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The Displacement Analysis of the Generalized Tracta Coupling[1]

The displacement analysis of spatial linkages has been the subject of a number of recent investigations, using a variety of mathematical approaches. Algebraic solutions have been developed principally, in cases in which the number of links, n, is less than or equal to 4. When $n > 4$, the complexity of the displacement analysis appears to increase by one or more orders of magnitude. In this paper we describe a method, which we call the geometric-configuration method, which we have used when $n > 4$. The method is illustrated with respect to the algebraic displacement analysis of a five-link spatial mechanism, which includes the Tracta joint as a special case. The Tracta joint is a spatial linkage of symmetrical proportions functioning as a constant-velocity universal joint for nonparallel, intersecting shafts (Myard, 1933). It has four turning or revolute pairs (R) and one plane pair (E), which is located symmetrically with respect to the input and output shafts. The generalization of this linkage, which we call the generalized Tracta coupling, is the R-R-E-R-R spatial linkage with general proportions. The displacement analysis of the general mechanism, for which we know of no previous solution, has been derived. An analysis of the effect of tolerances in the Tracta joint has been included.

Introduction

Single-degree-of-freedom, single-circuited spatial mechanisms may have up to seven links in the general case. The displacement analysis of this class of linkages, especially for $n \leq 4$, have been treated by a variety of approaches among which we may mention dual vectors, Dimentberg [8, 9];[2] matrices, Denavit, [6, 7]; Uicker, Denavit, and Hartenberg, [17], Uicker [18], Blesch, [3], etc.; quaternions, Yang [20–22]; vectors, Chace [4, 5]; tensors, Ho [11]; Sandor and Bisshopp [16]; the Mayor-Mises mapping, Beyer [2], Lowen [12], and others. Some of these are concerned primarily with numerical methods and others with algebraic methods. Both can be highly useful. We shall be concerned with the latter. Algebraic solutions tend to be particularly valuable in providing insight of a general nature. For example, the effects of parameter changes and tolerances, the determination of the number of circuits or ways to assemble the mechanism, ranges of motion, locking conditions, overclosure, etc. Numerical solutions are of course invaluable for many purposes, and the foregoing remarks are explanatory, rather than comparative.

With respect to algebraic solutions there seems to be at least an order-of-magnitude difference between $n = 4$ and $n > 4$. Most of the approaches just mentioned are successful in the former and much less or not at all (so far) in the latter case. The reasons are described in Wallace [19]. The only general algebraic solutions known to the authors, Dimentberg [8, 9]; Yang [22], for $n > 4$, are for five-link linkages with turning and cylindrical pairs. The general seven-link (R^7) linkage, of which most others are special cases, remains an unattained goal. The geometric configuration method, which we shall describe, is intended to circumvent some of the difficulties encountered when $n > 4$.

Among the motivations for this research is the application of multilink ($n \geq 5$) spatial mechanisms in shaft couplings. Ideally the coupling would be constant-velocity regardless of the linear and angular offset between the shafts. The double-Hooke joint, Duditsa [10], Mabie and Ocvirk [13], can have a constant

[1] Based on the doctoral dissertation of the first author in partial fulfillment of the requirements for the degree of Doctor of Engineering Science in the Department of Mechanical Engineering, Columbia University, New York.

[2] Numbers in brackets designate References at end of paper.

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velocity ratio under certain conditions. A numerical error analysis of this joint can be found in Austin, et al. [1]. Many other universal, multilink ($n > 4$) shaft couplings are known, which function as constant-velocity joints for intersecting shafts in view of symmetrical proportions (Clemens, Myard, Tracta, Voss, Wachter, and Riegel, Southwestern Industries, etc.). The Tracta joint is manufactured by Girling Ltd. in Birmingham, England, whose literature describes the constructional details.

The displacement analysis of the generalized Tracta coupling may shed some light not only on the geometric-configuration method, but also on the order of magnitude of the complexity of the displacement analysis of the general seven-link R^7 linkage, of which it (and many others) may be regarded as a special case.

Geometric-Configuration Method

This is a geometrical approach in which the linkage is first considered as disassembled into two configurations and then reassembled under appropriate constraints. This idea is not new; indeed it was used with extraordinary success by Dimentberg [8, 9] in a different way.

In the present approach, we consider the displacements of the input and output members as prescribed. This converts the mechanism into a structure or geometric configuration and the problem from a kinematic to a geometric one. We then consider the structure as made up of two substructures or configurations. The first, or "*frame configuration*" consists of the input link, the frame, and the output link. The second, or "*floating link configuration*" consists of the remainder of the mechanism. Each configuration can be characterized by a condition expressing the relative orientation of its terminal links. The two configurations are then reassembled with the constraint that the relative orientation of both sets of terminal links must be compatible. The success of the method hinges on the possibility of expressing the configuration conditions in a form which does not require the elimination of unwanted motion parameters. In the case of the *R-R-S-R-R* linkage (which did not yield to dual-vector methods so far [8, 9] and the *R-R-E-R-R* linkage, this method has been successful. We shall describe the analysis of the *R-R-E-R-R* linkage. A more detailed account can be found in [19].

