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Dissipative solitons*

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Abstract

A generalization of the Korteweg–de Vries equation (KdVE) is considered in which additional terms belonging to the Kuramoto–Sivashinsky equation (KSE) are incorporated to account for a production–dissipation (input–output) energy balance. Two different situations are thoroughly investigated.

First, the production–dissipation part of the equation is taken as a small perturbation to the KdVE, proportional to a smallness parameter ε. It is shown that within times limited by ε^{-1} (and beyond) the KdVE *seches* interact similarly to the Zabusky and Kruskal's findings, save the "aging" they experience. At longer times the localized solutions adopt the terminal shape and phase velocity, and different humps can form bound states. The increase of the production–dissipation parameter exaggerates the effects through reducing the "practical infinity" for the time scale. For $\varepsilon > 2$ with the rest of parameters equal to unity, the solution goes chaotic. These results outline the region where the long enough *transients* can be approximately considered as solitons, albeit *imperfect* ones.

The second situation is when the KS part of the equation is predominant. This happens either when ε is not small enough or for very long times ($t \to \infty$ or $t \gg \varepsilon^{-1}$) when strictly permanent shapes are attained which are in fact short waves and the dissipation (higher-order derivative) is dominant. It is shown that the solution to KSE of homoclinic shape (a hump) does not qualify for a wave-particle/soliton since it does not persist as a permanent shape after collision and yields to a chaotic régime. The heteroclinic shapes (kinks/bores/hydraulic jumps/shocks) do behave as particles but the interactions appear to be completely inelastic. After two such wave-particles collide they stick to each other and deform to produce a single structure of the same kind which carries the total momentum of the system. This kind of (really imperfect) solitons may be called "clayons" to emphasize the fact that upon collisions they behave as clay balls.

Thus the Zabusky and Kruskal's soliton concept is extended in two directions: to "long" transients practically "permanent" and solitonic in the time scale ε^{-1} set by production–dissipation processes and to true permanent wave-particles with, however, inelastic behaviour upon collisions.

* *Dedicated to Prof. Norman J. Zabusky and to Prof. Morikazu Toda*

1. Introduction

In the years 1952–1953 Fermi, Pasta and Ulam [22] (referred further as FPU), with the help of Mary Tsingou and the MANIAC I computer of Los Alamos National Laboratory (USA), investigated the "energy-sharing" properties of the linear normal modes in certain nonlinear lattice models. They were interested in the process of relaxation to thermal equilibrium. Nonlinear lattices with cubic, quartic or

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broken-linear nonlinearity seemed to be ideal systems for such purpose. Relaxation of some non-equilibrium distribution back to the equipartition of energy in the interacting modes was expected because of the presumed ergodic nature of a nonlinear lattice (today we know that this assumption is not correct. For a detailed discussion of the FPU problem in modern context and an overview of nonlinear problems, solitons and all that see, e.g., [1,8]). The question which FPU sought to investigate was how long it takes for a finite chain (e.g., with 16, 32 and 64 atoms), to reach the relaxation to equilibrium. The original objective had been to see at what rate the energy of the chain, initially put into a single mode like a harmonic wave, would gradually become a "mess" both in the form of the chain and in the way the energy is distributed among the modes. If all the energy is initially in the first normal mode one would expect the (nonlinear) coupling to generate a (slow) energy flow into the higher modes until equipartition with some small fluctuation is achieved. As $N \to \infty$ these fluctuations should vanish. Rather than that, what FPU discovered was recurrence to initial state. Fermi felt that their result "... constituted a minor/little discovery" although he intended to talk about this as a prestigious invited speaker (Gibbs lecture) at the annual meeting of the American Mathematical Society, as Ulam later recounted [22]. Fermi died (November 1954) prior to the writing of their report (May 1955). Although Ulam did talk about their findings at various scientific meetings, this report was only published as a Los Alamos report (1955) and in the open literature years later in Fermi's collected works (1965) [22]. It was the year that several authors returned to the same problem and realised the major importance of such "minor/little discovery". It was also realized that a continuum limit of the nonlinear lattice studied by FPU led to the long-time forgotten Korteweg–de Vries equation (KdVE) [40, 75] which was known to possess solitary waves and (nonlinear) cnoidal wave trains very much like experimentally observed by J. Scott Russell in August 1834 [63].

In 1965 N.J. Zabusky and M.D. Kruskal [84,83] reported on their numerical studies of the dynamical behaviour of the KdVE. Their motivation referred to a collisionless plasma which does not involve dissipation the latter being brought into the picture by the collisions. On the other hand, for lattices, the thermodynamic limit or "realistic" interaction with boundaries also brings irreversibility. They found that KdVE not only sustains solitary propagating disturbances, but that large-amplitude waves tend to break up into a spatial series of pulses with different amplitudes and velocities.

The second discovery of Zabusky and Kruskal was that these pulses retained their identity after they interacted, i.e., when they went through one another (or absorbed and emitted) they recovered their initial shapes. The only effect of their interaction is a shift in their space-time lines, corresponding to a temporary "acceleration". During their interaction their "combined" amplitude appears to be smaller than the sum of the initial amplitudes (linear superposition). Thus Zabusky and Kruskal coined the term *soliton* to describe a solitary, uniformly propagating localized disturbance (pulse), which preserves its structure and velocity after an interaction with another soliton, i.e. solitons had the property of (stable) particles. When initially the energy resides in a "soliton", it will always remain in that state and will not be shared or thermalized [84,83]. Solitons in a nonlinear system are the analogues of normal modes of linear problems.

At about the same time (1965 onward) Visscher and collaborators [55,54,60] considered wave propagation in a nonlinear lattice, and using a Lennard-Jones interaction potential studied energy transfer in thermal conduction. For one- and two-dimensional lattices with many impurities of different masses they found that energy transfer was generally enhanced by the introduction of nonlinear interaction terms. Such result can be understood by assuming that energy is accumulated in the form of localized packets/pulses or solitons which propagate without being hindered much by the impurities. Thus lattices and continuum models (some nonlinear PDEs) were linked together in the opening of a new area in Physics and in Mathematics [25].

