COMPRESSIBLE FLOW SOLVERS FOR LOW MACH NUMBER FLOWS – a review

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Abstract

The low Mach number setting is a singular limiting situation in compressible flows. As Mach number approaches zero, compressible (density-based) flow solvers suffer severe deficiencies, both in efficiency and accuracy. There are two main approaches advocated in the development of algorithms for the computation of low Mach number flows; first, There is the modification of compressible solvers (density-based) downward to low Mach numbers; second, extending incompressible solvers (pressure-based) towards this regime. Here, we present a brief review of the literature in this area. This addresses the modifications necessary to effectively apply density-based schemes and develop compressible pressure-based schemes to such low Mach number configurations.

1. DENSITY BASED METHODS

Density-based methods represent a large class of schemes adopted for compressible flows. Turkel *et al.* (1997) and Guillard and Viozat (1999) have identified that, in the low Mach number limit, the discretized solution of the compressible fluid flow equations may fail to provide an accurate approximation to the incompressible equations (quoting Guillard and Viozat (1999) in particular). As a 'rule-of-thumb', compressible schemes without modification become impractical for Mach numbers lower than around 0.3 (Roller and Munz, 2000), where, (Mach number, *Ma* is the ratio of speed of sound to speed of flow).

Time-marching density-based schemes are employed widely in computational fluid dynamics for computation of steady and transient transonic, supersonic and hypersonic flows, where switch of type occurs here, as *Ma* passes through unity. In the subsonic regime, when the magnitude of the flow-velocity is small, in comparison with the acoustic wave-speed, dominance of convection terms within the time-dependent equation system renders the system stiff and solvers converge slowly (Choi and Merkle, 1993). Time-marching procedures may suffer severe stability and accuracy restrictions and become inefficient for low Mach-number flow regimes. Here, for explicit schemes, the time-step must satisfy the Courant-Freidrichs-Lewy (CFL) conditions, where numerical stability considerations lead to small time-steps, due to the prevailing acoustic wave-speeds. On the other hand, implicit methods suffer from stiffness due to large disparity in the eigenvalues of the system. There, the condition number is high and eigenvalues may vary by orders of magnitude (Roller and Munz, 2000). As a consequence, the unpreconditioned algebraic system is ill-conditioned, rendering iterative solutions excessively time consuming. The effect of system stiffness on solution convergence is well known, for both explicit and implicit schemes.

Two distinct techniques have been proposed to capture solution convergence for low Mach-number regimes, preconditioning and asymptotic. Both techniques achieve rescaling of system condition numbers. The first technique is to pre-multiply time-derivatives by a suitable preconditioning matrix. Effectively, this scales the eigenvalues of the system to similar orders of magnitude and removes the disparity in wave-speeds, leading to a well-conditioned system (Turkel *et al.*, 1997). The second

technique introduces a perturbed form of the equations. This is known as the asymptotic method. Here, specific terms are discarded, so that the physical acoustic waves are replaced by pseudo-acoustic modes.

1.1. Preconditioning Schemes

For preconditioning schemes, Turkel (1987) has introduced a family of preconditioners for low Mach-number flows. Similarly, Van Leer *et al.* (1991) derived a symmetric preconditioner for the two-dimensional Euler equations. The preconditioning can be utilised for either compressible or incompressible flows, to accelerate convergence towards a steady-state solution. The main drawback to preconditioning methods is that the governing equations themselves switch in type, due to the additional transient term appended. The modified equations have only steady-state solutions in common with the original system (hence, are devoid of true transients). A further drawback is the lack of robustness near stagnation points. This may be due to artificial dissipation, where solution eigenvectors become almost parallel (Wong *et al*., 2001, Darmofal and Schmid, 1996). For the application of these methods to time-dependent problems, the 'dual-time-stepping' technique has emerged, where the physical time-derivative terms are treated as source terms. During each physical time-step, the system of pseudo-temporal equations is advanced in artificial time to reach a pseudo-steady-state, so that ultimately, a divergence-free constraint on the velocity field is satisfied (Liu and Liu, 1993).

