Implied Recovery*

Sanjiv R. Das
Santa Clara University
Santa Clara, CA 95053

Paul Hanouna
Villanova University
Villanova, PA 19085

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Abstract

In the absence of forward-looking models for recovery rates, market participants tend to use exogenously assumed constant recovery rates in pricing models. We develop a flexible jump-to-default model that uses observables: the stock price and stock volatility in conjunction with credit spreads to identify implied, endogenous, dynamic functions of the recovery rate and default probability. The model in this paper is parsimonious and requires the calibration of only three parameters, enabling the identification of the risk-neutral term structures of forward default probabilities and recovery rates. Empirical application of the model shows that it is consistent with stylized features of recovery rates in the literature. The model is flexible, i.e., it may be used with different state variables, alternate recovery functional forms, and calibrated to multiple debt tranches of the same issuer. The model is robust, i.e., evidences parameter stability over time, is stable to changes in inputs, and provides similar recovery term structures for different functional specifications. Given that the model is easy to understand and calibrate, it may be used to further the development of credit derivatives indexed to recovery rates, such as recovery swaps and digital default swaps, as well as provide recovery rate inputs for the implementation of Basel II.

JEL Codes: G0, G1.

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Abstract

In the absence of forward-looking models for recovery rates, market participants tend to use exogenously assumed constant recovery rates in pricing models. We develop a flexible jump-to-default model that uses observables: the stock price and stock volatility in conjunction with credit spreads to identify implied, endogenous, dynamic functions of the recovery rate and default probability. The model in this paper is parsimonious and requires the calibration of only three parameters, enabling the identification of the risk-neutral term structures of forward default probabilities and recovery rates. Empirical application of the model shows that it is consistent with stylized features of recovery rates in the literature. The model is flexible, i.e., it may be used with different state variables, alternate recovery functional forms, and calibrated to multiple debt tranches of the same issuer. The model is robust, i.e., evidences parameter stability over time, is stable to changes in inputs, and provides similar recovery term structures for different functional specifications. Given that the model is easy to understand and calibrate, it may be used to further the development of credit derivatives indexed to recovery rates, such as recovery swaps and digital default swaps, as well as provide recovery rate inputs for the implementation of Basel II.

1 Introduction

As the market for credit derivatives expands, the opportunity to extract forward-looking credit information from traded securities increases. Just as the growth of equity option markets resulted in the extraction of forward-looking implied volatilities, this paper develops a method to extract and identify the implied forward curves of default probabilities and recovery rates for a given firm on any date, using the extant credit default swap spread curve.

The paucity of data on recoveries (about a thousand defaults in the past 25 years or so) has made the historical modeling of recovery rates somewhat tenuous, though excellent studies exist on explaining realized recovery (see Altman, Brady, Resti and Sironi (2005); Acharya, Bharath and Srinivasan (2007)). As yet, no model exists that provides forecasted forward-looking recovery rate functions for the pricing of credit derivatives. By developing a model in which "implied" forward recovery rate term structures may be extracted using data from the credit default swap market and the equity market, our model makes possible the pricing of recovery related products such as recovery swaps and digital default swaps. Further, the recent regulatory requirements imposed by Basel II require that banks use recovery rate assumptions in their risk models. Thus, our model satisfies important trading and risk management needs.

While models for default likelihood have been explored in detail by many researchers[1], the literature on recovery rate calibration is less developed. Academics and practitioners have often assumed that the recovery rate in their models is constant, and set it to lie mostly in the 40-50% range for U.S. corporates, and about 25% for sovereigns. Imposing constant recovery may be practically exigent, but is unrealistic given that recovery rate distributions evidence large variation around mean levels (see Hu (2004) for many examples of fitted recovery rate distributions; Moody's reports that recovery rates may vary from as little as 7.8% for junior subordinated debt to as high as 83.6% for senior secured over the 1982-2004 period[2]). The rapid development of the credit default swap (CDS) market has opened up promising possibilities for extracting implied default rates and recovery rates so that the class of models developed here will enable incorporating realistic recovery rates into pricing models.

If the recovery rate is supplied exogenously, as in current practice, then the term structure of CDS spreads may be used to extract the term structure of risk-neutral default probabilities, either using a structural model approach as in the model of `CreditGrades` (Finger, Finkelstein, Lardy, Pan, Ta and Tierney (2002)), or in a reduced-form framework, as in Jarrow, Lando and Turnbull (1997), Duffie and Singleton (1999), Jarrow (2001), Madan, Guntay and Unal (2003), or Das and Sundaram (2007). However, assuming recovery rates to be static is an impractical imposition on models, especially in the face of mounting evidence that recovery rates are quite variable over time. For instance, Altman, Brady, Resti

[1] See, amongst others, Merton (1974), Leland (1994), Jarrow and Turnbull (1995), Longstaff and Schwartz (1995), Madan and Unal (1995), Leland and Toft (1996), Jarrow, Lando and Turnbull (1997), Duffie and Singleton (1999), Sobehart, Stein, Mikityanskaya and Li (2000), Jarrow (2001) and Duffie, Saita and Wang (2005))

[2] Based on "Default and Recovery Rates of Corporate Bond Issuers, 1920-2004", from Moody's, New York, 2005, Table 27.

and Sironi (2005) examined the time series of default rates and recovery levels in the U.S. corporate bond market and found both to be quite variable and correlated. The model we develop in this paper makes recovery dynamic, not static, and allows for a range of relationships between the term structures of risk-neutral forward default probabilities and recovery rates. We find overall, that recovery rates and default rates are inversely related, though this is not necessary on an individual firm basis. Though theoretically unconnected to the result of Altman et al, which is under the physical measure, this conforms to economic intuition that high default rates occur concurrently with low resale values of firm's assets.[3]

Extracting recovery rates has proven to be difficult owing to the existence of an identification problem arising from the mathematical structure of equations used to price many credit derivative products. Credit spreads (C) are approximately the result of the product of the probability of default (λ) and the loss rate on default ($L = 1 - \phi$), where ϕ is the recovery rate, i.e. $C = \lambda(1 - \phi)$. Hence, many combinations of λ and ϕ result in the same spread. Our model resolves this identification problem using stock market data in addition to spread data. We do this in a dynamic model of default probabilities and recovery rates.

The identification problem has been addressed in past work. Zhang (2003) shows how joint identification of default intensities and recovery rates may be carried out in a reduced form model. He applies the model to Argentine sovereign debt and finds that recovery rates are approximately 25%, the number widely used by the market. See Christensen (2005) for similar methods. Pan and Singleton (2005), using a panel of sovereign spreads on three countries (Mexico, Russia and Turkey), also identify recovery rates and default intensities jointly assuming recovery of face value rather than market value. Exploiting information in both the time series and cross-section they find that recovery rates may be quite different from the widely adopted 25% across various process specifications. Song (2007) uses a no-arbitrage restriction to imply recovery rates in an empirical analysis of sovereign spreads and finds similar recovery levels. Bakshi, Madan and Zhang (2001) (see also Karoui (2005)) develop a reduced-form model in which it is possible to extract a term structure of recovery using market prices; they show that it fits the data on BBB U.S. corporate bonds very well. They find that the recovery of face value assumption provides better fitting models than one based on recovery of Treasury. Madan, Guntay and Unal (2003) finesse the identification problem by using a sample of firms with both junior and senior issues.

Other strands of the estimation literature on recovery rates are aimed at explaining recovery rates in the cross-section and times series. Acharya, Bharath and Srinivasan (2007) find that industry effects are extremely important in distinguishing levels of recovery in the cross-section of firms over a long period of time. Chava, Stefanescu and Turnbull (2006) jointly estimate recovery rates and default intensities using a large panel data set of defaults.

In contrast to these times-series dependent *econometric* approaches to recovery rate extraction, this paper adopts a *calibration* approach, applicable to a single spread curve at a single point in time. The model eschews the use of historical data given its goal is to extract forward looking implied recovery rates. It uses the term structure of CDS spreads,

[3]An interesting point detailed by an anonymous referee suggests also that a negative relation between default probability and recovery under the risk-neutral measure may simply be an artifact of risk premia. Since risk premia on default probabilities drive these values higher under the risk neutral measure, and drive recovery rates lower, when risk premia are high (as in economic downturns), it induces a negative correlation between the two term structures.

and equity prices and volatility, at a single point in time, to extract both, the term structure of hazard rates and of recoveries, using a simple least-squares fit. The model extracts recovery functions, not just point estimates of recovery. Hence, it also provides insights into the dynamics of recovery.

The main differences between this approach and the econometric ones of Zhang (2003) and Pan and Singleton (2005) are two-fold. First, only information on a given trading day is used, in much the same way traders would calibrate any derivatives model in practice; in the empirical approaches, time series information is also required. Second, rather than extract a single recovery rate, our model delivers an entire *forward term structure* of recovery and a functional relation between recovery and state variables, thereby offering a dynamic model of recovery.

Other approaches closer to ours, though static, have also been attempted. A previous paper that adopts a calibration approach is by Chan-Lau (2008). He used a curve-fitting approach to determine the *maximal* recovery rate; this is the highest constant recovery rate that may be assumed, such that the term structure of default intensities extracted from CDS spreads admits economically acceptable values. (Setting recovery rates too high will at some point, imply unacceptably high default intensities, holding spreads fixed). This paper offers three enhancements to Chan-Lau's idea. First, it fits an entire term structure of recovery rates, not just a single rate (i.e. flat term structure). Second, it results in exact term-dependent recovery rates, not just upper bounds. Third, our approach also extracts the dynamics of the recovery rate because we obtain implied default functions and not just point estimates of the term structure.

It is also possible to find implied recovery rates if there are two contracts that permit separating recovery risk from the probability of default. Berd (2005) shows how this is feasible using standard CDS contracts in conjunction with digital default swaps (DDS), whose payoffs are function of only default events, not recovery (they are based on predetermined recovery rates); such pairs enable disentanglement of recovery rates from default probabilities in a static manner. Using these no-arbitrage restrictions is also exploited by Song (2007) in a study of sovereign spreads. This forms the essential approach in valuing a class of contracts called recovery swaps, which are options on realized versus contracted recovery rates. In Berd's model, calibrated default rates and recovery rates are positively correlated (a mathematical necessity, given fixed CDS spreads and no remaining degrees of freedom).[4] In the model in this paper, the functional relation between default probability and recovery does not impose any specific relationship. However, the empirical literature mostly evidences a negative correlation between the term structures of default probabilities and recovery rates.

Further, since default swap spreads are for a specified reference instrument, the recovery rate may be extracted separately for each reference instrument in a given capital structure.

The summary of the paper is as follows:

1. We explain the recovery rate identification problem (Section 2). In Section 3 We develop a flexible "jump-to-default" model that uses additional data, the stock price

[4]Empirically, there is some work on the historical correlation of realized default rates and recoveries. There is no technical relation between realized rates and implied ones; however, the empirical record shows that realized default rates and recovery are in fact negatively correlated (Altman, Brady, Resti and Sironi (2005); Gupton and Stein (2005) find a positive correlation between credit quality and recovery, and this relation persists even after conditioning on macroeconomic effects).

S and stock volatility σ in conjunction with credit spreads and the forward risk free rate curve to identify not just the implied values of default probability λ and recovery rate ϕ, but instead, the parameterized functional forms of these two inputs.

2. The model in this paper is parsimonious and requires the calibration of only three parameters. Calibration is fast and convergent, and takes only a few seconds (Section 4.2).

3. We apply the model to data on firm-level CDS spread curves sorted into 5 quintiles over 31 months in the period from January 2000 to July 2002 (see Section 4.1). In Section 4.2 we calibrate the model for each quintile and month, and then use the calibrated parameters to identify the risk-neutral term structures of forward default probabilities and recovery rates (Section 4.3).

4. The model is also illustrated on individual names and calibrates easily. Calibration may be undertaken to as few as three points on the credit spread curve (given a three-parameter model), but may also fit many more points on the term structure (Section 4.4).