A special place of honor occupies Toda's exponential lattice, which in one limit yields the hard sphere gas and in another, the harmonic crystal [71,72,69,70,68]. It is the first many-body problem constructed and exactly solved. Toda benefited from a deep knowledge of the KdVE related to elliptic functions.

Today we understand how under certain conditions nonlinearity and dispersion balance each other and localized solutions, stationary propagating waves (permanent solitary waves) appear in a dissipationless medium. The discovery goes back to Lagrange, Boussinesq [4,6,5] and Rayleigh [58] (see, however [74]). The KdVE was the first to undergo numerical investigation [84,83] unravelling the particle-like behaviour of the localized solutions. Since then a variety of conservation properties have been proved and a good deal of analytical techniques for solitons is now available for the KdVE and other soliton-bearing, integrable equations [20,69,70]. Thus the KdVE was not just an isolated curiosity.

Completely different is the case when dissipation is taken into account even as a small perturbation to the original model. The presence of dissipation spoils immediately integrability of the model and little can be done analytically, but who cares about integrability [64]? Yet this is the smaller evil. The problem is that very often dissipation is represented by higher-order spatial derivatives which means that considering it as a perturbation yields singular expansions that are only valid either for short times or on short distances. As far as the long transients (practically permanent in the time scale ε^{-1}) or stationary solutions are concerned the higher-order derivatives cannot be treated as small perturbations. These are the obstacles on the way of systematic treatment and exhaustive investigation of dissipative systems, e.g., the possible solitonic behaviour of their localized solutions or of the crests in wave trains such as the periodic cnoidal wave. The importance of the self-sustained localized solutions (coherent structures – as called in the field of turbulence) stems from the fact that they can retain their individuality for considerably longer time intervals than the characteristic times of the small-scale disturbances. Studying their shapes and collision dynamics can provide insight into the properties of the particular system under consideration.

The main purpose of the present work is to show that the (input-dissipation) energy balance can add to the balance between nonlinearity and dispersion to sustain particle-like solutions of dissipative Nonlinear Evolution Equations (NEE). This attitude is substantiated by a case study of the wave régimes of a nonlinear equation modelling different capillary flows in thin liquid layers – a dissipation modified Korteweg–de Vries equation. Mathematically speaking this equation is intrinsically dissipative – a generalized parabolic equation because of the highest-order derivative present. However, in ε^{-1} times it is much closer to a (conservative) wave equation for pulse propagation in dissipative fluids and some reaction–diffusion systems in chemistry and related fields. On the other hand following the steps of Zabusky and Kruskal we envisage extending the *soliton* concept to dissipative systems when one certainly does not care about integrability [64]. This is of interest in view of recently available experimental results [81,46,47,79].

2. The dissipation-modified KdV equation

2.1. Heuristic background and a model problem

The capillary flow in thin viscous films falling down an inclined plane (e.g., vertical wall) appears to be a model case for a continuous dynamical system which has low-dimensional phase space. Under the assumption of thin layer the flow in the bulk can be considered laminar and the influence of the bulk is reduced to the coupled drag force acting upon the surface. Thus an approximate equation containing only the surface variables can be derived for the elevation of the free surface. Since the works of Kapitza [34–36], it is known that the thin-film flow exhibits all major types of behaviour: laminar, periodic and turbulent. The main point here is that the latter is a "surface turbulence" – a chaotic behaviour of the long capillary waves on the surface while the bulk is perfectly laminar with Poiseuille flow. The evolution is governed by the Reynolds and Weber numbers[2]

$$\mathrm{Re} = \frac{UL}{\nu}, \quad \mathrm{We} = \frac{\sigma}{\rho g h_0^2}, \tag{1}$$

where σ is the surface tension, ρ – the density, g – the gravity acceleration, h_0 – the thickness of the undisturbed film, ν – kinematic coefficient of viscosity and U is the characteristic velocity. The wave number is defined as the ratio $\alpha = h_0/L$, where L is the characteristic wave length.

Retaining terms up to order $\mathcal{O}(\alpha)$ (see [3,2,32,44,45,49,66]) Gjevik [26] obtained the following $(1+1)D$ NEE (we use the notations of [27]):

$$h_t + (h^3)_x + \alpha\left\{\frac{6}{5}\left(h^6 - \frac{\mathrm{Re}_c}{\mathrm{Re}} h^3\right)h_x + \tfrac{1}{3}\,\alpha^2 \mathrm{We}\, h^3 h_{xxx}\right\}_x = 0, \tag{2}$$

where $\mathrm{Re}_c = \frac{5}{6}\cot\beta$ accounts for the inclination of the layer on angle β. This equation can produce singularities in finite time ([57,61]).

If the characteristic length of the solution under consideration is not long enough, the term responsible for the surface tension contains also geometrical nonlinearities inherent in the expression for the curvature of the surface (see, e.g., [32,53]). It was Homsy [27] who showed that in the weakly-nonlinear approximation (see, also the later works, e.g., [41], [67]), the appropriate simplification of (2) gives

$$\eta_t + 3\eta\eta_x + \alpha\,\mathrm{Re}\left\{3(\eta^3)_x + \frac{6}{5}\left(\frac{\mathrm{Re} - \mathrm{Re}_c}{\mathrm{Re}}\right)\eta_{xx} + \tfrac{1}{3}\,P\eta_{xxxx}\right\} = 0, \tag{3}$$

where $h = 1 + \alpha\,\mathrm{Re}\,\eta$ and $P = \alpha^2\mathrm{We}$.

After re-scaling of variables the above equation can be recast as follows:

$$\varphi_t + \varphi\varphi_x + \varphi_{xx} + \varphi_{xxxx} = 0, \tag{4}$$

or

$$\psi_t + (\psi_x)^2 + \psi_{xx} + \psi_{xxxx} = 0, \quad \psi_x \equiv \varphi. \tag{5}$$

The last form of the NEE under consideration was also obtained in [43] for the evolution of reaction fronts for states close to a transition point and small wave amplitude. Nowadays the equations (3) or (4) are referred to as Kuramoto–Sivashinsky equation (KSE – for brevity). The only nonlinearity in KSE is the convective term (called in the framework of formulation (4) – "eikonal" nonlinearity [28,29]). Despite the assumptions valid in principle only near the threshold of instability, the model equation (3), (4) turns out to be suitable for nonlinear states well beyond the threshold (see [10]).