Efficiency in preconditioner performance is known to be highly affected by the eigenvalue-spectrum of the system, which must be taken into account within the design of the preconditioner. This arises for example, when simulating combustion problems at low Mach numbers. However, finding suitable preconditioners with optimised properties for complex problems is far from straightforward. Darmofal and Schmid (1996) analysed the influence of eigenvector properties on the effectiveness of some preconditioners. Both theoretically and numerically, Darmofal & Schmid have demonstrated, that due to the lack of eigenvector orthogonality, small perturbations in the linearised evolution problem could be significantly amplified over short time-scales. The long-time or asymptotic behaviour of the linearised system is governed by the eigenvalue spectrum. However, for practical applications to nonlinear problems, this short-time non-normal growth may completely alter the mean-state, to the extent that the predicted long-time asymptotic behaviour may be lost. Darmofal & Schmid have demonstrated, through nonlinear preconditioned Euler predictions, that non-normal amplification does arise, and in practice, generates a significant lack of robustness, particularly near stagnation points.

1.2. Asymptotic schemes

With the second technique, the asymptotic or perturbation approach, a perturbed form of the equations is employed to eliminate system stiffness. Here, a Taylor series expansion of variables in power terms of the Mach number is introduced. This decouples the physical acoustic waves from the equations, replacing them by a set of

pseudo-acoustic forms, whose speeds are comparable to the fluid velocity (Choi and Merkle, 1993, Tomboulides and Orzag, 1998). Application of perturbation methods to extend compressible flow solvers to slightly compressible instances is straightforward, particularly for reactive flows. One may consider, for example the combustion setting where, due to the transient reactive terms, preconditioning schemes require a complex analysis. Although perturbation procedures are highly robust and applicable for both viscous and inviscid flows, the nature of the perturbation limits their usage, particularly with respect to mixed compressible-incompressible flows.

The basic philosophy behind asymptotic methods is, to decrease the numerical representation of the speed of sound artificially, by subtracting a constant pressure P_0 across the entire domain. In modifying away from the true speed of sound, the numerical scheme may enjoy larger time steps (for more details see (Jenny and Muller, 1999)). From a theoretical point of view, the situation is now well-understood in the inviscid limit: if the initial pressure field P scales with the square of the Mach number Ma, $P(x,0) = P_0 + Ma^2 P_2(x)$. Additionally, if the initial velocity field (at $t = 0$) is almost solenoidal: $u(x,0) = u_0(x) + Ma\, u_1(x)$ with $div(u_0) = 0$; then, the compressible flow solution remains uniformly bounded as the Mach number tends to zero. In the $(Ma \to 0)$ limit, the solution satisfies the 'reduced' equation system for the incompressible state (Roller and Munz, 2000).

A significant source of error at low Mach number arises due to the fact that the pressure term is of order $1/Ma^2$, which introduces considerable inaccuracy as Mach number approaches zero. In this regime, compressibility effects have little influence on momentum transfer, since, pressure becomes only a weak function of density. To prevent inaccuracy in the computation of pressure-gradients within the momentum equation, the pressure can be decomposed into two contributions (Choi and Merkle, 1993, Tomboulides and Orzag, 1998): $P(x,t) = P_o(t) + P(x,t)$, with $P_o(t)/P_o = O(1)$ and $P(x,t)/P_o = O(Ma^2)$. Here, $P_o(t)$ and $P(x,t)$ are termed the 'thermodynamic pressure' and the 'hydrodynamic pressure', respectively and P_o is simply a reference pressure. With this variable decomposition, only the thermodynamic pressure appears in the equations of energy and state. In the momentum equation, the gradient of the thermodynamic pressure vanishes, leaving only the gradient of hydrodynamic pressure.

2. PRESSURE-BASED METHODS

In contrast, pressure-based methods were originally conceived to solve incompressible flows, adopting pressure as a primary variable. With this approach, pressure variation remains finite, irrespective of Mach number, rendering computation tractable throughout the entire spectrum of Mach number (Karki and Patankar, 1989), hence circumventing the shortcomings of density-based methods. The first implementation of pressure-based schemes for compressible flow is widely attributed to the early contribution of Harlow and Amsden (1968, 1971), based on a semi-implicit finite difference algorithm.

Pressure-correction, or projection methods, are pressure-based fractional-staged schemes with correction for velocity and pressure (Peyret and Taylor, 1983), introduced through the pioneering work of Chorin (1968) and Temam (1969). Such methods have been employed effectively within several finite volume implementations, say through the SIMPLE (Semi-Implicit Pressure Linked Equations) family of schemes (Patankar, 1980). Karki and Patankar (1989) developed the SIMPLER method for compressible flows, applicable for a wide range of problem-speeds. These SIMPLE methods are first-order in time. Munz *et al.* (2003) extended the SIMPLE scheme for low Mach number flow employing multiple pressure variables, each being associated with different physical response. Similar procedures have been adopted by others (Bijl and Wesseling, 1998, Mary *et al*., 2000, Roller and Munz, 2000). Pressure-correction was taken forward within finite differences to a second-order by Van Kan (1986). Alternatively, within finite elements, Donea *et al.* (1982) introduced a pressure-correction fractional-step method, designed to significantly reduce computational overheads in transient incompressible viscous flow situations.

