WARFIELD DUALITIES INDUCED BY SELF-SMALL MIXED GROUPS

SIMION BREAZ

Abstract. For a self-small abelian group A of torsion-free rank 1, we characterize A-reflexive abelian groups which are induced by the contravariant functor $\mathrm{Hom}(-, A)$ in two cases: the range of $\mathrm{Hom}(-, A)$ is the category of all abelian groups, respectively the range of $\mathrm{Hom}(-, A)$ is the category of all left E-modules, where E is the endomorphism ring of A.

1. Introduction

Let A be an abelian group with E its endomorphism ring. It induces the following pairs of adjoint contravariant functors:

$$W(-) = \mathrm{Hom}(-, A) : Ab \to Ab : \mathrm{Hom}(-, A) = W(-)$$

$$\Delta(-) = \mathrm{Hom}(-, A) : Ab \to E\text{-Mod} : \mathrm{Hom}_E(-, A) = \Delta(-)$$

together the natural transformations $\nu : 1_{Ab} \to W^2$, respectively $\delta : 1_{Ab} \to \Delta^2$. An abelian group C is called *A-W-reflexive* (*A-Δ-reflexive*) if ν_C is an isomorphism (respectively δ_C is an isomorphism). The classes of A-W-reflexive, respectively A-Δ-reflexive, groups are maximal classes such that the restrictions of the functors W, respectively Δ, to these classes are dualities. In abelian groups theory, this kind of dualities are called *Warfield dualities*.

The study of Warfield dualities is an important tool. Results obtained by Warfield (for finite rank torsion-free A-reflexive groups, where A is torsion-free of rank 1) in its seminal paper [23] were extended, respectively used, by many authors during the last 40 years, especially in the theory of (finite-rank) torsion-free groups or in theories of torsion-free modules over some commutative domains. For example, Warfield's theory presented in [23, Section 3] was extended in [19], [20] and [21], to some more general classes of (locally-free) finite rank torsion-free groups. These generalizations, together with results concerning Butler groups (which exhibit another kind of duality in finite rank torsion-free setting), were used to explain the existence of some properties of finite rank torsion-free groups which are dual each to other. Moreover, in [14] the authors show that every duality on categories of torsion-free modules of finite rank over Dedekind domains, which satisfies some natural conditions, is Warfield. The reader can find more complete details for theories concerning Warfield dualities in [4] and [13].

In the present paper we will study A-W-reflexive (respectively A-Δ-reflexive) groups in the case A is a (mixed) self-small group of torsion-free rank 1. This

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study is motivated by the fact that many properties of finite rank torsion-free groups can be extended to some classes of self-small mixed groups (e.g. in [3] a Steinitz's Theorem concerning cancellation properties of tffr groups was extended to qd-groups, a class of self-small groups, in [9] it is proved that every self-small group of finite torsion-free rank has a unique Krull-Schmidt quasi-decomposition, in [15] the authors extended Arnold's duality [7] from torsion-free qd-groups to mixed qd-groups etc.). In many cases these extensions request new ideas for proofs (see [3] or [15]). Moreover, there are many results concerning tffr groups which are not valid for self-small mixed groups (e.g. [24]). Such differences are also exhibited in the present paper: in Remark 3.2 and Example 3.6 it is shown that some Warfield's results about finitely A-cogenerated groups, where A is a rank 1 torsion-free group, are not valid in the case A is a proper mixed self-small group of torsion-free rank 1, and in Theorem 5.4 it is proved that, in contrast with the torsion-free case, the notions "A-W-reflexive" and "A-Δ-reflexive" do not coincide in the mixed case (even they are not so far each to other).

In Section 2 we present, for reader's convenience, a brief summary for properties of finite torsion-free rank self-small groups. Section 3 is dedicated to finitely A-cogenerated groups, where A is a self-small group torsion-free rank 1. The structure of some special finitely A-cogenerated groups is described in Proposition 3.1, where it is proved that every finitely A-cogenerated group is a direct sum of a torsion group and a self-small group with some special properties. This result suggests us to split the study of A-W-reflexive and A-Δ-reflexive groups (Section 4, respectively Section 5) in two cases: torsion groups (Theorem 4.1, respectively Theorem 5.2) and self-small groups (Theorem 4.2 and Theorem 4.4, respectively Theorem 5.4). In the end of Section 4 we show that our results concerning the structure of A-W-reflexive groups cannot be extended to the case when A has p-rank at most 1 for all primes p, but it is not self-small.

In this paper all groups all abelian groups. The set of all primes is denoted by $\mathbb{P}$, and $\mathbb{Z}(p^k)$ denotes the cyclic group $\mathbb{Z}/p^k\mathbb{Z}$. If G is a group then $T_p(G)$ denotes the p-component of G, $T(G)$ is the torsion part of G, $\overline{G} = G/T(G)$. If p is a prime, the $\mathbb{Z}(p)$-dimension of G/pG is called the *p-rank* of G. A subgroup $F \leq G$ is *full* if G/F is a torsion group. If G is a torsion group then G_p denotes the p-component of G. If G is torsion-free $G_p = G \otimes \mathbb{Z}_p$ is the localization of G at the prime p. $S(G)$ denotes the set of all primes p such that $T_p(G) \neq 0$, and $D(G)$ is the set of all primes p such that $\overline{G}$ is p-divisible. If G is a finite rank torsion-free group, $OT(G)$ denotes the outher type of G. We say that a set U is *quasi-contained* in a set V if $V \setminus U$ is finite, and we denote this by $U \dot{\subseteq} V$. U and V are *quasi-equal* sets, denoted by $U \doteq V$, if each is quasi-contained in the other. All unexplained notions and notations can be found in [6], [12], [13] or [17].

2. Self-small groups of finite torsion free rank

An abelian group A is *self-small*, [8], if $\mathrm{Hom}(A, -)$ commutes with direct sums of copies of A. By [8], every endomorphic image of a self-small group is self-small and a torsion group is self-small if and only if it is finite, hence we have the following lemma:

Lemma 2.1. *If A is a self-small group then* $\mathrm{Hom}(A, T(A))$ *is a torsion group.*

For the finite torsion-free rank case we have the following characterizations for self-small groups.

Theorem 2.2. *Let A be a group of finite torsion-free rank. The following affirmations are equivalent:*

a) *A is self-small;*
b) [8, Theorem 3.6]
 i) *$T_p(A)$ is finite for all $p \in \mathbb{P}$,*
 ii) *There exists a full free subgroup $F \leq A$ such that A/F is p-divisible for almost all p with $T_p(A) \neq 0$;*
c) [5, Theorem 2.4]
 i) *$T_p(A)$ is finite for all $p \in \mathbb{P}$,*
 ii) *$Hom(A, T(A))$ is a torsion group;*
d) [1, Section 4] *$A = B \oplus H$ such that B is a finite group and H is an extension of group X by a group Y with the following properties:*
 i) *X is a finite rank torsion-free $S(A)$-divisible group,*
 ii) *Y is torsion-free or Y is an $S(A)$-pure subgroup of a direct product of (finitely generated) p-adic modules $\prod_{p\in S(A)} M(p)$ such that Y satisfies the following projection condition: there is a full free subgroup of finite rank $F \subset Y$ such that, for every $p \in S(A)$, the natural projection $\pi'_p(F)$ of F into $M(p)$ generates $M(p)$ as a $\widehat{\mathbb{Z}}_p$- module.*

Remark 2.3. If A is a self-small group of finite torsion-free rank, the condition described in b)ii) is valid for all full-free subgroups of A.

Convention 2.4. If A is a self-small group of finite torsion-free rank (or, more generally, a group whose primary components are direct summands) and U is a finite set of primes then $T_U(A) = \oplus_{p\in U} T_p(A)$ is a direct summand of A, hence $A = T_U(A) \oplus A(U)$. The direct complement $A(U)$ of $T_U(A)$ is unique if only if $\overline{A}$ is p-divisible for all $p \in U$ with $T_p(A) \neq 0$. However, $A(U)$ is unique up to an isomorphism. If the choice of the direct decomposition is not important we will speak about $A(U)$ assuming implicit that a direct decomposition $A = T_U(A) \oplus A(U)$ is fixed.