General Tracta Coupling or *R-R-E-R-R* Linkage

The linkage is shown in Fig. 1 with frame $A_1A_{51}A_5A_{45}$; input link A_1A_{12}; floating links $A_{12}A_2E_{23}$, $A_4A_3E_{23}$; and output link A_4A_{45}, input axis 1 and output axis 5; revolute pairs at A_1, A_{12}, A_4, A_{45}; and a plane pair at E_{23}. The rotation, θ_1, of the input link about axis No. 1 is communicated by the linkage into a rotation, θ_5, of the output link about axis No. 5. The dimensions of the linkage are the linear and angular offsets between revolute pairs $[(a_1, \alpha_1), (a_4, \alpha_4), (a_5, \alpha_5)]$; the distances along the revolute axes to the feet of the common perpendiculars between the axes (S_1, S_2, S_4, S_5); and the angle between the plane pair, E_{23}, and revolute axes 2 and 4 (α_2, α_3). The displacement equation is the functional relationship between angles θ_1 and θ_5.

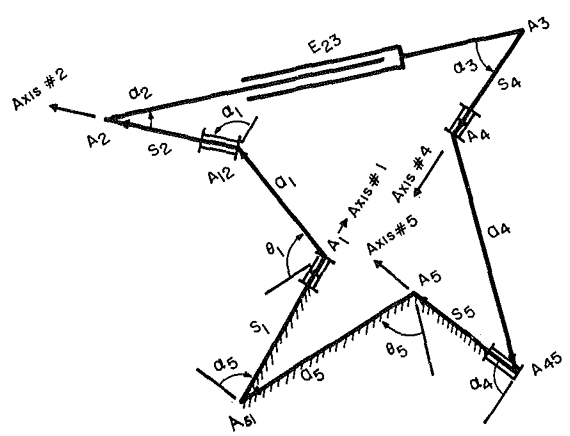

Fig. 1 Generalized Tracta coupling (*R-R-E-R-R*) spatial linkage

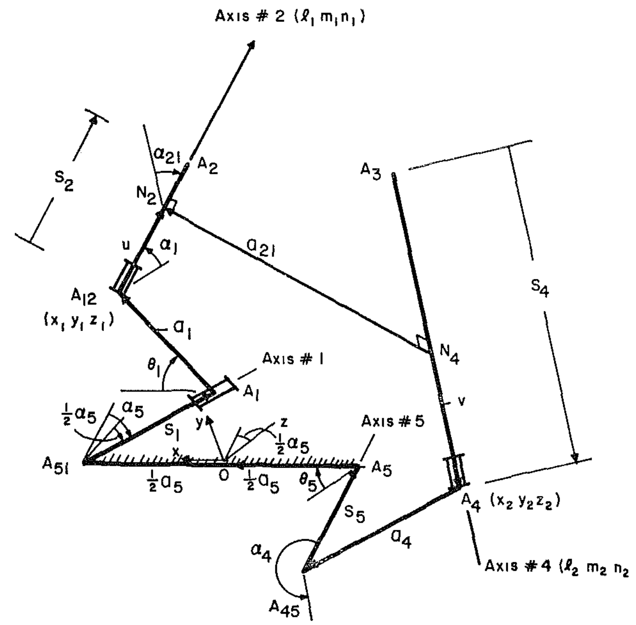

Fig. 2 Frame configuration

Frame Configuration *R-R*

The configuration is shown in Fig. 2. The configuration condition is the relative orientation between axes 2 and 4. This is defined by the four quantities:

a_{21} = common perpendicular between axes 2 and 4,

α_{21} = angle between axes 2 and 4,

$u = \mathbf{A}_{12}\mathbf{N}_2$,

$v = \mathbf{A}_4\mathbf{N}_4$.

The given quantities (i.e., quantities readily expressible as functions of θ_1, θ_5 and the linkage proportions) are:

l_i, m_i, n_i ($i = 1, 2$) = direction cosines of axes 2, 4;

x_1, y_1, z_1 = coordinates of A_{12},

x_2, y_2, z_2 = coordinates of A_4.

The origin and xyz-axes are as shown in the figure (origin, O, midway between A_{51} and A_5; x-axis along a_5 toward A_{51}; z-axis bisects the angle between the planes $A_1A_{51}A_5$ and $A_{51}A_5A_{45}$).

Then

$$\cos \alpha_{21} = l_1l_2 + m_1m_2 + n_1n_2. \quad (1)$$

Let u' be the running variable along axis 2, through $A_{12}(x_1, y_1, z_1)$ with (l_1, m_1, n_1) as the positive direction of axis 2, and define v' in a similar manner on axis 4 through $A_4(x_2, y_2, z_2)$. If Q_1 is any point on axis No. 2 and Q_2 is any point on axis No. 4, then the distance, S, between Q_1 and Q_2 is given by

$$S^2 = (x_1 + l_1u' - x_2 - l_2v')^2 + (y_1 + m_1u' - y_2 - m_2v')^2 + (z_1 + n_1u' - z_2 - n_2v')^2. \quad (2)$$

