LAW OF THE ABSORPTION TIME OF POSITIVE SELF-SIMILAR MARKOV PROCESSES

P. PATIE

Abstract. Let X be a positive self-similar Markov process with 0 as an absorbing state. The purpose of this paper is to describe the law of the absorption time, say T_0, which might occurs continuously or by a jump. We start by showing that the distribution function of T_0 can be expressed in terms of an increasing invariant function for a specific transient Ornstein-Uhlenbeck process associated to X. Furthermore, specializing on the spectrally negative case, we suggest an original methodology to get a power series or an integral representation of this invariant function. Then, by means of probabilistic arguments, we deduce some interesting analytical properties satisfied by these functions, which include, for instance, several types of hypergeometric functions. We end the paper by detailing some known and new examples. In particular, we offer an alternative proof of the recent result obtained by Bernyk et al. [2] regarding the law of the maximum of regular spectrally positive stable processes.

1. Introduction

Let $X = ((X_t)_{t\geq 0}, (\mathbb{Q}_x)_{x>0})$ be a self-similar Hunt process with values in $[0,\infty)$. It means that X is a right-continuous strong Markov process with quasi-left continuous trajectories and there exists $\alpha > 0$ such that X enjoys the following self-similarity property: for each $c > 0$ and $x \geq 0$,

$$\text{the law of the process } (c^{-1}X_{c^{\alpha}t})_{t\geq 0}, \text{under } \mathbb{Q}_x, \text{ is } \mathbb{Q}_{x/c}.$$

α is called the index of self-similarity. This class of processes has been introduced and studied by Lamperti [24]. During the last two decades, they have been intensively studied, to name just a few, we mention the works [43], [15], [7], [11], [40], [17] and [37]. Furthermore, we recall that Lamperti's main motivation was the fact that self-similar Markov processes arise as the limit of properly normalized stochastic processes. It is then not very surprising that these processes appear in different settings, such as, for instance, coagulation-fragmentation processes [5], and continuous state branching processes with immigration [35]. We also mention that, recently, in the SLE context, Alberts and Sheffield [1] describe a measure-valued function supported on the intersection of a chordal SLE(κ) curve with $\mathbb{R}$, $4 < \kappa < 8$, in terms of the law of the absorption

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time T_0 of some Bessel processes. Next, Lamperti [24] states the following zero-one laws: if

$$T_0 = \inf\{s > 0; X_s = 0\}$$

then, for all $x > 0$, either $\mathbb{Q}_x(T_0 < \infty) = 1$ or $\mathbb{Q}_x(T_0 < \infty) = 0$. In this paper, our purpose is to describe the distribution function of T_0 in the former case. Unfortunately, beside some isolated cases the distribution of T_0 is not attainable. We mention the papers [14], [19] and [35] where such examples can be found and refer to the survey paper [8] for a description of these cases. Besides, two notable exceptions might be worth mentioning: when X is a Bessel process of negative index and when X is a regular spectrally negative Lévy process killed upon entering the negative half-line. In the former case, several proofs can be found in the literature, see for instance the excellent monograph of Yor [45] and the more recent survey papers of Matsumoto and Yor [26] and [27]. However, most of the proofs rely on the knowledge of the semigroup of Bessel processes. For the second case, Bernyk et al. [2] derive a representation of the distribution of T_0 by inverting, in a non trivial way, the known expression of the Wiener-Hopf factorization of regular stable one-sided Lévy processes. Our approach will differ from these two cases. Indeed, in general, we do not have access neither to the semigroup of X nor to the Laplace transform of T_0. We shall first show that the distribution function of T_0 can be described as an invariant function for a specific Ornstein-Uhlenbeck process associated to X. Then, specializing on the case when X has only negative jumps, we suggest a methodology, relying merely on probabilistic arguments, to represent this invariant function as a power series. It turns out that finding the distribution function of T_0 boils down to compute merely an exponential moment for the first passage time above for X.

We now take X to be realized as the coordinate process $X_t(\omega) = \omega(t)$ on the Skohorod space Ω of càdlàg functions from $[0, \infty[$ to itself. We assume that 0 is an absorbing state for X, so that each of the laws $\mathbb{Q}_x$ governing X is carried by $\{\omega \in \Omega : \omega(t) = 0, \forall t \geq T_0(\omega)\}$. The natural filtration on Ω is $(\mathcal{F}_t)_{t\geq 0}$, and $\mathcal{F}_\infty = \bigcup_{t\geq 0} \mathcal{F}_t$. Lamperti proved that for each fixed $\alpha > 0$, there is a bijective correspondence between $[0, \infty)$-valued self-similar Markov processes with index α and real-valued Levy processes, i.e. processes with stationary and independent increments. More specifically, by introducing the additive functional

$$\Sigma_t = \inf\{s > 0;\ A_s = \int_0^s X_r^{-\alpha} dr > t\}, \quad t \geq 0,$$

Lamperti [24] showed that the process $\xi = (\xi_t)_{t\geq 0}$, defined by

$$\xi_t \;=\; \log\left(X_{\Sigma_t}\right), \quad t \geq 0, \tag{1.1}$$

is a Lévy process. Observe that the additive functional $(A_t)_{t\geq 0}$ is then determined by

$$A_t = \inf\{s > 0;\ \Sigma_s = \int_0^s e^{\alpha \xi_r} dr > t\}, \quad t \geq 0.$$

We write $\mathbb{P}$ for the law of the Lévy process with $\xi_0 = 0$. Moreover, let Ψ be the characteristic exponent of ξ. It is well known, see [3] and [22] for background on Lévy processes, that it admits the following Lévy-Khintchine representation, for any $u \in i\mathbb{R}$, the imaginary line,

$$\Psi(u) = \bar{b}u + \frac{\sigma}{2}u^2 + \int_{\mathbb{R}} (e^{ur} - 1 - ur\mathbb{I}_{\{|r|<1\}})\nu(dr) - q \tag{1.2}$$

where the parameters $q \geq 0, \bar{b} \in \mathbb{R}, \sigma \geq 0$ and the measure ν which satisfies the integrability condition $\int_{\mathbb{R}}(1 \wedge r^2)\nu(dr) < +\infty$ uniquely determined the law of ξ_1. We shall refer to ξ (resp. Ψ)

as the underlying Lévy process (resp. characteristic exponent) of X. Furthermore, in the case $\mathbb{Q}_x(T_0 < \infty) = 1$, Lamperti [24] explained that, either $q > 0$ and the absorption time occurs by a jump, i.e.

$$\mathbb{Q}_x(X_{T_{0-}} > 0, T_0 < \infty) = 1, \quad \forall x > 0,$$

or ξ drifts to $-\infty$ and T_0 occurs continuously, i.e.

$$\mathbb{Q}_x(X_{T_{0-}} = 0, T_0 < \infty) = 1, \quad \forall x > 0.$$

By means of the strong law of large numbers for Lévy processes we gather these two possibilities in the following.

H0. *Either $q > 0$ or $\lim_{t\to\infty} \xi_t = -\infty$ a.s. and $q = 0$.*

We emphasize that throughout the paper this condition does not hold necessarily, that is we may also consider the case when $\mathbb{Q}_x(T_0 < \infty) = 0, x > 0$.

The remaining part of the paper is organized as follows. In the next Section, we show that the distribution function of T_0 can be expressed in terms of an increasing invariant function for a transient Ornstein-Uhlenbeck process associated to X. We also recall several connections between the law of T_0 and some other important positive random variables. In Section 3, specializing on the case when X has only negative jumps, we derive a power series and an integral representation of the distribution of T_0. The proof and several consequences of this result are also presented in this part. Finally, in the last Section, we detail some known and new examples. We also mention that some of the results stated in Theorem 3.2 below were announced without proofs in the note [38].

2. A general result and some interesting connections

Throughout this Section, we assume that **H0** holds. We define the Ornstein-Uhlenbeck process $U = (U_t)_{t\geq 0}$ associated to X by

$$U_t = e^{\tilde{\alpha}t} X_{(1-e^{-t})},\ t \geq 0, \tag{2.1}$$

where $\tilde{\alpha} = \alpha^{-1}$. Next, we introduce the stopping time

$$H_0 = \inf\{s > 0;\ U_s = 0\} \tag{2.2}$$

and the function

$$1 - P(x) \quad = \quad \mathbb{Q}_x(H_0 < \infty),\ x > 0.$$

We are now ready to state the following.

Theorem 2.1. *For any $x > 0$ and $t > 0$, we have*

$$P(xt^{-\tilde{\alpha}}) \quad = \quad \mathbb{Q}_x(T_0 \geq t) \tag{2.3}$$

and P is increasing on $\mathbb{R}^+$ with $\lim_{x\to\infty} P(x) = 1$ and $P(0) = 0$.

Proof. First, a simple time change yields the following identity in distribution

$$H_0 \overset{(d)}{=} -\log(1 - T_0 \wedge 1).$$

Thus, we deduce that

$$\begin{aligned} 1 - P(x) &= \mathbb{Q}_x(H_0 < \infty) \\ &= \mathbb{Q}_x(T_0 < 1). \end{aligned}$$

Then, invoking the self-similarity property of X we obtain the identity

$$\mathbb{Q}_x(T_0 \geq t) = \mathbb{Q}_{xt^{-\tilde{\alpha}}}(T_0 \geq 1)$$

from which we deduce the identity (2.3) and the properties stated on P. □

We proceed by providing an heuristic analytical argument explaining the identity (2.3). We need to introduce the notation

$$Q(t,x) = \mathbb{Q}_x(T_0 \geq t),\ x,t > 0,$$

and to state the following result which is found in [24, Theorem 6.1].

Proposition 2.2. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be such that $f(x), xf'(x)$ and $x^2 f''(x)$ are continuous functions on $\mathbb{R}^+$, then f belongs to the domain, $\mathbb{D}(\mathbf{L})$, of the characteristic operator $\mathbf{L}$ of $(X,\mathbb{Q})$ which is given, for any $x > 0$, by*

$$\begin{aligned} \mathbf{L}f(x) &= x^{-\alpha}\Big(\frac{\sigma}{2}x^2 f''(x) + bxf'(x) \\ &+ \int_0^\infty f(rx) - f(x) - xf'(x)\log(r)\mathbb{I}_{\{|\log(r)|\leq 1\}}\nu(d\log(r)) - qf(x)\Big). \end{aligned} \tag{2.4}$$

Moreover, if **H0** *holds, then $f \in \mathbb{D}(\mathbf{L})$ if and only if $f(0) = 0$.*

Note that the process U defined in (2.1) is a transient Hunt process on $[0,\infty)$ and its infinitesimal generator, denoted by $\mathbf{A}$, has the following form, see e.g. Carmona et al. [14],

$$\mathbf{A}f(x) = \mathbf{L}f(x) + \tilde{\alpha}xf'(x), \quad x > 0,$$

for a function $f \in \mathbb{D}(\mathbf{L})$. Next, let us assume that the mapping $(t,x) \mapsto Q(t,x)$ defined on $\mathbb{R}^+ \times \mathbb{R}^+$ is smooth enough and thus can be expressed as the solution to the following initial boundary value problem

$$\begin{cases} \frac{\partial}{\partial t}Q(t,x) &= \mathbf{L}Q(t,x) \quad \text{on } (t,x) \in (0,\infty)\times(0,\infty), \\ Q(t,0) &= 0,\ t > 0, \\ Q(0,x) &= 1,\ x > 0. \end{cases} \tag{2.5}$$

As above, from the self-similarity property of X, one gets that

$$Q(t,x) = Q(1, xt^{-\tilde{\alpha}}) = Q(x^{-\alpha}t, 1),\ t,x > 0.$$

Thus, one may set

$$P(xt^{-\tilde{\alpha}}) = Q(1, xt^{-\tilde{\alpha}}),\ x,t > 0.$$

Next, observing that

$$\frac{\partial}{\partial t}Q(t,x) = -\tilde{\alpha}xt^{-\tilde{\alpha}-1}P'(xt^{-\tilde{\alpha}}) \text{ and } \frac{\partial}{\partial x}Q(t,x) = t^{-\tilde{\alpha}}P'(xt^{-\tilde{\alpha}}),$$

one gets

$$\frac{\partial}{\partial t}Q(t,x) = -\tilde{\alpha}xt^{-1}\frac{\partial}{\partial x}Q(t,x)$$

and the problem (2.5) is equivalent to

$$\begin{cases} \mathbf{L}Q(t,x) + \tilde{\alpha} x t^{-1} \frac{\partial}{\partial x} Q(t,x) = 0 & \text{on } (t,x) \in (0,\infty) \times (0,\infty), \\ Q(t,0) = 0, \ t > 0, \\ Q(0,x) = 1, \ x > 0. \end{cases}$$

Note that the linear operator $\mathbf{L}$ acts on the space variable only. Hence, one may fix $t = 1$ and solve

$$\begin{cases} \mathbf{A}P(x) = 0 & \text{on } x \in (0,\infty), \\ P(0) = 0, \\ P(\infty) = 1. \end{cases}$$

Therefore, we have transformed a parabolic integro-differential initial-boundary value problem into an integro-differential equation with boundary conditions. Actually, such a transformation is well-known for some specific partial differential equations. Indeed, by using a linear combination of the one-parameter groups of scaling symmetries of the heat equation, one obtains as reduced equation the Weber's differential equation, see [30] for more details. The solutions in such a case are called the scale-invariant solutions.