When the Marangoni effect is considered on the surface of the thin layer then an additional nonlinearity of the form $(\psi\psi_x)_x$ (see [24,39]) appears. The influence on the dynamical behaviour of this additional term as a destabilizing factor (according to its sign in the equations) was elucidated in [28].

[2] Nowadays we use the name Bond number for the inverse Weber number while the latter now refers to a balance between inertia and dissipation.

On the other hand in a series of papers, Velarde and collaborators (see [23,51]) showed the consistent way of incorporating the Marangoni effect [77] into the one-way long-wave assumption and obtained for the case opposite to the Bénard convection [52,76], i.e., when heating the liquid layer from above (from the air side), the following equation[3]:

$$u_t + 2\alpha_1 u u_x + \alpha_2 u_{xx} + \alpha_3 u_{xxx} + \alpha_4 u_{xxxx} + \alpha_5 (u u_x)_x = 0\,, \tag{6}$$

containing both the dispersion term u_{xxx} (as KdVE does) and additional nonlinear term $(uu_x)_x$ accounting for the Marangoni effect. There is also an extension of this description to the 3D case with waves propagating in arbitrary directions [51]. We call in what follows Eq. (6) the Korteweg–de Vries–Kuramoto–Sivashinsky–Velarde equation (KdV–KSVE). For the sake of completeness here we must mention that eq. (6) for the case $\alpha_5 = 0$ was considered in many other works (e.g., [21,37,42,48], etc., but with few exceptions only as a mathematical object without much concern about its physical relevance which shows the ubiquity of KdVE (Boussinesq Paradigm [19]) for description of shallow liquid layers and many other problems. The influence of the additional Marangoni nonlinearity on the shapes of the solitary waves was investigated in [18].

Thus, although the derivation of Eq. (6) provided by Velarde and colleagues [23,51] was part of a study of long-wave-length surface waves in shallow liquid layer subjected to Marangoni stresses due to imposed thermal gradients like with a shallow liquid layer heated from the ambient air or the derivation of other authors like Janiaud et al. [30] has a sound physical content in the study of Eckhaus instability for traveling waves we shall treat Eq. (6) as mathematical object and hence as a model problem in itself. This out-of-specific-context approach to Eq. (6) permits exploration of a wide class of solutions.

On the one hand Eq. (6) which contains the original KdVE balance between nonlinearity and dispersion also captures the basic elements of a driven system with instability threshold and saturation elements, i.e., an energy balance with input–output (production–dissipation) which permits the onset and nonlinear evolution of a solitary wave decaying at both infinities or a shock (kink/bore/hydraulic jump) as well as solitonic wave trains.

On the other hand KSE as a mathematical object is prototype of a dissipative system being a (generalized) parabolic (heat-conduction) equation. The only difference here is that diffusion is represented by the fourth spatial derivative while the second spatial derivative has the opposite sign to the second-order diffusion equation, hence accounting for energy production (pumping). The same role plays the Marangoni nonlinearity (with the improper sign). This intricate interplay is characteristic for the long-wave instability of KS when the fourth order dissipation fails to bound the second-order pumping. Although apparently simple, the KSE possesses a rich phenomenology (see [29]). As it will be seen later on, the KdV–KSVE has a reacher one because in the short time scales it retains a lot of traits of a conservative, dissipationless wave equation.

2.2. Localized solutions of KdV–KSVE and energy balance

We shall not consider here periodic wave trains but only solitary waves as ‘localized’ solutions. By definition the ‘localized’ solutions are supposed to asymptotically reach constant values at both infinities, namely:

[3] When bottom friction is fully incorporated there also appears a term $\alpha_6 u$ which brings wave trains as the most relevant solutions [59]. It is not included here.

$$u \xrightarrow{x\to-\infty} u_{-\infty} \quad \text{and} \quad u \xrightarrow{x\to+\infty} u_{+\infty} . \tag{7}$$

Due to their asymptotic nature these conditions yield trivial conditions for the spatial derivatives

$$u_x = u_{xx} = u_{xxx} \xrightarrow{x\to-\infty} 0 \quad \text{and} \quad u_x = u_{xx} = x_{xxx} \xrightarrow{x\to+\infty} 0 . \tag{8}$$

When stationary waves are considered in the moving frame $\xi = x - ct$, then (6) reduces to an ODE which can be integrated once to obtain

$$-cu + \alpha_1 u^2 + \alpha_2 u' + \alpha_3 u'' + \alpha_4 u''' + \alpha_5 uu' = C_1 , \tag{9}$$

where the prime stands for a differentiation with respect to ξ. The integration constant C_1 is defined from the asymptotic boundary conditions (7), (8) and must satisfy the following two conditions:

$$-cu_{-\infty} + u^2_{-\infty} = C_1 = -cu_{+\infty} + u^2_{+\infty} . \tag{10}$$

They are consistent when

$$u_{-\infty} = u_{+\infty} = 0 , \tag{11}$$

which is the condition for 'humps': surface elevations/positive solutions, or alternatively – dips/depressions/negative solutions, from an undisturbed level in the vicinity of the origin of the coordinate system and asymptotically decay to zero at infinities. The hump is, in fact, the homoclinics of the ODE equation (7) (see also [50,78]).

When the boundary constants do not obey Eq. (10) and do not adopt trivial values, the solution is called "kink" (bore/hydraulic jump/heteroclinics). In this case, the two conditions for the integration constant C_1 are consistent iff

$$c = \alpha_1(u_{-\infty} + u_{+\infty}) \quad \text{and} \quad C_1 = -\alpha_1 u_{-\infty} u_{+\infty} . \tag{12}$$

Before turning to the specifics of the numerical implementation it is important to address the issue of integral balances for the boundary value problems under consideration. It appears that those differ for the humps and kinks.