To find the common perpendicular, $a_{21} = N_2N_4$, it is necessary that $(\partial S/\partial u') = (\partial S/\partial v') = 0$. If $S \neq 0$, this will lead to two simultaneous equations in u' and v'. These can be solved for u and v and hence for a_{12}. The results are:

$$u = \frac{q_1 - q_2 \cos \alpha_{12}}{\sin^2 \alpha_{12}}, \tag{3}$$

$$v = \frac{q_1 \cos \alpha_{12} - q_2}{\sin^2 \alpha_{12}}, \tag{4}$$

$$a_{12}^2 = Q - \frac{(q_1^2 - 2q_1q_2 \cos \alpha_{12} + q_2^2)}{\sin^2 \alpha_{12}}, \tag{5}$$

where

$$\begin{aligned} Q &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2, \\ q_1 &= l_1(x_2 - x_1) + m_1(y_2 - y_1) + n_1(z_2 - z_1), \\ q_2 &= l_2(x_2 - x_1) + m_2(y_2 - y_1) + n_2(z_2 - z_1). \end{aligned} \tag{6}$$

From the coordinate system, shown in Fig. 2, we can express (x_i, y_i, z_i) and (l_i, m_i, n_i) as functions of θ_1, θ_5 and the linkage constants. These define $\cos \alpha_{12}$ Q, q_1, and q_2 and in turn the configuration-condition parameters $(a_{21}, \alpha_{21}, u, v)$ as functions of θ_1, θ_5 and the linkage parameters.

It will be sufficient at this stage to give the expressions for Q, $\cos \alpha_{21}$, q_1, and q_2 as functions of θ_1 and θ_5. The latter will be expressed in half-tangent form. After a substantial amount of algebra [19], these expressions are obtained as follows:

Let $u_1 = \tan \frac{1}{2} \theta_i$ and $u_5 = \tan \frac{1}{2} \theta_5$

$$\begin{aligned} A_1 &= (a_5 - a_4 - a_1) & \phi_1 &= (\alpha_5 - \alpha_4 - \alpha_1) \\ A_2 &= (a_5 - a_4 + a_1) & \phi_2 &= (\alpha_5 - \alpha_4 + \alpha_1) \\ A_3 &= (a_5 + a_4 - a_1) & \phi_3 &= (\alpha_5 + \alpha_4 - \alpha_1) \\ A_4 &= (a_5 + a_4 + a_1) & \phi_4 &= (\alpha_5 + \alpha_4 + \alpha_1) \end{aligned} \tag{7}$$

$$\begin{aligned} Q(1 + u_5^2)(1 + u_1^2) &= u_5^2\{u_1^2(A_1^2 + S_1^2 + 2S_1S_5 \cos \alpha_5 + S_5^2) \\ &\quad + u_1(4a_1S_5 \sin \alpha_5) + (A_2^2 + S_1^2 + 2S_1S_5 \cos \alpha_5 + S_5^2)\} \\ &\quad + u_5\{u_1^2(4a_4S_1 \sin \alpha_5) + u_1(-8a_1a_4 \cos \alpha_5) + 4a_4S_1 \sin \alpha_5\} \\ &\quad + \{u_1^2(A_3^2 + S_1^2 + 2S_1S_5 \cos \alpha_5 + S_5^2) + u_1(4a_1S_5 \sin \alpha_5) \\ &\quad + (A_4^2 + S_1^2 + 2S_1S_5 \cos \alpha_5 + S_5^2)\} \end{aligned} \tag{8}$$

$$\begin{aligned} \cos \alpha_{21}(1 + u_1^2)(1 + u_5^2) &= u_5^2\{u_1^2(-\cos \phi_1) - \cos \phi_2\} + u_5u_1(-4 \sin \alpha_4 \sin \alpha_1) \\ &\quad + u_1^2(-\cos \phi_3) - \cos \phi_4 \end{aligned} \tag{9}$$

$$\begin{aligned} q_1(1 + u_1^2)(1 + u_5^2) &= u_5^2\{u_1^2[-S_1 \cos \alpha_1 - S_5 \cos (\alpha_5 - \alpha_1)] \\ &\quad + u_1(-2(a_5 - a_4) \sin \alpha_1) - S_1 \cos \alpha_1 - S_5 \cos (\alpha_5 + \alpha_1)\} \\ &\quad + u_5\{u_1^2(-2a_4 \sin (\alpha_5 - \alpha_1) - 2a_4 \sin (\alpha_5 + \alpha_1)\} \\ &\quad + \{u_1^2[-S_1 \cos \alpha_1 - S_5 \cos (\alpha_5 - \alpha_1)] \\ &\quad + u_1[-2(a_5 + a_4) \sin \alpha_1] - S_1 \cos \alpha_1 - S_5 \cos (\alpha_5 + \alpha_1)\} \end{aligned} \tag{10}$$

$$\begin{aligned} q_2(1 + u_1^2)(1 + u_5^2) &= u_5^2\{u_1^2[S_5 \cos \alpha_4 + S_1 \cos (\alpha_5 - \alpha_4)] \\ &\quad + u_1[2a_1 \sin (\alpha_5 - \alpha_4)] + S_5 \cos \alpha_4 + S_1 \cos (\alpha_5 - \alpha_4)\} \\ &\quad + u_5\{u_1^2[2(a_5 - a_1) \sin \alpha_4] + 2(a_5 + a_1) \sin \alpha_4\} \\ &\quad + \{u_1^2[S_5 \cos \alpha_4 + S_1 \cos (\alpha_5 + \alpha_4)] \\ &\quad + u_1[2a_1 \sin (\alpha_5 + \alpha_4)] + S_5 \cos \alpha_4 + S_1 \cos (\alpha_5 + \alpha_4)\} \end{aligned} \tag{11}$$

Equations (8)–(11) in conjunction with equations (1), (3)–(5) define the frame-configuration condition.