More recently in the finite element context, Zienkiewicz and coworkers (Zienkiewicz *et al*., 1999, see Zienkiewicz and Codina, 1995, Zienkiewicz *et al*., 1995, Zienkiewicz and Taylor, 2000) have introduced the characteristic-based-split procedure (CBS). This implementation is a Taylor-Galerkin/Pressure-Correction scheme, suitable for both incompressible and compressible flow regimes. The crux here, is to split the equation system into two parts: a part of convection-diffusion type (discretised via a characteristic-Galerkin procedure) and one of self-adjoint type. With the CBS-scheme, one may solve both parts of the system in an explicit manner. Alternatively, one may use a semi-implicit scheme for the first part, allowing for much larger time-steps, and solve the second part implicitly, with its advantage of unconditional stability. The CBS procedure has been tested successfully on a number of scenarios, for example, transonic and supersonic flows, low Mach number flows with low and high viscosity, and in addition, on shallow-water wave problems.

In the incompressible viscoelastic regime, computational methods have matured significantly over the last two decades or so (Saramito and Piau, 1994, Guénette and Fortin, 1995, Baaijens, 1998, Walters and Webster, 2003). Here, it is desirable to extend the methodology into the weakly-compressible regime, and particularly so for viscous polymeric liquid flows. In this regard, density-based preconditioning or asymptotic methods often demand significant recoding. On the other hand, extending an existing incompressible flow code to accommodate compressibility would appear somewhat more straightforward. This is the thesis and starting point adopted for implementation throughout the current study. Precisely, our aim is to modify a pressure-correction technique for incompressible polymeric flows to accommodate weakly-compressible, yet highly-viscous, flows of low Mach number. This presents a natural extension to our earlier incompressible flow studies for viscous (Hawken *et al*., 1990), inelastic (Ding *et al*., 1995, three-dimensional) and viscoelastic (Matallah *et al*., 1998, Wapperom and Webster, 1998) fluids, where we have developed a hybrid schema to attain second-order accuracy.

3. GOVERNING FLOW EQUATIONS FOR VISCOUS LIQUIDS

The conservation of mass and momentum equations employed in the simulation of compressible steady Newtonian fluid under isothermal conditions may be considered as follows:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho U) = 0 \tag{1}$$

$$\rho \frac{\partial U}{\partial t} = [\nabla.\tau - \rho U.\nabla U - \nabla P] \tag{2}$$

where independent variables are (*t*,***x***), time and space, and dependent field variables are ρ, U, τ, P, density, velocity vector, stress tensor and pressure, respectively. Stress is related to the kinematic field through a constitutive law, which is defined for compressible Newtonian fluids as: $\tau_{ij} = \mu\left(2D_{ij} - \frac{2}{3}(\nabla.U)\delta_{ij}\right)$ (3)

where μ is the viscosity, δ_{ij} is Kronecker delta tensor and $D = \left(\frac{\nabla U + \nabla U^T}{2}\right)$ is the rate of deformation tensor (here, superscript *T* denotes tensor transpose). To extract non-dimensionalized governing equations, we define the following quantities, which relate the physical variables, such as velocity, dimension, density, pressure, stresses and time, respectively, to their non-dimensionalized counterparts (notation *):

$$U = U^o U^*;\ x = x^* L\ ;\rho = \rho^* \rho^o;\ P = \left(\frac{\mu U^o}{L}\right)P^*;\ \tau = \left(\frac{\mu U^o}{L}\right)\tau^*;\ t = t^* \frac{L^2 \rho^o}{\mu}\ . \tag{4}$$