For the humps one can define the *mass*, and *energy* of wave as

$$M_h = \int_{-\infty}^{+\infty} u \, dx , \quad E_h = \int_{-\infty}^{+\infty} \tfrac{1}{2} u^2 \, dx , \tag{13}$$

and thus

$$\frac{d}{dt} E_h = \int_{-\infty}^{+\infty} (\alpha_2 + \alpha_5 u) u_x^2 \, dx - \int_{-\infty}^{+\infty} \alpha_4 u_{xx}^2 \, dx , \tag{14}$$

where we see that energy supply is significant at rather long wave lengths while the dissipation dominates at shorter ones.

The above definitions are inappropriate for the kink case, because then the integrals in definitions (13) diverge. In this case, we can define *mass* of wave as integral of the first spatial derivative (which has a hump shape), namely

$$M_k = \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} dx \equiv u_{-\infty} - u_{\infty} . \tag{15}$$

Respectively the *momentum* and (*pseudo*)*energy* of the wave are defined as

$$P_k = \int_{-\infty}^{+\infty} \tfrac{1}{2}u_t = \alpha_1(u_{-\infty}^2 - u_{\infty}^2) \equiv M_k c\,, \quad E_k \equiv \int_{-\infty}^{+\infty} \tfrac{1}{2}u_x^2\,. \tag{16}$$

We use the terminology *(pseudo)energy*, because the quantity E_k has no physical meaning as true energy of the wave, yet it is a quadratic functional associated with the latter. The "conservation laws" for the so defined *mass* and *momentum* of the wave are a matter of b.c. The balance law for the kink *(pseudo)energy* reads

$$\frac{\mathrm{d}}{\mathrm{d}t}E_k = \int_{-\infty}^{+\infty} (\alpha_2 + \alpha_5 u)u_{xx}^2\,\mathrm{d}x - \int_{-\infty}^{+\infty} \alpha_4 u_{xxx}^2\,\mathrm{d}x + \int_{-\infty}^{+\infty} \alpha_1 u_x^3\,\mathrm{d}x\,. \tag{17}$$

It is seen that for the *(pseudo)energy* we have similar balance law, with however the presence of the last term which could provide either input or dissipation, depending on the particular shape of the wave.

3. Numerical exploration of KdV–KSVE

3.1. Difference scheme

It is convenient to render (6) into a system of two equations of second-order with respect to the spatial variable, namely

$$u_t + \alpha_1 u u_x + \alpha_5(u u_x)_x + \alpha_2 q + \alpha_3 q_x + \alpha_4 q_{xx} = 0\,, \quad u_{xx} = q\,. \tag{18}$$

The presentation as a system has significance for the numerical implementation allowing us to solve at each time step a system with better-conditioned matrix in comparison with the one that could have resulted from the direct difference approximation of the fourth-order equation. The details can be found in [14].

Consider the set functions u_i, $q_{i+1/2}$ on the regular mesh in the interval $[-L_1, L_2]$ with spacing h, i.e.,

$$x_i = -L_1 + (i-1)h\,, \quad h = \frac{L_1 + L_2}{N-1}\,, \tag{19}$$

where N is the total number of grid points in the said interval. Note that the mesh for function q_i is staggered with respect to the main mesh where the function u is defined.

For the long-time evolution, it is preferable to have implicit scheme with very high order of approximation with respect to time in order to be able to use large time increments. For this reason we use a four-stage scheme which provides third order of approximation in time. As far as a multistage scheme requires more initial conditions than available, then at the initial stages it is simply a two-stage or three-stage scheme. The latter means that we approximate the time derivative as follows:

$$\left.\frac{\delta u}{\mathrm{d}t}\right|_{i-1/2}^{n+1} \equiv \frac{1}{2}\left(\frac{11u_i^{n+1} - 18u_i^n + 9u_i^{n-1} - 2u_i^{n-2}}{6\tau} + \frac{11u_{i-1}^{n+1} - 18u_{i-1}^n + 9_{i-1}^{n-1} - 2u_{i-1}^{n-2}}{6\tau}\right),$$

$$\left.\frac{\delta u}{\delta t}\right|_{i-1/2}^{2} \equiv \frac{1}{2}\left(\frac{3u_i^2 - 4u_i^1 + u_i^0}{2\tau} + \frac{3u_{i-1}^2 - 4u_{i-1}^1 + u_{i-1}^0}{2\tau}\right),$$

$$\left.\frac{\delta u}{\delta t}\right|_{i-1/2}^{1} \equiv \frac{1}{2}\left(\frac{u_i^1 - u_i^0}{2\tau} + \frac{u_{i-1}^1 - u_{i-1}^0}{2\tau}\right). \tag{20}$$

Employing Newton's quasilinearization we arrive at the following difference scheme:

$$\left.\frac{\delta u}{dt}\right|_{i-1/2}^{n+1} + \frac{\alpha_4}{h^2}(q_{i+1/2}^{n+1} - 2q_{i-3/2}^{n+1} + q_{i-3/2}^{n+1}) + \frac{\alpha_3}{2h}(q_{i+1/2}^{n+1} - q_{i-1/2}^{n+1}) + \alpha_2 q_{i-1/2}^{n+1} + 4\alpha_1 \frac{u_i^n u_i^{n+1} - u_{i-1}^n u_{i-1}^{n+1}}{h}$$
$$= 2\alpha_1 \frac{u_i^{n^2} - u_{i-1}^{n^2}}{h} + \mathcal{O}(\tau^3 + h^2)\,, \quad i = 2, \ldots, N-2\,, \tag{21}$$

$$\frac{1}{h^2}(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) = \tfrac{1}{2}(q_{i+1/2}^{n+1} + q_{i-1/2}^{n+1})\,, \quad i = 2, \ldots, N-1\,, \tag{22}$$

coupled with the following b.c.:

$$u_1 = u_2 = u_{-\infty}\,, \quad u_{N-1} = u_N = u_\infty\,.$$

3.2. Long-time transients of KdV–KSVE. "Aging" solitons

As already pointed out, the immediate physical significance of the KdV–KSVE equation, appropriate to the assumptions used in its derivation, is when the input and dissipation terms are smaller in comparison with the inertia and dispersion ([23]). Through introducing appropriate scaling of the variables one can reduce the equation to one with $\alpha_2 = \alpha_4 = \varepsilon$ and $\alpha_3 = 1$. Respectively, the coefficient of the convective nonlinearity can be rendered to any value by means of re-scaling the amplitude of function u. For illustration we take $\alpha_1 = 3$, $\alpha_5 = 0$. The parameters to be varied are ε and the phase velocities of the structures imposed as initial condition.