5. Several experiments on individual names show that the parameters tend to be relatively stable and change in a smooth manner over time (Section 4.4.1).

6. The model is shown to work well for different recovery rate specifications and the extracted term structures are robust to changes in model specification (Section 4.4.2).

7. The calibration of the model is not sensitive to small changes in the inputs. Hence, the mathematical framework of the model is stable (Section 4.4.3).

8. The model may be extended to fitting multiple debt tranches of the same issuer, such that the term structure of default probabilities remains the same across tranches, yet allowing for multiple term structures of recovery rates, one for each debt tranche (Section 4.4.4). The same approach also works to extend the model to fitting multiple issuers (within the same rating class for example) simultaneously to better make use of the information across issuers.

Since the model framework admits many possible functional forms, it is flexible, yet parsimonious, allowing for the pricing of credit derivatives linked to recovery such as recovery swaps and digital default swaps. Concluding comments are offered in Section 5.

2 Identification

It is well understood in the literature that separate identification of recovery rates and default intensities is infeasible using only the term structure of credit default swap (CDS) spreads (for example, see the commentary in Duffie and Singleton (1999) and Pan and Singleton (2005)). Here, we introduce the notation for our model and clarify the exact nature of the identification problem. The dynamic model of implied recovery will be introduced in the next section.

Assume there are N periods in the model, indexed by $j = 1, \ldots, N$. Without loss of generality, each period is of length h, designated in units of years. Thus, time intervals in the model are $\{(0, h), (h, 2h), \ldots, ((N-1)h, Nh)\}$. The corresponding end of period maturities are $T_j = jh$.

Risk free forward interest rates in the model are denoted $f((j-1)h, jh) \equiv f(T_{j-1}, T_j)$, i.e. the rate over the jth period in the model. We write these one period forward rates in short form as f_j, the forward rate applicable to the jth time interval. The discount functions (implicitly containing zero-coupon rates) may be written as functions of the forward rates, i.e.

$$D(T_j) = \exp\left\{-\sum_{k=1}^{j} f_k h\right\}, \tag{1}$$

which is the value of \$1 received at time T_j.

For a given firm, default is likely with an intensity denoted as $\xi_j \equiv \xi(T_{j-1}, T_j)$, constant over forward period j. Given these intensities, the survival function of the firm is defined as

$$Q(T_j) = \exp\left\{-\sum_{k=1}^{j} \xi_k h\right\} \tag{2}$$

We assume that at time zero, a firm is solvent, i.e. $Q(T_0) = Q(0) = 1$.

Default Swaps

For the purposes of our model, the canonical default swap is a contract contingent on the default of a bond or loan, known as the "reference" instrument. The buyer of the default swap purchases credit protection against the default of the reference security, and in return pays a periodic payment to the seller. These periodic payments continue until maturity or until the reference instrument defaults, in which event, the seller of the swap makes good to the buyer the loss on default of the reference security. Extensive details on valuing these contracts may be found in Duffie (1999).

We denote the periodic premium payments made by the buyer to the seller of the N period default swap as a "spread" C_N, stated as an annualized percentage of the nominal value of the contract. Without loss of generality, we stipulate the nominal value to be \$1. We will assume that defaults occur at the end of the period, and given this, the premiums will be paid until the end of the period. Since premium payments are made as long as the reference instrument survives, the expected present value of the premiums paid on a default swap of maturity N periods is as follows:

$$A_N = C_N\ h\ \sum_{j=1}^{N} Q(T_{j-1}) D(T_j) \tag{3}$$

This accounts for the expected present value of payments made from the buyer to the seller.

The other possible payment on the default swap arises in the event of default, and goes from the seller to the buyer. The expected present value of this payment depends on the recovery rate in the event of default, which we will denote as $\phi_j \equiv \phi(T_{j-1}, T_j)$, which is the recovery rate in the event that default occurs in period j. The loss payment on default is then equal to $(1-\phi_j)$, for every \$1 of notional principal. This implicitly assumes the "recovery of par" (RP) convention is being used. This is without loss of generality. Other conventions such as "recovery of Treasury" (RT) or "recovery of market value" (RMV) might be used just as well.

The expected loss payment in period j is based on the probability of default in period j, conditional on no default in a prior period. This probability is given by the probability of surviving till period $(j-1)$ and then defaulting in period j, given as follows:

$$Q(T_{j-1}) \; \left(1 - e^{-\xi_j h}\right) \tag{4}$$

Therefore, the expected present value of loss payments on a default swap of N periods equals the following:

$$B_N = \sum_{j=1}^{N} Q(T_{j-1}) \; \left(1 - e^{-\xi_j h}\right) D(T_j) \; (1-\phi_j) \tag{5}$$

The fair pricing of a default swap, i.e. a fair quote of premium C_N must be such that the expected present value of payments made by buyer and seller are equal, i.e. $A_N = B_N$.

Identification

In equations (3) and (5), the premium C_N and the discount functions $D(T_j)$ are observable in the default risk and government bond markets respectively. Default intensities ξ_j (and the consequent survival functions $Q(T_j)$), as well as recovery rates ϕ_j are not directly observed and need to be inferred from the observables.

Since there are N periods, we may use N default swaps of increasing maturity each with premium C_j. This mean there are N equations $A_N = B_N$ available, but $2N$ parameters: $\{\xi_j, \phi_j\}, j = 1, 2, \ldots, N$ to be inferred. Hence the system of equations is not sufficient to result in an identification of all the required parameters.

How is identification achieved? There are two possible approaches.

1. We assume a functional form for recovery rates, thereby leaving only the default intensities to be identified. For example, we may assume, as has much of the literature, that recovery rates are a constant, exogenously supplied value, the same for all maturities. The calculations to extract default intensities under this assumption are simple and for completeness are provided in Appendix A. We may also assume the recovery rate term structure is exogenously supplied. Again, the simple calculations are provided in Appendix B. Both these methods are bootstrapping approaches which assume that recovery rates are time-deterministic (static) and not dynamic.

2. The more general approach is to assume a dynamic model of recovery rates. This approach specifies recovery rates as functions of state variables, and therefore, recovery becomes dynamic, based on the stochastic model for the underlying state variable. In this class of model we extract dynamic implied functions for both, forward default intensities and forward recovery rates. It is this approach that we adopt in the paper.

The specific dynamic model that we use is denoted the "jump-to-default" model. It resides within the class of reduced form models of default, and uses additional information from the equity markets, namely stock prices and volatilities to identify default intensities and recovery rates. We describe this model next.

3 The Jump-to-Default Model

Our model falls within the *hybrid* model class of Das and Sundaram (2007) (DS), extended here to extracting *endogenous* implied recovery term structures from CDS spreads. The DS model permits stochastic term structures of risk free rates, modeled using a forward rate model. As a result, the default hazard function in that model is very general, where in addition to equity state variables, interest rates are also permitted to be state variables. The process of default in DS has dynamics driven by both equity and interest rates. In this paper, we do not make interest rates stochastic. Instead we generalize the recovery rate model. In DS, recovery rates are static, constant and exogenous. Here, we make a significant extension where recovery rates are dynamic, stochastic and endogenous. The techniques introduced here result in identification of the recovery rate term structures and the dynamics of recovery rates as functions of the stochastic process of equity.

The inputs to the model are the term structure of forward CDS spreads, $C_j, j = 1, \ldots, N$; forward risk free interest rates $f_j, j = 1, \ldots, N$, the stock price S and its volatility σ (these last two inputs are the same as required in the implementation of the Merton (1974) model). The outputs from the model are (a) implied functions for default intensities and recovery rates, and (b) the term structures of forward default probabilities (λ_j) and forward recovery rates (ϕ_j).

The single driving state variable in the model is the stock price S. We model its stochastic behavior on a Cox, Ross and Rubinstein (1979) binomial tree with an additional feature: the stock can jump to default with probability λ, where λ is state-dependent. Hence, from each node, the stock will proceed to one of three values in the ensuing period:

$$S \rightarrow \begin{cases} S\,u = Se^{\sigma\sqrt{h}} & \text{w/prob } q(1-\lambda) \\ S\,d = Se^{-\sigma\sqrt{h}} & \text{w/prob } (1-q)(1-\lambda) \\ 0 & \text{w/prob } \lambda \end{cases}$$

The stock rises by factor $u = e^{\sigma\sqrt{h}}$, or falls by factor $d = e^{-\sigma\sqrt{h}}$, conditional on no default. We have assumed that recovery on equity in the event of default is zero (the third branch). This third branch creates the "jump-to-default" (JTD) feature of the model. $\{q, 1-q\}$ are the branching probabilities when default does not occur. If f is the risk free rate of interest for the period under consideration, then under risk-neutrality, the discounted stock price must be a martingale, which allows us to imply the following jump-compensated risk-neutral probability

$$q = \frac{R/(1-\lambda) - d}{u - d}, \quad R = e^{fh}.$$

We note that this jump-to-default tree provides a very general model of default. For instance, if we wish to value a contract that pays one dollar if default occurs over the next 5 years, then we simply attach a value of a dollar to each branch where default occurs, and then, by backward recursion, we accumulate the expected present value of these default cashflows to get the fair value of this claim. Conversely, if we know the value of this claim, then we may infer the "implied" value of default probability λ that results in the model value that matches the price of the claim. And, if we know the value of the entire term structure of these claims, we can "imply" the term structure of default probabilities. By generalizing this model to claims that are recovery rate dependent, we may imply the term structure of recovery rates.

We denote each node on the tree with the index $[i, j]$, where j indexes time and i indexes the level of the node at time j. The initial node is therefore the $[0, 0]$ node. At the end of the first period, we have 2 nodes $[0, 1]$ and $[1, 1]$; there are three nodes at the end of the second period: $[0, 2]$, $[1, 2]$ and $[2, 2]$. And so on. We allow for a different default probability $\lambda[i, j]$ at each node. Hence, the default intensity is assumed to be time- and state-dependent, i.e. dynamic. Further, for any reference instrument, we apply a recovery rate at each node, denoted $\phi[i, j]$, which again, is dynamic over $[i, j]$. We define functions for the probability of default and the recovery rate as follows:

$$\lambda[i,j] = 1 - e^{-\xi[i,j]\, h}, \quad \xi[i,j] = \frac{1}{S[i,j]^b}, \tag{6}$$

$$\phi[i,j] = N(a_0 + a_1 \lambda[i,j]) \tag{7}$$

$$S[i,j] = S[0,0]\, u^{j-1}\, d^i$$

where $N(\cdot)$ is the cumulative normal distribution. Thus, the default probabilities and recovery rates at each node are specified as functions of the state variable $S[i, j]$, and are parsimoniously parameterized by three variables: $\{a_0, a_1, b\}$. Note that our functional specifications for λ and ϕ result in values that remain within the range $(0, 1)$. The intermediate variable ξ is the default "intensity" or hazard rate of default. When the stock price goes to zero, the hazard rate of default ξ becomes infinite, i.e. immediate default occurs. And as the stock price gets very high, ξ tends towards zero. One may envisage more complicated functional forms for λ and ϕ with more parameters as needed to fit the market better.

