\lfloor\frac{|sb|}{R}\rfloor+\lfloor\frac{|sc|}{R}\rfloor-2. \tag{3}$$

Proof. By steinerizing the optimum Steiner tree, we get a solution of $STP-MSP$ on $\{a,b,c\}$ with exactly $\lceil\frac{|sa|}{R}\rceil+\lceil\frac{|sb|}{R}\rceil+\lceil\frac{|sc|}{R}\rceil-2$ Steiner points.

Let $|T|$ be the total length of T, which is the sum of the length of edges of T. Then

$$(k-1)\cdot R+3R\geq|T|\geq|sa|+|sb|+|sc|,$$

i.e.,

$$k+2\geq\frac{|sa|+|sb|+|sc|}{R}\geq\lfloor\frac{|sa|+|sb|+|sc|}{R}\rfloor\geq\lfloor\frac{|sa|}{R}\rfloor+\lfloor\frac{|sb|}{R}\rfloor+\lfloor\frac{|sc|}{R}\rfloor.$$

Therefore, (3) holds. ∎

Lemma 5.7 gives an upper bound on the cost of (a,b,c).

Lemma 5.8 *[16] Testing whether three circles has a point in common can be done in constant time.*

For any given points a,b and c on on the Euclidean plane, one can find the minimum Steiner tree on $\{a,b,c\}$ in constant time. Let $q_{a,b,c}$ be the number of Steiner points used to steinerize the optimum Steiner tree on $\{a,b,c\}$, and $q_P=\max\{q_{a,b,c}|\{a,b,c\}\subset P\}$. Then, by Lemma 5.7 and Lemma 5.8, the weight W of $H_3(V,F,W)$ can be calculated in $O(n^3q_P^2)$ time. By Theorem 5.4 and Theorem 5.6, we have

Theorem 5.9 *Given a set P of n terminals and a positive constant R, there exists a randomized algorithm that computes a solution of $STP-MSP$ on P such that the number of Steiner points is at most $\frac{5}{2}$ times of the optimum running in poly(n,q_P) time with probability at least* 0.5.

The complete algorithm is given in Figure 17.

6 Conclusion

In this paper, we studied how relay sensors can influence the minimum power topology in wireless sensor networks. With the introduction of 1 or 2 relay sensors, the total consumed power for maintaining minimum power topology and the maximum degree for the minimum power topology are decreased. These reductions are more significant for sparse topology. If we restrict the transmission power for each sensor to some small value and use relay sensors to guarantee connectivity, we achieve similar results. This problem is formulated to STP-MSP, a NP-hard network optimization problem. This paper also proposed two approximate algorithms (with performance analysis) for STP-MSP.

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