One sees that the higher-order derivative of the equation under study is multiplied by a small parameter. Thus one is inevitably faced with a singular expansion. Then, the solutions of the dissipation modified equation should be expected to be reasonably well approximated by the solutions of the original KdVE only on a long but finite time interval of order of $\mathcal{O}(\varepsilon^{-1})$. One of the objectives of the present subsection is to verify this assertion.

Consider the well known *sech*-solution of the KdVE:

$$u = \frac{3c}{2\alpha_1}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{c}{\alpha_3}}(x - ct)\right), \tag{23}$$

where c is the phase velocity of the sech.

The first important issue to be addressed when generalizing the notion of soliton to the long-time transient of the dissipation-modified KdVE, is the problem of collision of two localized solutions. We begin with the sech solutions of KdVE. The sech is not a solution of the full KdV–KSVE but for small ε the mismatch is supposed to be small and its impact should be felt at very long times only, i.e. beyond ε^{-1}.

In Fig. 1 we present the over-taking collision of two seches of phase velocities $c = 10$ and $c = 5$, respectively. The result is very instructive in the sense that it confirms the intuitive expectation that on time intervals shorter than the dissipation time ε^{-1} the interaction must be essentially similar to the KdVE. Indeed Fig. 1b shows the trajectories of centres of coherent structures. The dotted lines are projected trajectories of stationary propagating seches. The dashed line is the trajectory of the accelerating sech (see Fig. 3). It is seen that the phase shift is essentially similar to the KdVE and the smaller sech experiences the larger phase shift. The phase shift is "negative" in the sense that the structures are temporarily accelerated during the interaction and re-appear at positions farther ahead in

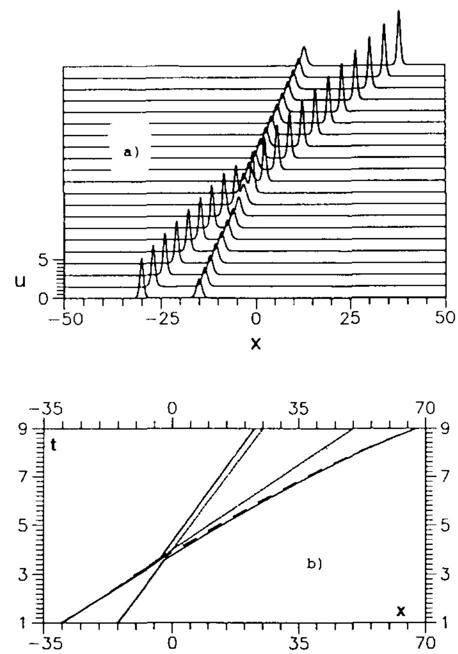

Fig. 1. Interaction of two seches with $c = 10$ and $c = 5$ in KdV–KSVE for small KSV part $\alpha_2 = \alpha_4 = 0.001$. (a) Evolution of shape with time. (b) Trajectories of centres of solitons.

the directions of their motion relative to the mass center, than the position they would have reached, were the interaction not to take place.

Yet there is a conspicuous difference between the KdVE and the KdV–KSVE. In Fig. 1a is seen that the total amplitude of the compound signal never exceeds the amplitude of the larger sech. It is well known from the analytical two-soliton solution of KdVE that the amplitude of the compound signal is smaller than the sum of the amplitudes of the two initial solitons, but it is still considerably larger than the amplitude of the bigger soliton. Thus this inelastic facet of the interaction of seches is significantly exaggerated by the presence of the KS part.

An important finding for small $\varepsilon = \alpha_2 = \alpha_4$ is the celerity selection that takes place after the collision of seches. Fig. 2 shows the evolution of the smaller sech from Fig. 1 (phase velocity $c = 5$ and amplitude 2.5) after it reemerges from the overtaking collision with the faster one.

What is seen in Fig. 2 is the slow "aging" of the solitary wave decreasing in amplitude but keeping its sech-like shape. After long enough dimensionless time $t \approx 8000$ the coherent structure approximately attains its terminal shape (Fig. 2a). This time interval is in fact almost an order of magnitude longer than $\varepsilon^{-1} = 1000$ scale. Just to be on the safe side we continued the calculations until $T = 15\,000$ (Fig. 2b) and the result is marginally refined. Its convergence to a terminal state is beyond any doubts. It is interesting to note that the shape is virtually equal to a sech with phase velocity $c = 1.399$. We made sure that the four digits reported here are correct. Consequently, the amplitude is 0.69924 with the same accuracy of four digits. Thus the production-dissipation mechanism being of order of $\varepsilon = 0.001$ does not act to quantitatively change the shape of solution, but it provides a qualitative effect on the spectrum of sech-like solutions. While for KdV equation the spectrum is continuous $c > 0$, here one has a single value $c = 1.399$, i.e. the dissipative terms play the role of celerity selection.

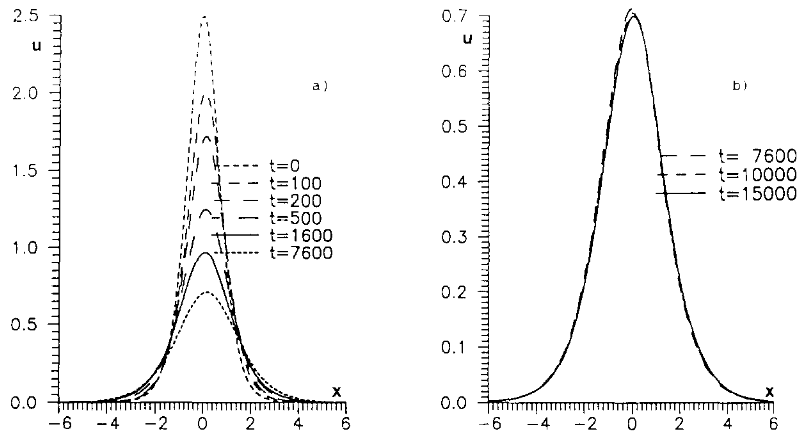

Fig. 2. "Aging" of a sech with $c = 5$ in KdV–KSVE for small KS part $\alpha_2 = \alpha_4 = 0.001$.