If Y satisfies the condition d) ii) in the previous theorem then we say that it satisfies *the p-adic projection condition*. It is not hard to see that if Y satisfies the p-adic projection condition and $F' \leq Y$ is a full free subgroup then $\pi'_p(F')$ generates M_p for almost all p. We recall from the proofs of [1, Section 4] that $X = \cap_{p\in S(A)} p^\omega A$ and M_p is the p-adic completion of $A/p^\omega A$ for all $p \in S(A)$.

From the previous theorem we obtain more properties for self-small groups of finite torsion-free rank.

Corollary 2.5. *Let A be a self-small group of finite torsion-free rank n.*

a) [10, Lemma 2.1] *If $F \leq A$ is a full free subgroup and $\pi_p : A \to T_p(A)$ are canonical projections corresponding to some direct decompositions $A = T_p(A) \oplus A(p)$, then $\pi_p(F) = T_p(A)$ for almost all p.*

b) *If $T_p(A)$ is of rank n for almost all $p \in S(A)$ then $\overline{A}$ is p-divisible for almost all $p \in S(A)$ and there exists an embedding $A \hookrightarrow \prod_{p\in S(A)} T_p(A)$ which is p-pure for almost all $p \in S(A)$.*

Proof. b) Let $F \leq A$ be a full free subgroup of A and p a prime such that $T_p(A)$ is of rank n and $D = A/F$ is p-divisible. Using the exact sequence $F/pF \to A/pA \to$

$D/pD \to 0$, where $D/pD = 0$, we observe that the $\mathbb{Z}(p)$-dimension of A/pA is at most n. Since $A/pA = T_p(A)/pT_p(A) \oplus A(p)/pA(p)$, it follows that $A(p)/pA(p) = 0$, hence $A(p)$ is p-divisible.

If $\mu : A \to \prod_{p \in S(A)} T_p(A)$ is the homomorphism induced by the canonical projections $\pi_p : A \to T_p(A)$, then $\mu(A)$ is a self-small group since $T(A) = T(\mu(A))$ and $\mathrm{Hom}(\mu(A), T(A))$ must be a torsion group. Since almost all primary components of $\mu(A)$ are of rank n, by a) we deduce that the torsion-free rank of $\mu(A)$ is n, hence μ is injective. Moreover, $\mu(A)/T(A)$ is p-divisible for almost all $p \in S(A)$, hence μ is p-pure for almost all $p \in S(A)$. □

The following result generalizes [2, Theorem 3.3]. We include the proof for reader's convenience.

Lemma 2.6. *Let A be a group such that every p-component of A is a direct summand. For every group C, the canonical embedding*

$$\Theta = \Theta_{CA} : \mathrm{Hom}(C, A)/\mathrm{Hom}(C, T(A)) \hookrightarrow \mathrm{Hom}(\overline{C}, \overline{A}),$$
$$f + \mathrm{Hom}(C, T(A)) \mapsto [\overline{f} : c + T(C) \mapsto f(c) + T(A)]$$

is pure.

Proof. We consider the exact sequence $0 \to T(A) \to A \to \overline{A} \to 0$, and applying $\mathrm{Hom}(C, -)$ we obtain a monomorphism $\varphi : \mathrm{Hom}(C, A)/\mathrm{Hom}(C, T(A)) \to \mathrm{Hom}(C, \overline{A})$. Moreover, applying $\mathrm{Hom}(-, \overline{A})$ to the exact sequence $0 \to T(C) \to C \to \overline{C} \to 0$ we obtain an isomorphism $\psi : \mathrm{Hom}(C, \overline{A}) \to \mathrm{Hom}(\overline{C}, \overline{A})$. It is not hard to see that $\Theta = \psi\varphi$, hence it is enough to prove that φ is a pure monomorphism.

Since $\mathrm{Coker}(\varphi) \leq \mathrm{Ext}(C, T(A))$, and $\mathrm{Ext}(C, T(A))$ has trivial p-components for all $p \notin S(A)$ by [17, Lemma 52.1], φ is p-pure for all $p \notin S(A)$.

Let $p \in S(A)$. We consider a direct decomposition $A = T_p(A) \oplus A(p)$. Let $\rho : A \to A(p)$ be the canonical projection, and $\iota : A(p) \to A$ be the inclusion map. As in the first part of the proof we have the canonical embedding $\varphi' : \mathrm{Hom}(C, A(p))/\mathrm{Hom}(C, T(A(p))) \to \mathrm{Hom}(C, \overline{A(p)})$. Using the direct decompositions $\mathrm{Hom}(C, A) = \mathrm{Hom}(C, A(p)) \oplus \mathrm{Hom}(C, T_p(A))$, respectively $\mathrm{Hom}(C, T(A)) = \mathrm{Hom}(C, T(A(p))) \oplus \mathrm{Hom}(C, T_p(A))$ and the isomorphism $\overline{A(p)} \cong \overline{A}$, it is not hard to construct a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(C, A)/\mathrm{Hom}(C, T(A)) & \xrightarrow{\varphi} & \mathrm{Hom}(C, \overline{A}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(C, A(p))/\mathrm{Hom}(C, T(A(p))) & \xrightarrow{\varphi'} & \mathrm{Hom}(C, \overline{A(p)}) \end{array}$$

such that the vertical arrows are isomorphisms. Moreover, φ' is p-pure since $p \notin S(A(p))$, hence φ is p-pure, and the proof is complete. □

Corollary 2.7. *Let A be a self-small group. If C is a group then the group $\mathrm{Hom}(C, A)/\mathrm{Hom}(C, T(A))$ is p-divisible for all $p \in D(A)$.*

Self-small groups of torsion-free rank 1 were studied in [1]. In the following results we summarize basic properties of this kind of groups:

Theorem 2.8. [1] *A group A is self-small of torsion-free rank 1 if and only if $A \cong B \oplus R$ with B a finite group and $0 \neq R \leq \mathbb{Q}$ or $A \cong B \oplus A(T, R)$, where B is a finite group and:*

a) $T = \oplus_{p\in S} T_p$ *is a torsion group such that* $S \subseteq \mathbb{P}$ *is an infinite set of primes and all primary components* T_p *are non-zero cyclic groups,*
b) $R \leq \mathbb{Q}$ *is a rational group which is* p*-divisible for all* $p \in S$*,*
c) $T \leq A(T,R) \leq \prod_{p\in S} T_p$ *such that*
   i) $A(T,R)/T \cong R$*,*
   ii) $A(T,R)$ *satisfies the projection condition: if* $a \in A(T,R)$ *is an infinite order element then* $T_p = \langle \pi_p(a)\rangle$*, where* $\pi_p : \prod_{p\in S} T_p \to T_p$ *are the canonical projections, for all* $p \in S$*.*

If $A \cong B \oplus A(T,R)$ then we will say that A is a mixed self-small group. Moreover, if $A \cong A(T,R)$ for some T and R then we will say that A is *standard*. Remark that A mixed self-small group of torsion-free rank 1 is standard if and only if $S(A) \subseteq D(A)$ and every p-component of A is cyclic. It is not hard to see that for every mixed self-small group of torsion-free rank 1 there exists a finite set of primes U such that $A(U)$ is standard.

Corollary 2.9. [1] *The set of pairs* (T,R)*, where* T *and* R *are groups which satisfy conditions a) and b) of Theorem 2.8, is a complete set of independent invariants for standard self-small groups of torsion-free rank 1.*

If A_1 and A_2 are self-small groups of torsion-free rank 1, the structure of the group $\mathrm{Hom}(A_1,A_2)$ can be more complicated in the mixed case than in the torsion-free case. Moreover, in general $A_1 \oplus A_2$ is not a self-small group (see [1]). In the present paper we are interested only in a particular case. The proof of the following lemma is a simple exercise.

Lemma 2.10. *Let* $A_1 = A(T,R_1)$ *and* $A_2 = A(T,R_2)$ *be self-small groups of torsion-free rank 1 with the same torsion part. The following are equivalent:*
i) $\mathrm{Hom}(A_1,A_2)$ *is not a torsion group;*
ii) $\mathrm{type}(R_1) \leq \mathrm{type}(R_2)$*;*
iii) *There exists an embedding* $A_1 \hookrightarrow A_2$*.*

Since we are interested about finitely A-cogenerated groups, we recall from [1] two results concerning finite index subgroups of finite powers A^n, where A is a self-small group of torsion-free rank 1. Quasi-isomorphic torsion groups are defined in [6, p.11] and characterized in [6, Exercise 1.10].