Our model implies some implicit connections between the parameters in order for the equivalent martingale measure to exist in this jump-to-default model. These are as follows. We begin with the probability function:

$$q = \frac{R/(1-\lambda) - d}{u - d} \tag{8}$$

If $0 \le q \le 1$, it implies the following two restrictions:

$$u \ge R/(1-\lambda), \qquad d \le R/(1-\lambda) \tag{9}$$

Substituting in the expressions for $u = e^{\sigma\sqrt{h}}$, $R = e^{fh}$, and $1 - \lambda = e^{-\xi h}$, we get

$$\sigma \ge (f + \xi)\sqrt{h} \ge -\sigma \tag{10}$$

The latter inequality is trivially satisfied since $f, \xi, \sigma \ge 0$. The first inequality is also satisfied, especially when $h \to 0$.[5] We also note that if we begin with the expression $\sigma \ge (f + \xi)\sqrt{h}$

[5] As an aside, we also note that this restriction is also implicit in the seminal paper by Cox, Ross and Rubinstein (1979) on binomial trees. In that paper, the risk-neutral probabilities are given by

$$q = \frac{e^{fh} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} \implies \sigma \ge f\sqrt{h} \ge -\sigma \tag{11}$$

which is exactly the same restriction as we have in equation (10) above when $\xi = 0$.

or $\sigma \geq (f + \frac{1}{S^b})\sqrt{h}$, then rearranging gives the slightly modified expression:

$$b \geq \frac{-\ln\left(\frac{\sigma}{\sqrt{h}} - f\right)}{\ln S} \tag{12}$$

We note that b is decreasing when S is increasing, when h is decreasing, when σ is increasing, or when f is decreasing. An additional restriction is that $b \geq 0$ so that default risk in the model increases when the stock price decreases. If we use values from the data in the paper, we find that these conditions on b are easily met. Hence, the existence of martingale measures for pricing default risky securities is assured.

Given the values of $\{a_0, a_1, b\}$, the tree may be used to price CDS contracts. This is done as follows. The fair spread on a CDS contract is that which makes the present value of expected premiums on the CDS, denoted $A[i,j]$, equal to the present value of expected loss on the reference security underlying the CDS, denoted $B[i,j]$. These calculations have been exposited earlier. In this case however, we use the values of $\lambda[i,j]$ and $\phi[i,j]$ on the tree to compute the fair CDS spreads by backward recursion. The operative recursive equations are (for the CDS with maturity of N periods):

$$\begin{aligned}
A[i,j] &= C_N/R + \\
&\quad \frac{1}{R}\left[q[i,j](1-\lambda[i,j])A[i,j+1] + (1-q[i,j])(1-\lambda[i,j])A[i+1,j+1]\right] \\
B[i,j] &= \lambda[i,j](1-\phi[i,j]) + \\
&\quad \frac{1}{R}\left[q[i,j](1-\lambda[i,j])B[i,j+1] + (1-q[i,j])(1-\lambda[i,j])B[i+1,j+1]\right] \\
&\quad \text{for all } N, i
\end{aligned}$$

The fair spread C_N is the one that makes the initial present value of expected premiums $A[0,0]$ equal to the present value of expected losses $B[0,0]$. The fair spread may be written as $C_j(a_0, a_1, b) \equiv C_j(S, \sigma, f; a_0, a_1, b)$, $j = 1 \ldots N$.

We proceed to fit the model by solving the following least-squares program:

$$\min_{b,a_0,a_1} \frac{1}{N} \sum_{j=1}^{N} \left[C_j(a_0, a_1, b) - C_j^0\right]^2 \tag{13}$$

where $\{C_j^0\}, j = 1 \ldots N$ are the observable market CDS spreads, and $C_j(a_0, a_1, b)$ are the model fitted spreads. This provides the root mean-squared (RMSE) fit of the model to market spreads by optimally selecting the three model parameters $\{a_0, a_1, b\}$. Note that each computation of $C_j(a_0, a_1, b)$ requires an evaluation on a jump-to-default tree, and the minimization above results in calling N such trees repeatedly until convergence is attained.[6] The model converges rapidly, in a few seconds.

Once we have the calibrated parameters, we are able to compute the values of $\lambda[i,j]$ and $\phi[i,j]$ at each node of the tree. The forward curve of default probabilities $\{\lambda_j\}$ and recovery

[6]The reader will recognize that this is analogous to equity options where we numerically solve for the implied volatility surface using binomial trees if American options are used. See for example Dumas, Fleming and Whaley (1998) where a least-squares fit is undertaken for implied tree models of equity options.

rates $\{\phi_j\}$ are defined as the set of expected forward values (term structure)

$$\phi_j = \sum_{i=0}^{j} p[i,j]\phi[i,j], \quad \forall j \tag{14}$$

$$\lambda_j = \sum_{i=0}^{j} p[i,j]\lambda[i,j], \quad \forall j \tag{15}$$

where we have denoted the total probability of reaching node $[i,j]$ (via all possible paths) on the tree as $p[i,j]$. The probabilities $p[i,j]$ are functions of the node probabilities $q[i,j] = (R/(1-\lambda[i,j]) - d)/(u-d)$ and the probabilities of survival $(1-\lambda[i,j])$ on the paths where default has not occurred prior to the period j.[7]

We note that no restrictions have been placed on parameters $\{a_0, a_1, b\}$ that govern the relationship between default probabilities, recovery rates and stock prices. If these reflect the stylized empirical fact that default rates and recovery rates are negatively correlated then $a_1 < 0$. Also if default rates are increasing when stock prices are falling, then $b > 0$. We do not impose any restrictions on the optimization in equation (13) when we implement the model. As we will see in subsequent sections the parameters that we obtain are all economically meaningful.

3.1 Mapping the model to structural models

Our reduced-form model uses the stock price as the driving state variable. In this subsection we show that, with this state variable, the model may be mapped into a structural model. Specifically, we will see that the model results in a jump-diffusion version of Merton (1974).[8]

Assume that the defaultable stock price S is driven by the following jump compensated stochastic process:

$$dS = (r+\xi)S\,dt + \sigma S\,dZ - S\,dN(\xi) \tag{16}$$

where $dN(\xi)$ is a jump (to default) Poisson process with intensity ξ. When the default jump occurs, the stock drops to zero. The same stochastic process is used in Samuelson (1972) as well. Define debt $D(S,t)$ (and correspondingly credit spreads) as a function of S and time

[7]We note that the forward recovery curve is the expected value of all the recovery rates $\phi[,j]$ across all states i at time j without any discounting. The expectation is taken using the probabilities of paths on the jump-to-default tree. This is the expected recovery rate in a dynamic model where recovery rates are stochastic, because they are functions of the stock price. It is not the same as the recovery rate term structure in a static model where recovery rates are not stochastic, i.e. there is a single recovery rate for each time j. It is possible to recover the static recovery term structure from the dynamic one, as the static term structure is a nonlinear function of the dynamic one. We prefer to use the dynamic term structure, consistent with our model. In practice the two term structures are likely to be quite similar. One way to recover the static term structures is to use the fitted dynamic function for recovery rates as a function of default probabilities, as in equation (7), in the expressions for λ in Appendix B, and numerically bootstrap both term structures.

[8]In a previous version of this paper, we had developed a static implied recovery procedure using the original Merton model, considering only diffusions. Hence, based on the mapping in this subsection, this version of the paper may also be interpreted as a generalization of the prior static model to a dynamic model based on jump-diffusions.

t. Using Ito's lemma we have

$$dD = \left[D_S(r+\xi)S + \frac{1}{2}D_{SS}\sigma^2 S^2 + D_t\right]\,dt + D_S\sigma S\,dZ + [\phi D - D]dN(\xi) \tag{17}$$

where ϕ is the recovery rate on debt. No restrictions are placed on ϕ, which may be stochastic as well. Therefore, in a structural model setting we may deduce the stochastic process for firm value $V = S + D$, i.e.

$$\begin{aligned} dV &= dS + dD \\ &= \left[(1+D_S)(r+\xi)S + \frac{1}{2}D_{SS}\sigma^2 S^2 + D_t\right]\,dt \\ &\quad +(1+D_S)\sigma S\,dZ + [D(\phi-1) - S]\,dN(\xi) \end{aligned} \tag{18}$$

If we define leverage as $L = D/V$, then equation (18) becomes

$$\begin{aligned} \frac{dV}{V} &= \left[(1+D_S)(r+\xi)(1-L) + \frac{1}{2}D_{SS}\sigma^2 S(1-L) + \frac{D_t}{V}\right]\,dt \\ &\quad +\sigma(1+D_S)(1-L)\,dZ + (\phi L - 1)\,dN(\xi) \end{aligned} \tag{19}$$

Therefore, there is a system of equations (16), (17), (19), that are connected, and imply the default intensity ξ and recovery rate ϕ. We may choose to model firm value V (the structural model approach of equation 19), or model S (the reduced-form approach of equation 16). Both approaches in this setting result in dynamic models of implied recovery. In the previous section, we chose to proceed with the reduced-form approach in discrete time. Our exposition here also shows the equivalence in continuous time. (See also Jarrow and Protter (2004) for connections between structural and reduced-form models in the context of information content of the models). Next, we undertake empirical and experimental illustrations of the methodology.

4 Analysis

4.1 Data

We obtained data from `CreditMetrics` on CDS spreads for 3,130 distinct firms for the period from January 2000 to July 2002. The data comprises a spread curve with maturities from 1 to 10 years derived from market data using their model. We used only the curve from 1 to 5 years, and incorporated half-year time steps into our analysis by interpolating the half-year spread levels from 1 to 5 years. There are a total of 31 months in the data. We sorted firms into five quintiles each month based on the expected default frequencies provided by `CreditMetrics` and then averaged values (equally-weighted) at each spread maturity for all firms and observations within quintile for the month. Hence, we obtain an average CDS spread curve per month for each quintile.

Summary data by quintile is provided in Table 1. The database also contained stock prices, and debt per share[9], and historical equity volatilities. Forward interest rate term

[9]Debt per share is based on the CreditGrades methodology (see Finger, Finkelstein, Lardy, Pan, Ta and Tierney (2002)). Debt comprises short-term and long-term debt plus half of other debt and zero accounts payables. Number of shares comprises both common and preferred shares.

Table 1: Summary data on stock values and CDS spreads. The data is from `CreditMetrics` on CDS spreads for 3,130 distinct firms for the period from January 2000 to July 2002. There are a total of 31 months in the data. We sorted firms into five quintiles each month based on the expected default frequencies provided by `CreditMetrics` and then averaged values (equally-weighted) at each spread maturity for all firms and observations within quintile for the month.

	Quintile 1	Quintile 2	Quintile 3	Quintile 4	Quintile 5
Stock price	23.20	20.76	18.57	16.36	14.26
Stock volatility	0.6176	0.6350	0.6579	0.6866	0.7375
Debt per share	9.39	10.08	10.38	10.81	11.61

	Market CDS Spreads (basis points)				
Maturity	Quintile 1	Quintile 2	Quintile 3	Quintile 4	Quintile 5
1 yr	3.48	10.86	32.90	98.86	376.48
2 yr	4.41	14.67	42.62	110.41	332.72
3 yr	6.46	22.48	60.08	137.70	337.33
4 yr	9.67	32.15	77.77	161.45	348.91
5 yr	13.16	41.47	92.88	179.47	358.12

structures are computed for each day by bootstrapping Treasury yields. This yield data is obtained from the Federal Reserve Historical data repository that is available on their web site.

4.2 Fitting the model

We apply the jump-to-default model to the data. Fitting the implied default intensity and recovery functions is carried out using the `R` statistical package. The model converges rapidly in less than 1 second in almost all cases. The parsimony of the model implementation is readily seen in the Excel VBA code we provide in Appendix C.

For each of the 31 months in the sample, we calibrated the CDS spreads to the three parameters $\{a_0, a_1, b\}$ using a least squares fit of the model spread curve to the quintile averaged market spreads in the data. The average percentage error in fitting the CDS spread curve across all quintiles and months is 4.55%. The standard deviation around this error is 1.5%.

Actual market spreads and fitted spreads by quintile and maturity across all periods are shown in Table 2 where the average values of the parameters $\{a_0, a_1, b\}$ are reported for each quintile. Parameter a_0 has no economically determined sign, but the other two parameters do, and we find that they conform to theory. We see that a_1 is less than zero, implying an inverse relation between default probabilities and recovery rates for all quintiles. Parameter b is greater than zero as required, implying that as stock prices fall, the probability of default becomes larger. Since $\lambda = 1/S^b$, the smaller the value of b, the greater the probability of default. As can be seen in Table 2, the value of this parameter is much smaller for the higher (low credit quality) quintile.