Floating-Link Configuration *R-E-R*

This is shown in Fig. 3. The configuration condition is once again the relative orientation between axes 2 and 4. The four quantities defining this orientation are

a_{42} = common perpendicular between axes 2 and 4

α_{42} = angle between axes 2 and 4

$S_2' = \mathbf{N_2A_2}$

$S_4' = \mathbf{N_4A_3}$.

The direction cosines of axes 2 and 4 are (l_2, m_2, n_2) and (l_4, m_4, n_4), respectively; angles α_2, α_3 are the fixed angles between the plane pair E_{23} and revolute axes 2, 4.

In Fig. 3, if the plane pair, E_{23}, is disconnected, a line of each half of the plane will generate a cone when allowed to rotate freely about the axis of the adjacent revolute pair. The configuration condition is the condition that these two cones have a common tangent plane. The general condition for the tangency of a quadric surface and a plane is complex [15]. When the quadric surface is a cone, however, the general condition fails. But two simpler conditions can be developed as follows. First, the plane must pass through the vertices of the cones, and second, a normal to the plane must form given angles with the cone axes.

The equation of a plane in normal form is

$$\underline{A}x + \underline{B}y + \underline{C}z + \underline{D} = 0, \tag{12a}$$

where

$$\underline{A}, \underline{B}, \underline{C}, \underline{D} \text{ are constants and } \underline{A}^2 + \underline{B}^2 + \underline{C}^2 = 1. \tag{12b}$$

Since the plane passes through the cone vertices $A_2(x_2, y_2, z_2)$ and $A_3(x_4, y_4, z_4)$, we have

$$\underline{A}x_i + \underline{B}y_i + \underline{C}z_i + \underline{D} = 0 \qquad (i = 2, 4). \tag{13a, b}$$

The condition that the plane forms a given angle, α_2, with axis 2 is

$$\underline{A}l_2 + \underline{B}m_2 + \underline{C}n_2 = \sin \alpha_2. \tag{13c}$$

Similar considerations with respect to axis 4 yields

$$\underline{A}l_4 + \underline{B}m_4 + \underline{C}n_4 = \sin \alpha_4. \tag{13d}$$

The set (13a, b, c, d) can be solved for $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$, which can then be substituted into equation (12a, b). When (l_i, m_i, n_i) and (x_i, y_i, z_i) $[i = 2, 4]$ in the resulting expression are expressed as functions of the linkage proportions using the coordinate system, shown in Fig. 3, the geometric configuration condition is obtained:

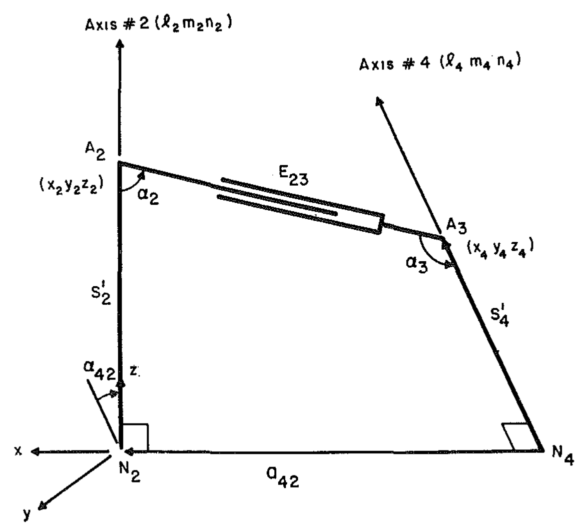

Fig. 3 Floating-link configuration

$$a_{42}^2(\sin^2\alpha_{42} - \sin^2\alpha_2 - \sin^2\alpha_3 + 2\sin\alpha_2\sin\alpha_3\cos\alpha_{42}) - \sin^2\alpha_{42}(S_4'\sin\alpha_3 - S_2'\sin\alpha_2)^2 = 0. \quad (14)$$

In the case when both cone angles (α_2, α_3) vanish, equation (14) implies that either $a_{42} = 0$ or $\sin\alpha_{42} = 0$. This is geometrically reasonable, because in order to pass a plane through two lines (the degenerate case of two cones with zero cone angles); either the lines must be parallel ($\sin\alpha_{42} = 0$), or they must intersect ($a_{42} = 0$). The case of intersecting lines is of interest in the Tracta joint.

R-R-E-R-R Displacement Equation

Reassembly of the frame and floating-link configurations requires the following compatibility equations, Fig. 4:

$$\begin{aligned} \alpha_{42} &= \alpha_{21}, \\ a_{42} &= a_{21}, \\ S_2' &= S_2 - u, \\ S_4' &= S_4 - v. \end{aligned} \quad (15)$$