This dimensionless form is very suitable for viscous-dominated flows, of present interest. In addition, we introduce the dimensionless group Reynolds number and the dimensionless gradient operator, respectively, as:

$$R_e = \frac{\rho^o U^o L}{\mu};\ \nabla = \frac{\nabla^*}{L}. \tag{5}$$

Following the above, the non- dimensionalized momentum equation may be expressed as:

$$\rho^* \frac{\partial U^*}{\partial t^*} = \nabla^* \cdot \tau^* - R_e \rho^* U^* \cdot \nabla U^* - \nabla^* P^*, \tag{6}$$

which, upon rearranging and discarding * notation for clarity leads to:

$$\rho \frac{\partial U}{\partial t} = \nabla.\tau - R_e \rho U \cdot \nabla U - \nabla P. \tag{7}$$

Correspondence in the continuity equation provides:

$$\frac{\partial \rho}{\partial t} + R_e \nabla.(\rho U) = 0 \,. \tag{8}$$

To complete the two sets of governing equations above, it is necessary to introduce an equation of state relating density to pressure. For liquids, we consider the modified Tait (Tait, 1888) equation of state, in the form

$$\frac{P + B}{P_0 + B} = \left(\frac{\rho}{\rho_0} \right)^m \tag{9}$$

where, *m* and *B* are parameters and P_o , ρ_o denote the reference value of pressure and density, respectively. Note, that this equation is applied only to isentropic change. Nevertheless, it can be utilise with reasonable accuracy in the general case, since m is independent of entropy and *B* and ρ_o are constants (Brujan, 1999).After rearranging and differentiating the equation of state and assuming isentropic condition, we gather:

$$\frac{\partial P}{\partial \rho} = mk\rho^{m-1} = \frac{m(P + B)}{\rho} = c^2_{(X,t)} \tag{10}$$

where $k = \dfrac{(P_0 + B)}{\rho_0^m}$ is a constant and $c_{(X,t)}$ is the speed of sound, a field parameter, distributed in space X and time t .

The next step is to incorporate the above theory within a discrete representation.

4. PRESSURE-CORRECTION SCHEME FOR COMPRESSIBLE FLOWS

Taylor-Galerkin (TG) schemes have emerged, via Taylor-series expansions, to provide high-order time-stepping schemes of various forms, see Donea (1984) and Löhner *et al.* (1984). The principle constructive methodology is to discretised advection-based equations first, in time, and second, in space (Galerkin). Time derivatives may be replaced by spatial equivalents, from the original differential equation (Lax-Wendroff (1960)). This introduces explicit or implicit-type schemes, of various orders of accuracy, and of one-step or two-step implementations (see Löhner *et al.* (1984) and Appendix B). Such schemes have been used widely to solve model problems to more complex flows (Hawken *et al.*, 1990, Ding *et al.*, , Baloch *et al.*, 1995, Townsend and Webster, 1987, Wapperom and Webster, 1998). Extension to include diffusion terms (viscous) demands care to retain stability, typically Crank-Nicolson discretisation, introducing implicitness to the formulation. To advance from advection-diffusion equations to Navier-Stokes, again requires further sophistication. Pressure-correction may deal with this in the incompressible regime, being a

fractional-staged procedure (Hawken *et al.*, 1990, Townsend and Webster, 1987). We outline below how this may be extended to the compressible regime.

To describe the two-step TG-scheme for compressible flows, let us first consider the momentum equation:

$$\rho \frac{\partial U}{\partial t} = \nabla.\tau - R_e \rho\, U.\nabla U - \nabla P\,. \tag{11}$$

For viscous fluids, the TG-scheme may be expressed through the doublet:

at a half-step (t=n+1/2)

$$\rho \frac{\Delta U^{\,n+\frac{1}{2}}}{\Delta t/2} = \nabla.\tau^{\,n+\frac{1}{4}} + \left[- R_e \rho U.\nabla U - \nabla P\right]^n \tag{12}$$

and the *correction-step (t=n+1)*

$$\rho\, \frac{\Delta\, U^{\,n+1}}{\Delta\, t} = \nabla .\tau^{\,n+\frac{1}{2}} + \left[- R_e \rho\, U\, .\nabla\, U\, - \nabla\, P\, \right]^{n+\frac{1}{2}} \tag{13}$$

where, the operator $\Delta(.)^{n+\theta}$ is defined as $\Delta(.)^{n+\theta} = (.)^{n+\theta} - (.)^n$ and semi-implicit representation of diffusion terms is implied, viz,

$$\nabla.\tau^{\,n+\frac{\alpha}{2}} = \frac{\nabla.\tau^{\,n+\alpha} - \nabla.\tau^{\,n}}{2} + \nabla.\tau^{\,n}\,. \tag{14}$$