Table 2: Actual and fitted spread term structures. The spread values represent the averages across 31 months of data for each quintile broken down by the maturity of the CDS contract into 1 to 5 years. The parameters $\{a_0.a_1, b\}$ are the three parameters for the default probability and recovery rate functions. We report the average of these values for the entire period by quintile.

Parameter	Quintile 1	Quintile 2	Quintile 3	Quintile 4	Quintile 5
a_0	2.0937	1.4679	0.9778	0.6622	7.1245
a_1	-4.1005	-5.0468	-3.2557	-1.1549	-11.9068
b	1.3421	1.3503	1.2594	1.0399	0.3519
	Market CDS Spreads in basis points				
Maturity	Quintile 1	Quintile 2	Quintile 3	Quintile 4	Quintile 5
1 yr	3.48	10.86	32.90	98.86	376.48
2 yr	4.41	14.67	42.62	110.41	332.72
3 yr	6.46	22.48	60.08	137.70	337.33
4 yr	9.67	32.15	77.77	161.45	348.91
5 yr	13.16	41.47	92.88	179.47	358.12
	Fitted CDS Spreads in basis points				
Maturity	Quintile 1	Quintile 2	Quintile 3	Quintile 4	Quintile 5
1 yr	3.20	10.12	30.12	90.35	352.71
2 yr	4.44	14.73	43.36	117.60	342.67
3 yr	6.55	22.75	61.89	143.25	347.60
4 yr	9.81	32.62	78.77	161.64	354.06
5 yr	13.05	41.13	91.39	173.80	359.22
RMSE/Avg Spread	0.0405	0.0351	0.0451	0.0625	0.0444
Stdev	0.0126	0.0082	0.0104	0.0065	0.0191

4.3 Calibrated term structures

Using the data in Table 1 on the "average" firm within each quintile, we calibrated the model. For each month and quintile, we used the calibrated values of $\{a_0, a_1, b\}$ to compute the term structures of forward probabilities of default and forward recovery rates by applying equations (14) and (15). These are displayed in Figures 1 and 2 respectively.

The term structures of forward default probabilities in Figure 1 show increasing term structures for the better quality credit qualities (quintiles 1 to 3). Quintile 4 has a humped term structure, and quintile 5 (the poorest credit quality) has a declining term structure of default probability. This conforms to known intuitions. For low quality firms, the short-run likelihood of default is high, and then declines, conditional on survival. Conversely, for high quality firms the short-run probability of default is low compared to longer maturities. The overall average risk-neutral probability of default is 5.86% (across all maturities in the term structure). We note that expected default frequencies under the statistical probability measure in prior work (see Das, Duffie, Kapadia and Saita (2007) for example) is around 1.5%, implying a ratio of risk-neutral to physical default probabilities of around 3 to 4, consistent with what is known in the literature, e.g. Berndt, Douglas, Duffie, Ferguson and Schranz (2005). We also note that the conditional forward default probabilities for Quintile 5 decline rapidly. This is expected, because conditional on survival, these currently weak firms may be viewed by the market as being more resilient in the long run.

The term structures of forward recovery rates are shown in Figure 2. Recovery term structures show declining levels as the quality of the firms declines from quintile 1 to quintile 5. Quintile 1 recovery rates are in the 90% range, whereas the worst quality quintile has recovery rates that drop to the 30% range. The average across all maturities and quintiles of the risk-neutral forward recovery rate in our data is 73.55%. The term structures are declining, i.e. the forward recovery is lower when the firm defaults later rather than sooner. There may be several reasons for this. First, firms that migrate slowly into default suffer greater dissipation of assets over time, whereas a firm that has a short-term surprise default may be able to obtain greater resale values for its assets. Second, sudden defaults of high-grade names that are associated with fraud result in loss of franchise value, though the assets of the firm are still valuable and have high resale values. Third, recovery rates of the poorest quintile firms are in the range seen in the empirical record. This is the quintile from which almost all defaults have historically occurred. In contrast, surprise defaults of high quality firms are likely to result in higher recovery rates. Finally, we note that in the Merton (1974) structural class of models, conditional recovery rates usually decline with maturity, since recovery in that model depends on the firm value at maturity; conditional on default, a firm is likely to have lost more of its asset value over a longer time frame.

In a separate exercise, we fitted the model to the average firm for each quintile within each month. We then averaged default probabilities and recovery rates across maturities and quintiles for each month to construct time series indices. The plot is shown in Figure 3. Over the period from 2000 to 2002, the credit markets worsened. This is evidenced in the increasing levels of implied probabilities of default and declining levels of recovery rates. The time series correlation of the two series is -0.56. While this relationship is obtained for the risk-neutral measure, similar, though technically unconnected results are obtained for the statistical measure – Altman, Brady, Resti and Sironi (2005) also found a strong negative

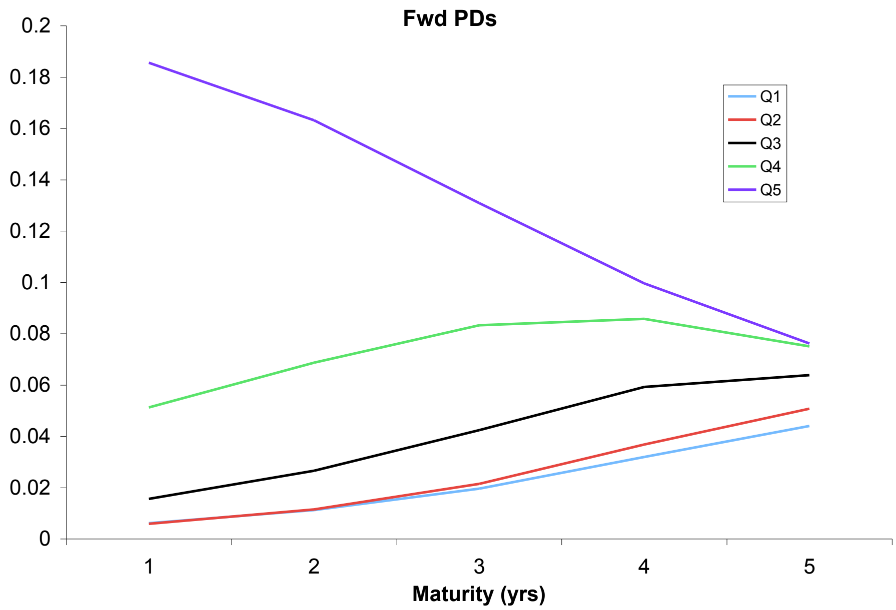


Figure 1: Term structure of Forward Probabilities of Default for each quintile using data averaged over 31 months (Jan 2000 to July 2002). All firms in the sample were divided into quintiles based on their expected default frequencies (EDFs). The average CDS spread curve in each quintile is used to fit the jump-to-default (JTD) model using the stock price and stock volatility as additional identification data. Fitting is undertaken using a two parameter function for the forward recovery rate (ϕ) and a one parameter function for the forward probability of default (λ). The overall average probability of default from all quintiles and across all months and maturities in the data is 5.86%.

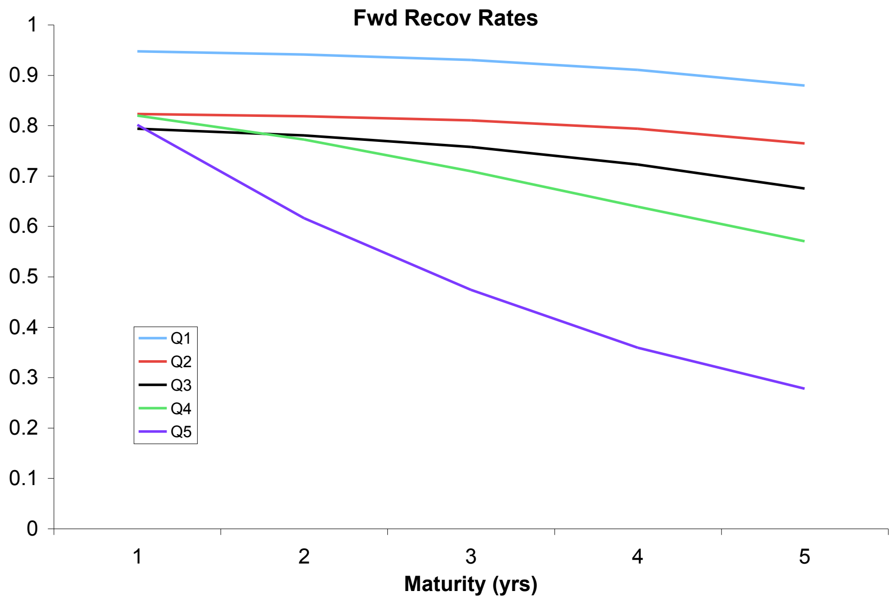


Figure 2: Term structure of Forward Probabilities of Default for each quintile using data averaged over 31 months (Jan 2000 to July 2002). All firms in the sample were divided into quintiles based on their expected default frequencies (EDFs). The average CDS spread curve in each quintile is used to fit the jump-to-default (JTD) model using the stock price and stock volatility as additional identification data. Fitting is undertaken using a two parameter function for the forward recovery rate (ϕ) and a one parameter function for the forward probability of default (λ). The overall average recovery rate from all quintiles and across all months in the data is 73.55%.

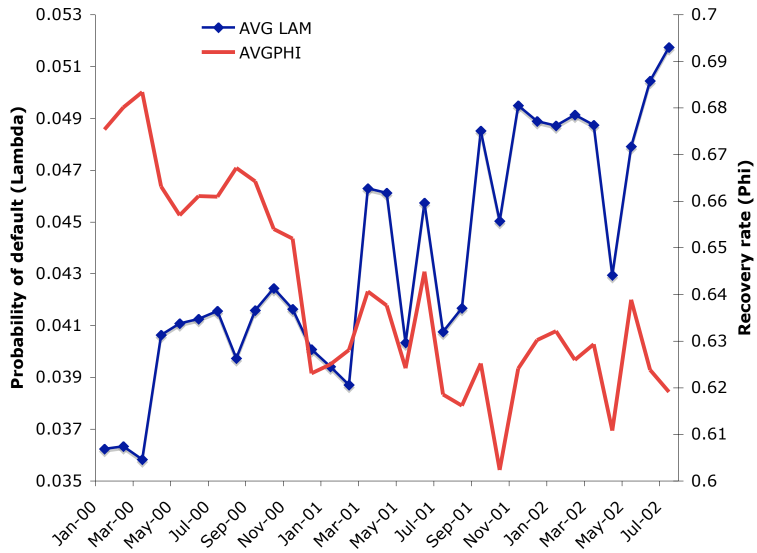


Figure 3: Average probability of default and recovery rates over the sample period. The Figure depicts the equally weighted average across all quintiles of the data. As credit risk increased in the economy from 2000 to 2002, we see that the probability of default increased whereas the implied recovery rate declined. The average is taken across all maturities of the term structure at a given point in time. The correlation between the time series of default probability and recovery rates is -0.56.

correlation using data on realized default and recovery rates. Similar levels of negative correlation are also provided in the results of Chava, Stefanescu and Turnbull (2006).

4.4 Individual Firm Calibration

The analysis of the model using quintile-aggregated data provides only a first cut assessment that the model is a reasonable one. In practice, the model is intended for application to individual names. Therefore in this sub-section, we explore three liquidly traded issuers over the period January 2000 to July 2002. The issues we examine are (a) parameter stability in the calibration of the model, (b) evolution of the term structures of forward default probabilities and recovery rates over this time period, and (c) the stability of the model calibration to different forms of the recovery function.

We chose three issuers for this exercise, of low, medium and high credit risk. The three firms are Sunoco (SUN), General Motors (GM) and Amazon (AMZN) respectively. For each firm month, we fitted the model to only 5 points on the average CDS spread curve for the month, i.e. to the 1,2,3,4,5 year maturities. The calibration exercise also uses the stock price, stock volatility, and the forward interest rate curve (averages for the month). The statistical package `R` was used for this exercise. We also replicated the results in Excel to make sure that both implementations provided the same results. The Excel VBA program is presented in the Appendix.