The variables on the right-hand side of (15) are the frame-configuration parameters (α_{21}, a_{21}, u, v) which were defined as functions of Q, q_1, and q_2 in equations (3), (4), (8)–(11). Hence, using equation (15) in conjunction with (8)–(11), we can express α_{42}, a_{42}, S_2', S_4' as functions of Q, q_1, q_2 and substitute the resulting expressions into equation (14), the floating-link configuration condition. This will give the displacement equation in a form involving only θ_1, θ_5 and the linkage dimensions. The algebra involved in carrying out this procedure is formidable, but feasible [19]. The resulting displacement equation is as follows:

$$\begin{aligned} -(Q + R\cos\alpha_{21}) &+ 2\sin\alpha_2\sin\alpha_3(R + q_1q_2) + \cos^2\alpha_2(Q - q_2)^2 \\ &+ \cos^2\alpha_3(Q - q_1)^2 - \sin^2\alpha_{21}(S_2\sin\alpha_2 - S_4\sin\alpha_3)^2 \\ &+ 2(S_2\sin\alpha_2 - S_4\sin\alpha_3)[(q_1 - q_2\cos\alpha_{21})\sin\alpha_2 \\ &- (q_1\cos\alpha_{21} - q_2)\sin\alpha_3] = 0. \quad (16) \end{aligned}$$

where $R = Q\cos\alpha_{21} - 2q_1q_2$, and expressed as a function of θ_1 and θ_5 is as follows:

$$\begin{aligned} R(1 + u_1^2)(1 + u_5^2) &= u_5^2\{u_1^2[-A_1^2\cos\phi_1 + S_1^2\cos\phi_2 + 2S_1S_5\cos(\alpha_4 - \alpha_1) \\ &+ S_5^2\cos\phi_3] + u_1[2S_5\{A_1\sin(\alpha_4 + \alpha_1) - A_2\sin(\alpha_4 - \alpha_1)\} \\ &- 2S_1\{A_1\sin\phi_1 - A_2\sin\phi_2\}] + [-A_2^2\cos\phi_2 \\ &+ S_1^2\cos\phi_1 + 2S_1S_5\cos(\alpha_4 + \alpha_1) + S_5^2\cos\phi_4]\} \\ &+ u_5\{u_1^2[-2S_5(A_1\sin\phi_1 - A_3\sin\phi_3) + 2S_1\{A_1\sin(\alpha_4 + \alpha_1) \\ &+ A_3\sin(\alpha_4 - \alpha_1)\}] + u_1[2A_1A_4\cos(\alpha_4 - \alpha_1) \\ &- 2A_2A_3\cos(\alpha_4 + \alpha_1) - 4\sin\alpha_4\sin\alpha_1\{S_1^2 + S_5^2 \\ &+ 2S_1S_5\cos\alpha_5\}] + [-2S_5(A_2\sin\phi_2 - A_4\sin\phi_4) \\ &+ 2S_1A_2\sin(\alpha_4 - \alpha_1) + 2S_1A_4\sin(\alpha_4 + \alpha_1)] \\ &+ \{u_1^2[-A_3^2\cos\phi_3 + S_1^2\cos\phi_4 + 2S_1S_5\cos(\alpha_4 + \alpha_1) \\ &+ S_5^2\cos\phi_1] + u_1[-2S_5A_3\sin(\alpha_4 - \alpha_1) \\ &+ 2S_5A_4\sin(\alpha_4 + \alpha_1) - 2S_1A_3\sin\phi_3 + 2S_1A_4\sin\phi_4] \\ &+ [-A_4^2\cos\phi_4 + S_1^2\cos\phi_3 + 2S_1S_5\cos(\alpha_4 - \alpha_1) \\ &+ S_5^2\cos\phi_2]\}. \quad (17) \end{aligned}$$

Equation (16), together with (8)–(11), (17), represents the displacement equation of the generalized Tracta coupling or *R-R-E-R-R* spatial linkage.

In arriving at this form of the displacement equation, considerable effort was made to obtain a relatively compact form, free of extraneous roots. The key to achieving this objective was the grouping of variables represented by the symbol R, equation (17). As was not apparent to us at first sight, R is reducible to

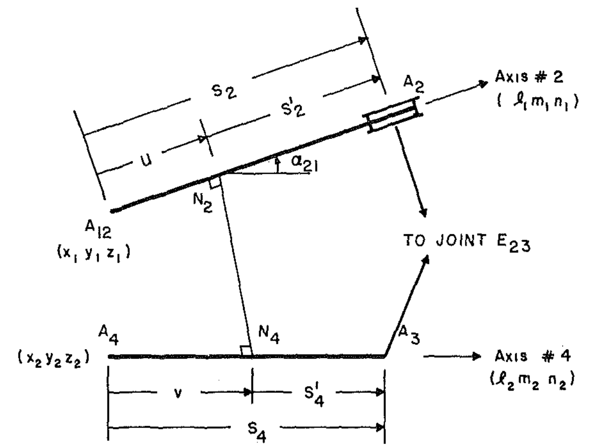

Fig. 4 Compatibility diagram

a function linear in the sines and cosines of θ_1 and θ_5. Examination of equation (16) readily yields the following conclusion:

Theorem. The degree of the input-output equation of the general Tracta coupling in half-tangent form is four. Hence, for a given position of the input shaft, the linkage may in general be assembled in a maximum of four different ways.