It is interesting to note that this selection can be predicted by simple application of the energy balance (14). For the set of parameters used here it can be recast as follows

$$\frac{\mathrm{d}}{\mathrm{d}t} E_h = \varepsilon \int_{-\infty}^{+\infty} [u_x^2 - u_{xx}^2]\,\mathrm{d}x\,. \tag{24}$$

Let us also assume that for long-length solutions the nature of the singular perturbation does not show up and hence let us seek for a regular perturbative solution in the form:

$$u = \tfrac{1}{2}c\,\mathrm{sech}^2[\tfrac{1}{2}\sqrt{c}(x - ct)] + \varepsilon u_1\,. \tag{25}$$

For convenience we introduce

$$u(x - ct) \equiv \tfrac{1}{2}cv(\zeta)\,, \quad \zeta = \tfrac{1}{2}\sqrt{c}(x - ct)\,, \quad v(\zeta) = \mathrm{sech}^2(\zeta)\,, \quad \mathrm{d}x = \tfrac{1}{2}\sqrt{c}\,\mathrm{d}\zeta\,. \tag{26}$$

Then

$$u_x = \frac{c^{3/2}}{4}\frac{\mathrm{d}v}{\mathrm{d}\zeta}\,, \quad u_{xx} = \frac{c^2}{8}\frac{\mathrm{d}^2v}{\mathrm{d}\zeta^2}\,,$$

and the balance law (24) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} E_h = \varepsilon\frac{\sqrt{c}}{2} \int_{-\infty}^{+\infty} \left[\frac{c^3}{16}\left(\frac{\mathrm{d}v}{\mathrm{d}\zeta}\right)^2 - \frac{c^4}{64}\left(\frac{\mathrm{d}^2v}{\mathrm{d}\zeta^2}\right)^2\right]\mathrm{d}x + \mathcal{O}(\varepsilon^2)\,, \tag{27}$$

Now we evaluate the integrals from the sech solution $v(\zeta)$ to get

$$\int_{-\infty}^{\infty} v_\zeta^2\,\mathrm{d}\zeta = \tfrac{16}{15}\,, \quad \int_{-\infty}^{\infty} v_{\zeta\zeta}^2\,\mathrm{d}\zeta = \tfrac{64}{21}\,.$$

Neglecting the higher order terms we obtain that in order to have a stationary propagating wave ($\mathrm{d}E_h/\mathrm{d}t = 0$) the r.h.s. (27) must vanish, i.e.,

$$\tfrac{1}{21}c - \tfrac{1}{15} = 0 \Rightarrow c = 1.4\,, \tag{28}$$

which agrees with our numerical finding $c = 1.399$. Naturally the amplitude $c/2 = 0.7$ does not compare that well (difference about 0.1%), because the real one obtained in the numerical simulation incorporates also the term $\varepsilon \sim 0.001$ which was not evaluated here.

So far we have provided *quantitative* numerical evidence of how the energy balance selects the phase velocity of the wave even when it is proportional to a small parameter hence not changing significantly the shape of the solitary wave.

At the same time, for the more intensive sech from Fig. 1 of phase velocity $c = 10$ (amplitude 5), the shape change due to the interaction triggers the production and the sech starts growing until it reaches the shape of higher amplitude that is a solution of the full KdV–KSVE equation (Fig. 3a). It is interesting to note that the process here is faster than the process of decrease in the previous case, being completed within a time of order 30, which is of order of $0.03\varepsilon^{-1}$. Since the KS part is the value which corresponds to the result presented in Fig. 3, $c = 17.64$, this result seems unexpected, because the energy balance clearly selects only a single value of celerity. Yet it is hard to believe that it is an artifact of the numerics – the same numerics that predicted the celerity in the case of Fig. 2 with accuracy of 0.01% and after 15 000 dimensionless time units or over a million time steps of the scheme. So we leave the question open about the second eigen value for celerity.

Elphick et al. [21] mention the effect of celerity selection and even give a numerical value for the eigen-parameter – phase velocity/celerity for one set of parameters. The problem was given a special treatment in our previous investigation [13,18] where the eigen-parameter was calculated with high

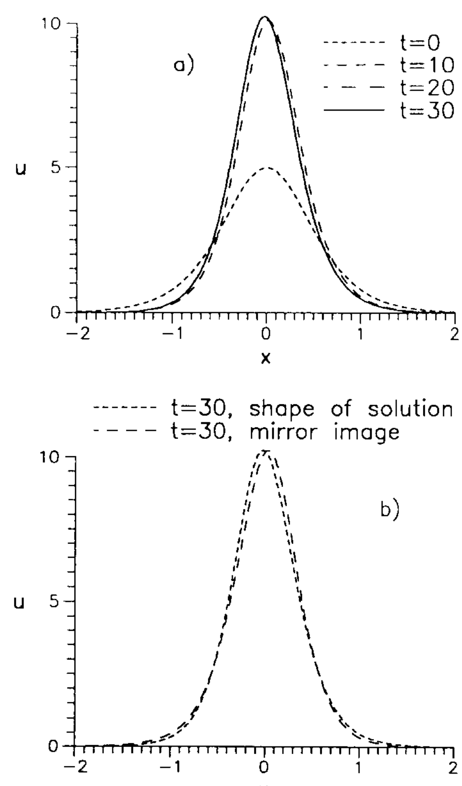

Fig. 3. Growth of a sech with $c = 10$ in KdV–KSVE for small KSV part $\alpha_2 = \alpha_4 = 0.001$. (a) Evolution of shape with time (b) Asymmetry of the permanent shape.