Proposition 2.11. [1] *Let* $A = A(T,R)$ *be a standard mixed self-small group of torsion-free rank 1 and* n *a positive integer.*

a) *If* $C \leq A^n$ *is a subgroup of finite index then* $C \cong A(T_1,R) \oplus \dots A(T_n,R)$*, where* $T_1,\dots,T_n$ *are torsion groups which are quasi-isomorphic to* T*. Moreover, if* $T(C) = T(A^n)$ *then we can choice* $T_1 = \cdots = T_n = T$*, hence* $C \cong A^n$*.*

b) *If* C *is a mixed self-small group of torsion-free rank* n *such that* $T(C) \cong T^n$ *and* $\overline{C} \cong R^n$ *then* $C \cong A^n$*.*

3. Finitely A-cogenerated groups

Proposition 3.1. *Let* A *be a mixed self-small group of torsion-free rank 1. For a group* C *of finite torsion-free rank* n*, the following are equivalent:*
a) C *is finitely* A*-cogenerated and* $S(A)$ *is quasi-contained in* $D(C)$*;*
b) $C = C' \oplus T$ *with the following properties:*
   i) T *is a finitely* A*-cogenerated torsion group,*

ii) $OT(\overline{C}) \leq \text{type}(\overline{A})$,
iii) C' *a self-small group such that* $T_p(C') \leq T_p(A)^n$ *for all* $p \in \mathbb{P}$, *and* $T_p(C') \cong T_p(A)^n$ *for almost all* $p \in \mathbb{P}$.

Proof. *a)* $\Rightarrow$ *b)* Let U be a finite set of primes such that $A(U)$ is standard mixed self-small group of torsion-free rank 1. Moreover, we can choose U such that $S(A(U)) \subseteq D(C)$. Let m be a positive integer such that $C \leq A^m$. Since every p-component of C is finite, we can consider a similar direct decomposition $C = T_U(C) \oplus C(U)$. If $\rho_U : A^m \to A(U)^m$ is the canonical projection then $\mathrm{Ker}(\rho_U) = T_U(A)^m$, hence the restriction $\rho_{U|C(U)} : C(U) \to A(U)$ is injective. Therefore, we can suppose that A is a standard mixed self-small group of torsion-free rank 1, and $C \leq A^m$, such that $S(A) \subseteq D(C)$.

Let $p \in S(A)$. We consider the direct decompositions $A = T_p(A) \oplus A(p)$ and $C = T_p(C) \oplus C(p)$. Since $A(p)$ is the maximal p-divisible subgroup of A and $C(p)$ is p-divisible, $C(p) \leq A(p)^n$, and it follows that the canonical projection $\pi_p^C : C \to T_p(C)$ is the restriction of the canonical projection $\pi_p : A^m \to T_p(A^m)$ to C, hence $\pi_p(C) \subseteq C$.

We fix $\langle c_1 \rangle \oplus \cdots \oplus \langle c_n \rangle$ a full free subgroup of C, and we complete it to a full free subgroup $\langle c_1 \rangle \oplus \cdots \oplus \langle c_n \rangle \oplus \langle a_{n+1} \rangle \oplus \cdots \oplus \langle a_m \rangle$ of A^m. There exists a finite set $W \subseteq \mathbb{P}$ such that for all $p \notin W$ we have

$$T_p(A^m) = \langle \pi_p(c_1), \ldots, \pi_p(c_n), \pi_p(a_{n+1}), \ldots, \pi_p(a_m) \rangle.$$

If $p \notin W$, since $T_p(A^m) \cong \mathbb{Z}(p^{k_p})^m$ for some k_p, it follows that $\langle \pi_p(c_1), \ldots, \pi_p(c_n) \rangle \cong \mathbb{Z}(p^{k_p})^n$, and it is a direct summand of $T_p(A_m)$. But we just proved that $\pi_p(c) \in C$ for all $c \in C$. Hence $\langle \pi_p(c_1), \ldots, \pi_p(c_n) \rangle$ is a direct summand of $T_p(C)$. For every $p \notin W$ we denote by L_p a direct complement of $\langle \pi_p(c_1), \ldots, \pi_p(c_n) \rangle$ in $T_p(C)$.

Let $L = (\oplus_{p \notin W} L_p) \oplus (\oplus_{p \in W} T_p(C))$. We consider a direct decomposition $C = (\oplus_{p \in W} T_p(C)) \oplus C_0$ and set $C' = \{c \in C_0 \mid \forall p \in \mathbb{P}\ \pi_p(c) \in \langle \pi_p(c_1), \ldots, \pi_p(c_n) \rangle\}$. Of course $C' \cap L = 0$. Moreover, it is easy to see that $T(C) = L \oplus T(C')$. If $c \in C$ is an infinite order element, then there exist integers $k \neq 0$, $k_1, \ldots, k_n$ such that $kc = k_1 c_1 + \cdots + k_n c_m$. If $p \notin W$ is a prime which is coprime with k, it is not hard to see that $\pi_p(c) \in \langle \pi_p(c_1), \ldots, \pi_p(c_n) \rangle$. If $p \notin W$ is a divisor for k then we write $\pi_p(c) = t_{1p} + t_{2p}$ with $t_{1p} \in \langle \pi_p(c_1), \ldots, \pi_p(c_n) \rangle$ and $t_{2p} \in L$. Then $c' = c - (\sum_{p|k, p \notin W} t_{2p}) - \sum_{p \in W} \pi_p(c) \in C'$, hence $L + C' = C$, and $C = L \oplus C'$.

To close the proof, it is enough to show that C' is self-small. We consider the homomorphism $\pi : C' \to \prod_{p \in S(A)} T_p(C')$, induced by the canonical projections π_p of $C' = T_p(C') \oplus C'(p)$ onto its p-components. Then $K = \mathrm{Ker}(\pi) = \cap_{p \in S(A)} C'(p)$ is an $S(A)$-divisible subgroup of A^m: if $x \in K$, $p \in S(A)$ and $y \in C'(p)$ such that $py = x$ (such an element y always exists since $\overline{C}$ is p-divisible, hence $C(p)$ is also p-divisible) then $y \in C'(q)$ for all $q \in S(A)$ since otherwise there exists $q \in S(A)$ such that $\pi_q(y) \neq 0$, hence $\pi_q(x) = p\pi_q(y) \neq 0$, and $x \notin C'(q)$, a contradiction. But it is not hard to see, using the projection condition, that A^n has not non-trivial $S(A)$-divisible subgroups, hence π is a monomorphism. Then we can view C' as a subgroup of $\prod_{p \in S(A)} T_p(C')$. By the construction C' satisfies the projection condition. Moreover, $C'/T(C')$ is $S(A)$-divisible, hence $C'/T(C')$ is an $S(A)$-pure subgroup of $(\prod_{p \in S(A)} T_p(C'))/T(C')$, hence C' is $S(A)$-pure in $\prod_{p \in S(A)} T_p(C')$. Using Theorem 2.2, we deduce that C' is self-small.

b) $\Rightarrow$ *a)* First, we observe that $S(A)$ is quasi-contained in $D(C)$ as a consequence of Corollary 2.5.

It is enough to suppose that C is a self-small group which satisfies the conditions ii) and iii). Let V be a finite set of primes such that $A(V)$ is a standard mixed self-small group of torsion-free rank 1 (which always exists), and $U = \{p \in \mathbb{P} \mid T_p(C) \neq T_p(A)^n\} \cup V$. Then U is a finite set, and if we consider direct decompositions $A = T_U(A) \oplus A(U)$ and $C = T_U(C) \oplus C(U)$, we observe that C is A-cogenerated if and only if $C(U)$ is $A(U)$-cogenerated. Therefore, we can suppose that $A \cong A(T, R)$ is a standard mixed self-small group of torsion-free rank 1 and $T_p(C) \cong T_p(A)^n$ for all $p \in \mathbb{P}$.

We consider the homomorphism $\varphi : C \to \prod_p T_p(C)$ which is induced by the projections $\pi_p : C \to T_p(C)$, $p \in S(A)$. If $C' = \varphi(C)$ then $T(C') = T(C)$, and it follows that C' is a self-small group since $\mathrm{Hom}(C', T(C'))$ is a torsion group because it can be embedded in $\mathrm{Hom}(C, T(C))$. Therefore, by Corollary 2.5 we have $r_0(C) = n$, and it follows that φ is injective, hence we can view C as a subgroup of $\prod_p T_p(C)$.