Case of Intersecting Floating-Revolute Axes

The case when axes 2 and 4 intersect is a special case of equation (16). It is easier, however, to start anew. In Fig. 2, the condition that a_{21} vanishes is obtained by setting S, equation (2), equal to zero. This requires that

$$\nabla = \begin{vmatrix} l_1 & l_2 & x_2 - x_1 \\ m_1 & m_2 & y_2 - y_1 \\ n_1 & n_2 & z_2 - z_1 \end{vmatrix} = 0. \quad (18)$$

When this determinant is expanded and the values of (l_i, m_i, n_i), (x_i, y_i, z_i), ($i = 1, 2$) derived from Fig. 2 are substituted into the expansion, the displacement equation is obtained:

$$\begin{aligned} u_5^2\{u_1^2A_1\sin\phi_1 &+ u_1[-2S_1\sin(\alpha_5 - \alpha_4)\sin\alpha_1 \\ &+ 2S_5\sin\alpha_4\sin\alpha_1] + A_2\sin\phi_1\} \\ &+ u_5\{u_1^2[2S_1\sin\alpha_4\sin\alpha_1 - 2S_5\sin(\alpha_5 - \alpha_1)\sin\alpha_4] \\ &+ u_1[-2(a_4 + a_1)\sin(\alpha_4 + \alpha_1) - (a_4 - a_1)\sin(\alpha_4 - \alpha_1)] \\ &+ [-2S_1\sin\alpha_4\sin\alpha_1 - 2S_5\sin(\alpha_5 + \alpha_1)\sin\alpha_4]\} \\ &+ \{u_1^2A_3\sin\phi_3 + u_1[-2S_1\sin(\alpha_5 + \alpha_4)\sin\alpha_1 \\ &- 2S_5\sin\alpha_4\sin\alpha_1] + A_4\sin\phi_4\} = 0 \quad (19) \end{aligned}$$

where u_i, A_j, and ϕ_j ($i = 1, 5$), ($j = 1, 2, 3, 4$) are as defined in equation (7). Equation (19) represents the input-output relationship for any *R-R-E-R-R* linkage with zero cone angles and further analysis is possible using this relationship. One result, however, is apparent at once: the linkage can be assembled in just two ways.

Tracta Joint and The Effect of Tolerances

The *R-R-E-R-R* linkage with the symmetrical proportions used in the constant-velocity coupling will be called the *Tracta joint*, rather than the Tracta coupling.

The nominal Tracta joint with nonparallel, but intersecting shafts has the following proportions:

$$\begin{aligned} S_2 &= S_4 = 0 & \alpha_3 &= \alpha_2 = 0 \\ a_1 &= a_4 = 0 & \alpha_4 &= \alpha_1 = 90\text{ deg} \\ S_1 &= S_5 & a_5 &= 0 \\ & & \alpha_5 &\neq 0 \text{ and possibly variable.} \end{aligned} \quad (20)$$

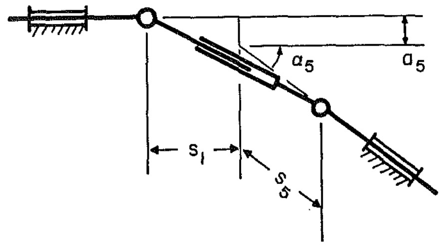

Fig. 5 Tracta joint with general housing errors

In the construction of the Tracta joint, a spherical housing encloses the mechanism, in order to assure the intersection of the coupled shafts regardless of variation of the angle between them. Hence, we define the *general housing error* as the case when $S_1 \neq S_5$ and $a_5 \neq 0$, the other parameters being nominal, as given in (20). This case is shown in Fig. 5. The coupled shafts are then nonparallel ($\alpha_5 \neq 0$) nonintersecting ($a_5 \neq 0$), and one shaft extends further into the housing than the other ($S_1 \neq S_5$).

When $\alpha_2 = \alpha_3 = 0$, as in equation (20), the general displacement equation (16) reduces to

$$Q \sin^2 \alpha_{21} + 2q_1q_2 \cos \alpha_{21} - q_1^2 - q_2^2 = 0. \qquad (21)$$

Or, another approach can be used. Equation (19) is applicable to the Tracta joint with housing errors, for as long as the cone angles remain zero, axes 2 and 4 intersect. If the dimensions $a_4 = a_1 = 0$, $\alpha_1 = \alpha_4 = 90$ deg and $\alpha_2 = \alpha_3 = 0$ are substituted into equation (19), the displacement equation for the Tracta joint with housing errors assumes the following form

$$\begin{aligned} u_5^2\{u_1^2(-a_5 \sin \alpha_5) + u_1[2(S_1 \cos \alpha_5 + S_5)] + a_5 \sin \alpha_5\} \\ + u_5\{u_1^2[2(S_1 + S_5 \cos \alpha_5)] - 2(S_1 + S_5 \cos \alpha_5)\} \\ + \{u_1^2 a_5 \sin \alpha_5 + u_1[-2(S_1 \cos \alpha_5 + S_5)] - a_5 \sin \alpha_5\} = 0. \end{aligned} \qquad (22)$$

Apparently equation (21) is equation (22) squared.

This equation can be converted into

$$(S_1 \cos \alpha_5 + S_5) \tan \theta_1 + (S_1 + S_5 \cos \alpha_5) \tan \theta_5 = -a_5 \sin \alpha_5. \qquad (23)$$

If $a_5 = 0$ and $S_5 = S_1$, equation (23) shows that

$$\tan \theta_5 = -\tan \theta_1,$$

or

$$\theta_5 = -\theta_1 \qquad \text{or} \qquad (-\theta_1 + \pi). \qquad (24)$$

Keeping in mind the sign convention we have used, it follows from equation (24) that the nominal Tracta joint transmits a unity, constant, angular-velocity ratio, a result derived in a different manner by Myard [14]. Equation (24) indicates that by assembling the joint in a different manner—for example, by removing the pin in either of the floating revolute pairs, rotating the output shaft 180 deg, and then reinstalling the pin—the same one-to-one coupling will be obtained.