2.1. **Connection between T_0 and some random variables.** In this part, we establish or recall some connections between the law of the positive random variable T_0 and the law of some interesting random variables. To this end, we introduce the following notation, which might be justified again by means of the self-similarity property of X,

$$S(tx^{-\alpha}) = P(xt^{-\tilde{\alpha}}) = \mathbb{Q}_x(T_0 \geq t), \quad t, x > 0, \tag{2.6}$$

where we recall that $\tilde{\alpha} = \alpha^{-1}$.

2.1.1. *Law of the maximum of stable Lévy processes.* Let Z be an α-stable Lévy process, with $0 < \alpha < 2$ and we exclude the case when Z is a subordinator. Then, let us denote by X the process Z killed upon entering the negative half-line. X is then a positive self-similar Feller process and the characteristic triplet of the underlying Lévy process has been computed by Caballero and Chaumont [12], see also [13] and [23], for an expression of its characteristic exponent in terms of the Pochhammer symbol. Next, we denote by $\hat{Z}$ the dual of Z, i.e. $\hat{Z} = -Z$ which is also an α-stable Lévy process. Then, by means of the translation invariance of Lévy processes, we deduce readily the following identities

$$\begin{aligned} \mathbb{Q}_x\left(T_0 \leq t\right) &= \mathbb{P}_x\left(\inf_{0<s\leq t} Z_s \leq 0\right) \\ &= \mathbb{P}\left(\max_{0<s\leq t} \hat{Z}_s \geq x\right) \end{aligned}$$

which can be written as follows

$$\mathbb{P}\left(\max_{0\leq s\leq t} \hat{Z}_s \geq x\right) = P(xt^{-\alpha}), \quad x, t > 0. \tag{2.7}$$

As a Lévy process, it is well known that the law of the maximum process of stable processes can be described in terms of the Wiener-Hopf factors which corresponds to its Laplace transform in time and space. The methodology we will develop requires merely to inverse the Laplace transform in space to obtain the law of the maximum process. We refer to Section 4.2 for more details on this case.

2.1.2. *Exponential functional of Lévy processes.* Let us recall the notation

$$\Sigma_t = \int_0^t e^{\alpha \xi_s} ds, \ t \geq 0.$$

It is plain, from the Lamperti transformation (1.1), that, in the case $q = 0$ and $\mathbb{E}[\xi_1] < 0$, one gets the identity in distribution

$$(T_0, \mathbb{Q}_x) \stackrel{(d)}{=} (x^{\alpha} \Sigma_{\infty}, \mathbb{P}).$$

Moreover, when $q > 0$, one has

$$(T_0, \mathbb{Q}_x) \stackrel{(d)}{=} (x^{\alpha} \Sigma_{\mathbf{e}_q}, \mathbb{P})$$

where $\mathbf{e}_q$ is an independent exponential random variable of parameter q. Hence, we have for any $t > 0$ and $q \geq 0$,

$$\mathbb{P}(\Sigma_{\mathbf{e}_q} \geq t) = S(t) \tag{2.8}$$

where we understand that $\mathbf{e}_0 = \infty$. As pointed out by several authors, see Carmona et al. [14], Rivero [40], and, Maulik and Zwart [28], the study of the exponential functional is also motivated by its connection to some interesting random affine equations. Indeed, from the strong Markov property for Lévy processes, which entails that for any finite stopping time T in the filtration $(\mathcal{F}_{\Sigma_t}, t \geq 0)$, the process $(\xi_{t+T} - \xi_T, t \geq 0)$ is independent of $\mathcal{F}_{\Sigma_T}$ and has the same distribution than ξ, we deduce readily that the random variable $\Sigma_{\mathbf{e}_q}$ is solution to the random affine equation

$$\Sigma_{\mathbf{e}_q} \stackrel{(d)}{=} \int_0^1 e^{-\xi_s} ds + e^{-\xi_1} \Sigma'_{\mathbf{e}_q} \tag{2.9}$$

where, on the right-hand side, $\Sigma'_{\mathbf{e}_q}$ is taken independent of $(\int_0^1 e^{-\xi_s} ds, e^{-\xi_1})$. This type of random equations has been deeply studied by Kesten [21] and Goldie [20]. Relying on Kesten's results, Rivero [40] shows that there exists a constant $C > 0$ such that one has the following asymptotic behavior

$$S(t) \sim C t^{-\tilde{\alpha}\theta} \quad \text{as} \ \ t \to \infty \tag{2.10}$$

($f(x) \sim g(x)$ as $x \to a$ means that $\lim_{x \to a} \frac{f(x)}{g(x)} = 1$ for any $a \in [0, \infty]$) whenever the Lévy process satisfies this set of conditions

(1) ξ is not arithmetic (i.e. its state space is not a subgroup of $r\mathbb{Z}$ for some $r > 0$),
(2) there exists $\theta > 0$ such that $\mathbb{E}[e^{\theta \xi_1} \mathbb{I}_{\{1 < \mathbf{e}_q\}}] = 1$,
(3) $\int_{x>1} x e^{\theta x} \nu(dx) < \infty$.

We refer to Theorem 3.2 and Proposition 3.5 for several characterizations of the Kesten's constant C in the case ξ is a spectrally negative Lévy process. We end up this part by recalling that the exponential functional of Lévy processes appears to be a key object in several fields. In particular, in mathematical finance, the law of $\Sigma_{\mathbf{e_q}}$ allows us to derive the Laplace transform in time of the price of the so-called Asian options, see [18] for the Brownian motion case and [33] for more general Lévy processes.

2.1.3. *Entrance law and stationary measures.* Let us denote by $\hat{X}$ the dual of X with respect to the reference measure $x^{\alpha-1}dx$. Bertoin and Yor [6] showed that $\hat{X}$ is a positive self-similar Markov process with underlying Lévy process $\hat{\xi} = -\xi$, the dual of ξ with respect to the Lebesgue measure. Moreover, assuming that ξ is not arithmetic and $\infty < E[\xi_1] < 0$, they show that $\hat{X}$ admits an entrance law at 0, i.e. the family of probability measure $(\mathbb{Q}_x)_{x>0}$ converges, as $x \to 0$, in the sense of finite-dimensional distribution, to a probability measure, denoted by $\mathbb{Q}_{0^+}$. In particular, they obtain the following identity, for any bounded borelian function f,

$$\mathbb{E}_{0^+}[f(\hat{X}_1^{\alpha})] = \frac{1}{\alpha|E[\xi_1]|}E[f(\Sigma_{\infty}^{-1})\Sigma_{\infty}^{-1}].$$

Hence, we obtain for any $x > 0$

$$\mathbb{Q}_{0^+}(\hat{X}_1 \in dx) = \frac{1}{|E[\xi_1]|x}S(dx^{-\alpha}).$$

We also mention that in [34], it is shown that the measure $\mathbb{Q}_{0^+}(\hat{X}_1 \in dx)$ is the stationary measure of the Ornstein-Uhlenbeck process defined by

$$\hat{U}_t^{(1)} = e^{-t}\hat{X}_{e^t},\ t \geq 0.$$

Finally, we define, under $\mathbb{P}$, the process $Y = (Y_t)_{t\geq 0}$, by

$$Y_t = e^{\alpha\xi_t}\left(Y_0 + \int_0^t e^{-\alpha\xi_s}ds\right),\ t \geq 0.$$

This generalization of the Ornstein-Uhlenbeck process is a specific instance of the continuous analogue of random recurrence equations, as shown by de Haan and Karandikar [16]. Carmona et al. [14] showed that Y is a homogeneous Markov process with respect to the filtration generated by ξ. Moreover, they showed, from the stationarity and the independency of the increments of ξ, that, for any fixed $t \geq 0$,

$$Y_t \overset{(d)}{=} Y_0 e^{\alpha\xi_t} + \int_0^t e^{\alpha\xi_s}ds.$$

Then, if $\mathbb{E}[\xi_1] < 0$, they deduced that $Y_t \overset{(d)}{\to} \Sigma_{\infty}$ as $t \to \infty$. Hence if $V_0 \overset{(d)}{=} \Sigma_{\infty}$, the process Y is stationary with stationary measure the law of Σ_{∞}.

3. The spectrally negative case

In this Section we consider the case when ξ is a *spectrally negative Lévy process*, i.e. $\nu(0,\infty) \equiv 0$ in (1.2) and we exclude the cases when ξ is degenerate, that is when ξ is the negative of a subordinator or a pure drift process. In this setting, it is well-known that the characteristic exponent Ψ admits an analytical continuation to the lower half-plane. Thus, we write simply $\psi(u) = \Psi(-iu), u \geq 0$, when $q = 0$ and $\overline{\psi}(u) = \psi(u) - q$ otherwise. It means that, for any $u \geq 0$, we have

$$\psi(u) = \bar{b}u + \frac{\sigma}{2}u^2 + \int_{-\infty}^0 (e^{ur} - 1 - ur\mathbb{I}_{\{|r|<1\}})\nu(dr). \tag{3.1}$$

Let us now recall some basic properties of the Laplace exponent ψ, which can be found for instance in Bingham [10]. First, it is plain that $\lim_{u\to\infty}\psi(u) = +\infty$ and by monotone convergence, one gets $E[\xi_1] = \bar{b} + \int_{-\infty}^{-1} r\nu(dr) \in [-\infty, \infty)$. Thus, by means of the strong law of large numbers for Lévy processes, the condition $\lim_{t\to\infty} = -\infty$ a.s. in **H0** is equivalent to the requirement $E[\xi_1] < 0$.

Moreover, the asymptotic behavior of ψ depends on its coefficients. Indeed, if $\sigma = 0$ and $\int_{-\infty}^0 1 \wedge |r|\nu(dr) < \infty$, that is the Lévy process ξ has paths of bounded variation, then, as we have excluded the degenerate cases, we have, see [3, Corollary VII.5], writing $b = \bar{b} - \int_{-1}^0 r\nu(dr)$,

$$\lim_{u\to\infty} \frac{\psi(u)}{u} = b > 0. \tag{3.2}$$

For the other cases, we have

$$\lim_{u\to\infty} \frac{\psi(u)}{u} = +\infty. \tag{3.3}$$

Differentiating ψ again, one observes that ψ is convex. Note that 0 is always a root of the equation $\psi(u) = 0$. However, in the case $\mathbb{E}[\xi_1] < 0$, this equation admits another positive root, which we denote by θ. This yields the so-called Cramér condition

$$\mathbb{E}[e^{\theta\xi_1}] = 1.$$

Moreover, for any $\mathbb{E}[\xi_1] \in [-\infty, \infty)$, the function $u \mapsto \psi(u)$ is continuous and increasing on $[\max(\theta, 0), \infty)$. Thus, it has a well-defined inverse function $\phi : [0, \infty) \to [\max(\theta, 0), \infty)$ which is also continuous and increasing. Finally, we set the following notation, for any $u \geq 0$,

$$\begin{aligned} \psi_\theta(u) &= \psi(u+\theta), \\ \overline{\psi}_{\phi(q)}(u) &= \overline{\psi}(u+\phi(q)). \end{aligned}$$

Recalling that $\psi(\theta) = 0$ and observing that $\overline{\psi}(\phi(q)) = 0$, the mappings $\psi_\theta, \overline{\psi}_{\phi(q)}$ are plainly Laplace exponents of conservative Lévy processes. We also point out that $\psi'_\theta(0^+) = \psi'(\theta^+) > 0$ and $\overline{\psi}'_{\phi(q)}(0^+) = \psi'(\phi(q)) = \frac{1}{\phi'(q)} > 0$. We denote the totality of all functions ψ of the form (3.1) with the additional condition that $\psi'(0^+) \geq 0$ by $\mathcal{LK}_+$. Note, from the stability of infinitely divisible distributions under convolution and convolution powers to positive real numbers, that $\mathcal{LK}_+$ forms a convex cone in the space of positive valued functions defined on $[0, \infty)$.