Let D be the group defined by the properties $C \leq D \leq \prod_p T_p(C)$ and $\overline{D} = D/T(C)$ is the pure hull of $C/T(C)$ in $(\prod_p T_p(C))/T(C)$. Hence $\overline{D}$ is torsion-free divisible of rank n, and using Proposition 2.11 we obtain $D = \oplus_{i=1}^n D_i$ with $D_i \cong A(T, \mathbb{Q})$ for all $i \in \{1, \ldots, n\}$.

Let $j \in \{1, \ldots, n\}$. We consider the canonical projection $\pi_j : D \to D_j$ and we denote $C_j = \pi_j(C) \leq D_j$. Then C_j is a mixed self-small group of torsion-free rank 1 and $C_j \cong A(T, X)$ with $\mathrm{type}(X) \leq OT(\overline{C})$ (X is a rank 1 epimorphic image of $\overline{C}$). Since $OT(\overline{C}) \leq \mathrm{type}(\overline{A})$, we obtain $\mathrm{type}(X) \leq \mathrm{type}(\overline{A})$, hence there exists a monomorphism $\varphi_j : C_i \to A$.

Therefore, for all $i \in \{1, \ldots, n\}$, there exist homomorphisms $\varphi_j \pi_j : C \to A$ with $\mathrm{Ker}(\varphi_j \pi_j) \leq \oplus_{i \neq j} D_i$. The family $\{\varphi_j \pi_j \mid j \in \{1, \ldots, n\}\}$ induces a homomorphism $\varphi : C \to A^n$ which is monic, hence C is finitely A-cogenerated. □

Remark 3.2. In the case A is torsion-free, the hypothesis C is A-cogenerated of finite rank implies C is finitely A-cogenerated, [23, Proposition 3 and Lemma 2]. This is not the valid if A is a proper mixed group: Let A be the pure subgroup of $\prod_{p \in \mathbb{P}} \mathbb{Z}(p) a_p$ which is generated by $\oplus_{p \in \mathbb{P}} \mathbb{Z}(p) a_p$ and the element $a = (a_p)_{p \in \mathbb{P}}$. If C is the pure subgroup of $\prod_{p \in \mathbb{P}} \mathbb{Z}(p) a_p$ which is generated by $\oplus_{p \in \mathbb{P}} \mathbb{Z}(p) a_p$, a and $b = ((p-1)a_p)_{p \in \mathbb{P}}$, then it is obvious that C is an A-cogenerated self-small group of torsion-free rank 2 (by Theorem 2.2). Moreover $D(A) = D(C) = S(C)$, but C does not verify the condition b)iii) in Proposition 3.1.

From the proof of Proposition 3.1 we obtain more informations about finitely A-cogenerated groups. The proof of *b*) ⇒ *a*) gives us the following

Corollary 3.3. *Let A be a mixed self-small group of torsion-free rank 1 and C a self-small group of torsion-free rank n which verify b)ii) and b)iii) (or, equivalently, the condition a)). Then there exists an embedding* $0 \to C \to A^n$.

Using the proof of *a*) ⇒ *b*), we obtain

Corollary 3.4. *Let A be a mixed self-small group of torsion-free rank 1. If $C \leq A^n$ is a subgroup of torsion-free rank n such that $\overline{C}$ is p-divisible for almost all $p \in S(A)$ then $T_p(C) = T_p(A^n)$ for almost all $p \in \mathbb{P}$ and C is self-small.*

Proof. From the proof of *a*) ⇒ *b*) we obtain $T_p(C) = T_p(A^n)$ for almost all $p \in \mathbb{P}$. Let B be a finite subgroup of A^n such that $T(C + B) = T(A^n)$. We observe that it is enough to prove that $C' = C + B$ is self-small, and we can suppose $T(C) = T(A^n)$. Therefore, we have an exact sequence $0 \to C \to A^n \to X \to 0$, where $X \cong \overline{A}^n/\overline{C}$ is a torsion group such that $X_p = 0$ for almost all $p \in S(A)$. Moreover, if, for a prime $p \in S(A)$, the p-component X_p is non-zero, then X_p is a direct sum of a finite group and a divisible group. We obtain the exact sequence $0 \to \mathrm{Hom}(X, T(A)) \to \mathrm{Hom}(A^n, T(A)) \to \mathrm{Hom}(C, T(A)) \to \mathrm{Ext}(X, T(A))$. Since $\mathrm{Hom}(X, T(A))$ and $\mathrm{Ext}(X, T(A)) \cong \oplus_{p \in S(A),\ X_p \neq 0} \mathrm{Ext}(X_p, T_p(A))$ are finite groups, the group $\mathrm{Hom}(C, T(A))$ is a torsion group, hence $\mathrm{Hom}(C, T(C)) \leq \mathrm{Hom}(C, T(A))^n$ is a torsion group, and it follows that C is self-small. □

We close this section with a result about the group $\mathrm{Hom}(C, A)$. Recall from [16, Example 6] that in general $\mathrm{Hom}(C, A)$ is not a self-small group for A and C self-small groups (even for the case $A = C$). However, this is not the case for A of torsion-free rank 1 and C finitely A-cogenerated. Moreover, in this hypothesis, the embedding Θ_{CA} from Proposition 2.6 is an isomorphism.

Proposition 3.5. *Let A be a mixed self-small group of torsion-free rank 1 and C a finitely A-cogenerated self-small group of torsion-free rank m. The following statements are true.*

a) $C^* = \mathrm{Hom}(C, A)$ *is a self-small finitely A-cogenerated group of torsion-free rank m.*

b) $T(C^*) = \mathrm{Hom}(C, T(A))$, *and the natural homomorphism* $\Theta_{CA} : \overline{C^*} \to \mathrm{Hom}(\overline{C}, \overline{A})$ *is an isomorphism.*

Proof. a) Let $C \leq A^n$. If $r_0(C) < n$ we can add to C a free subgroup $F_0 \leq A^n$ such that $C + F_0 = C \oplus F_0$ is of torsion free rank n, and we observe that $\mathrm{Hom}(C, A)$ is self-small of torsion-free rank m whenever $\mathrm{Hom}(C \oplus F_0, A) \cong \mathrm{Hom}(C, A) \oplus A^{n-m}$ is self-small of torsion-free rank n since the class of self-small groups is closed with respect direct summands. Moreover, $C \oplus F_0$ is self-small, hence we can suppose $r_0(C) = n$.

Let $F = \langle c_1 \rangle \oplus \cdots \oplus \langle c_n \rangle$ be a full free subgroup of C. We claim that C/F is p-divisible for almost all $p \in S(A)$.

Since C is self-small, C/F is p-divisible for almost all $p \in S(C)$. Moreover, $\langle \pi_p(c_1), \ldots, \pi_p(c_n) \rangle = T_p(A)^n \cong \mathbb{Z}(p^{k_p})^n$, and we obtain $\langle \pi_p(c_1), \ldots, \pi_p(c_n) \rangle = \langle \pi_p(c_1) \rangle \oplus \cdots \oplus \langle \pi_p(c_n) \rangle$ for almost all $p \in S(A)$. If p is such a prime, then $h_p(\pi_p(c_i)) = 0$ for all $i \in \{1, \ldots, n\}$. Therefore, if $i \in \{1, \ldots, n\}$ then $h_p(c_i) = 0$ for almost all $p \in S(A)$. Let

$$U = \{p \in S(A) \mid \langle \pi_p(c_1), \ldots, \pi_p(c_n) \rangle = \langle \pi_p(c_1) \rangle \oplus \cdots \oplus \langle \pi_p(c_n) \rangle = T_p(A)^n\}.$$

Then $U \doteq S(A)$ and $h_p(c_i) = 0$ for all $p \in U$ and $i \in \{1, \ldots, n\}$.

Let $p \in U \backslash S(C)$ and suppose $(C/F)_p \neq 0$. There exists an element $c \in C \backslash F$ with $pc \in F$, hence there exist integers $l_1, \ldots, l_n$ such that $pc = l_1 c_1 + \cdots + l_n c_n$. Then $1 \leq h_p(\pi_p(pc)) = h_p(\pi_p(l_1 c_1 + \cdots + l_n c_n) = \min\{h_p(\pi_p(l_1 c_1)), \ldots, h_p(\pi_p(l_n c_n))\}$. This is possible only if p divides l_i for all $i \in \{1, \ldots, n\}$. Hence there exists $c' \in F$ such that $pc = pc'$, and we obtain $p \in S(C)$, a contradiction. Therefore $C/F = 0$ for all $p \in U$, and the claim is proved.