For pressure gradient at $t = n+\theta$, we may adopt the θ-representation ($\theta = 1/2$, Crank-Nicolson)

$$\nabla P^{\,n+\theta} = \nabla P^{\,n} + \theta\nabla\left(P^{\,n+1} - P^{\,n}\right) = \nabla P^{\,n} + \theta\nabla\left(\Delta P^{\,n+1}\right), \tag{15}$$

so that Eq.(13) becomes:

$$\rho\, \frac{\Delta U^{\,n+1}}{\Delta t} = \nabla.\tau^{\,n+\frac{1}{2}} - \left[R_e \rho U.\nabla U\right]^{n+\frac{1}{2}} - \nabla P^{\,n} - \theta\nabla\left(\Delta P^{\,n+1}\right). \tag{16}$$

To introduce the projection method into the above (for more details see (Townsend and Webster, 1987)), it is convenient to utilise an auxiliary variable U^*, such as:

$$U^{\,n+1} = U^{\,*} - \left(\frac{\Delta t}{\rho}\right)\theta\nabla\left(\Delta P^{\,n+1}\right), \tag{17}$$

from which, using notation $\Delta U^* = U^* - U^n$, we observe:

$$\rho \frac{\Delta U^{n+1}}{\Delta t} = \rho \frac{\Delta U^{*}}{\Delta t} - \theta \nabla \left(\Delta P^{n+1} \right). \tag{18}$$

Substitution of Eq.(18) into Eq.(16) provides

$$\rho \frac{\Delta U^{*}}{\Delta t} = \nabla . \tau^{n+\frac{1}{2}} - \left[R_e \rho U . \nabla U \right]^{n+\frac{1}{2}} - \nabla P^{n} . \tag{19}$$

Taking the divergence of both sides of Eq.(17), one gathers:

$$\nabla . \left(\rho U^{n+1} \right) = \nabla . \left(\rho U^{*} \right) - \Delta t \theta \nabla^{2} \left(\Delta P^{n+1} \right) \tag{20}$$

from which, by appealing to the continuity Eq.(8), one extracts:

$$R_e^{-1} \frac{\Delta \rho^{n+1}}{\Delta t} + \nabla . \left(\rho U^{*} \right) = \Delta t \theta \nabla^{2} \left(\Delta P^{n+1} \right) . \tag{21}$$

In addition, by employing the chain rule upon Eq.(10) and taking difference operations, we may relate density increment to pressure increment through,

$$\frac{\Delta \rho^{n+1}}{\Delta t} = \frac{1}{c^{2}_{(X,t)}} \frac{\Delta P^{n+1}}{\Delta t} , \tag{22}$$

where $c_{(X,t)}$ is defined in Section(2). To obtain Eq(10), we assumed isentropic conditions, following Karki and Patankar (1989), Zienkiewicz and Condina (1995) and Brujan (1999). Other alternatives assumptions may be adopted, such as isenthalpic (1993) or homenthalpic (Munz *et al*., 2003). Note, under steady-state conditions, pressure changes (hence pressure) will vanish. Consequently, steady solution will be independent of any of the above assumptions. However, this may affect transient results and convergence properties of the associated schemes. Finally, we substitute Eq.(22) into Eq.(21) to realise a compressible temporal evolutionary expression for pressure, of the form:

$$\frac{1}{R_e c^{2}_{(X,t)}} \frac{\Delta P^{n+1}}{\Delta t} - \Delta t \theta \nabla^{2} \left(\Delta P^{n+1} \right) = -\nabla . \left(\rho U^{*} \right) . \tag{23}$$

This is the new equation that we introduce into the incompressible TG-formulation at stage 2 (see below). In summary, the TG-solution strategy would encompass:

stage 1-a: solution of Eq.(12) to yield velocity at half-step $U^{n+1/2}$

(momentum equation)

stage 1-b: solution of Eq.(19) to yield predicted U^{*}

(momentum equation)

stage 2: solution of Eq.(23) to yield pressure difference ΔP^{n+1}

(continuity equation)

stage 3: solution of Eq.(17) to yield corrected U^{n+1}

(pseudo equation)

5. CONCLUSION

We have effectively covered the relevant material from the literature on numerical solvers for compressible flow, where we are particularly interested in low Mach number scenarios. In this manner, we have laid out current thinking on density-based and pressure-based approaches. We also highlight our own derivation of a pressure-correction scheme we propose for compressible flows.

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