We proceed by introducing a few further notation. First, let $\psi \in \mathcal{LK}_+$ and set $a_0 = 1$ and for any $n = 1, 2, \ldots$,

$$a_n(\psi; \alpha) = \left(\prod_{k=1}^n \psi(\alpha k) \right)^{-1}.$$

In [37], the author introduces the following power series

$$\mathcal{I}_\psi(z) = \sum_{n=0}^\infty a_n(\psi; \alpha) z^n \tag{3.4}$$

and shows by means of classical criteria that the mapping $z \mapsto \mathcal{I}_\psi(z)$ is an entire function. We refer to [37] for interesting analytical properties enjoyed by these power series and also for connections with well-know special functions, such as, for instance, the modified Bessel functions and several generalizations of the Mittag-Leffler function. The author also proves that for any $r > 0$, the mapping $x \mapsto \mathcal{I}_\psi(x^\alpha)$ is an 1-eigenfunction for X and

$$\mathbb{E}_x\left[e^{-T_a}\right] = \frac{\mathcal{I}_\psi(x^\alpha)}{\mathcal{I}_\psi(a^\alpha)}, \quad 0 \leq x \leq a,$$

where $T_a = \inf\{s > 0;\ X_s = a\}$. Next, let G_κ be a Gamma random variable independent of X, with parameter $\kappa > 0$. Its density is given by $g(dy) = \frac{e^{-y}y^{\kappa-1}}{\Gamma(\kappa)}dy, y > 0$, with Γ the Euler gamma function. Then, in [34], the author suggested the following generalization

$$\begin{aligned} \mathcal{I}_\psi(\kappa; z) &= \mathbb{E}\left[\mathcal{I}_\psi\left(G_\kappa z\right)\right] \\ &= \frac{1}{\Gamma(\kappa)}\int_0^\infty e^{-t}t^{\kappa-1}\mathcal{I}_\psi\left(tz\right)dt. \end{aligned} \tag{3.5}$$

By means of the integral representation of the Gamma function $\Gamma(\rho) = \int_0^\infty e^{-s}s^{\rho-1}ds, \mathfrak{Re}(\rho) > 0$, and an argument of dominated convergence, one obtains the following power series representation

$$\mathcal{I}_\psi(\rho; z) = \frac{1}{\Gamma(\rho)}\sum_{n=0}^{\infty} a_n(\psi;\alpha)\Gamma(\rho+n)z^n \tag{3.6}$$

which is easily seen to be valid for any $|z| < R = \lim_{u\to\infty}\frac{\psi(\alpha u)}{u} > 0$. Moreover, for any $|z| < R$, the mapping $\rho \mapsto \mathcal{I}_\psi(\rho; z)$ is a meromorphic function defined for all complex numbers ρ except at the poles of the Gamma function, that is at the points $\rho = 0, -1, \ldots$ However, they are removable singularities. Indeed, for any $|z| < R$ and any integer $N \in \mathbb{N}$, one has, by means of the recurrence relation $\Gamma(z+1) = z\Gamma(z)$, $\mathcal{I}_\psi(0; z) = 1$ and

$$\mathcal{I}_\psi(-N; z) \ = \ \sum_{n=0}^{N}(-1)^n\frac{\Gamma(N+1)}{\Gamma(N+1-n)}a_n(\psi;\alpha)z^n.$$

Thus, by uniqueness of the analytical extension for any $|z| < R$, $\mathcal{I}_\psi(\rho; z)$ is an entire function in ρ. Note also that for $\rho = 0, -1, \ldots$, as a polynomial, $\mathcal{I}_\psi(-\rho; z)$ is an entire function in z. We point out that when ψ is the Laplace exponent of a standard Brownian motion with a positive drift, $(\mathcal{I}_\psi(-N; e^{i\pi}z))_{N\in\mathbb{N}}$ is expressed in terms of the sequence of Laguerre polynomials. In the following, we summarize the above results and provide an analytical continuation of $\mathcal{I}_\psi(\rho; z)$ in the case $R = \alpha b$, that is when ξ is with paths of bounded variation.

Proposition 3.1. *(1) If $\lim_{u\to\infty}\frac{\psi(u)}{u} = \infty$, then $\mathcal{I}_\psi(\rho; z)$ is an entire function in both arguments z and ρ.*

(2) If $\lim_{u\to\infty}\frac{\psi(u)}{u} = b > 0$, then $\mathcal{I}_\psi(\rho; z)$ is analytic in the disc $|z| < \alpha b$ and for any fixed $\rho = 0, -1, \ldots$, $\mathcal{I}_\psi(\rho; z)$, as a polynomial, is an entire function.

For any $\rho \neq 0, -1, \ldots$, $\mathcal{I}_\psi(\rho; z)$ admits an analytical continuation in the entire complex plane cut along the positive real axis given by

$$\mathcal{I}_\psi(\rho; z) = \frac{1}{2i\pi\Gamma(\rho)}\int_{-i\infty}^{i\infty} a_s(\varphi;\alpha)\Gamma(s+\rho)\Gamma(-s)\left(-\frac{z}{\alpha}\right)^s ds, \quad |\arg(-z)| < \pi, \tag{3.7}$$

where the contour is indented to ensure that all poles (resp. nonnegative poles) of $\Gamma(\rho+s)$ (resp. $\Gamma(-s)$) lie to the left (resp. right) of the intended imaginary axis and for any $\mathfrak{Re}(s) > -1$

$$a_s(\varphi;\alpha) = \prod_{k=1}^{\infty}\frac{\varphi(\alpha(k+s+1))}{\varphi(\alpha k)}$$

with $\varphi(s) = b - \widehat{v}(s)$ and $\widehat{v}(s) = \int_0^\infty e^{-sr}\nu(-\infty, -r)dr$ is the Laplace transform of the tail of the Lévy measure ν. Moreover, for any $\rho \in \mathbb{C}$, $\mathcal{I}_\psi(\rho; z)$ admits, in the half-plane

$\mathfrak{Re}(z) < \frac{\alpha b}{2}$, *the following power series representation*

$$\mathcal{I}_\psi(\rho;z) = \left(1-\frac{z}{\alpha b}\right)^{-\rho}\sum_{n=0}^{\infty}\mathcal{I}_\psi(-n;\alpha b)\frac{\Gamma(\rho+n)}{n!\Gamma(\rho)}\left(\frac{z}{z-\alpha b}\right)^n. \tag{3.8}$$

Finally, for any fixed $\mathfrak{Re}(z) < \frac{\alpha b}{2}$, $\mathcal{I}_\psi(\rho;z)$ *is an entire function in the argument* ρ.

Proof. The first item and the first part of the second one follow from the discussion preceding the Proposition. Thus, let us assume that $\lim_{u\to\infty}\frac{\psi(u)}{u} = b > 0$ and $\rho \neq 0,-1,\ldots$ Since $\psi \in \mathcal{LK}_+$, ψ is well-defined and analytic in the positive right half-plane and $\psi(u) > 0$ for any $u > 0$. Our first aim is to extend the coefficients $a_n(\psi;\alpha)$ to a function of the complex variable. For the paths of the Lévy process ξ is of bounded variation, its Laplace exponent ψ has the following form, see [3, Sec. VII.3],

$$\psi(u) = u(b - \widehat{v}(u)).$$

Thus, for any $n \geq 0$, we have

$$a_n(\psi;\alpha) = \frac{1}{\Gamma(n+1)\alpha^n}a_n(\varphi;\alpha)$$

with $a_n(\varphi;\alpha)^{-1} = \prod_{k=1}^n \varphi(\alpha k)$ and $a_0(\varphi;\alpha) = 1$. It is plain that the mapping $\widehat{v}$ is analytic in $F_0 = \{s \in \mathbb{C};\ \mathfrak{Re}(s) > 0\}$ and $\widehat{v}(u)$ is decreasing on $\mathbb{R}^+$ with $0 < \widehat{v}(0) < \alpha b$ since $\psi \in \mathcal{LK}_+$. Then, we may write

$$\begin{aligned}
a_s(\psi;\alpha) &= \frac{1}{\Gamma(s+1)\alpha^s}a_s(\varphi;\alpha)\\
&= \frac{1}{\Gamma(s+1)\alpha^s}\prod_{k=1}^{\infty}\frac{\varphi(\alpha(k+s+1))}{\varphi(\alpha k)}
\end{aligned}$$

where the infinite product is easily seen to be absolutely convergent for any $\mathfrak{Re}(s) > 0$ by taking the logarithm and noting that $|\widehat{v}(s)| \leq \widehat{v}(\mathfrak{Re}(s))$. Moreover, $a_s(\varphi;\alpha)$ satisfies the functional equation

$$a_{s+1}(\varphi;\alpha) = \frac{1}{\varphi(\alpha(s+1))}a_s(\varphi;\alpha)$$

which shows that $a_s(\varphi;\alpha)$ is analytic in the half-plane $F_{-1} = \{s \in \mathbb{C};\ \mathfrak{Re}(s) > -1\}$. Consequently, $a_s(\varphi;\alpha)$ is bounded on any closed subset of F_{-1}. Then, we set $G(s) = \Gamma(s+\rho)\Gamma(-s)a_s(\varphi;\alpha)$ and define

$$\mathfrak{I}_{\mathfrak{L}_R} = -\frac{1}{2i\pi\Gamma(\rho)}\int_{\mathfrak{L}_R} G(s)\left(-\frac{z}{\alpha}\right)^s ds$$

where the integral is take in a clockwise direction round the contour $\mathfrak{L}_R$, consisting of a large semi-circle, of center the origin and radius R, lying to the right of the imaginary axis. This contour is intended to ensure that all poles (resp. nonnegative poles) of $\Gamma(\rho+s)$ (resp. $\Gamma(-s)$) lie to the left (resp. right) of the intended imaginary axis. This contour is always possible since we have assumed that $\rho \neq 0,-1,\ldots$ We can split $\mathfrak{I}_{\mathfrak{L}_R}$ up into two integrals, $\mathfrak{I}_{\mathfrak{A}_{iR}}$ along the imaginary axis and, writing $s = Re^{i\theta}$,

$$\mathfrak{I}_{\mathfrak{C}_R} = -\frac{1}{2\pi i}\int_{-\pi/2}^{\pi/2} G(Re^{i\theta})\left(-\frac{z}{\alpha}\right)^{Re^{i\theta}} Re^{i\theta}d\theta.$$

Recalling the following well-known asymptotic formulae, see e.g. [31, Section 2.4], as $|s| \to \infty$,

$$\begin{aligned}
\Gamma(s+\rho) &\sim \sqrt{2\pi} e^{-Re^{i\theta}} R^{Re^{i\theta}+\rho-\frac{1}{2}} e^{i\theta(Re^{i\theta}+\rho-\frac{1}{2})}, \quad |\theta|<\pi, \\
\Gamma(-s) &\sim e^{-\pi R|\sin\theta|} e^{Re^{i\theta}} R^{-Re^{i\theta}-\frac{1}{2}} e^{i\theta(-Re^{i\theta}-\frac{1}{2})}, \quad |\theta|<\pi
\end{aligned}$$

and

$$\left|\left(-\frac{z}{\alpha}\right)^s\right| \sim |\alpha z|^{R\cos\theta} e^{-R\sin\theta \arg(-z)},$$

we deduce that as $|s| \to \infty$

$$\left|G(s)\left(-\frac{z}{\alpha}\right)^s\right| \sim aR^{\mathfrak{Re}(\rho)-1}|\alpha z|^{R\cos\theta} \begin{cases} e^{-R|\sin\theta|(\pi+\arg(-z))}, & 0<\theta\leq \pi/2 \\ e^{-R|\sin\theta|(\pi-\arg(-z))}, & -\pi/2 \leq \theta < 0, \end{cases} \tag{3.9}$$

where a is a positive constant. On the one hand, along the path $\mathfrak{A}_{iR}$ we have $\theta = \pm\frac{\pi}{2}$ and thus as $|z| \to \infty$

$$\left|G(s)\left(-\frac{z}{\alpha}\right)^s\right| \sim aR^{\mathfrak{Re}(\rho)-1} e^{\pm\frac{\pi}{2}\Im(\rho)} \begin{cases} e^{-R(\pi+\arg(-z))} & \theta = \pi/2 \\ e^{-R(\pi-\arg(-z))} & \theta = -\pi/2. \end{cases}$$