On the other hand, if $\sin \alpha_5 = 0$, equation (23) also reduces to equation (24). Thus the Tracta joint will function as a constant-velocity joint under either of the following conditions:

1 $a_5 = 0$, $S_5 = S_1$; α_5 arbitrary, but $\cos \alpha_5 \neq -1$.

2 $\alpha_5 = 0$, a_5; S_5 and S_1 arbitrary.

The usual operation of the Tracta joint corresponds to case 1, while case 2 corresponds to a condition for which an Oldham coupling can also be used.

Numerical Results

The case when the floating revolute axes intersect, equation (19), which includes the Tracta joint with housing errors, has

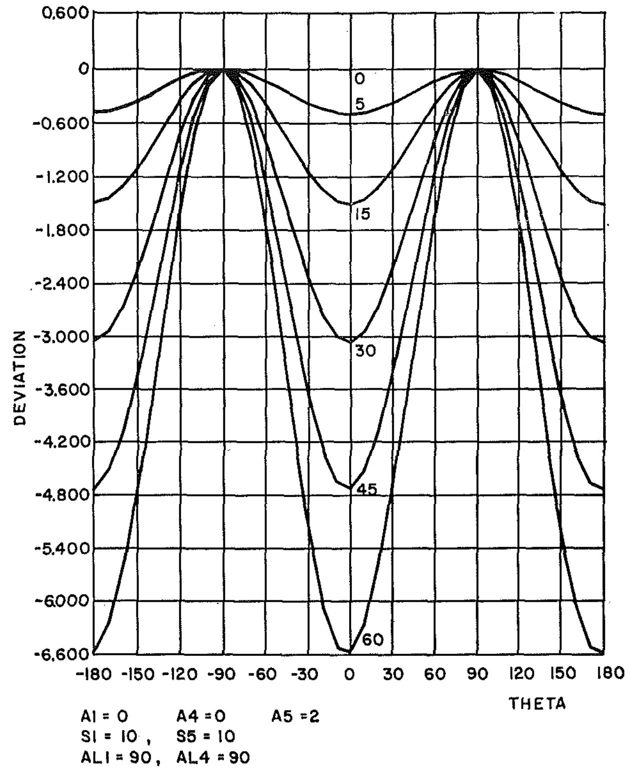

Fig. 6 Tracta joint shaft-displacement error with nonintersecting shafts

been programmed (IBM 7094) and the results plotted on a Stromberg-Carlson plotter. The program and numerical results are given in reference [19].

The effect of tolerances giving rise to housing errors were investigated. All computations were carried out with $S_1 = 10$, $a_1 = a_4 = 0$, $\alpha_1 = \alpha_4 = 90$ deg and $\alpha_2 = \alpha_3 = 0$ deg. For the nominal Tracta joint, $\theta_5 = -\theta_1$. Hence, we may define the angular-displacement error or deviation $\delta = \theta_1 + \theta_5$. The curves, Figs. 6–8, show δ as a function of input rotation θ_1. A family of curves is shown on each graph. Each curve corresponds to a particular value of α_5 (0, 5, 15, 30, 45, and 60 deg).

Fig. 6 shows a case of symmetrically disposed input and output shafts ($S_1 = S_5 = 10$), which are not intersecting ($a_5 = 2$). The maximum deviation, δ_{max}, for $\alpha_5 = 60$ deg is equal to 6.60 deg. The maximum deviation varies almost linearly with offset, a_5, so that for $a_5 = 1.0$, for example, δ_{max} is approximately equal to 3.30 deg.

Fig. 7 shows a case of intersecting ($a_5 = 0$), but unsymmetrically disposed input and output shafts ($S_1 = 10$, $S_5 = 12$). In this case the maximum deviation for $\alpha_5 = 60$ deg is equal to approximately 1.7 deg—a rather small error under the circumstances. By way of comparison, a single Hooke joint with a 60-deg shaft angle would have a maximum deviation of 35.3 deg, assuming its construction would permit its functioning without mechanical interference. The maximum deviation of the Tracta joint in this case is almost directly proportional to the nonsymmetry of the shaft positions relative to the housing ($S_5 - S_1$). Thus, for $\alpha_5 = 60$ deg, $S_1 = 10$ and $S_5 = 10.1$, $\delta_{max} = 0.093$ deg, while for the same values of α_5 and S_1, but with $S_5 = 11.0$, $\delta_{max} = 0.9$ deg.

Fig. 8 shows corresponding curves for combined housing errors, i.e., nonintersecting ($a_5 \neq 0$) and unsymmetrically disposed shafts ($S_1 \neq S_5$). The particular dimensions for this case are $a_5 = 1.0$ and $S_5 = 11.0$ and it can be seen that the effect of the nonintersection predominates over the nonsymmetry of the shaft positions relative to the housing.