It follows that $\mathrm{Hom}(C/F, A)$ is bounded, hence the kernel of the natural homomorphism $\varphi : \mathrm{Hom}(C, A) \to \mathrm{Hom}(F, A) \cong A^n$ is bounded. Moreover, since in the exact sequence $0 \to \mathrm{Hom}(A^n/C, A) \to \mathrm{Hom}(A^n, A) \to \mathrm{Hom}(C, A)$ the first group is finite, we deduce that the group $\mathrm{Hom}(C, A)$ is of torsion-free rank n. Then

$C' = \mathrm{Im}(\varphi) \cong \mathrm{Hom}(C, A)/\mathrm{Ker}(\varphi)$ is a subgroup of torsion-free rank n of A^n such that $\overline{C}$ is p-divisible for almost all $p \in S(A)$. By Corollary 3.4, C' is self-small. Therefore, $C^* = \mathrm{Hom}(C, A)$ is self-small of torsion free rank n.

Moreover, there exists a direct decomposition $C^* = T_V(C^*) \oplus C^*(V)$, where V is a finite set of primes, such that $\mathrm{Ker}(\varphi) \le T_V(C^*)$, and it is not hard to observe that both $T_V(C^*)$ and $C^*(V) \le C'$ are finitely A-cogenerated groups, hence C^* is a finitely A-cogenerated group.

b) Observe, by the proof of a), that if F is a full free subgroup of C then C/F is p-divisible for almost all $p \in S(A)$, and it follows that $T(\mathrm{Hom}(C, A)) = \mathrm{Hom}(A, T(C))$. Moreover, using a) we obtain $r_0(C) = r_0(C^*) = r_0(\mathrm{Hom}(\overline{C}, \overline{A}))$, and the embedding $\Theta_{CA} : \mathrm{Hom}(A, C)/\mathrm{Hom}(A, T(C)) \hookrightarrow \mathrm{Hom}(\overline{C}, \overline{A})$ is pure by Proposition 2.6. Therefore this embedding is an isomorphism. □

Example 3.6. In contrast with Warfield's result [23, Proposition 3], if A or C are a proper mixed groups the hypothesis $\mathrm{Hom}(C, A)$ is self-small of torsion-free rank n does not imply that C is (finitely-)A-cogenerated: If $A = A(\oplus_{p\in\mathbb{P}}\mathbb{Z}(p), \mathbb{Q})$ and $C = A(\oplus_{p\in\mathbb{P}}\mathbb{Z}(p^2), \mathbb{Q})$ are self-small mixed groups of torsion-free rank 1 then $\mathrm{Hom}(C, A) \cong A$ but C is not A-cogenerated.

Corollary 3.7. *Suppose that A is a mixed self-small group of torsion-free rank 1 and C is a finitely A-cogenerated group.*

Let $\Phi : C^{**} \to \overline{C}^{**}$, where $\overline{C}^{**} = \mathrm{Hom}(\mathrm{Hom}(\overline{C}, \overline{A}), \overline{A})$, be the homomorphism defined by $\Phi(f) : \alpha' \mapsto f(\Theta_{CA}^{-1}(\alpha')) + T(A)$ for all $\alpha' \in \mathrm{Hom}(\overline{C}, \overline{A})$. Then the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(C) & \longrightarrow & C & \xrightarrow{\rho} & \overline{C} & \longrightarrow & 0 \\
 & & \big\downarrow \nu & & \big\downarrow \nu_C & & \big\downarrow \nu'_{\overline{C}} & & \\
0 & \longrightarrow & T(C^{**}) & \longrightarrow & C^{**} & \xrightarrow{\Phi} & \overline{C}^{**} & \longrightarrow & 0
\end{array}$$

is commutative with exact sequences, where $\nu'_{\overline{C}}$ is the natural map, ν is the restriction of ν_C to $T(C)$, and $\rho : C \to \overline{C}$ is the canonical epimorphism.

Proof. By Proposition 3.5a), C^* is finitely A-cogenerated. Let $\pi : C^{**} \to \overline{C^{**}}$ be the canonical epimorphism. Then $\Phi = \mathrm{Hom}(\Theta_{CA}, \overline{A})\Theta_{C^*A}\pi$, and using Proposition 3.5b) we observe that $\mathrm{Hom}(\Theta_{CA}, \overline{A})$ and Θ_{C^*A} are isomorphisms. Hence Φ is epic and $\mathrm{Ker}(\Phi) = T(C^{**})$. Therefore, the diagram is with exact sequences and the left square is commutative.

In order to prove that the right square is commutative, let $c \in C$. Then $\Phi\nu_C(c), \nu_{\overline{C}}\rho(c) : \mathrm{Hom}(\overline{C}, \overline{A}) \to \overline{A}$, and $\Phi\nu_C(c)(\alpha') = \nu_C(c)(\Theta_{CA}^{-1}(\alpha')) + T(A) = \Theta_{CA}^{-1}(\alpha')(c) + T(A) = \alpha'(c + T(A)) = \nu_{\overline{C}}(c + T(A))(\alpha')$ for all $\alpha' \in \mathrm{Hom}(\overline{C}, \overline{A})$, hence $\Phi\nu_C(c) = \nu_{\overline{C}}(c+T(A)) = \nu_{\overline{C}}\rho(c)$ for all $c \in C$, and the proof is complete. □

4. A-W-reflexive groups of finite torsion-free rank

Theorem 4.1. *Let A be a mixed self-small group of torsion-free rank 1. A finitely A-cogenerated torsion group T is A-W-reflexive if and only if $T_p = 0$ for all primes p with $T_p(A)$ non-cyclic.*

Proof. Consider a torsion group $T \le A^n$, for some positive integer n. Since $S(T) \subseteq S(A)$, we can suppose $T = \oplus_{p\in S(A)}T_p$. We denote $X_p = \mathrm{Hom}(T_p, T_p(A))$, and we observe that X_p is a finite p-group for all p. Moreover, $\mathrm{Hom}(T,A) = \mathrm{Hom}(T,T(A)) \cong \prod_{p\in S(A)} X_p$.

Let $f : \mathrm{Hom}(T,A) \to A$ be a group homomorphism. By Theorem 2.8, we can view A as an $D(A)$-pure subgroup of $\prod_{p\in S(A)} T_p(A)$ and $f : \prod_{p\in S(A)} X_p \to \prod_{p\in S(A)} T_p(A)$. Hence we can identify $f = (f_p)$, where $f_p : X_p \to T_p(A)$ are group homomorphisms. Suppose that there exists $x = (x_p) \in \prod_{p\in S(A)} X_p$ such that $f(x)$ is of infinite order. Then $f_p(x_p) \neq 0$ for infinitely many $p \in S(A)$. Let $S = \{p \in S(A) \mid f_p(x_p) \neq 0\}$ and for every subset $U \subseteq S$ we consider the element $x^U = (x^U_p) \in \prod_{p\in S(A)} T_p$ with $x^U_p = x_p$ if $p \in U$ and $x^U_p = 0$ if $p \notin U$. It is not hard to see that $f(x^U) \neq f(x^V)$ whenever $U, V \subseteq S$ with $U \neq V$. Then $\mathrm{Im}(f)$ is uncountable, hence $\mathrm{Im}(f) \nsubseteq A$, a contradiction. Therefore $\mathrm{Hom}(\prod_{p\in S(A)} X_p, A) = \mathrm{Hom}(\prod_{p\in S(A)} X_p, T(A))$. Since $\prod_{p\in S(A)} X_p$ is self-small by [8, Corollary 1.3] and [22, Corollary 1.8], $\mathrm{Hom}(\prod_{p\in S(A)} X_p, \oplus_{p\in S(A)} T_p)$ is a torsion group by Lemma 2.1, hence the image of every homomorphism $f : \prod_{p\in S(A)} X_p \to T(A)$ is finite. We have the isomorphisms

$$\mathrm{Hom}(\mathrm{Hom}(T,A),A) \cong \mathrm{Hom}(\prod_{p\in S(A)} X_p, T(A)) \cong$$

$$\bigoplus_{p\in S(A)} \mathrm{Hom}(\prod_{p\in S(A)} X_p, T_p(A)) \cong \bigoplus_{p\in S(A)} \mathrm{Hom}(X_p, T_p(A))$$

Therefore, T is A-W-reflexive if and only if T_p is $T_p(A)$-W-reflexive for all p. Using [17, Corollary 43.3], we observe that T is A-W-reflexive if and only if $T_p = 0$ for all primes p with $T_p(A)$ non-cyclic. □

The following result reduces the study of A-reflexive non-torsion groups to the case when A is a standard mixed self-small group of torsion-free rank 1.