For the integral (3.7) to converge absolutely therefore requires that $|\arg(-z)| < \pi$. On the other hand, the asymptotic estimate (3.9) gives that as $R \to \infty$

$$\mathfrak{I}_{\mathfrak{C}_R} \to 0 \quad \text{if } |z|<1 \text{ and } |\arg(-z)|<\pi.$$

Thus, as $R \to \infty$,

$$\mathfrak{I}_{\mathfrak{L}_R} \to -\frac{1}{2i\pi}\int_{-i\infty}^{i\infty} G(s)\left(-\frac{z}{\alpha}\right)^s ds.$$

Finally, evaluating $\mathfrak{I}_{\mathfrak{L}_R}$ by the Cauchy integral theorem and letting $R \to \infty$, we get

$$\frac{1}{2i\pi}\int_{-i\infty}^{i\infty} G(s)\left(-\frac{z}{\alpha}\right)^s ds = \frac{1}{\Gamma(\rho)}\sum_{n=0}^{\infty} a_n(\psi;\alpha)\Gamma(\rho+n)z^n, \quad |z|<1 \text{ and } |\arg(-z)|<\pi.$$

Therefore, the integral (3.7) offers an analytical continuation of the mapping $z \mapsto \mathcal{I}(\rho; z)$ in the entire complex plane cut along the positive real axis. Moreover, we deduce from such an analytical extension that the power series (3.6) has an unique singularity on the circle $|z| = \alpha b$ located at the point $z = \alpha b > 0$. Now, following a device developed for hypergeometric series, see e.g. [29], we introduce the function $\mathcal{H}$ defined for some $a \in \mathbb{C}$ by

$$\mathcal{H}_{\psi,a}(\rho; z) = (1-z)^{-\rho}\mathcal{I}_{\psi}\left(\rho; \frac{a\alpha b z}{z-1}\right).$$

Note that

$$\mathcal{I}_{\psi}(\rho; az) = \left(1-\frac{z}{\alpha b}\right)^{-\rho}\mathcal{H}_{\psi,a}\left(\rho; \frac{z}{z-\alpha b}\right). \tag{3.10}$$

Thus, denoting by $(b_n)_{n\geq 0}$ the coefficients of the power series $\mathcal{H}_{\psi,a}(\rho; z)$, we have $b_0 = a_0$ and by means of residues calculus, with $\mathfrak{C}$ a circle around 0 of small radius and with positive orientation, we have for $n \geq 1$

$$\begin{aligned} b_n &= \frac{1}{2\pi i}\int_{\mathfrak{C}} \frac{\mathcal{H}_{\psi,a}(\rho;z)}{z^{n+1}}dz \\ &= (-1)^n\frac{1}{2\pi i}\int_{\mathfrak{C}}(1-z)^{-\rho}\,\mathcal{I}_{\psi}\left(\rho;\frac{a\alpha bz}{z-1}\right)\frac{dz}{z^{n+1}}dv \\ &= \frac{1}{\Gamma(\rho)}\sum_{k=0}^{n}(-a\alpha b)^k a_k(\psi;\alpha)\frac{\Gamma(\rho+n)}{\Gamma(n-k+1)}. \end{aligned}$$

Thus, one gets

$$(1-z)^{-\rho}\mathcal{I}_{\psi}\left(\rho;\frac{a\alpha bz}{z-1}\right) = \sum_{n=0}^{\infty}\mathcal{I}_{\psi}(-n;a\alpha b)\frac{(\rho)_n}{n!}z^n.$$

From Weierstrass's double series theorem, the above identity is true if $|z| < \frac{1}{1+|a|}$. Moreover, the function on the left-hand side has a singularity at $z=1$ and $z=\frac{1}{1-a}$. Thus, the series on the right-hand side is convergent if $|z|<1$ and $|z(1-a)|<1$. By choosing $a=1$, we conclude by observing that the series on the right-hand side of (3.10) is convergent for $\mathfrak{Re}(z) < \frac{\alpha b}{2}$. $\square$

Next, we introduce the following notation, for any $z \in \mathbb{C}$,

$$\begin{aligned} \mathcal{O}_{\psi}(z) &= \mathcal{I}_{\psi}(e^{i\pi}z) \\ &= \sum_{n=0}^{\infty}(-1)^n a_n(\psi;\alpha)z^n, \end{aligned}$$

and, similarly, for any $\rho \in \mathbb{C}$,

$$\mathcal{O}_{\psi}(\rho;z) = \mathcal{I}_{\psi}(\rho;e^{i\pi}z), \quad |\arg(z)|<\pi.$$

Finally, in order to present our next result in a compact form, when **H0** holds, we write

$$\gamma = \begin{cases} \phi(q) & \text{if } q>0, \\ \theta & \text{otherwise,} \end{cases} \tag{3.11}$$

and, for any $u \geq 0$,

$$\psi_{\gamma}(u) = \begin{cases} \overline{\psi}_{\phi(q)}(u) & \text{if } q>0, \\ \psi_{\theta}(u) & \text{otherwise.} \end{cases} \tag{3.12}$$

We are now ready to state the main result of this part.

Theorem 3.2. *Assume that* **H0** *holds and recall that* $S(tx^{-\alpha}) = P(xt^{-\tilde{\alpha}}) = \mathbb{Q}_x(T_0 \geq t), x,t>0$. *Then, there exists a constant* $C_{\gamma}>0$ *such that*

$$\mathcal{O}_{\psi_{\gamma}}(\tilde{\alpha}\gamma;x) \sim \frac{x^{-\tilde{\alpha}\gamma}}{C_{\gamma}} \quad \text{as } x\to\infty \tag{3.13}$$

and

$$S(t) = C_{\gamma}t^{-\tilde{\alpha}\gamma}\mathcal{O}_{\psi_{\gamma}}(\tilde{\alpha}\gamma;t^{-1}),\ t>0. \tag{3.14}$$

Moreover, the law of T_0 *under* $\mathbb{Q}_1$ *is absolutely continuous with a density, denoted by* s, *given by*

$$s(t) = \tilde{\alpha}\gamma C_{\gamma}t^{-\tilde{\alpha}\gamma-1}\mathcal{O}_{\psi_{\gamma}}(1+\tilde{\alpha}\gamma;t^{-1}),\ t>0.$$

More generally, for any integer m, writing $s^{(m)} = \frac{d^m}{dt^m}s$*, we have, for any* $t > 0$*,*

$$s^{(m)}(t) \;=\; (-1)^m \frac{\Gamma(m+1+\tilde{\alpha}\gamma)}{\Gamma(\tilde{\alpha}\gamma)} C_\gamma t^{-\tilde{\alpha}\gamma-1-m} \mathcal{O}_{\psi_\gamma}(m+1+\tilde{\alpha}\gamma; t^{-1}).$$

We postpone the proof of the Theorem until the Section 3.1 and offer instead a few consequences of this result. First, we deduce readily from the above Theorem the following asymptotic behaviors for large values.

Corollary 3.3. *With the notation used and introduced in Theorem 3.2, we have*

$$S(t) \;\sim\; C_\gamma t^{-\tilde{\alpha}\gamma} \quad \text{as } t \to \infty \tag{3.15}$$

and, for any $m = 0, 1 \ldots$*,*

$$s^{(m)}(t) \;\sim\; (-1)^m C_\gamma \frac{\Gamma(m+1+\tilde{\alpha}\gamma)}{\Gamma(\tilde{\alpha}\gamma)} t^{-\tilde{\alpha}\gamma-1-m} \quad \text{as } t \to \infty. \tag{3.16}$$

In the case $\lim_{u\to\infty} \frac{\psi(u)}{u} = \infty$, by easily checking that the mapping $x \mapsto \mathcal{I}_\psi(\rho; x)$ is increasing on $\mathbb{R}^+$, for any $\mathfrak{Re}(\rho) > 0$, one deduces, again from the above Theorem the following.

Corollary 3.4. *Let us assume that* $\lim_{u\to\infty} \frac{\psi(u)}{u} = \infty$*. With the notation used and introduced in Theorem 3.2, the entire function* $z \mapsto \mathcal{O}_{\psi_\gamma}(\tilde{\alpha}\gamma; z)$ *has no real zeros.*

Next, we observe that the function $P \in \mathbb{D}(\mathbf{A})$, hence $P(xt^{-\tilde{\alpha}}), x, t > 0$, is solution to the initial boundary value problem (2.5). We do not aim to pursue in this direction, nevertheless let us point out that the uniqueness of the solution can be proved by invoking an argument involving the Martin boundary associated to the process X. Further details are provided in [39]. Next, let us recall that in [37], the expression of the Laplace transform of T_0, in the case $\mathbb{E}[\xi_1] < 0$, $q = 0$ and $\theta < \alpha$, is given, for any $r, x \geq 0$, as follows

$$\mathbb{E}_x\left[e^{-rT_0}\right] \;=\; \mathcal{N}_{\psi,\theta}(rx^\alpha) \tag{3.17}$$

where

$$\mathcal{N}_{\psi,\theta}(r) = \mathcal{I}_\psi(r) - C(\theta) r^{\tilde{\alpha}\theta} \mathcal{I}_{\psi_\theta}(r)$$

and the positive constant $C(\theta)$ is characterized by

$$\mathcal{I}_\psi(r) \sim C(\theta) r^{\tilde{\alpha}\theta} \mathcal{I}_{\psi_\theta}(r) \quad \text{ as } r \to \infty.$$

Thus, Theorem 3.2 offers a formula for the inversion of this Laplace transform. More specifically, one has

$$\mathcal{N}_{\psi,\theta}(rx^\alpha) = x^{-\alpha} \int_0^\infty e^{-rt} s(tx^{-\alpha}) dt.$$

From the expression of s, it appears that a term-by-term inversion does not seem trivial.

Moreover, as we have excluded the case when $-\xi$ is a subordinator, it is not difficult to verify that the Lévy processes we consider in this Section satisfy the conditions listed below (2.10). Hence, from (2.10) and (3.15), we get that the asymptotic behavior (3.13) offers a characterization of the Kesten's constant. In what follows, we provide some representations of this normalizing constant in terms of the Laplace exponent. To this end, let us observe, in the case $\lim_{u\to\infty} \frac{\psi_\gamma(u)}{u} = \infty$, as $0 < \psi_\gamma'(0^+) < \infty$, we have, for any $u > 0$

$$\begin{aligned} \psi_\gamma(\alpha u) \;&=\; \hat{b}\alpha u + \frac{\sigma}{2}(\alpha u)^2 + \int_{-\infty}^0 (e^{\alpha u r} - 1 - \alpha u r) e^{\gamma r} \nu(dr) \\ &=\; (\alpha u)^2 \bar{\varphi}_\gamma(\alpha u) \end{aligned}$$

where $\hat{b} = \bar{b} + \sigma\gamma + \int_{-\infty}^{0} \left(e^{\gamma r} - \mathbb{I}_{\{|r|<1\}}\right) r\nu(dr)$ and

$$\bar{\varphi}_{\gamma}(\alpha u) = \frac{\hat{b}}{\alpha u} + \frac{\sigma}{2} + \int_0^{\infty} e^{-\alpha u r} \int_{-\infty}^{-r} \int_{-\infty}^{-s} e^{\gamma v} \nu(dv) ds dr.$$

Thus, as above, one may define the function

$$\begin{aligned} a_s(\psi_\gamma;\alpha) &= \frac{1}{\alpha^2\Gamma^2(s+1)} a_s(\bar{\varphi}_\gamma;\alpha) \\ &= \frac{1}{\alpha^2\Gamma^2(s+1)} \prod_{k=1}^{\infty} \frac{\bar{\varphi}_\gamma(\alpha(k+s+1))}{\bar{\varphi}_\gamma(\alpha k)} \end{aligned}$$

and observe the identity

$$a_{s+1}(\bar{\varphi}_\gamma;\alpha) = \frac{1}{\bar{\varphi}_\gamma(\alpha(s+1))} a_s(\bar{\varphi}_\gamma;\alpha)$$

with $a_0(\bar{\varphi}_\gamma;\alpha) = 1$. Hence $a_s(\bar{\varphi}_\gamma;\alpha)$ is a meromorphic function in $F_{-\gamma} = \{s \in \mathbb{C}; \mathfrak{Re}(s) > -\gamma - 1\}$ with simple poles at the points $s_k = -k-1$ for $k = 0, 1, \ldots$ and $s_k > -\gamma - 1$.