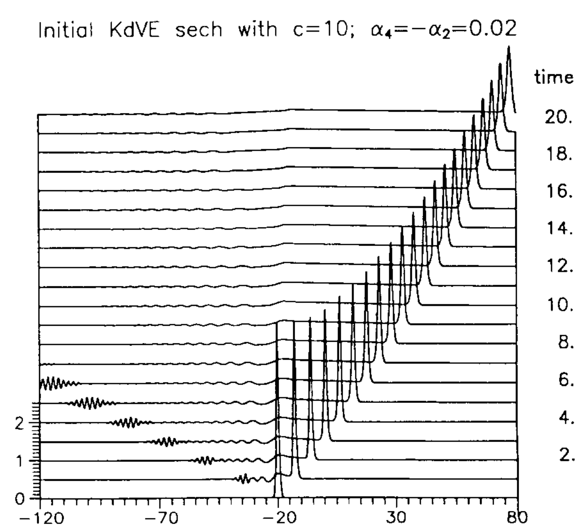

Fig. 4. Evolution of a sech with $c = 10$ in KdV–KSVE for $\varepsilon = \alpha_2 = \alpha_4 = 0.02$.

accuracy for different sets of governing parameters. When a time dependent solution is considered the latter being numerically shifted together with the moving frame, the accuracy could never be as high as for the specialized methods of solution [13,18] tailored to tackle the inverse nature of the problem. Yet, the solutions of present work show a very high degree of stationarity and the first four digits of the phase velocity are reliable.

In order to clarify the influence of the KS part we return again to the case of a single sech taken as an initial condition. We calculated the evolution for different values of $\varepsilon = \alpha_2 = \alpha_4$. Fig. 4 depicts the case with small $\varepsilon = 0.02$, but not so small as the previously considered case. The sech gradually decreases in amplitude and decelerates to the eigen value for the particular case $c = 6.79$. Some signals are created that escape in direction opposite to the direction of motion of the main sech. An interesting feature is the occurrence of an envelope which moves to the left with velocity of order of 40. Clearly, this signal is related to the fastest growing mode of long-wave linear instability of the equation under consideration but its localized appearance suggests that it is not just the balance between the linear terms (energy input–output) but rather it is sustained by an appropriate balance between these and the nonlinearity. We have found numerically that at the beginning it grows almost linearly in time but then turns explosively unstable (perhaps, blowing up in finite time, as far as one can judge from numerical experiments). For this reason at a certain moment, when the envelop is far enough we cut down the region and continue the simulations with the rest of the wave. This is the way how the solution in Fig. 4 is obtained for longer times.

We continued the numerical experiments for the same value of the dispersion parameter and increasing ε. The scenario is essentially the same as for the small ε, i.e., the main hump gradually decreases to the characteristic eigen amplitude and the radiation escapes rapidly in the opposite direction. The only qualitative difference in the range $0.1 \leq \varepsilon \leq 1$ is that now a second hump develops which in the beginning is smaller and slower, but gradually increases and speeds up.

Beyond $\varepsilon = 2$ we found the realm of chaotic solutions of the dissipative system under consideration. For the KSE transition to chaos takes place for arbitrary small values of ε, because there is no dispersion there. Hence dispersion plays a *laminarizing* role (see, also, [11,21,37]). Although chaotic, the solution of the equation under consideration is dominated by the presence of large deterministic structure, in the sense that a significant part of the energy of pulsations is connected with these structures. This intermittent chaotic/deterministic behaviour occurs in turbulence, where the coinage "coherent structure" was introduced and methods of phase averaging were developed for pattern recognition of structures (see, e.g., [7]).

An appropriate mathematical model for treating intermittent deterministic–chaotic régimes is the so-called random point functions that are random trains of similar shapes the latter being randomly distributed throughout the space or in time. The random point model allowed effective approximate closure of the cascade equations of turbulence and was instrumental in predicting the statistical characteristic for different chaotic motions, e.g., Lorenz attractor, Poiseuille flows, plane mixing layer [12,17]. As shown in [15], the chaotic solution of KSE pertains to the same class being fairly well approximated by a random point function. The predicted correlation functions, higher-order moments, etc. statistical characteristics were in very good quantitative agreement with the direct numerical experiment [65] (for more details, see also the monograph [17]).

Let us now resume investigation of overtaking collisions. In Figs. 5a–c we present the overtaking of a small sech of initial phase velocity $c = 2$ by a bigger one with $c = 10$ for $\varepsilon = 0.1$. Taking such large value for the parameter ε allows us to observe significant effect within a shorter dimensionless time (and still reasonable computational time). In Fig. 5a the two seches are separated enough and within the