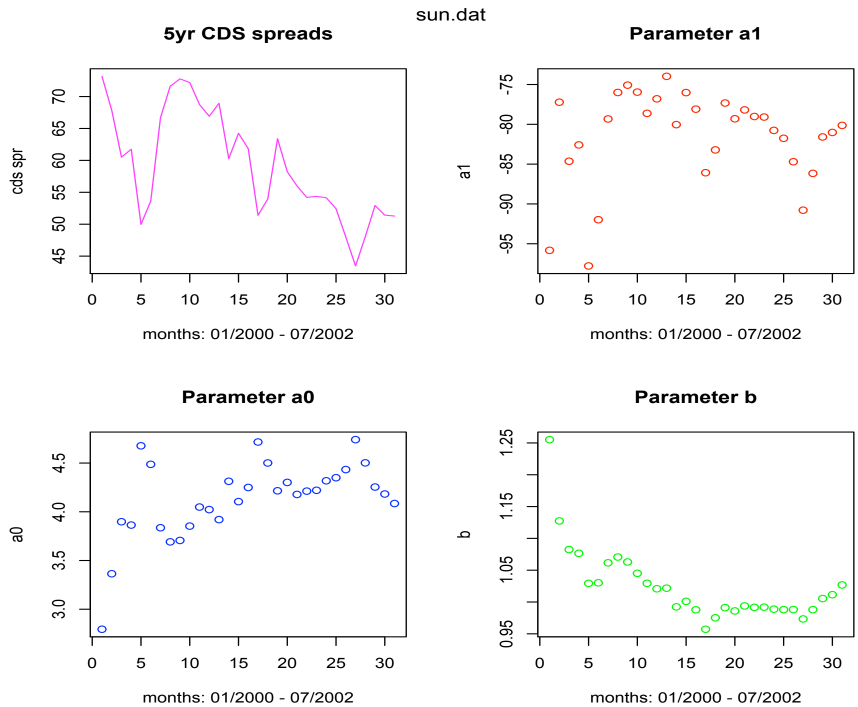


Figure 4: Calibration Time Series for Sunoco. This graph shows the time series of CDS spreads and calibrated parameters (a_0, a_1, b) for the period Jan 2000 to July 2002.

4.4.1 Time series

We begin by examining the results for the low risk firm Sunoco. Figure 4 shows the history of the 5 yr CDS spread. We see that spreads for Sunoco declined over time, even though the overall change for the 5 year maturity is just 25 basis points, which is very small. The figure also shows that the three parameters (a_0, a_1, b) of the λ and ϕ functions are relatively stable over time. The term structures of forward default probabilities and recovery rates are shown in Figure 5. The term structures have the same shapes as those seen in the case of the quintile analysis of the previous section.

The medium risk firm that we analyzed is GM, results for which are shown in Figures 6 and 7. Whereas with Sunoco, the 5 year CDS spread ranged from 45 to 75 basis points, in the case of GM, the range is from 150 to 450 basis points. We note that the term structure of CDS spreads is inverted, so that the short term spreads are higher than long term spreads. The shape of the spread curve is indicative of the market's concern with short-run default. Correspondingly, we see that the parameter b is much lower than that experienced with Sunoco. This is also evidence in a downward sloping term structure of default probability. In the case of GM, spreads increased over the time period, whereas with Sunoco, the opposite occurred. Also, the 5 year recovery rate is much lower for GM (in the range of 20-30%) versus Sunoco (in the range of 70-80%).

As an example of a firm with very high risk, we look at Amazon. Result for AMZN are shown in Figures 8 and 9. In this case, spreads ranged from 450 to 1100 basis points, and

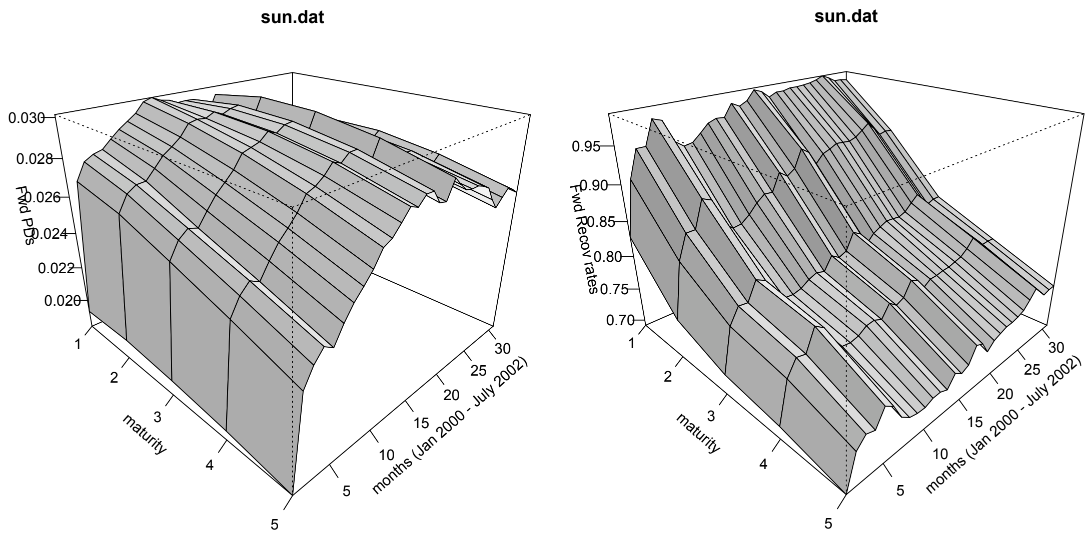


Figure 5: Calibration Time Series for Sunoco. This graph shows the time series of implied term structures of forward default probabilities and recovery rates for the period Jan 2000 to July 2002.

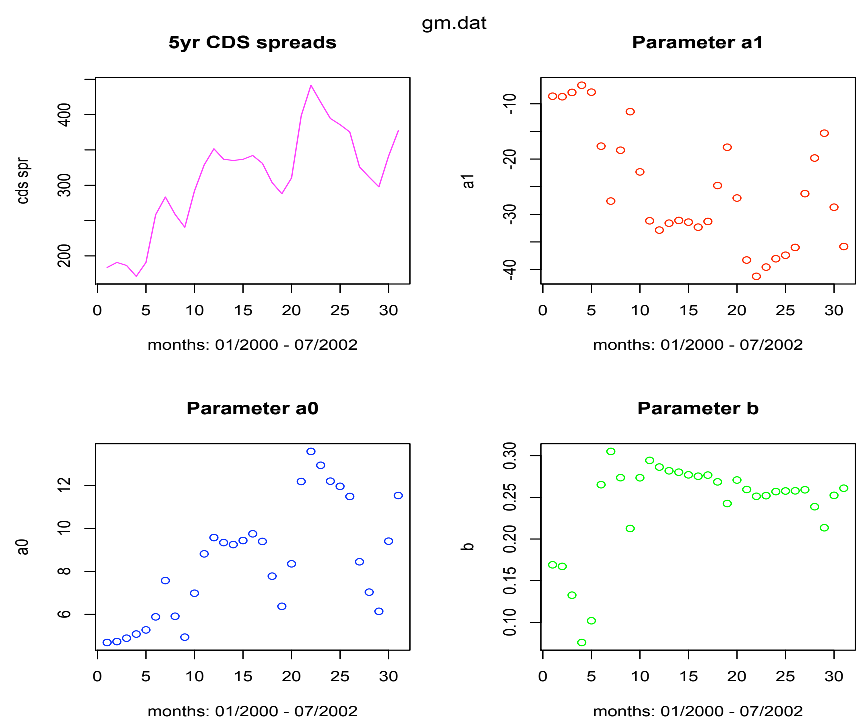


Figure 6: Calibration Time Series for General Motors. This graph shows the time series of CDS spreads and calibrated parameters (a_0, a_1, b) for the period Jan 2000 to July 2002.

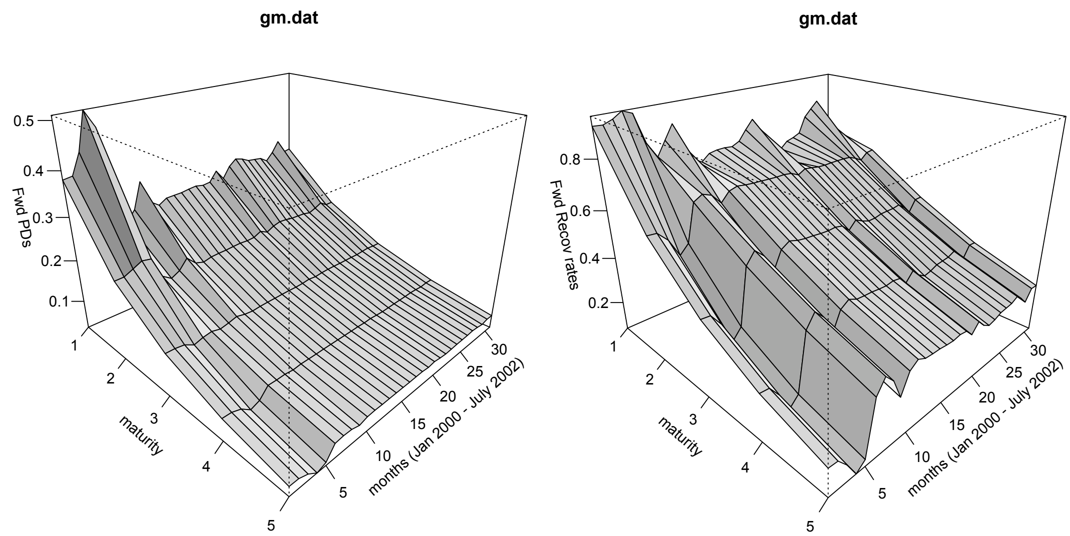


Figure 7: Calibration Time Series for General Motors. This graph shows the time series of implied term structures of forward default probabilities and recovery rates for the period Jan 2000 to July 2002.

increased rapidly over the time period as market conditions worsened. The probabilities of default increased and then declined somewhat. However, in a clear indication of worsening conditions, the recovery rates declined rapidly over the time period, suggestive of a low implied resale value of Amazon's assets. We see that the parameters vary over time but in a smooth manner. In the case of all three firms, parameter evolution appears to be quite smooth (with a range). Hence, there is some "stickiness" to the parameters, suggesting that credit model structure, even for individual firms, does not change in a drastic manner. The indication we obtain from these single-issuer analyses is that our dynamic modeling of the term structures of default probabilities and recovery rates appears to work well with forward interest rates, stock prices and stock volatilities as driving state variables.

4.4.2 Alternative Recovery Function Specifications

The recovery model in the paper is essentially a probit function specification. To recall, the recovery rate is modeled as $\phi = N(a_0 + a_1\lambda)$, where $N(\cdot)$ is the normal distribution function. We note that this specification ensures that the recovery rate remains bounded in the $(0, 1)$ range.