Proposition 3.5. *If* $\lim_{u\to\infty} \frac{\psi(u)}{u} = b$ *then*

$$C_\gamma = \alpha^{\tilde{\alpha}\gamma} a_{-\tilde{\alpha}\gamma}(\varphi_\gamma;\alpha).$$

Otherwise, we have, writing $\psi(\alpha u) = \alpha u \varphi(\alpha u)$,

$$C_\gamma = \begin{cases} \psi'_\gamma(0^+) & \text{if } \tilde{\alpha}\gamma = 1 \\ \alpha^n \psi'_\gamma(0^+) \left(\prod_{k=1}^{n} \varphi(\alpha k)\right)^{-1} & \text{if } \tilde{\alpha}\gamma = n+1,\ n = 1, 2 \ldots \\ \frac{\alpha^{2\tilde{\alpha}\gamma}}{\Gamma(1-\tilde{\alpha}\gamma)} a_{-\tilde{\alpha}\gamma}(\bar{\varphi}_\gamma;\alpha) & \text{otherwise.} \end{cases}$$

Finally, if $q = 0$ *and* $0 < \theta < \alpha$. *Then, the constant* C_θ *is such that*

$$\mathcal{I}_\psi(r) \sim \Gamma(1-\tilde{\alpha}\theta) C_\theta r^{\tilde{\alpha}\theta} \mathcal{I}_{\psi_\theta}(r) \qquad \text{as } r \to \infty.$$

Proof. Let us start by pointing out that it is not difficult to check that we have, in all cases, $C_\gamma > 0$. Moreover, let us first assume that $\lim_{u\to\infty} \frac{\psi(u)}{u} = b$. From Proposition 3.1, we have

$$\mathcal{O}_{\psi_\gamma}(\tilde{\alpha}\gamma; z) = \frac{1}{2i\pi\Gamma(\tilde{\alpha}\gamma)} \int_{-i\infty}^{i\infty} a_s(\varphi_\gamma;\alpha)\Gamma(s+\tilde{\alpha}\gamma)\Gamma(-s)\left(\frac{z}{\alpha}\right)^s ds, \quad |\arg(z)| < \pi.$$

Hence, upon displacement of the path to the left in order to include the first pole of $\Gamma(s+\tilde{\alpha}\gamma)$ we obtain, from Theorem 3.2 and a residue computation, that

$$\mathcal{O}_{\psi_\gamma}(\tilde{\alpha}\gamma; z) = \alpha^{\tilde{\alpha}\gamma} a_{-\tilde{\alpha}\gamma}(\varphi_\gamma;\alpha) z^{-\tilde{\alpha}\gamma} + o(z^{-\tilde{\alpha}\gamma})$$

which gives the characterization of C_γ in this case.

For the other case, one may follow a similar line of reasoning than in the proof of Proposition 3.1 to obtain, writing $\bar{G}(s) = \frac{a_s(\bar{\varphi}_\gamma;\alpha)}{\Gamma(s+1)}\Gamma(s+\tilde{\alpha}\gamma)\Gamma(-s)$, the following identity

$$\mathcal{O}_{\psi_\gamma}(\tilde{\alpha}\gamma; z) = \frac{1}{2i\pi\Gamma(\tilde{\alpha}\gamma)} \int_{-i\infty}^{i\infty} \bar{G}(s)\left(\frac{z}{\alpha^2}\right)^s ds$$

which is now valid in the sector $|\arg(z)| < \pi/2$. As above, after a displacement of the path to the left in order to include the first pole of $\Gamma(s+\tilde{\alpha}\gamma)$ we obtain, from Theorem 3.2 and a residue computation, that

$$\mathcal{O}_{\psi_\gamma}(\tilde{\alpha}\gamma; z) = \frac{1}{\Gamma(\tilde{\alpha}\gamma)} \left(\sum_{k=1}^{[\tilde{\alpha}\gamma]} \frac{\Gamma(k)\mathrm{Res}_{j=-k} a_j(\bar{\varphi}_\gamma;\alpha)}{\Gamma(1-k)} \left(\frac{z}{\alpha^2}\right)^{-k} + \mathrm{Res}_{s=-\tilde{\alpha}\gamma} \bar{G}(s) \left(\frac{z}{\alpha^2}\right)^{-s} \right) + o(z^{-\tilde{\alpha}\gamma})$$

where the sum is 0 if $[\tilde{\alpha}\gamma]$, the integer part of $\tilde{\alpha}\gamma$, is lower than 1. Since $a_s(\widehat{\varphi}_\gamma, \alpha)$ has a simple pole at $j = -1, \ldots, -[\tilde{\alpha}\gamma]$, the terms in the sum vanishes. Hence, if $\tilde{\alpha}\gamma$ is not an integer $\bar{G}(s)$ has a simple pole at $-\tilde{\alpha}\gamma$ and the expression of C_γ follows readily in this case. If $\tilde{\alpha}\gamma = n+1$, then $\bar{G}(s)$ has a double pole at $-(n+1)$ and using the recurrence relations of both the gamma function and $a_s(\bar{\varphi}_\gamma;\alpha)$, we deduce that

$$\begin{aligned}
\mathrm{Res}_{s=-(n+1)}\bar{G}(s) &= \lim_{s\to -n-1} \frac{d}{ds}\left((s+n+1)^2 \bar{G}(s)\right) \\
&= \lim_{s\to -n-1} \frac{d}{ds}\left(\alpha^{-n-2}\prod_{k=1}^{n} \varphi_\gamma(\alpha(s+k))\psi_\gamma(\alpha(s+n+1)) a_{s+n+1}(\bar{\varphi}_\gamma;\alpha)\right) \\
&= \alpha^{-n-2}\Gamma(n+1)\psi'_\gamma(0^+)\prod_{k=1}^{n}\varphi(\alpha k)
\end{aligned}$$

and the result follows. The second part of the Proposition is proved as follows. Let us write $\hat{F}(r) = \mathbb{E}_1\left[1 - e^{-rT_0}\right]$. Then, from (3.17), one deduces easily that

$$\hat{F}(r) \sim C(\theta) r^{\tilde{\alpha}\theta} \quad \text{as } \ r \to 0,$$

which is equivalent, according to Bingham et al. [9, Corollary 8.1.7], to

$$S(t) \sim \frac{C(\theta)}{\Gamma(1-\tilde{\alpha}\theta)} t^{-\tilde{\alpha}\theta} \quad \text{as } \ t \to \infty$$

which completes the proof. □

Note that the constant C_θ also appears in the study of the rates of convergence of transient diffusions in spectrally negative Lévy potentials. Indeed, Singh [42, Theorem 1] characterized the rates of convergence of these diffusions hitting times in terms of scaled one-sided stable distributions. It is easily verified that his scaling factors can be expressed in terms of C_θ. Thus, Theorem 3.2 and Proposition 3.5 offer some explicit representations of these factors in terms of the Laplace exponent of the Lévy potential.

3.1. **Proof of Theorem 3.2.** According to Theorem 2.1, our aim is to derive an expression for the function $P(x) = 1 - \mathbb{Q}_x(H_0 < \infty)$, $x > 0$. We split the proof into two main stages. First, we show how to construct some time-space invariant functions for the semigroup of X when X does not reach 0 a.s., i.e. X is associated via the Lamperti mapping with a $\psi \in \mathcal{LK}_+$. Then, we develop some devices for relating the previous result to the construction of the invariant function for the Ornstein-Uhlenbeck process defined in (2.1). To achieve this, we will consider separately the cases $q > 0$, and, $q = 0$ and $E[\xi_1] < 0$.

We use the notation introduced in the Theorem 3.2 and take first X with underlying Lévy exponent ψ_γ. We denote its law (resp. its expectation operator) by $\mathbb{Q}^{(\gamma)}$ (resp. $\mathbb{E}^{(\gamma)}$). We recall that $\psi_\gamma \in \mathcal{LK}_+$ and hence the condition **H0** does not hold. Next, we simply write

$Q^{(\gamma)} = (Q_t^{(\gamma)})_{t\geq 0}$ for the semigroup of X, i.e. for any bounded Borelian function g and $t, x > 0$, one has

$$Q_t^\gamma g(x) = \mathbb{E}_x^{(\gamma)}\left[g(X_t)\right].$$

From [6], we have that Q^γ is a Feller semigroup. Then, we say, for any $r \in \mathbb{R}$, that a function I is r-invariant for Q^γ if

$$e^{-rt} Q_t^\gamma I(x) = I(x), \ x > 0.$$

We start with the following Lemma which is a slight generalization of [37, Theorem 1].

Lemma 3.6. *For any $x > 0$, the mapping $x \mapsto \mathcal{O}_{\psi_\gamma}(x^\alpha)$ is an -1-eigenfunction for X, i.e., with the obvious notation, we have*

$$\mathbf{L}^\gamma \mathcal{O}_{\psi_\gamma}(x^\alpha) + \mathcal{O}_{\psi_\gamma}(x^\alpha) = 0.$$

Hence, for any $r > 0$, the mapping $x \mapsto \mathcal{O}_{\psi_\gamma}(rx^\alpha)$ is $-r$-invariant for Q^γ.

Proof. First, recalling that the mapping $z \mapsto \mathcal{O}_{\psi_\gamma}(z)$ is an entire function, it is plain, from Proposition 2.2, that the function $\mathcal{O}_{\psi_\gamma}(x^\alpha) \in \mathbb{D}(\mathbf{L})$. Next, note that, for any $\beta > 0$, $x^\beta \in \mathbb{D}(\mathbf{L}^\gamma)$ and

$$\mathbf{L}^\gamma x^\beta = x^{\beta-\alpha}\psi_\gamma(\beta). \tag{3.18}$$

Then, for any positive integer N, writing $\mathcal{O}_{\psi_\gamma}^N(x^\alpha) = \sum_{n=0}^N (-1)^n a_n(\psi_\gamma;\alpha) x^{\alpha n}$ and using successively the linearity of the operator $\mathbf{L}$ and (3.18), we get, for any $x > 0$,

$$\begin{aligned}\mathbf{L}^\gamma \mathcal{O}_{\psi_\gamma}^N(x^\alpha) &= \sum_{n=1}^N a_n(\psi_\gamma;\alpha)(-1)^n \mathbf{L}^\gamma x^{\alpha n} \\ &= -\sum_{n=0}^{N-1} a_n(\psi_\gamma;\alpha)(-1)^n x^{\alpha n}.\end{aligned}$$

The series being absolutely continuous, the right-hand side of the previous identity convergences as $N \to \infty$. Hence, by dominated convergence, we get

$$\mathbf{L}^\gamma \mathcal{O}_{\psi_\gamma}(x^\alpha) = -\mathcal{O}_{\psi_\gamma}(x^\alpha), \quad x > 0,$$

which completes the proof of the first statement. Then, on the one hand, observe from the self-similarity property of X, that, for any $r > 0$, the mapping $x \mapsto f(x) = \mathcal{O}_{\psi_\gamma}(rx^\alpha)$ is an $-r$-eigenfunction. On the other hand, we have, for any $r, t, x > 0$,

$$\frac{d}{dt} e^{rt} Q_t^\gamma f(x) = r e^{rt} Q_t^\gamma f(x) + e^{rt}\frac{d}{dt} Q_t^\gamma f(x). \tag{3.19}$$

Since Q^γ is a Feller semigroup, $Q_0^\gamma f(x) = f(x)$ and

$$\frac{d}{dt} Q_t^\gamma f(x) = Q_t^\gamma \mathbf{L}^\gamma f(x).$$

Thus, integrating (3.19) yields

$$e^{rt} Q_t^\gamma f(x) - f(x) = r\int_0^t e^{rs} Q_s^\gamma f(x) ds + \int_0^t e^{rs} Q_s^\gamma \mathbf{L}^\gamma f(x) ds.$$

The proof is completed by recalling that $\mathbf{L}^\gamma f(x) = -rf(x)$. □

Following a device developed by the author in [34], we now show how to construct some specific time-space invariant functions for Q^γ in terms of the r-invariant function of X. We emphasize that this methodology may be applied to any self-similar Markov processes, that is also for the ones having two-sided jumps. Indeed, the proof of the following result, which generalizes a device suggested by Shepp [41] in the case of the Brownian motion, relies merely on the self-similarity property of X. Finally, we write, for any $\lambda \in \mathbb{R}$,

$$\zeta^\lambda = \begin{cases} +\infty & \text{if } \lambda \geq 0, \\ \frac{1}{\alpha|\lambda|} & \text{otherwise.} \end{cases} \tag{3.20}$$

We are now ready to state the following result which is a generalization of [34, Theorem 1].