Theorem 4.2. *Let A be a mixed self-small group of torsion-free rank 1 and $U \subseteq S(A)$ the minimal set of primes such that $A(U)$ is a standard mixed self-small group of torsion-free rank 1. If C is a non-torsion finitely A-cogenerated group, the following are equivalent.*

a) *C is A-W-reflexive;*

b) i) *$S(A) \subseteq D(A)$*

   ii) *$T_p(C) = 0$ for all $p \in U$, and*

   iii) *C is $A(U)$-W-reflexive.*

Proof. a)⇒b) Suppose that $C \neq T(C)$ is a finitely A-cogenerated group which is A-W-reflexive. Then every p-component of C is A-W-reflexive, and using Theorem 4.1 we deduce $T_p(C) = 0$ whenever $T_p(A)$ is not a cyclic group. Suppose that there exists a prime p such that $T_p(A) \neq 0$ and $\overline{A}$ is not p-divisible. Then $\overline{C}$ is not p-divisible (since $\overline{C}$ is $\overline{A}$-cogenerated), and if we consider a direct decomposition $C = T_p(C) \oplus C(p)$ then $C(p)$ is a A-W-reflexive group which is not p-divisible and whose p-component is 0. But, since $C(p) \neq pC(p)$, $\mathrm{Hom}(C(p), T_p(A))$ is a non-zero bounded p-group, hence $T_p(C(p)^{**}) \neq 0$, a contradiction. Then $S(A) \subseteq D(A)$, and U is the set of primes p such that $T_p(A)$ is not a cyclic group. Using again [17, Corollary 43.3] we deduce that ii) is valid.

Since $U \subseteq S(A)$, $\overline{A}$ is p-divisible for all $p \in U$. Then $A(U)$ is p-divisible for all $p \in U$. Moreover $C \cong C^{**}$, and using ii) together with Corollary 2.7, we deduce that C is p-divisible for all $p \in U$, hence $\mathrm{Hom}(C, A) = \mathrm{Hom}(C, A(U))$ and $\mathrm{Hom}(C^*, A) = \mathrm{Hom}(C^*, A(U))$. Therefore C is $A(U)$-W-reflexive.

b)⇒a) As in the last part of the proof for a)⇒b) C is p-divisible for all $p \in U$ and $C^{**} \cong \mathrm{Hom}(\mathrm{Hom}(C, A(U)), A(U))$. □

We have seen in Remark 3.2 that if A is a proper mixed group, a finite torsion-free rank self-small group which is A-cogenerated is not necessarily finitely A-cogenerated. This is not the case for A-reflexive groups.

Lemma 4.3. *Let A be a standard mixed self-small group of torsion-free rank 1. If C is an A-W-reflexive self-small group of finite torsion-free rank, then*

a) *C^* is self-small;*
b) *C is finitely A-cogenerated.*

Proof. a) Suppose that C^* is not a self-small group. Since every p-component of C^* is finite, it follows that $\mathrm{Hom}(C^*, T(C^*))$ is not a torsion group. But $S(C^*) = S(C) \subseteq S(A)$, hence $\mathrm{Hom}(C^*, T(A))$ is not a torsion group. Then $\mathrm{Hom}(C^*, T(A))$ is uncountable, hence $C \cong C^{**}$ is uncountable, a contradiction.

b) If $c \in C$ is an infinite order element then $\nu_C(c)$ is also of infinite order. Moreover, $\mathrm{Hom}(C^*, T(A))$ is a torsion group, hence there exists $f_c \in \mathrm{Hom}(C, A)$ such that $f_c(c) \notin T(A)$.

Let $c_1, \dots, c_n \in C$ be a maximal independent system of infinite order elements. By what we just proved, there exist $f_1, \dots, f_n \in \mathrm{Hom}(C, A)$ such that $f_i(c_i) \notin T(A)$ for all $i \in \{1, \dots, n\}$. Let $f : C \to A^n$ be the homomorphism induced by these homomorphisms. Then $K = \mathrm{Ker}(f) = \cap_{i=1}^n \mathrm{Ker}(f_i)$ is a torsion group, and $C' = C/K$ is a finitely A-cogenerated self-small group of torsion-free rank $n = r_0(C)$ such that $\overline{C'}$ is p-divisible for all $p \in S(A)$. Using Corollary 3.4 we deduce that $T_p(C') \cong T_p(A)^n$ for almost all p, hence $T_p(C)/K_p \cong T_p(A)^n \cong \mathbb{Z}(p^{k_p})^n$ for almost all p. But $T_p(C) = \langle \pi_p(c_1), \dots, \pi_p(c_n)\rangle$ for almost all p, as a consequence of Corollary 2.5 (here $\pi_p : C \to T_p(C)$ is the canonical projection corresponding to the direct decomposition $C = T_p(C) \oplus C(p)$). Moreover, $\mathrm{ord}(\pi_p(c_i)) \leq p^{k_p}$ for all $i \in \{1, \dots, n\}$. Therefore $|T_p(C)| \leq p^{nk_p}$. It follows that $K_p = 0$ (and $T_p(C) \cong T_p(A)^n$) for almost all p. Hence K is a finite group, and it follows that it can be embedded in a finite direct summand $T_U(C)$, where U is a finite set of primes. Therefore $C' \cong T_U(C)/K \oplus C(U)$, and we obtain that $C(U)$ is finitely A-cogenerated. Moreover $T_U(C)$ is finitely A-cogenerated, hence C is finitely A-cogenerated. □

Theorem 4.4. *Let A be a standard mixed self-small group of torsion-free rank 1. The following conditions are equivalent for a self-small group C of torsion-free rank $n > 0$:*

a) *C is A-W-reflexive;*
b) i) *$T_p(C) = T_p(A)^n$ for almost all $p \in S(A)$;*
   ii) *If $T_p(A) \cong \mathbb{Z}(p^{k_p})$ then $p^{k_p} T_p(C) = 0$ for all primes p;*
   iii) *$\overline{C}$ is $\overline{A}$-W-reflexive.*

Proof. a)⇒b) By Lemma 4.3, C is finitely A-cogenerated. Hence i) and ii) follows from Proposition 3.1, and iii) is a consequence of Corollary 3.7.

b)⇒a) Using Proposition 3.1 we deduce that C is finitely A-cogenerated. Let $p \in S(A)$. Then $A = T_p(C) \oplus C(p)$ with $C(p)$ a p-divisible group with trivial p-component, and it is not hard to see that $C^{**} = T_p(C)^{**} \oplus C^{**}(p)$, where $T_p(C)^{**} =$

$\mathrm{Hom}(\mathrm{Hom}(T_p(C), A), A) \cong T_p(C)$ and $C^{**}(p) = \mathrm{Hom}(\mathrm{Hom}(C(p), A), A)$ is a group with trivial p-component. Since every p-component of C is finite, the restriction $\nu : T(C) \to T(C^\star)$ of ν_C to $T(C)$ is an isomorphism. Using again the commutative diagram provided by Corollary 3.7, we obtain that ν_C is an isomorphism, hence C is A-W-reflexive. □

In the following example we will show that the previous characterization cannot be extended to the general case when A is of p-rank 1 for all primes p.

Example 4.5. Let A be the pure subgroup of $\prod_{p\in\mathbb{P}} \mathbb{Z}(p^2)x_p$ which is generated by $\oplus_{p\in\mathbb{P}}\mathbb{Z}(p^2)x_p$ and $a = (px_p)$. Then A is of p-rank 1 for all $p \in \mathbb{P}$ and $A/T(A) \cong \mathbb{Q}$. By Corollary 2.5, A is not self-small. Moreover, we consider C the pure subgroup of $\prod_{p\in\mathbb{P}} \mathbb{Z}(p)y_p$ which is generated by $\oplus_{p\in\mathbb{P}}\mathbb{Z}(p)y_p$ and the element $c = (y_p)$. Then C is a mixed self-small group of torsion-free rank 1.