We now examine two other specifications. These are:

1. Logit. The functional form is

$$\phi = \frac{1}{1 + \exp(a_0 + a_1\lambda)}$$

2. Arctan. The functional form here is as follows:

$$\phi = \frac{1}{2}\left[\arctan(a_0 + a_1\lambda)\,\frac{2}{\pi} + 1\right]$$

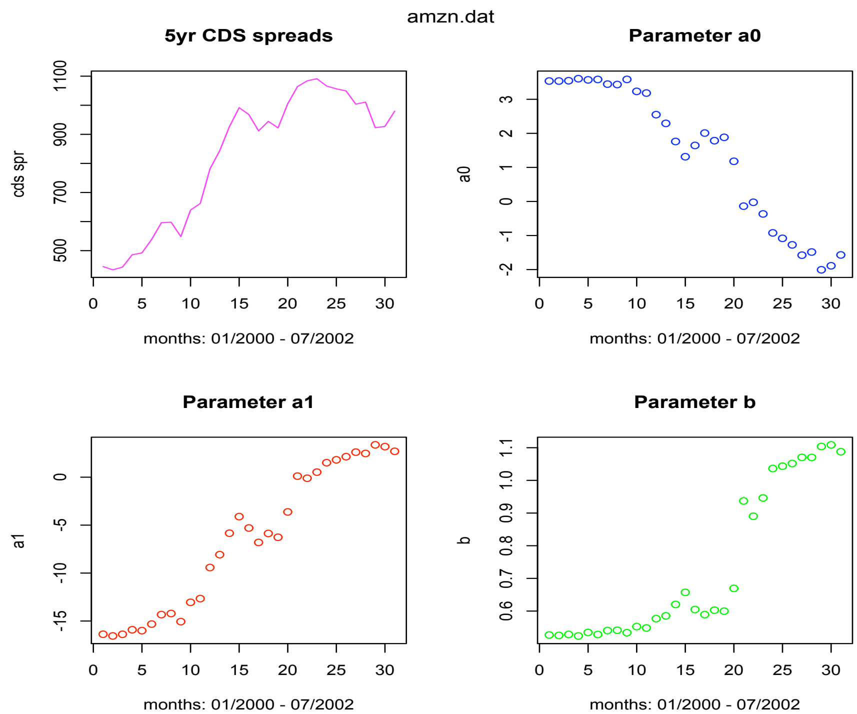


Figure 8: Calibration Time Series for Amazon. This graph shows the time series of CDS spreads and calibrated parameters (a_0, a_1, b) for the period Jan 2000 to July 2002.

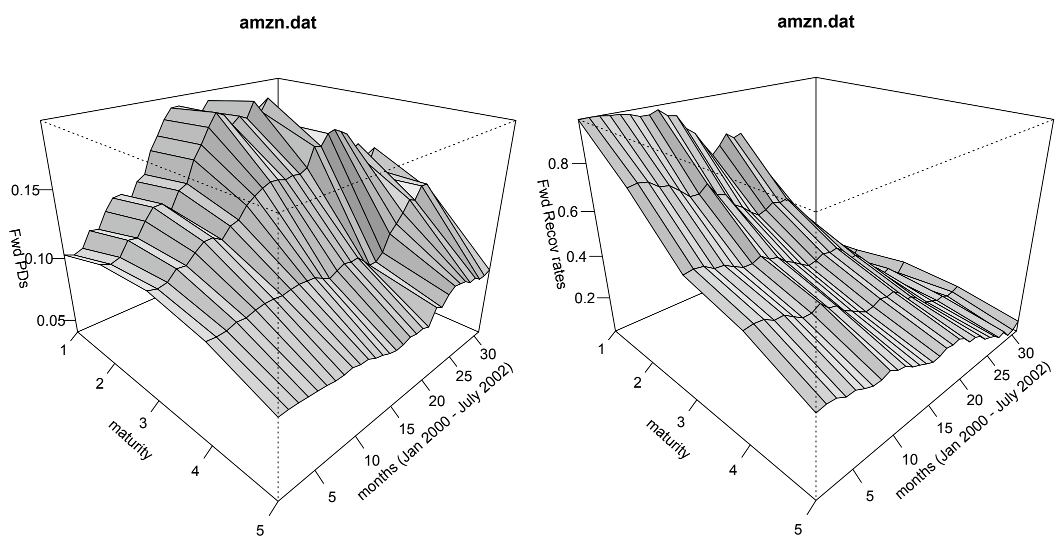


Figure 9: Calibration Time Series for Amazon. This graph shows the time series of implied term structures of forward default probabilities and recovery rates for the period Jan 2000 to July 2002.

We note that under both these specifications as well, the recovery function produces values that remain in the $(0, 1)$ interval.

We re-calibrated the model under these specifications and compared the extracted term structures of default and recovery for all three of our sample firms. The results are presented in Table 3. We used the calibration of the model for the month of September 2001 as an illustration. We note the following features of these results. First, the fitting error (RMSE) is very small, ranging from less than 1% of average CDS spreads (in the case of AMZN) to 8.2% (in the arctan model for SUN). The model appears to fit better for poor credit quality firms than for good ones (because spreads are larger for poor firms, the percentage error tends to be lower). Second, the calibrated term structures of default probabilities and recovery rates are similar across the three recovery function specifications. They are the closest for AMZN and least similar for SUN, although not that different. Therefore, model specification does not appear to be much of an issue. Of course, the fitted parameters are different, but the fact that the models deliver similar results suggests that the framework is general and may be implemented in different ways. The overall conclusion is that the model is insensitive to the parametric form of the recovery function, and tends to fit the data better for high risk firms than low risk ones.

Table 3: Calibration of the model for alternative recovery function specifications. For each of the three illustrative firms - SUN, GM, AMZN - we calibrated the model for September 2001 using three specifications for the recovery function: (a) Probit: $\phi = N(a_0 + a_1\lambda)$, (b) Logit: $\phi = \frac{1}{1+\exp(a_0+a_1\lambda)}$, and (c) Arctan: $\phi = \frac{1}{2}\left[\arctan(a_0 + a_1\lambda)\ \frac{2}{\pi} + 1\right]$. The results are presented in the three panels below, one for each firm. The market spreads are shown, along with the fitted spreads, so as to give an idea of the quality of fit. For each recovery function, we report the fitted spread and the calibrated term structure of forward default probabilities and recovery rates. The percentage RMSE is also reported (it is the root mean-squared error divided by the average CDS spread across maturities for the month).

SUN		Probit Model: %RMSE=4.8			Logit Model: %RMSE=0.2			Arctan Model: %RMSE=8.17		
Maturity (yrs)	Mkt CDS spr	Fitted Spr	Fwd PD	Fwd Recov	Fitted Spr	Fwd PD	Fwd Recov	Fitted Spr	Fwd PD	Fwd Recov
1	6.74	6.23	0.0278	0.9776	6.76	0.1209	0.9944	9.97	0.0314	0.9683
2	15.40	14.99	0.0285	0.9100	15.40	0.1016	0.8569	12.14	0.0319	0.9305
3	28.98	31.08	0.0287	0.8209	28.93	0.0853	0.7194	31.31	0.0318	0.8808
4	43.08	43.89	0.0285	0.7844	43.17	0.0718	0.5958	43.17	0.0312	0.7934
5	55.99	53.78	0.0278	0.7438	55.95	0.0606	0.4983	54.03	0.0302	0.7305
Parameters	(a_0, a_1, b):	4.178	-78.189	0.994	6.521	-96.774	0.570	21.344	-360.64	0.959
GM		Probit Model: %RMSE=3.4			Logit Model: %RMSE=3.8			Arctan Model: %RMSE=5.1		
Maturity (yrs)	Mkt CDS spr	Fitted Spr	Fwd PD	Fwd Recov	Fitted Spr	Fwd PD	Fwd Recov	Fitted Spr	Fwd PD	Fwd Recov
1	1038.12	1045.11	0.2816	0.6288	1045.67	0.2753	0.6202	1049.91	0.2465	0.5741
2	735.34	697.81	0.1869	0.6590	691.93	0.1845	0.6673	674.56	0.1723	0.7040
3	532.72	536.99	0.1287	0.5332	538.19	0.1280	0.5353	545.66	0.1239	0.5481
4	447.25	463.44	0.0911	0.4015	465.54	0.0913	0.4080	468.90	0.0912	0.4486
5	398.49	421.44	0.0661	0.3102	422.58	0.0666	0.3192	424.08	0.0685	0.3600
Parameters	(a_0, a_1, b):	12.724	-44.026	0.248	-23.330	82.950	0.254	95.694	-387.23	0.282
AMZN		Probit Model: %RMSE=0.1			Logit Model: %RMSE=0.1			Arctan Model: %RMSE=0.1		
Maturity (yrs)	Mkt CDS spr	Fitted Spr	Fwd PD	Fwd Recov	Fitted Spr	Fwd PD	Fwd Recov	Fitted Spr	Fwd PD	Fwd Recov
1	749.92	749.86	0.1381	0.4571	749.87	0.1382	0.4576	749.86	0.1381	0.4572
2	942.44	942.52	0.1876	0.3957	942.50	0.1876	0.3959	942.51	0.1876	0.3957
3	1048.50	1048.66	0.1713	0.3102	1048.69	0.1713	0.3103	1048.67	0.1713	0.3102
4	1054.55	1054.48	0.1014	0.2307	1054.49	0.1014	0.2308	1054.48	0.1014	0.2307
5	1071.10	1070.98	0.0910	0.1845	1070.99	0.0909	0.1845	1070.98	0.0910	0.1845
Parameters	(a_0, a_1, b):	-0.116	0.063	0.931	0.183	-0.094	0.930	-0.146	0.079	0.931

4.4.3 Calibration sensitivity to changes in inputs

We now examine whether the implied framework is overly sensitive to the inputs. Are the extracted term structures of default and recovery so sensitive to the inputs that small changes in the inputs result in dramatic changes in the resultant term structures?

To assess this question, we used the calibration setting for our three sample firms in the month of September 2001, as before. We changed each input to the model by increasing them individually by 10% and then examined the resultant changes in the calibrated parameters and the term structures. The results are portrayed in Table 4. Across all the experiments conducted, the sensitivity of the calibrated parameters and the implied term structures does not display any sudden changes. In fact, the change in implied values does not exceed 10%. The evidence clearly suggests that the model framework is very stable in the calibration process.

4.4.4 Calibrating multiple spread curves

For a single firm, there may be multiple debt issues with different features. For example, the debt tranches may differ in their seniority. CDS contracts written on different reference instruments of the same firm will have the same default probability term structure, but different recovery rate term structures.

It is possible to calibrate our model to multiple debt tranches of the same firm. Without loss of generality, assume that there are two debt tranches in the firm. Calibration is undertaken by fitting both term structures of CDS spreads to five parameters: $\{a_{01}, a_{11}, a_{02}, a_{12}, b\}$, where a_{lj} is the parameter $a_l, l = \{0, 1\}$ for the j-th term structure of credit spreads. Of course, since the term structure of default probabilities is the same across the debt tranches, only a common parameter b is necessary.

To illustrate, we take the example of Amazon for September 2001. Using the same data as in the previous subsection, we created two term structures of CDS spreads by setting one term structure to be higher than that of the original data and setting the second one to be lower. One may imagine the latter term structure to relate to debt with higher seniority than the former term structure. Each term structure is annual and hence we calibrate ten observations to five parameters by minimizing the root mean-squared error (RMSE) across both term structures. Convergence is rapid and results in a single term structure of default probabilities and two term structures of recovery rates. The inputs and outputs of this calibration exercise are shown in Figure 10. As expected, the higher quality tranche results in recovery rates that are higher than that of the lower quality tranche.

Analogously, it may be of interest to calibrate the spread curves of multiple issuers jointly such that they all have the same default probability term structure, but different recovery term structures. The approach would be the same – the parameter b is will be the same across issuers, but separate parameters will be used for fitting the individual recovery term structures.

Table 4: Sensitivity of implied term structures of forward default probabilities (FwdPD) and recovery rates (FwdRecov) to changes in input variables. Keeping forward rates and CDS spreads fixed (initial data), the base case inputs and calibrated term structures of default probabilities and recovery rates are presented first. In the lower half of the panel for each firm, we show the input stock price and volatility, as well as the three parameters (a_0, a_1, b) as calibrated for best fit. The fitting error (RMSE as a percentage of average CDS spread) is also reported. Calibration is then undertaken after shifting the stock price up by 10%. Results are reported under the heading "Stock Shift". Next, the "Volatility Shift" case shows the calibration results when volatility is shifted up by 10%. Finally, the "CDS Spr Shift" columns show the results when the entire spread curve experiences a parallel upward shift of 10%.