Lemma 3.7. *Assume that, for any $r < 0$, the mapping $x \mapsto I(rx^\alpha)$ is r-invariant for Q^γ and that $I \in L^1(g(dy))$. Then, writing $\chi = \alpha\lambda$ and recalling that G_κ is a Gamma random variable with parameter $\kappa > 0$, independent of $\mathcal{F}_\infty$, the mapping $x \mapsto I_\kappa(x)$ defined by*

$$I_\kappa(x) = \Gamma(\kappa)\mathbb{E}\left[I\left(\chi G_\kappa x^\alpha\right)\right], \; x > 0, \tag{3.21}$$

satisfies the identity, for any $0 \leq t < \zeta^\lambda$,

$$(1+\chi t)^{-\kappa} Q_t^\gamma \left(d_{(1+\chi t)^{-\tilde{\alpha}}} I_\kappa\right)(x) = I_\kappa(x), \; x > 0, \tag{3.22}$$

where d stands for the dilatation operator, i.e. $d_c f(x) = f(cx)$.

Proof. Let us assume that $I \in L^1(g(dy))$. Then, for any $x > 0$ and $0 \leq t < \zeta^\lambda$, we have, from the definition of I_κ, that

$$(1+\chi t)^{-\kappa} Q_t^\gamma \left(d_{(1+\chi t)^{-\tilde{\alpha}}} I_\kappa\right)(x) \;=\; (1+\chi t)^{-\kappa} \mathbb{E}_x^{(\gamma)}\left[\int_0^\infty I\left(\chi y (1+\chi t)^{-1} X_t^\alpha\right) e^{-y} y^{\kappa-1} dy\right].$$

Performing the change of variable $z = y(1+\chi t)^{-1}$ and using successively Fubini theorem and the invariance property of I, we get

$$\begin{aligned}
(1+\chi t)^{-\kappa} Q_t^\gamma \left(d_{(1+\chi t)^{-\tilde{\alpha}}} I_\kappa\right)(x) &= \mathbb{E}_x^{(\gamma)}\left[\int_0^\infty e^{-z\chi t} I\left(\chi z X_t^\alpha\right) e^{-z} z^{\kappa-1} dz\right] \\
&= \int_0^\infty I\left(\chi z x^\alpha\right) e^{-z} z^{\kappa-1} dz \\
&= I_\kappa(x)
\end{aligned}$$

which completes the proof. □

Next, we introduce the stopping times, $T_a^{(\lambda)}$ and $H_a^{(\lambda)}$, defined, for any $\lambda \in \mathbb{R}$ and $a > 0$, by

$$\begin{aligned}
T_a^{(\lambda)} &= \inf\{s > 0;\; X_s = a(1+\alpha\lambda s)^{\frac{1}{\alpha}}\} \wedge \zeta^\lambda, \\
H_a^{(\lambda)} &= \inf\{s > 0;\; e^{-\lambda s} X_{\tau_\lambda(s)} = a\}
\end{aligned}$$

where $\tau_\lambda(s) = (e^{\alpha\lambda s} - 1)/\alpha\lambda$. Writing $(a)_+ = \max(a, 0)$, we have

$$e^{\lambda\alpha H_a^{(\lambda)}} \overset{(d)}{=} \left(1 + \alpha\lambda T_a^{(\lambda)}\right)_+ \tag{3.23}$$

and, in particular, for $a = 0$, since $T_0^{(\lambda)} = T_0 \wedge \zeta^\lambda$, we obtain

$$e^{\lambda\alpha H_0^{(\lambda)}} \overset{(d)}{=} \left(1 + \alpha\lambda T_0\right)_+ .$$

In particular, we have $H_0 = H_0^{(-\tilde{\alpha})}$ with H_0 defined in (2.2). Next, for any $a > 0$, we set

$$\begin{aligned}\kappa^+_\psi(a) &= \inf\{\kappa \in \mathbb{R}^+;\ \mathcal{O}_\psi(\kappa; a^\alpha) = 0\}\\ \kappa^-_\psi(a) &= \inf\{\kappa \in \mathbb{R}^+;\ \mathcal{I}_\psi(-\kappa; a^\alpha) = 0\},\end{aligned}$$

with the usual convention that $\inf\{\varnothing\} = \infty$, for the smallest positive real zero of the function $\mathcal{O}_\psi(.; a^\alpha)$ (resp. $\mathcal{I}_\psi(-.; a^\alpha)$). We are now ready to state the following.

Corollary 3.8. *Let $0 \leq x \leq a$ and recall that $\chi = \alpha\lambda$.*

(1) Then, for any $\lambda < 0$ and any $\rho \in \mathbb{C}$ with $\mathfrak{Re}(\rho) < \kappa^+_{\psi_\gamma}(|\chi|^{\tilde{\alpha}}a)$, we have

$$\mathbb{E}^{(\gamma)}_x\left[(1+\chi T^{(\lambda)}_a)^{-\rho}_+\right] = \frac{\mathcal{O}_{\psi_\gamma}(\rho; |\chi| x^\alpha)}{\mathcal{O}_{\psi_\gamma}(\rho; |\chi| a^\alpha)}.$$

Consequently, for any real κ such that $\kappa < \kappa^+_{\psi_\gamma}(a)$, the mapping $x \mapsto \mathcal{O}_\psi(\kappa; |\chi| x^\alpha)$ is positive on $\mathbb{R}^+$.

(2) Similarly, for any $\lambda > 0$ and any $\rho \in \mathbb{C}$ with $\mathfrak{Re}(\rho) > \kappa^-_{\psi_\gamma}(\chi^{\tilde{\alpha}}a)$, we have

$$\mathbb{E}^{(\gamma)}_x\left[(1+\chi T^{(\lambda)}_a)^{-\rho}\right] = \frac{\mathcal{I}_{\psi_\gamma}(\rho; \chi x^\alpha)}{\mathcal{I}_{\psi_\gamma}(\rho; \chi a^\alpha)}$$

where we assume that $\lambda a^\alpha < b$ if $\lim_{u\to\infty} \frac{\psi_\gamma(u)}{u} = b$.

Proof. Let us first deal with the case $\lambda < 0$ and by the self-similarity property of X, we set, without loss of generality, $\chi = -1$. Moreover, since X is a Feller process on $[0, \infty)$, we can start by fixing $x = 0$ and $a > 0$. Then, recalling that $\mathcal{O}_{\psi_\gamma}(0, a^\alpha) = 1$, we observe that $\mathcal{O}_{\psi_\gamma}(\kappa; a^\alpha)$ is positive for any $0 \leq \kappa < \kappa^+_{\psi_\gamma}(a)$ reals. The existence of such an interval follows from the fact that the zeros of a non constant holomorphic function are isolated. Thus, by means of the Dynkin formula combined with Lemmae 3.6 and 3.7, we deduce, for any $0 \leq \kappa < \kappa^+_{\psi_\gamma}(a)$, that

$$\mathbb{E}^{(\gamma)}_0\left[(1-T^{(\lambda)}_a)^{-\kappa}_+\right] = \frac{1}{\mathcal{O}_{\psi_\gamma}(\kappa; a^\alpha)}. \tag{3.24}$$

Next, we recall, from the identity (3.23), that

$$e^{\kappa H^{(\lambda)}_a} \stackrel{(d)}{=} \left(1 - T^{(\lambda)}_a\right)^{-\kappa}_+.$$

Since $H^{(\lambda)}_a$ is a positive random variable, as a Laplace transform, the left-hand side on the identity (3.24), is analytic in the half-plane $\{\rho \in \mathbb{C};\ \mathfrak{Re}(\rho) < \kappa^+_{\psi_\gamma}(a)\}$ and positive on $\mathbb{R}^+$, see e.g. [44, Chap. II]. Then, let us assume that there exists a complex number $\rho(a)$ in the strip $0 \leq \mathfrak{Re}(\rho(a)) < \kappa^+_{\psi_\gamma}(a)$ such that $\mathcal{O}_{\psi_\gamma}(\rho(a); a^\alpha) = 0$. However, as the left-hand side of (3.24) is analytical with respect to the argument κ in this strip, we deduce, by the principle of analytical continuation, that this is not possible. Moreover, we get that $\mathcal{O}_{\psi_\gamma}(\rho; a^\alpha)$ has no zeros on $\{\rho \in \mathbb{C} :\ \mathfrak{Re}(\rho) < \kappa^+_{\psi_\gamma}(a)\}$ and is positive on $\{\kappa \in \mathbb{R} :\ \kappa < \kappa^+_{\psi_\gamma}(a)\}$. Finally, let us consider a real number a_1 such that $0 < a_1 \leq a$. Clearly, $\mathbb{Q}^{(\gamma)}_0$-a.s. $(1 - T^{(\lambda)}_{a_1})^{-\kappa}_+ \leq (1 - T^{(\lambda)}_a)^{-\kappa}_+$, for any $0 \leq \kappa < \kappa^+_{\psi_\gamma}(a) \wedge \kappa^+_{\psi_\gamma}(a_1)$. Then, we deduce from (3.24), for any $0 \leq \kappa < \kappa^+_{\psi_\gamma}(a) \wedge \kappa^+_{\psi_\gamma}(a_1)$, that

$$0 < \frac{1}{\mathcal{O}_{\psi_\gamma}(\kappa; a_1^\alpha)} \leq \frac{1}{\mathcal{O}_{\psi_\gamma}(\kappa; a^\alpha)}.$$

Thus, it is not difficult to see that $\kappa^+_{\psi_\gamma}(a_1) \geq \kappa^+_{\psi_\gamma}(a)$. Therefore, since $\kappa^+_{\psi_\gamma}(x) \geq \kappa^+_{\psi_\gamma}(a)$, for any $0 \leq x \leq a$, the strong Markov property and the absence of positive jumps of X complete the proof in the case $\lambda < 0$. The case $\lambda > 0$ is proved by following a similar reasoning. We point out that in [34, Corollary 3.2], the restriction $\lambda a^\alpha < b$ if $\lim_{u\to\infty} \frac{\psi_\gamma(u)}{u} = b$ has been omitted. □

We are now able to proceed to the proof of Theorem 3.2 which we now split into two parts: the case when T_0 occurs continuously, i.e. $q = 0$ and $E[\xi_1] < 0$ and the case when T_0 occurs by a jump, i.e. $q > 0$.

3.1.1. *Continuous killing.* Here, we assume that $q = 0$ and $E[\xi_1] < 0$. Thus, there exist $\theta > 0$ such that $\psi(\theta) = 0$ and $\gamma = \theta$. We recall that $\psi_\theta(u) = \psi(\theta + u)$ with $\psi'_\theta(0^+) > 0$ and thus $\psi_\theta \in \mathcal{LK}_+$.