We claim that $\mathrm{Hom}(C, A) \cong C$. In order to prove this we will use techniques developed in [5] and [18] for endomorphism rings. Applying the covariant functor $\mathrm{Hom}(-, A)$ to the exact sequence $0 \to T(C) \to C \to \overline{C} \to 0$, we can view $\mathrm{Hom}(C, A)$ as a subgroup of $\mathrm{Hom}(T(C), T(A)) = \prod_{p\in\mathbb{P}} \mathrm{Hom}(\mathbb{Z}(p)y_p, \mathbb{Z}(p^2)x_p) \cong \prod_{p\in\mathbb{P}} \mathbb{Z}(p)$, since $\overline{C}$ is divisible and A is reduced.

For every prime p we consider the homomorphism $\varphi_p : \mathbb{Z}(p)y_p \to \mathbb{Z}(p^2)x_p$, $\varphi(y_p) = px_p$. Then we obtain a homomorphism $\varphi = (\varphi_p) : \prod_{p\in\mathbb{P}} \mathbb{Z}(p)y_p \to \prod_{p\in\mathbb{P}} \mathbb{Z}(p^2)x_p$ with $\varphi(c) = a$ and $\varphi(T(C)) \subseteq T(A)$. If $c' \in C$ is an infinite order element, then $nc' = mc$ for some integers, hence $n\varphi(c') = ma$. Using [11, Exercise S 3.25], we obtain $\varphi(c') \in A$. Then $\varphi(C) \subseteq A$, hence we can consider φ as an element of $\mathrm{Hom}(C, A)$. It follows that $C^* = \mathrm{Hom}(C, A)$ is of torsion-free rank 1. Moreover, $\overline{C^*}$ is divisible by Lemma 2.6. Therefore $C^* = \mathrm{Hom}(C, A)$ is the pure subgroup of $\prod_{p\in\mathbb{P}} \mathrm{Hom}(\mathbb{Z}(p)y_p, \mathbb{Z}(p^2)x_p) \cong \prod_{p\in\mathbb{P}} \mathbb{Z}(p)$ which is generated by φ and $\oplus_{p\in\mathbb{P}}\mathrm{Hom}(\mathbb{Z}(p)y_p, \mathbb{Z}(p^2)x_p)$. Moreover, φ satisfies the projection condition described in Theorem 2.8 c). Then C^* is a standard mixed self-small group of torsion-free rank 1 and $C^* \cong C$ as a consequence of Corollary 2.9.

5. A-Δ-Reflexive groups

In this section we will prove that the study of A-Δ-reflexive groups is reduced to the case when A is a standard mixed self-small group of torsion-free rank 1, and in this hypothesis A-Δ-reflexive groups and A-W-reflexive groups coincide.

Lemma 5.1. *Let A be a finite group. Then every finite group T which is A-cogenerated is A-Δ-reflexive.*

Proof. Observe that $\Delta(T)$ is finitely generated as an $\mathrm{End}(A)$-module. By [12, Lemma 4.2.2] there exists an exact sequence $0 \to T \to A^n \to B \to 0$, where n is a positive integer which stays exact under Δ. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & A^n & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta_T} & & \downarrow{\scriptstyle \delta_{A^n}} & & \downarrow{\scriptstyle \delta_B} & & \\ 0 & \longrightarrow & \Delta^2(T) & \longrightarrow & \Delta^2(A^n) & \longrightarrow & \Delta^2(B) & & \end{array} .$$

Since δ_{A^n} is an isomorphism and B is A-cogenerated, hence δ_B is a monomorphism, it follows that δ_T is an isomorphism. □

Theorem 5.2. *Let A be a mixed self-small group of torsion-free rank 1. If T is a finitely A-cogenerated torsion group then T is A-Δ-reflexive.*

Proof. By what we proved in Theorem 4.1, $\Delta^2(T) \leq \mathrm{Hom}(\mathrm{Hom}(T, A), A)$ is a torsion group and

$$\Delta^2(T) \cong \bigoplus_{p\in S(A)} \mathrm{Hom}_E(X_p, T_p(A)) \cong \bigoplus_{p\in S(A)} \mathrm{Hom}_{E(T_p(A))}(X_p, T_p(A)),$$

hence T is A-Δ-reflexive if and only if every p-component T_p is $T_p(A)$-Δ-reflexive, and this follows from Lemma 5.1. □

Lemma 5.3. *Let A be a standard mixed self-small group of torsion-free rank 1 with the endomorphism ring E. Then $\mathrm{Hom}_E(M, N) = \mathrm{Hom}(M, N)$ for all left E-modules M and N.*

Proof. We note that the ring E is, as an abelian group, of torsion-free rank 1. Moreover, for all $p \in S(A)$ the group A has a unique direct decomposition $A = T_p(A) \oplus A(p)$, where $A(p)$ is a p-divisible group, hence $E = T_p(E) \times E(p)$, where $T_p(E) = E(T_p(A))$ and $E(p) = E(A(p))$. Therefore, every p-component of E is cyclic and $\overline{E}$ is p-divisible for all $p \in S(A) = S(E)$. Therefore, every left E-module M has a decomposition $M = T_p(M) \times M(p)$, where $T_p(M)$ is an $T_p(E)$-module and $M(p)$ is an $E(p)$-module. Since $E(p)$ is p-divisible with trivial p-component, $M(p)$ has the same properties (there exists an exact sequence $E(p)^{(J)} \xrightarrow{f} E(p)^{(I)} \to M(p) \to 0$, and $\mathrm{Im}(f)$ a p-pure subgroup of $E(p)^{(I)}$ since $E(p)^{(J)}$ is p-divisible).

Let M, N be left E-modules and let $\varphi : M \to N$ be a $\mathbb{Z}$-homomorphism. For every $p \in S(A)$ the homomorphism φ induces a homomorphism $\varphi_p : T_p(M) \to T_p(N)$, which is a $T_p(E)$-homomorphism, hence an E-homomorphism. Therefore, the restriction $\varphi_1 : T(M) \to T(N)$ of φ to $T(M)$ is an E-homomorphism. Moreover, $\overline{M}$ and $\overline{N}$ are canonically $\overline{E}$-modules. Then the induced homomorphism $\overline{\varphi} : \overline{M} \to \overline{N}$, $\overline{\varphi}(m + T(M)) = \varphi(m) + T(N)$ is an $\overline{E}$-homomorphism, and it follows that $\overline{\varphi}$ is an E-homomorphism.

To prove that φ is an E-homomorphism we consider $\alpha \in E$ and $x \in M$. Since $\overline{\varphi}$ is an E-homomorphism, $\overline{\varphi}(\alpha(x + T(M)) = \alpha\overline{\varphi}(x + T(M))$, hence $\varphi(\alpha x) - \alpha\varphi(x) \in T(N)$. Then there exists an integer k such that $U = \{p \in \mathbb{P} \mid p \text{ divides } k\} \subseteq S(A)$ and $k\varphi(\alpha x) - k\alpha\varphi(x) = 0$. We consider direct decompositions $M = T_U(M) \oplus M(U)$ and $E = T_U(E) \times E(U)$, and write $x = x_U + x(U)$ with $x_U \in T_U(M)$ and $x(U) \in M(U)$, respectively $\alpha = \alpha_U + \alpha(U)$ with $\alpha_U \in T_U(E) = E(T_U(A))$ and $\alpha(U) \in E(U) = E(A(U))$. Then $\varphi(\alpha x(U)) = \alpha\varphi(x(U))$. Moreover, since the restriction $\varphi_{|T_U(M)} : T_U(M) \to T_U(M)$ is an E-homomorphism, we have $\varphi(\alpha x_U) = \alpha\varphi(x_U)$, and it follows that φ is an E-homomorphism. □

Theorem 5.4. *Let A be a mixed self-small group of torsion-free rank 1 and U a finite set of primes such that $A(U)$ is a standard mixed self-small group of torsion-free rank 1. A non-torsion finitely A-cogenerated group C is A-Δ-reflexive if and only if $C(U)$ is $A(U)$-W-reflexive.*

Proof. We consider direct decompositions $A = T_U(A) \oplus A(U)$ and $C = T_U(C) \oplus C(U)$. By Theorem 5.2 and [25, 47.4(1)], C is A-Δ-reflexive if and only if $C(U)$ is A-Δ-reflexive. Therefore we can suppose $T_U(C) = 0$. Moreover, we will assume that $D(C) = D(A)$ since in both hypotheses (C is A-Δ-reflexive or C is $A(U)$-Δ-reflexive) this property is valid, and by Proposition 3.1 we can suppose that C is a self-small group which satisfies the conditions b)i) and b)ii) of Theorem 4.4.