SUN	Initial Data		Base Case		Stock Shift		Volatility Shift		CDS Spr Shift	
T	FwdRt	Mkt Spr	FwdPD	FwdRecov	FwdPD	FwdRecov	FwdPD	FwdRecov	FwdPD	FwdRecov
1	0.0282	6.74	0.0278	0.9776	0.0288	0.9787	0.0209	0.9662	0.0297	0.9782
2	0.0341	15.40	0.0285	0.9100	0.0293	0.9101	0.0225	0.9049	0.0303	0.9041
3	0.0412	28.98	0.0287	0.8209	0.0295	0.8204	0.0238	0.8177	0.0304	0.8095
4	0.0478	43.08	0.0285	0.7844	0.0291	0.7837	0.0246	0.7739	0.0300	0.7776
5	0.0545	55.99	0.0278	0.7438	0.0283	0.7424	0.0250	0.7372	0.0291	0.7322
			Inputs	Params	Inputs	Params	Inputs	Params	Inputs	Params
Stk Price	a_0		36.293	4.179	39.922	4.339	36.293	3.328	36.293	4.394
Stk Vol	a_1		0.338	-78.190	0.338	-80.220	0.372	-71.636	0.338	-79.919
	b			0.994		0.958		1.073		0.974
	RMSE%			4.808		5.062		2.304		6.071

GM	Initial Data		Base Case		Stock Shift		Volatility Shift		CDS Spr Shift	
T	FwdRt	Mkt Spr	FwdPD	FwdRecov	FwdPD	FwdRecov	FwdPD	FwdRecov	FwdPD	FwdRecov
1	0.0282	1038.12	0.2816	0.6288	0.2819	0.6292	0.3023	0.6542	0.2847	0.5961
2	0.0341	735.34	0.1869	0.6590	0.1873	0.6588	0.1952	0.6527	0.1881	0.6501
3	0.0412	532.72	0.1287	0.5332	0.1290	0.5329	0.1310	0.5041	0.1290	0.5317
4	0.0478	447.25	0.0911	0.4015	0.0914	0.4006	0.0907	0.3697	0.0911	0.3974
5	0.0545	398.49	0.0661	0.3102	0.0663	0.3089	0.0644	0.2798	0.0659	0.3052
			Inputs	Params	Inputs	Params	Inputs	Params	Inputs	Params
Stk price	a_0		87.256	12.726	95.982	12.965	87.256	13.602	87.256	13.026
Stk Vol	a_1		0.324	-44.034	0.324	-44.828	0.356	-43.678	0.324	-44.905
	b			0.248		0.242		0.229		0.245
	RMSE%			3.362		3.345		4.094		3.251

AMZN	Initial Data		Base Case		Stock Shift		Volatility Shift		CDS Spr Shift	
T	FwdRt	Mkt Spr	FwdPD	FwdRecov	FwdPD	FwdRecov	FwdPD	FwdRecov	FwdPD	FwdRecov
1	0.0282	749.92	0.1381	0.4571	0.1196	0.3722	0.1750	0.5729	0.1382	0.4030
2	0.0341	942.44	0.1876	0.3957	0.1737	0.3460	0.2180	0.4608	0.1876	0.3491
3	0.0412	1048.50	0.1713	0.3102	0.1734	0.2872	0.1766	0.3363	0.1713	0.2738
4	0.0478	1054.55	0.1014	0.2307	0.1048	0.2096	0.1013	0.2423	0.1014	0.2035
5	0.0545	1071.10	0.0910	0.1845	0.0938	0.1711	0.0856	0.1840	0.0910	0.1628
			Inputs	Params	Inputs	Params	Inputs	Params	Inputs	Params
Stk price	a_0		7.756	-0.116	8.531	-0.409	7.756	0.255	7.756	-0.255
Stk Vol	a_1		0.972	0.063	0.972	0.690	1.069	-0.406	0.972	0.069
	b			0.931		0.961		0.805		0.930
	RMSE%			0.011		0.286		0.308		0.011

5 Discussion

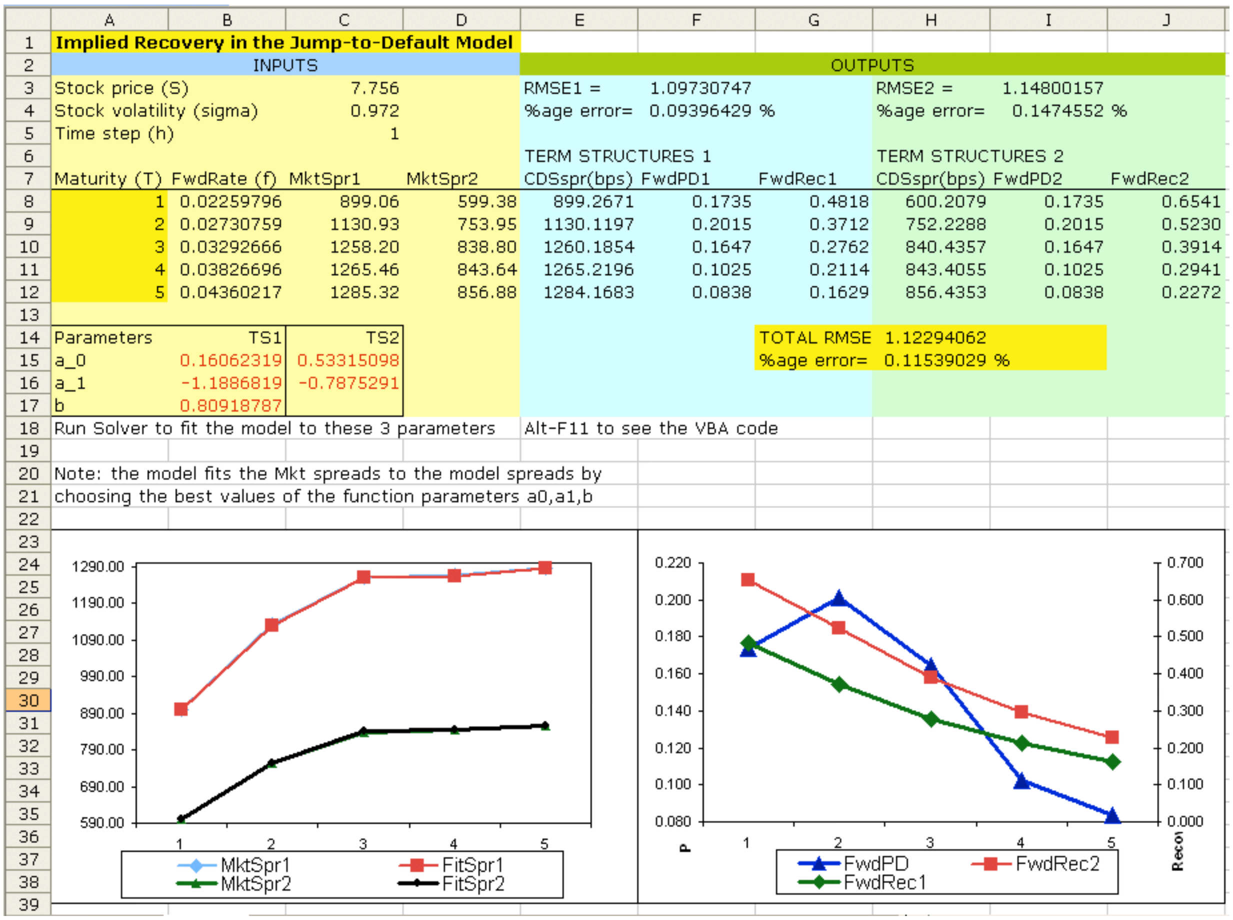
There is no extant model for determining forward looking recovery rates, even though recovery rates are required in the pricing of almost all credit derivative products. This market deficiency stems from an identification problem emanating from the mathematical structure of credit products. Market participants have usually imposed recovery rates of 40% or 50% (for U.S. corporates) in an ad-hoc manner in their pricing models.

The paper remedies this shortcoming of existing models. We develop a flexible jump-to-default model that uses additional data, the stock price S and stock volatility σ in conjunction with credit spreads to identify not just the implied values of default probability λ and recovery rate ϕ, but instead, the parameterized functional forms of these two inputs. The model in this paper is parsimonious and requires the calibration of only three parameters $\{a_0, a_1, b\}$.

We illustrate the application of the model using average firm data on CDS spread curves for 5 quintiles over 31 months in the period from January 2000 to July 2002. We calibrate the model for each quintile and month, and then use the calibrated parameters to identify the risk-neutral term structures of forward default probabilities and recovery rates.

We also examined the behavior of the model on low, medium and high risk firms, with upward and downward sloping credit spread term structures. Many useful results are obtained. First, the model calibrate very well to individual firms, with very low mean-squared errors. Second, fitting the model month by month, we find that there is stability in the estimated parameters. Hence, the parameters do change in a smooth manner over time, and the dynamics of the extracted term structures of default probability and recovery are driven by changes in the state variables (equity prices and volatilities). Third, we assessed the model for different recovery rate specifications and found that the extracted term structures are robust to changes in model specification. Fourth, we demonstrate that the calibration of the model is not sensitive to small changes in the inputs. Hence, the mathematical framework of the model is stable. Fifth, we extended the model to fitting multiple debt tranches of the same issuer, such that the term structure of default probabilities remains the same across tranches, yet we obtain multiple term structures of recovery rates, one for each debt tranche. The same approach also works to extend the model to fitting multiple issuers (within the same rating class for example) simultaneously to better make use of the information across issuers.

A natural question that arises in this class of models is whether the model depends critically on integration of the equity and credit markets (see Kapadia and Pu (2008) for evidence of weak integration). We note that integration of markets is a very strong condition that is sufficient but not necessary in our framework. The model uses the stock price and volatility as state variables that drive the dynamics of default and recovery. As long as there is an established empirical link between the dynamics of equity and credit risk, the model is on a good footing. Indeed, the same is assumed in all structural models of default risk, and we have shown this linkage theoretically in the paper. There is a growing body of empirical evidence that CDS spreads are well-related to equity dynamics, as in the work of Berndt, Douglas, Duffie, Ferguson and Schranz (2005), Duffie, Saita and Wang (2005), and Das and Hanouna (2007). Another body of evidence shows that that default and recovery prediction models are also well grounded on distance-to-default, a measure of credit risk extracted from equity prices and volatilities (see Finger, Finkelstein, Lardy, Pan, Ta and Tierney (2002);



	A	B	C	D	E	F	G	H	I	J
1	**Implied Recovery in the Jump-to-Default Model**									
2	INPUTS				OUTPUTS					
3	Stock price (S)		7.756		RMSE1 =	1.09730747		RMSE2 =	1.14800157	
4	Stock volatility (sigma)		0.972		%age error=	0.09396429 %		%age error=	0.1474552 %	
5	Time step (h)		1							
6					TERM STRUCTURES 1			TERM STRUCTURES 2		
7	Maturity (T)	FwdRate (f)	MktSpr1	MktSpr2	CDSspr(bps)	FwdPD1	FwdRec1	CDSspr(bps)	FwdPD2	FwdRec2
8	1	0.02259796	899.06	599.38	899.2671	0.1735	0.4818	600.2079	0.1735	0.6541
9	2	0.02730759	1130.93	753.95	1130.1197	0.2015	0.3712	752.2288	0.2015	0.5230
10	3	0.03292666	1258.20	838.80	1260.1854	0.1647	0.2762	840.4357	0.1647	0.3914
11	4	0.03826696	1265.46	843.64	1265.2196	0.1025	0.2114	843.4055	0.1025	0.2941
12	5	0.04360217	1285.32	856.88	1284.1683	0.0838	0.1629	856.4353	0.0838	0.2272
13										
14	Parameters	TS1	TS2				TOTAL RMSE	1.12294062		
15	a_0	0.16062319	0.53315098				%age error=	0.11539029 %		
16	a_1	-1.1886819	-0.7875291							
17	b	0.80918787								
18	Run Solver to fit the model to these 3 parameters				Alt-F11 to see the VBA code					
19										
20	Note: the model fits the Mkt spreads to the model spreads by									
21	choosing the best values of the function parameters a0,a1,b									

Figure 10: Simultaneous calibration of multiple CDS spread curves. CDS contracts written on different reference instruments of the same firm will have the same default probability term structure, but different recovery rate term structures. We take the example of Amazon for September 2001. We created two term structures of CDS spreads by setting one term structure to be higher than that of the original data and we set the second one to be lower. Each term structure is annual and we calibrate ten observations to five parameters. Calibration is undertaken by fitting both term structures of CDS spreads to five parameters: $\{a_{01}, a_{11}, a_{02}, a_{12}, b\}$, where a_{lj} is the parameter a_l for the j-th term structure. Since the term structure of default probabilities is the same across the debt tranches, only a common parameter b is necessary. The screen shot shows the fit of the model in Excel. The input spread curves are denoted "MktSpr1" and "MktSpr2". The fitted spread curves are "CDSspr". The forward default probability and recovery rate curves are denoted "FwdPD" and "FwdRec" respectively.