Lemma 3.9. *(1) Let $\lambda < 0$. Then, $\kappa^+_\theta(|\chi|^{\tilde\alpha}a) = \kappa^+_{\psi_\theta}(|\chi|^{\tilde\alpha}a) - \tilde\alpha\theta > 0$ and for any $\kappa < \kappa^+_\theta(|\chi|^{\tilde\alpha}a)$ and $0 < x \leq a$, we have*

$$\mathbb{E}_x\left[(1+\chi T_a^{(\lambda)})^{-\kappa}\mathbb{I}_{\{T_a^{(\lambda)} < T_0 \wedge \zeta^\lambda\}}\right] \;=\; \frac{x^\theta}{a^\theta}\frac{\mathcal{O}_{\psi_\theta}(\kappa+\tilde\alpha\theta;\chi x^\alpha)}{\mathcal{O}_{\psi_\theta}(\kappa+\tilde\alpha\theta;\chi a^\alpha)}.$$

In particular, for any $0 < x \leq a$, we have

$$\mathbb{Q}_x\left[T_a^{(\lambda)} < T_0 \wedge \zeta^\lambda\right] \;=\; \frac{x^\theta}{a^\theta}\frac{\mathcal{O}_{\psi_\theta}(\tilde\alpha\theta;|\chi| x^\alpha)}{\mathcal{O}_{\psi_\theta}(\tilde\alpha\theta;|\chi| a^\alpha)}. \tag{3.25}$$

(2) Similarly, for any $\lambda > 0$ and $0 < x \leq a$,

$$\mathbb{Q}_x\left[T_a^{(\lambda)} < T_0\right] \;=\; \frac{x^\theta}{a^\theta}\frac{\mathcal{I}_{\psi_\theta}(\tilde\alpha\theta;\chi x^\alpha)}{\mathcal{I}_{\psi_\theta}(\tilde\alpha\theta;\chi a^\alpha)} \tag{3.26}$$

where we assume again that $\lambda a^\alpha < b$ if $\lim_{u\to\infty}\frac{\psi_\theta(u)}{u} = b$.

Remark 3.10. From (3.25) and (3.26), we observe, for any $\lambda < 0$ and $0 < x \leq a$, the following identity

$$\mathbb{Q}_x\left[T_a^{(\lambda)} < T_0 \wedge \zeta^\lambda\right] \;=\; \mathbb{Q}_{e^{i\tilde\alpha\pi}x}\left[T^{(|\lambda|)}_{e^{i\tilde\alpha\pi}a} < T_0\right].$$

It would be interesting to prove such a formula directly from the definition of $T_a^{(\lambda)}$.

Proof. Let us first consider $\lambda < 0$ and as above we fix $\chi = -1$. We start by using the fact that the function $x \mapsto x^{-\theta}$ is excessive for $Q_t^{(\theta)}$, see e.g. [40]. More precisely, one has, for any $t > 0$ and for any F a $\mathcal{F}_t$-measurable and bounded random variable,

$$\mathbb{E}_x^{(\theta)}[F] = \mathbb{E}_x\left[X_t^\theta F, t < T_0\right], \quad x > 0.$$

Note that this relation also holds for any $\mathcal{F}_\infty$-stopping time. Moreover, proceeding as in the proof of Corollary 3.8, one gets that the Mellin transform of the positive random variable $(1 - T_a^{(\lambda)})_+$ is well defined for any real κ such that $\kappa \leq 0$. Thus, since X has no positive jumps, one obtains by means of both Corollary 3.8 and the optional stopping theorem, for any $\kappa \leq 0$,

$$\begin{aligned}\mathbb{E}_x\left[(1-T_a^{(\lambda)})_+^{-\kappa}\mathbb{I}_{\{T_a^{(\lambda)}<T_0\}}\right] &= \frac{x^\theta}{a^\theta}\mathbb{E}_x^{(\theta)}\left[(1-T_a^{(\lambda)})_+^{-(\kappa+\tilde\alpha\theta)}\right]\\ &= \frac{x^\theta}{a^\theta}\frac{\mathcal{O}_{\psi_\theta}(\kappa+\tilde\alpha\theta;x^\alpha)}{\mathcal{O}_{\psi_\theta}(\kappa+\tilde\alpha\theta;a^\alpha)}.\end{aligned}$$

We deduce that $\kappa^+_\theta(|\chi|^{\tilde{\alpha}}a) > 0$ and the proof of the item (1) is completed by letting $\kappa \to 0$. The last statement follows again by adapting the above arguments to the case $\lambda > 0$. $\square$

We are now ready to complete the proof of Theorem 3.2 in the case $\gamma = \theta$. Let us fix $\lambda = -\tilde{\alpha}$ and observe, from (3.20), that $\zeta^{-\tilde{\alpha}} = 1$, then one gets that

$$\begin{aligned}\mathbb{Q}_x\left[T_a^{(-\tilde{\alpha})} < T_0 \wedge 1\right] &= \mathbb{Q}_x\left[\tau_{-\tilde{\alpha}}(H_a) < \tau_{-\tilde{\alpha}}(H_0) \wedge 1\right] \\ &= \mathbb{Q}_x\left[H_a < H_0\right]\end{aligned}$$

since the mapping $t \mapsto \tau_{-\tilde{\alpha}}(t)$ is increasing and $\tau^{-1}_{-\tilde{\alpha}}(1) = \infty$. Thus, as X has no positive jumps, one deduces that

$$\begin{aligned}\lim_{a\to\infty} \mathbb{Q}_x\left[T_a^{(-\tilde{\alpha})} < T_0\right] &= \mathbb{Q}_x\left[H_0 = \infty\right] \\ &= P(x).\end{aligned}$$

Hence, as we have learnt from Corollary 3.8 and Lemma 3.9 that the mapping $x \mapsto \mathcal{O}_{\psi_\theta}(\tilde{\alpha}\theta; x^\alpha)$ is positive on $\mathbb{R}^+$, there exists a constant $C_\theta > 0$ such that

$$\mathcal{O}_{\psi_\theta}(\tilde{\alpha}\theta; x^\alpha) \quad \sim \quad C_\theta^{-1} x^{-\theta}.$$

Finally, recalling that $\lim_{x\to\infty} P(x) = 1$, we obtain that

$$P(x) = C_\theta x^\theta \mathcal{O}_{\psi_\theta}(\tilde{\alpha}\theta; x^\alpha).$$

Hence, we deduce the expression of S by recalling that $S(t) = P(t^{-\tilde{\alpha}})$. Finally, the series $\mathcal{O}_{\psi_\theta}(\tilde{\alpha}\theta; x^\alpha)$ being absolutely continuous, the expression of the density s is obtained by differentiating terms by terms. Indeed, one has

$$\begin{aligned}s(t) &= -\frac{d}{dt}S(t) \\ &= C_\theta t^{-\tilde{\alpha}\theta-1}\frac{1}{\Gamma(\tilde{\alpha}\theta)}\sum_{n=0}^{\infty}(-1)^n a_n(\psi)(\tilde{\alpha}\theta+n)\Gamma(\tilde{\alpha}\theta+n)t^{-n} \\ &= \frac{\Gamma(\tilde{\alpha}\theta+1)}{\Gamma(\tilde{\alpha}\theta)} C_\theta t^{-\tilde{\alpha}\theta-1}\mathcal{O}_{\psi_\theta}(1+\tilde{\alpha}\theta; a^\alpha).\end{aligned}$$

The expression of the successive derivatives are obtained by an induction argument.

3.1.2. *Killed by a jump.* Throughout this part, we assume that ξ is a spectrally negative Lévy process killed at some independent exponential time of parameter $q > 0$. Recall that, for any $u \geq 0$, $\overline{\psi}(u) = \psi(u) - q$, ϕ is such that $\psi \circ \phi(u) = u$ and the mapping $\overline{\psi}_{\phi(q)}(u) = \overline{\psi}(u + \phi(q)) \in \mathcal{LK}_+$ as $\overline{\psi}'_{\phi(q)}(0^+) = \frac{1}{\phi'(q)} > 0$.

Lemma 3.11. *Let* $\lambda < 0$. *Then,* $\kappa^+_{\phi(q)}(|\chi|^{\tilde{\alpha}}a) = \kappa^{\pm}_{\overline{\psi}_{\phi(q)}}(|\chi|^{\tilde{\alpha}}a) - \tilde{\alpha}\phi(q) > 0$ *and for any* $\kappa < \kappa^{\pm}_{\overline{\psi}_{\phi(q)}}(|\chi|^{\tilde{\alpha}}a)$ *and* $0 < x \leq a$, *we have*

$$\mathbb{E}_x\left[(1+\chi T_a^{(\lambda)})_+^{-\kappa}\mathbb{I}_{\{T_a^{(\lambda)}<T_0\}}\right] = \frac{x^{\phi(q)}}{a^{\phi(q)}}\frac{\mathcal{O}_{\overline{\psi}_{\phi(q)}}(\kappa+\tilde{\alpha}\phi(q); \chi x^\alpha)}{\mathcal{O}_{\overline{\psi}_{\phi(q)}}(\kappa+\tilde{\alpha}\phi(q); \chi a^\alpha)}.$$

In particular,

$$\mathbb{Q}_x\left[T_a^{(\lambda)} < T_0 \wedge \zeta^\lambda\right] = \frac{x^{\phi(q)}}{a^{\phi(q)}} \frac{\mathcal{O}_\psi(\tilde{\alpha}\phi(q); \chi x^\alpha)}{\mathcal{O}_\psi(\tilde{\alpha}\phi(q); \chi a^\alpha)}.$$

Similarly, for any $\lambda > 0$ *and* $0 < x \leq a$*, we have*

$$\mathbb{Q}_x\left[T_a^{(\lambda)} < T_0\right] = \frac{x^{\phi(q)}}{a^{\phi(q)}} \frac{\mathcal{I}_\psi(\tilde{\alpha}\phi(q); \chi x^\alpha)}{\mathcal{I}_\psi(\tilde{\alpha}\phi(q); \chi a^\alpha)}$$

where we assume that $\lambda a^\alpha < b$ *if* $\lim_{u\to\infty} \frac{\psi_{\phi(q)}(u)}{u} = b$.

Remark 3.12. From the Lemma, we also observe, for any $\lambda < 0$ and $0 < x \leq a$, the following identity

$$\mathbb{Q}_x\left[T_a^{(\lambda)} < T_0 \wedge \zeta^\lambda\right] = \mathbb{Q}_{e^{i\tilde{\alpha}\pi}x}\left[T_{e^{i\tilde{\alpha}\pi}a}^{(|\lambda|)} < T_0\right].$$

Proof. Let us observe from the Lamperti mapping (1.1) that the semigroup, $(Q_t)_{t\geq 0}$ of X, is given, for a function f positive and measurable on $\mathbb{R}^+$, by

$$Q_t f(x) = \mathbb{E}_x^{(q)}\left[e^{-qA_t} f(X_t)\right], \quad t \geq 0, x > 0,$$

where $\mathbb{E}^{(q)}$ stands for the expectation operator associated to the law of X with underlying Laplace exponent ψ. This identity could also be derived from the Feynman-Kac formula. Thus, for any $\mathcal{F}_\infty$-stopping time T, one has

$$\mathbb{E}_x\left[f(X_T)\right] = \mathbb{E}_x^{(q)}\left[e^{-qA_T} f(X_T)\right].$$

Moreover, as ξ has independent increments, it is plain that the process $(e^{-qt+\phi(q)\xi_t})_{t\geq 0}$ is $\mathbb{P}^{(q)}$-martingale, where $\mathbb{P}^{(q)}$ stands for the law of the Lévy process with Laplace exponent ψ. By time change, one deduces that the process $(X_t^{\phi(q)} e^{-qA_t})_{t\geq 0}$ is a $\mathbb{Q}_1$-martingale. Thus, one can define a new probability measure, which we denote by $\mathbb{Q}^{(\phi(q))}$, as follows, for any $t > 0$ and for any F a $\mathcal{F}_t$-measurable and bounded random variable,

$$\mathbb{E}_x^{(\phi(q))}\left[F\right] = \mathbb{E}_x^{(q)}\left[X_t^{\phi(q)} e^{-qA_t} F\right],\ x > 0.$$

It is easily seen that the underlying Laplace exponent of X, under $\mathbb{Q}^{(\phi(q))}$, is $\overline{\psi}_{\phi(q)}$. Hence, one gets by the absence of positive jumps for X and an application of the optional stopping theorem, that, for any $0 < x \leq a$ and $\kappa \leq 0$,

$$\begin{aligned}
\mathbb{E}_x\left[(1+\chi T_a^{(\lambda)})_+^{-\kappa} \mathbb{I}_{\{T_a^{(\lambda)} < T_0\}}\right] &= \mathbb{E}_x^{(q)}\left[e^{-qA_{T_a^{(\lambda)}}} (1+\chi T_a^{(\lambda)})_+^{-\kappa} \mathbb{I}_{\{T_a^{(\lambda)} < T_0\}}\right] \\
&= \left(\frac{x}{a}\right)^{\phi(q)} \mathbb{E}_x^{(\phi(q))}\left[(1+\chi T_a^{(\lambda)})_+^{-(\kappa+\tilde{\alpha}\phi(q))}\right] \\
&= \left(\frac{x}{a}\right)^{\phi(q)} \frac{\mathcal{O}_{\overline{\psi}_{\phi(q)}}(\kappa+\tilde{\alpha}\phi(q); |\chi| x^\alpha)}{\mathcal{O}_{\overline{\psi}_{\phi(q)}}(\kappa+\tilde{\alpha}\phi(q); |\chi| a^\alpha)}
\end{aligned}$$

where the last line follows from Corollary 3.8 since $\overline{\psi}'_{\phi(q)}(0^+) > 0$. The first statement is proved. The second one is obtained by means of the same devices. □

The proof of the Theorem is completed by following a line of reasoning similar to the previous case.