Suppose that C is A-Δ-reflexive. Observe that $\delta_C = \iota_C \mu_C$, where $\iota_C : \Delta^2(C) \to C^{**}$ is the inclusion map, and $\overline{E} = E(\overline{A})$ (E denotes the endomorphism ring of A). Then we can replace the vertical arrows and the bottom row in the diagram provided by Corollary 3.7 such that we obtain the following commutative diagram with exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(C) & \longrightarrow & C & \xrightarrow{\rho} & \overline{C} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta_C} & & \downarrow{\scriptstyle \delta'_{\overline{C}}} & & \\
0 & \longrightarrow & T(\Delta^2(C)) & \longrightarrow & \Delta^2(C) & \xrightarrow{\Phi} & {\Delta'}^2(\overline{C}) & \longrightarrow & 0
\end{array},$$

where Δ' denotes both covariant functors $\mathrm{Hom}(-, \overline{A})$ and $\mathrm{Hom}_{\overline{E}}(-, \overline{A})$, and $\delta'_{\overline{C}} : \overline{C} \to {\Delta'}^2(\overline{C})$ is the canonical map. Then $\overline{C}$ is $\overline{A}$-W-reflexive, hence C is $A(U)$-W-reflexive as a consequence of Theorem 4.4.

Conversely, suppose that C is $A(U)$-W-reflexive. Let $f : \mathrm{Hom}(C, A) \to A$ be an E-homomorphism. We will look at f as a group homomorphism. Since $\mathrm{Hom}(C, A) = \mathrm{Hom}(C, T_U(A)) \oplus \mathrm{Hom}(C, A(U))$, we can view f as a matrix $f = \begin{pmatrix} f_1 & f_2 \\ 0 & f_3 \end{pmatrix}$, where $f_1 : \mathrm{Hom}(C, T_U(A)) \to T_U(A)$, $f_2 : \mathrm{Hom}(C, A(U)) \to T_U(A)$ and $f_3 : \mathrm{Hom}(C, A(U)) \to A(U)$. It is not hard to see that f_3 is an $E(A(U))$-homomorphism. Moreover, since f is an E-homomorphism, $f(\varphi\alpha) = \varphi f(\alpha)$ for all $\varphi \in E$ and all $\alpha \in \mathrm{Hom}(C, A)$. We identify $\varphi = \begin{pmatrix} \varphi_1 & \varphi_2 \\ 0 & \varphi_3 \end{pmatrix}$, where $\varphi_1 : T_U(A) \to T_U(A)$, $\varphi_2 : A(U) \to T_U(A)$, $\varphi_3 : A(U) \to A(U)$, and $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ with $\alpha_1 \in \mathrm{Hom}(C, T_U(A))$ and $\alpha_2 \in \mathrm{Hom}(C, A(U))$. For $\varphi_1 = 1_{T_U(A)}$, $\varphi_2 = 0$ and $\varphi_3 = 0$ we obtain $f_2 = 0$.

We consider the homomorphism

$$\Phi : \mathrm{Hom}_E(\mathrm{Hom}(C, A), A) \to \mathrm{Hom}_{E(A(U))}(\mathrm{Hom}(C, A(U)), A(U)),\ \Phi(f) = f_3,$$

and we claim that Φ is monic.

To prove this claim we observe that there exists an exact sequence

$$(\star)\ \ 0 \to C \xrightarrow{\mu} A(U)^n \to X \to 0$$

which exists since C is finitely A-cogenerated and $T_U(C) = 0$. Moreover, $\overline{C}$ is p-divisible for all $p \in D(A)$, and using Corollary 3.3, we can suppose $n = r_0(C)$ in $(\star)$. In this hypothesis, X is a torsion group. If $p \in U \cap D(A)$ then C is p-divisible, hence μ is p-pure, and $X_p = 0$, and if $p \in U \setminus D(A)$ the p-component $T_p(X)$ is finite as a consequence of [6, Section 1]. Therefore $X_U = \oplus_{p \in U} X_p$ is finite. Let $\mu(C) \leq H \leq A(U)^n$ such that $H/\mu(C) = X(U)$. By Lemma 2.11 we deduce $H \cong A(U)^n$ (since H is of finite index in $A(U)^n$ and $T(H) = T(A(U)^n)$). Therefore, we can choose the sequence $(\star)$ such that $X_U = 0$. In this context, applying the contravariant functor $\mathrm{Hom}(-, T_U(A))$, and using $\mathrm{Ext}(X, T_U(A)) = 0 = \mathrm{Hom}(X, T_U(A))$ we deduce that

$$\mathrm{Hom}(\mu, T_U(A)) : \mathrm{Hom}(A(U)^n, T_U(A)) \to \mathrm{Hom}(C, T_U(A))$$

is an isomorphism. We fix the n-uple $(\mu_1, \ldots, \mu_n) \in \mathrm{Hom}(C, A(U))^n$ which is induced by μ and the canonical projections $A(U)^n \to A(U)$. Therefore for every homomorphism $\beta \in \mathrm{Hom}(C, T_U(A))$ there exists a unique n-uple $(\beta_1, \dots, \beta_n) \in \mathrm{Hom}(A(U), T_U(A))^n$ such that $\beta = \sum_{i=1}^n \beta_i\mu_i$. We identify each β_i with the corresponding endomorphism of A: $\beta_i = \begin{pmatrix} 0 & \beta_i \\ 0 & 0 \end{pmatrix}$. Then $\begin{pmatrix} f_1(\beta) \\ 0 \end{pmatrix} = f(\beta) = f(\sum_{i=1}^n \beta_i\mu_i) = \sum_{i=1}^n \begin{pmatrix} 0 & \beta_i \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f_1 & 0 \\ 0 & f_3 \end{pmatrix}\begin{pmatrix} 0 \\ \mu_i \end{pmatrix} = \sum_{i=1}^n \beta_i f_3(\mu_i)$. Therefore, for every $f_3 \in \mathrm{Hom}_{E(A(U))}(\mathrm{Hom}(C, A(U)), A(U))$, there exists a unique $f_1 : \mathrm{Hom}(C, T_U(A)) \to T_U(A)$ such that the matrix $\begin{pmatrix} f_1 & 0 \\ 0 & f_3 \end{pmatrix}$ represents an element from $\mathrm{Hom}_E(\mathrm{Hom}(C, A), A)$, and the claim is proved.

Now we consider the canonical homomorphisms $\delta_C : C \to \mathrm{Hom}_E(\mathrm{Hom}(C, A), A)$ and $\delta'_C : C \to \mathrm{Hom}_{E(A(U))}(\mathrm{Hom}(C, A(U)), A(U))$. To complete the proof it is enough to show $\delta'_C = \Phi\delta_C$. Let $c \in C$. Then $\delta_C(c) : \mathrm{Hom}(C, A) \to A$, $\delta_C(c)(\alpha) = \alpha(c)$. As in the beginning of the proof, we write $\delta_C(c) = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_3 \end{pmatrix}$, where $\delta_1 : \mathrm{Hom}(C, T_U(A)) \to T_U(A)$, $\delta_3 : \mathrm{Hom}(C, A(U)) \to A(U)$, and $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ with $\alpha_1 \in \mathrm{Hom}(C, T_U(A))$ and $\alpha_2 \in \mathrm{Hom}(C, A(U))$. Then $\Phi\delta_C(c) = \delta_3 : \alpha_2 \mapsto \alpha_2(c)$ for all $\alpha_2 \in \mathrm{Hom}(C, A(U))$, hence $\delta'_C = \Phi\delta_C$. Since δ'_C is an isomorphism Φ must be epic, hence Φ is an isomorphism, and it follows that δ_C is an isomorphism. □

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"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, Str. Mihail Kogălniceanu 1, 400084 Cluj-Napoca, Romania
E-mail address: bodo@math.ubbcluj.ro