Gupton and Stein (2005); Jarrow (2001); Sobehart, Stein, Mikityanskaya and Li (2000); and Das, Hanouna and Sarin (2006)). Nevertheless, the model is agnostic as to the state variable that may be used to drive credit spread dynamics. Instead of equity, any other variable that relates to credit risk may be used such as macroeconomic variables and ratings, as long as there are sufficient inputs that allow the variable's dynamics to be represented on the model tree.

The model is flexible in the manner in which it may be calibrated. Instead of equity, options may be used to calibrate the model. We note that the volatility input to the model currently captures the information from options. However, one deficiency of the model is that it does not use the information in the option smile, which is partially related to credit risk (it is easy to show that in a jump-to-default model of equity, a negative skew on tree from the default jump results in an implied volatility skew). To accommodate the volatility skew in our model we would need to change the pricing tree from the current one, i.e. the Cox, Ross and Rubinstein (1979) tree, to the local volatility tree model of Derman and Kani (1994). This would enhance the complexity of the model appreciably and we leave it for future research. However, this is an important extension, because it allows the entire volatility surface to be used in the calibration exercise, raising the information content of the model.

Given that the model is easy to understand and calibrate, it may be used to further the development of credit derivatives indexed to recovery rates, such as recovery swaps and digital default swaps. It will also provide useful guidance to regulators in their recovery specifications for the implementation of Basel II.

A Appendix: Constant Recovery Rates

In practice, a common assumption is to fix the recovery rate to a known constant. If we impose the condition that $\phi_j \equiv \phi$, $\forall j$, it eliminates N parameters, leaving only the N default intensities, λ_j. Now, we have N equations with as many parameters, which may be identified in a recursive manner using bootstrapping. To establish ideas, we detail some of the bootstrapping procedure.

Starting with the one-period ($N = 1$) default swap, with a premium C_1 per annum, we equate payments on the swap as follows:

$$\begin{aligned} A_1 &= B_1 \\ C_1\ h\ Q(T_0)D(T_1) &= \left(1 - e^{-\lambda_1 h}\right) D(T_1)(1-\phi) \\ C_1\ h &= \left(1 - e^{-\lambda_1 h}\right)(1-\phi) \end{aligned}$$

This results in an identification of λ_1, which is:

$$\lambda_1 = -\frac{1}{h} \ln \left[\frac{1 - \phi - C_1\ h}{1 - \phi} \right], \tag{20}$$

which also provides the survival function for the first period, i.e. $Q(T_1) = \exp(-\lambda_1\ h)$.

We now use the 2-period default swap to extract the intensity for the second period. The premium for this swap is denoted as C_2. We set $A_2 = B_2$ and obtain the following equation which may be solved for λ_2.

$$C_2\ h\ \sum_{j=1}^{2} Q(T_{j-1}) D(T_j) = \sum_{j=1}^{2} Q(T_{j-1}) \left(1 - e^{-\lambda_j h}\right) D(T_j)(1-\phi) \tag{21}$$

Expanding this equation, we have

$$\begin{aligned} & C_2\ h\ \{Q(T_0)D(T_1) + Q(T_1)D(T_2)\} \\ = \ & Q(T_0)\left(1 - e^{-\lambda_1 h}\right) D(T_1)(1-\phi) \\ & + Q(T_1)\left(1 - e^{-\lambda_2 h}\right) D(T_2)(1-\phi) \end{aligned}$$

Re-arranging this equation delivers the value of λ_2, i.e.

$$\lambda_2 = -\frac{1}{h} \ln[L_1/L_2] \tag{22}$$

$$\begin{aligned} L_1 \equiv \ & Q(T_0)\left(1 - e^{-\lambda_1 h}\right) D(T_1)(1-\phi) \\ & + Q(T_1)D(T_2)(1-\phi) - C_2\ h\ [D(T_1) + Q(T_1)D(T_2)] \end{aligned} \tag{23}$$

$$L_2 \equiv Q(T_1)D(T_2)[1-\phi] \tag{24}$$

and we also note that $Q(T_0) = 1$.

In general, we may now write down the expression for the kth default intensity:

$$\lambda_k = \frac{\ln\left[\frac{Q(T_{k-1})D(T_k)(1-\phi) + \sum_{j=1}^{k-1} G_j - C_k\ h\ \sum_{j=1}^{k} H_j}{Q(T_{k-1})D(T_k)(1-\phi)}\right]}{-h} \tag{25}$$

$$G_j \equiv Q(T_{j-1})\left(1 - e^{-\lambda_j h}\right) D(T_j)(1-\phi) \tag{26}$$

$$H_j \equiv Q(T_{j-1})D(T_j) \tag{27}$$

Thus, we begin with λ_1 and through a process of bootstrapping, we arrive at all $\lambda_j, j = 1, 2, \ldots, N$.

B Time-Dependent Recovery Rates

The analysis in the previous appendix is easily extended to the case where recovery rates are different in each period, i.e. we are given a vector of ϕ_js. The bootstrapping procedure remains the same, and the general form of the intensity extraction equation becomes (for all k)

$$\lambda_k = \frac{-1}{h} \ln\left[\frac{Q(T_{k-1})D(T_k)(1-\phi_j) + \sum_{j=1}^{k-1} G_j - C_k\ h\ \sum_{j=1}^{k} H_j}{Q(T_{k-1})D(T_k)(1-\phi_j)}\right] \tag{28}$$

$$G_j \equiv Q(T_{j-1})\left(1 - e^{-\lambda_j h}\right) D(T_j)(1-\phi_j) \tag{29}$$

$$H_j \equiv Q(T_{j-1})D(T_j) \tag{30}$$

This is the same as equation (25) where the constant ϕ is replaced by maturity-specific ϕ_js.

C Excel VBA Code

In this appendix, we provide a sample program that we developed for spreadsheet illustration of the model. This code is written Visual Basic for Applications, a standard add-in for Excel spreadsheets. The code is for illustration only, and actual implementations may be undertaken using code that is much more optimized and works for much smaller time steps. However, this code runs very well in Excel, and may be used to calibrate the model. The solver function in Excel is able to calibrate the three model parameters $\{a_0.a_1, b\}$ to observed spreads in a few seconds, when supplied with the following inputs (as may be noticed in the function call below): stock price (S), volatility (σ), maturity of the longest CDS spread T, forward curve of risk free rates (f), and time step (h). The output of the model comprises three term structures, that of the spreads, default probabilities and recovery rates.

```
'FUNCTION TO COMPUTE CDS SPREADS
'The model only returns (spr,lam,phi) for a specific maturity t
Function jtd(s0, v, t, f, h, a0, a1, b)

'Set up basic values
n = t / h
dt = h
u = Exp(v * Sqr(dt))
d = Exp(-v * Sqr(dt))

'Set up the stock price, pd, phi, prob trees
ReDim s(n + 1, n + 1)
ReDim xi(n, n)
ReDim lam(n, n)
ReDim r(n)
ReDim q(n, n)
ReDim phi(n, n)

'Root node values
s(1, 1) = s0
xi(1, 1) = 1 / s0 ^ b
lam(1, 1) = 1 - Exp(-xi(1, 1) * h)
phi(1, 1) = phi_fn(a0, a1, s(1, 1), lam(1, 1))
r(1) = Exp(f(1) * h)
q(1, 1) = (r(1) / (1 - lam(1, 1)) - d) / (u - d)

'Generate trees
If (n > 1) Then
    For i = 2 To n
        r(i) = Exp(f(i) * h)
        s(i, 1) = s(i - 1, 1) * u
```

```
        xi(i, 1) = 1 / s(i, 1) ^ b
        lam(i, 1) = 1 - Exp(-xi(i, 1) * h)
        'Check here that lambda is within range,
        'because it is not guaranteed for large h
        'In production systems, h is always small
        If (lam(i, 1) > 0.99) Then
            lam(i, 1) = 0.99    'The correct fix here is to make time step h small
        End If
        phi(i, 1) = phi_fn(a0, a1, s(i, 1), lam(i, 1))
        q(i, 1) = (r(i) / (1 - lam(i, 1)) - d) / (u - d)

        For j = 2 To i
            s(i, j) = s(i - 1, j - 1) * d
            xi(i, j) = 1 / s(i, j) ^ b
            lam(i, j) = 1 - Exp(-xi(i, j) * h)
            If (lam(i, j) > 0.99) Then
                lam(i, j) = 0.99
            End If
            phi(i, j) = phi_fn(a0, a1, s(i, j), lam(i, j))
            q(i, j) = (r(i) / (1 - lam(i, j)) - d) / (u - d)
        Next j
    Next i
End If

'Backward recursion for cds spread
ReDim AA(n + 1, n + 1) 'PV of default payments at end of period
ReDim BB(n + 1, n + 1)  'PV of $1 premium payments at end of period
For j = 1 To n
        AA(n + 1, j) = 0
        BB(n + 1, j) = 0
Next j
For i = n To 1 Step -1
    For j = 1 To i
        AA(i, j) = (1 - lam(i, j)) * (q(i, j) * AA(i + 1, j)
                      + (1 - q(i, j)) * AA(i + 1, j + 1)) / r(i)
                      + lam(i, j) * (1 - phi(i, j)) / r(i)
        BB(i, j) = (1 - lam(i, j)) * (q(i, j) * BB(i + 1, j)
                      + (1 - q(i, j)) * BB(i + 1, j + 1)) / r(i) + 1 / r(i)
    Next j
Next i

'Backward recursion for forward Lambda and Phi
ReDim LL(n + 1)   'Forward Lambda
ReDim PP(n + 1)   'Forward Lam*(1-Phi)
For j = 1 To n
```

```
        LL(j) = lam(n, j)
        PP(j) = phi(n, j)
Next j
For i = n - 1 To 1 Step -1
    For j = 1 To i
        LL(j) = (1 - lam(i, j)) * (q(i, j) * LL(j) + (1 - q(i, j)) * LL(j + 1))
        PP(j) = (1 - lam(i, j)) * (q(i, j) * PP(j) + (1 - q(i, j)) * PP(j + 1))
    Next j
Next i


'Return the spread value
spr = AA(1, 1) / BB(1, 1) * 10000 / h
jtd = Array(spr, LL(1), PP(1))

End Function

'RECOVERY FUNCTION (3 Different Functional forms)
Function phi_fn(a0, a1, s, lam)
    'PROBIT
    phi_fn = WorksheetFunction.NormSDist(a0 + a1 * lam)
    'LOGIT
    'phi_fn = 1 / (1 + Exp(a0 + a1 * lam))
    'ARCTAN
    'phi_fn = 0.5 * (WorksheetFunction.Atan2(1, a0 + a1 * lam) * 2
                   / WorksheetFunction.Pi + 1)
End Function
```

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