4. Some illustrative examples

4.1. **The Bessel processes.** We consider ξ to be a 2-scaled Brownian motion with drift $2b \in \mathbb{R}$ and killed at some independent exponential time of parameter $q > 0$, i.e. $\overline{\psi}(u) = 2u^2 + 2bu - q$ and $2\phi(q) = \sqrt{2q + b^2} - b$. Note that $\overline{\psi}_{\phi(q)}(u) = 2u^2 + (2b + \phi(q))u$. Its associated self-similar process X is well known to be a Bessel process of index b killed at a rate $q \int_0^t X_s^{-2} ds$. Moreover, we obtain, setting $\varrho = b + 2\phi(q)$,

$$\begin{aligned} \mathcal{O}_{\overline{\psi}_{\phi(q)}}(\rho; x) &= \frac{\Gamma(\varrho+1)}{\Gamma(\rho)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\rho+n)}{n!\Gamma(n+\varrho+1)} (x/2)^n \\ &= \Phi\left(\rho, \varrho+1; -x/2\right) \end{aligned}$$

where Φ stands for the confluent hypergeometric function. We refer to Lebedev [25, Section 9] for useful properties of this function. Next, using the following asymptotic

$$\Phi\left(\rho, \varrho+1; -x\right) \sim \frac{\Gamma(\varrho+1)}{\Gamma(\varrho+1-\rho)} x^{-\rho} \quad \text{as } x \to \infty,$$

we get that $C_{\phi(q)} = \frac{\Gamma(\varrho+1-\phi(q))}{2^{\phi(q)}\Gamma(\varrho+1)}$. Thus, we obtain, recalling that, for any $q > 0$, $\varrho - \phi(q) = b + \phi(q) > 0$,

$$\begin{aligned} s_{\phi(q)}(t) &= \phi(q) \frac{\Gamma(\varrho+1-\phi(q))}{2^{\phi(q)}\Gamma(\varrho+1)} t^{-\phi(q)-1} \Phi\left(1+\phi(q), \varrho+1; -(2t)^{-1}\right) \\ &= \frac{b+\phi(q)}{2^{\phi(q)}\Gamma(\phi(q))} t^{-\phi(q)-1} \int_0^1 e^{-\frac{u}{2t}} (1-u)^{\varrho-\phi(q)-1} u^{\phi(q)} du \end{aligned}$$

which is the expression [45, (5.a) p.105]. Considering now the case $q = 0$ and $b < 0$, we obtain readily that $\theta = -b$ and

$$\begin{aligned} s_{\theta}(t) &= \frac{2^b}{\Gamma(-b)} t^{b-1} \Phi\left(1-b, 1-b; -(2t)^{-1}\right) \\ &= \frac{2^b}{\Gamma(-b)} t^{b-1} e^{-\frac{1}{2t}}. \end{aligned}$$

Hence, we deduce the well-known identity $(T_0, \mathbb{Q}_1) \stackrel{(d)}{=} \frac{1}{2G_{-b}}$ where we recall that G_{-b} stands for a Gamma random variable of parameter $-b > 0$.

4.2. **Law of the maximum of regular spectrally positive stable Lévy processes.** Here X is a regular spectrally negative α-stable, $\alpha \in (1, 2)$, Lévy process killed upon entering into the negative half-line. Caballero and Chaumont [11] characterized the characteristic triplet of the underlying Lévy process. In [35], the author obtained the expression for its Laplace exponent. Instead of using this expression, we follow an alternative route. Indeed, in [32], the author computed the unique increasing invariant function, say P_λ, of the Ornstein-Uhlenbeck process defined by

$$U_t = e^{-\lambda t} X_{(e^{\alpha\lambda t}-1)/\alpha\lambda},\ t \geq 0,$$

for $\lambda > 0$. The function P_λ, is given, with C a constant to be determined and recalling that $\tilde{\alpha} = 1/\alpha$, by

$$
\begin{aligned}
(4.1) \qquad P_\lambda(x) &= Cx^{\alpha-1}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-\tilde{\alpha})}{\Gamma(\alpha n+\alpha)}(\alpha\lambda)^n x^{\alpha n} \\
&= Cx^{\alpha-1}{}_2\Psi_1\left(\begin{array}{c}(1,1),(1,1-\tilde{\alpha})\\(\alpha,\alpha)\end{array}\middle|\,\alpha\lambda x^{\alpha}\right), \quad x \geq 0,
\end{aligned}
$$

where ${}_2\Psi_1$ stands for the Wright hypergeometric function. We mention that $P_\lambda(x)$ is obtained from the identity (3.5) with $\mathcal{I}_\psi$ the so-called q-scale function of a spectrally negative α-stable process which was computed by Bertoin [4] in terms of the Mittag-Leffler function. More precisely, we have

$$
P_\lambda(x) = C\lim_{\kappa\to 0}\Gamma(\kappa)\mathbb{E}\left[\mathcal{I}_\psi\left(G_{\frac{\kappa}{\alpha\lambda}}(\alpha\lambda)^{\tilde{\alpha}}x\right)\right]
$$

with

$$
\mathcal{I}_\psi(x) = x^{\alpha-1}\sum_{n=0}^{\infty}\frac{x^{\alpha n}}{\Gamma(\alpha(n+1))}.
$$

From Remark 3.12, we have $P(x) = P_{\tilde{\alpha}}(e^{i\tilde{\alpha}\pi}x)$. Note that $P(0) = 0$ and using the large asymptotic of the function ${}_2\Psi_1$, detailed can be found in [36], we get, as $x \to \infty$,

$$
{}_2\Psi_1\left(\begin{array}{c}(1,1),(1,1-\tilde{\alpha})\\(\alpha,\alpha)\end{array}\middle|\,-x^{\alpha}\right) \sim \left(\frac{\sin(\tilde{\alpha}\pi)}{\pi}\right)^{-1}x^{1-\alpha}.
$$

Hence, by setting $C = \frac{\sin(\tilde{\alpha}\pi)}{\pi}$, we obtain the required condition $\lim_{x\to\infty}P(\infty) = 1$ and

$$
P(x) = \frac{\sin(\tilde{\alpha}\pi)}{\pi}x^{\alpha-1}{}_2\Psi_1\left(\begin{array}{c}(1,1),(1,1-\tilde{\alpha})\\(\alpha,\alpha)\end{array}\middle|\,-x^{\alpha}\right).
$$

Next, from the identity (2.7), we find that

$$
\mathbb{P}\left(\max_{0\leq s\leq 1}\hat{Z}_s \geq x\right) = P(x)
$$

where $\hat{Z}$ is a spectrally positive stable process of index α. Thus, by differentiating, one gets the following expression for the density

$$
p(x) = \frac{\sin(\tilde{\alpha}\pi)}{\pi}x^{\alpha-2}{}_2\Psi_1\left(\begin{array}{c}(1,1),(1,1-\tilde{\alpha})\\(\alpha,\alpha-1)\end{array}\middle|\,-x^{\alpha}\right),
$$

which is the expression found by Bernyk et al. [2, Theorem 1].

4.3. **The self-similar saw-tooth processes.** Finally, we consider the so-called saw-tooth process introduced and deeply studied by Carmona et al. [15]. It is a self-similar positive Markov process of index $\alpha = 1$ with underlying Lévy process the sum of a drift of parameter $b = 1$ and the negative of a compound Poisson process of parameter $\beta > 0$ whose jumps are exponentially distributed with parameter $\delta + \beta - 1 > 0$, i.e.

$$
\psi(u) = u\frac{u+\delta-1}{u+\delta+\beta-1}, \quad u \geq 0.
$$

Moreover, in [15], the authors show that

$$\phi(q) = \frac{1}{2}\left(q - (\delta - 1) + \bar{\phi}(q)\right), \quad q \geq 0,$$

where $\bar{\phi}(q) = \sqrt{(q-(\delta-1))^2 + 4(\delta+\beta-1)q}$. Let us proceed with the case $q = 0$. Note, for $1-\beta < \delta < 1$, that $\gamma = 1-\delta$ and

$$\psi_{1-\delta}(u) = u\frac{u+1-\delta}{u+\beta}.$$

Thus

$$a_n(\psi_{1-\delta}, 1) = \frac{\Gamma(n+1+\beta)\Gamma(2-\delta)}{\Gamma(1+\beta)\Gamma(n+1)\Gamma(n+2-\delta)}, \quad a_0 = 1,$$

and for $|z| < 1$

$$\begin{aligned}
\mathcal{O}_{\psi_{1-\delta}}(\rho; z) &= \frac{\Gamma(2-\delta)}{\Gamma(\rho)\Gamma(1+\beta)}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(\rho+n)\Gamma(n+1+\beta)}{\Gamma(n+2-\delta)n!}z^n \\
&= {}_2F_1(\rho, 1+\beta, 2-\delta; -z)
\end{aligned}$$

where ${}_2F_1(a,b;x)$ stands for the hypergeometric function, see Lebedev [25, Section 9] for a detailed account on this function. Next, recalling the identity

$${}_2F_1(-n, 1+\beta, \delta; 1) = \frac{\Gamma(2-\delta)\Gamma(n+1-\delta-\beta)}{\Gamma(2-\delta+n)\Gamma(1-\delta-\beta)},$$

we recover from (3.8) the well-known identity

$${}_2F_1(\rho, 1+\beta, 2-\delta; z) = (1-z)^{-\rho}{}_2F_1\left(\rho, 1-\delta-\beta, \delta; \frac{z}{z-1}\right),$$

which provides an analytical continuation of the hypergeometric function into the half plane $\mathfrak{Re}(z) < \frac{1}{2}$. Finally, using the following asymptotic

$${}_2F_1(\rho, 1+\beta, 2-\delta; -x) \sim \frac{\Gamma(2-\delta)\Gamma(1+\beta-\rho)}{\Gamma(2-\delta-\rho)\Gamma(1+\beta)}x^{-\rho} \quad \text{as } x \to \infty,$$

one obtains

$$S(t) = \frac{\Gamma(1+\beta)}{\Gamma(2-\delta)\Gamma(\beta+\delta)}t^{\delta-1}{}_2F_1\left(1-\delta, 1+\beta, \delta; -t^{-1}\right).$$

Moreover, after some easy computations, one gets for $\gamma = \phi(q), q > 0$,

$$\bar{\psi}_{\phi(q)}(u) = u\frac{u+\phi(q)}{u+\beta+\delta+\phi(q)-1}.$$

Thus, proceeding as above, we obtain

$$\mathcal{O}_{\bar{\psi}_{\phi(q)}}(\rho; z) = {}_2F_1(\rho, \beta+\delta+\phi(q), 1+\phi(q); -z)$$

and

$$S(t) = \frac{\Gamma(\beta+\delta+\phi(q))\Gamma(1+\bar{\phi}(q)-\phi(q))}{\Gamma(1+\bar{\phi}(q))\Gamma(\beta+\delta)}t^{-\phi(q)}{}_2F_1\left(\phi(q), \beta+\delta+\phi(q), 1+\bar{\phi}(q); -t^{-1}\right).$$

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Département de Mathématiques, Université Libre de Bruxelles, Boulevard du Triomphe, B-1050, Bruxelles, Belgique.

E-mail address: ppatie@ulb.ac.be