Professor Sanders has shown that the general housing error for the Tracta joint can be found directly when second-order

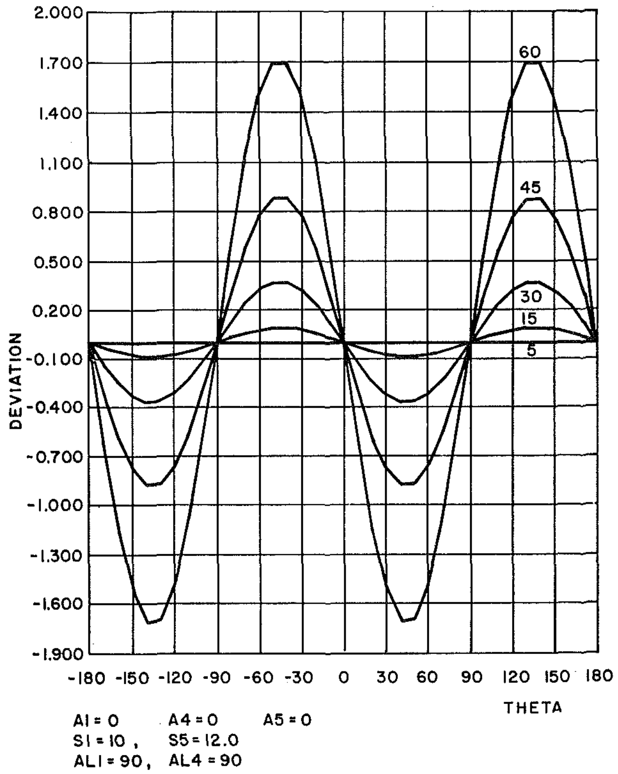

Fig. 7 Tracta joint shaft-displacement error with unsymmetrically disposed shafts

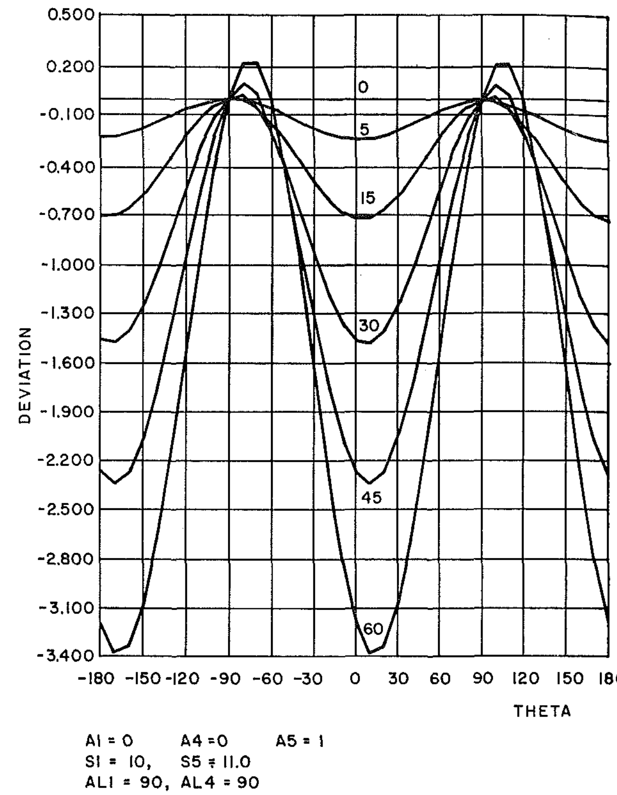

Fig. 8 Tracta joint shaft-displacement error for combined nonintersecting and unsymmetrically disposed shafts

tolerance effects are neglected. For, setting $\theta_5 = -\theta_1 + \delta$ (as before) and assuming δ to be small, we have

$$\tan \tfrac{1}{2}\,\theta_5 \cong -\tan \tfrac{1}{2}\,\theta_1 + \tfrac{1}{2}\,\delta\,(1 + \tan^2 \tfrac{1}{2}\,\theta_1).$$

Letting $S_5 = S_1 + \epsilon$, where ϵ is small, and neglecting terms in δ, a_5, and/or ϵ of order 2 and up, equation (22) can then be converted into the following form:

$$\delta = -\frac{a_5}{2S_1}\tan \tfrac{1}{2}\,\alpha_5\,(1 + \cos 2\theta_1) - \frac{(S_5 - S_1)}{2S_1}\tan^2 \tfrac{1}{2}\,\alpha_5 \sin 2\theta_1, \tag{25}$$

all angles being expressed in radians.

The results obtained from equation (25) correspond closely to those shown in Figs. 6–8. The equation also permits an approximate determination of $\delta_{\max}$ by setting $(d\delta/d\theta) = 0$. The result is

$$\delta_{\max} = -\frac{\tan \frac{1}{2}\,\alpha_5}{2S_1}\,[a_5 \pm \sqrt{a_5^2 + (S_5 - S_1)^2 \tan^2 \tfrac{1}{2}\,\alpha_5}] \tag{26}$$

This maximum occurs at $\theta_1 = \theta_{1\,\max}$, where

$$2\theta_{1\,\max} = \tan^{-1}\left[\left(\frac{S_5 - S_1}{a_5}\right)\tan \tfrac{1}{2}\,\alpha_5\right] \tag{27}$$

The case when $\alpha_2 = \alpha_3 \neq 0$ can be found in reference [19].

Conclusions

The displacement analysis of the general *R-R-E-R-R* spatial linkage has been derived using a method, which we have called the geometric-configuration method. The effect of tolerances (housing errors) on the displacements of the Tracta joint has been evaluated analytically and numerically. The method can be applied also to other multilink ($n > 4$) spatial linkages.

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