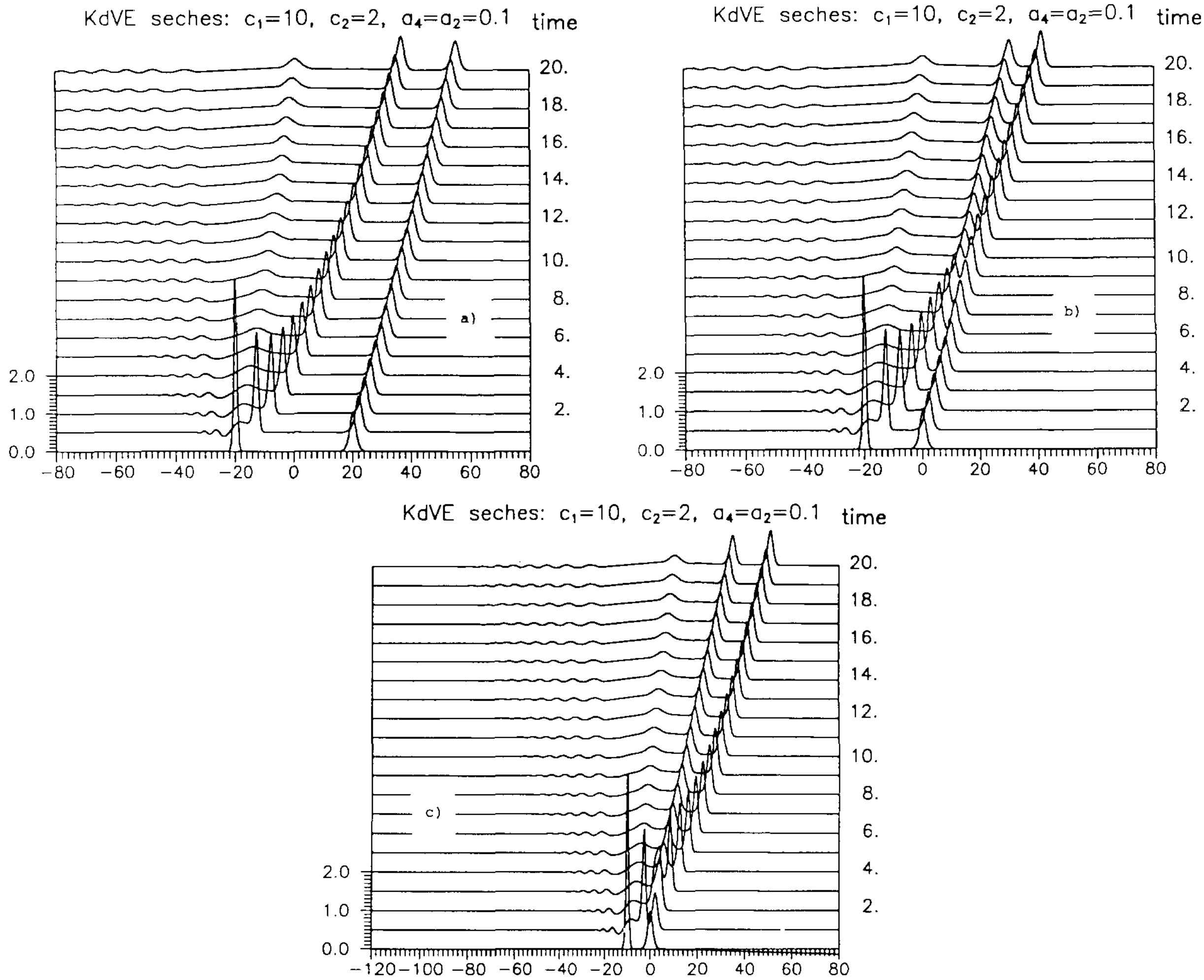

Fig. 5. Overtaking interactions of a sech with $c_1 = 10$ and $c_2 = 2$ in KdV–KSVE for $\varepsilon = \alpha_2 = \alpha_4 = 0.1$. (a) Large initial separation. (b) Moderate initial separation. (c) Small initial separation.

characteristic time $\varepsilon^{-1} = 0.1^{-1} = 10$ they could not collide. Because of their "aging", they attain the permanent shape (and hence the terminal phase velocity) before they become close enough to interact. The amplitude of the large soliton diminishes twice for dimensionless time equal to one! Approximately twice is the reduction of the phase velocity. It is not an effect of order $\varepsilon = 0.1$ which is another facet of the singular asymptotic nature of the problem under consideration. Much less violent is the evolution of the smaller sech which is much closer to the eigen-solution of the stationary KdV–KSVE for this particular value of ε.

The result of the next experiment is presented in Fig. 5b where the initial separation of the seches is reduced twice in comparison with Fig. 5a. This distance is short enough and even after the faster crest has lost half of its speed it is still able to get close enough to the slower one so that both feel each other. However, they can not pass through each other. They reach a distance when the "repelling" part of the interaction potential becomes significant and the *underdog* slows its motion even further due to the repulsive force from the forerunner. The energy of the former is transferred to the latter which grows and becomes somewhat faster. Then the two structures slowly separate to a distance compatible with one of the local minima of the potential of interaction (as said in [37] one structure "nests" in the local minimum of the other). By then, they are already of permanent shape and move precisely with the same phase velocity. Thus they form a *bound state* which is *metastable* in the sense that it can be broken only by strong enough disturbance but not by an infinitesimal one. Further reduction of the initial separation reveals a genuine clash of the two structures during which they form a single hump (see the line corresponding to $t = 2$ in Fig. 5c). Hereafter the scenario is the same as the case in Fig. 5b. We proceeded keeping track on the evolution of the signal from Fig. 5c. The two large humps eventually attain the same terminal shape and phase velocity. They are rather similar to a sech shape but form a bound state. The third hump is growing on a much slower time scale than the one presented in the figure. It eventually reaches the same terminal speed and shape, but after a longer time and it lags far behind the other two. Here no triplet bound state is formed in the wake of the overtaking collision (Fig. 5c).

For larger ε the terminal shape becomes more complicated exhibiting deeper local minima. Then one expects tighter[4] bound states. Indeed, the results presented in Figs. 6a–d confirm this idea for the case $\varepsilon = 1$. Following overtaking the two structures form a tight bound state that moves as a single coherent structure with constant phase velocity.

The same effect is observed also for $\varepsilon = 2$ but the picture is much more complicated. As already mentioned, the value $\varepsilon = 2$ can be considered as the threshold of chaotic régime and beyond it the dynamics is richer. After overtaking different signals are excited and part of them form bound states with different degrees of "tightness". Especially instructive is the portion of the signal shown in Fig. 7 where one can see three fully developed permanent shapes: a single hump and two two-hump bound states with different distances between the humps. Obviously, the tighter bound state is formed when the forerunner structure "nests" in a local minimum that is closer to the centre of the trailing structure. The exhaustive taxonomy of the bound states and the investigation of the margins of their stability (metastability) go well beyond the scope of the present work and will be given elsewhere.

The results of the present subsection allow us to claim the coherent structures of dissipation modified KdVE as solitons. We extend thus the Zabusky and Kruskal's definition to waves that are not of rigorously permanent shape. There is nothing wrong in doing this, because even the *proton* is known to be a long lived though unstable particle that eventually decays. The only problem is the time scale, because physically speaking there is no "infinite" time for which the waves (or particles) could attain the truly permanent shapes. What we claim is that in times shorter than "practical infinity" (the latter represented by the inverse of the small parameter ε^{-1}), the interactions of solitary waves (or wave crests) is almost solitonic with tolerable degree of inelasticity. A difference with Zabusky and Kruskal's study is that here a recurrence of the initial state is impossible due to the dissipative nature of our equation. In very many experimental situations both in the lab [34–37,46,47,62,79–81] and in nature [63] the experimental time is short enough and what are observed are actually long lived transients of the above discussed type whose "aging" is slow enough to allow calling them solitons.

[4] One should be aware that words like "tighter", "looser", etc., only have meaning relatively to the scale of the solution. We appeal here to their intuitive meaning without going into quantitative details.

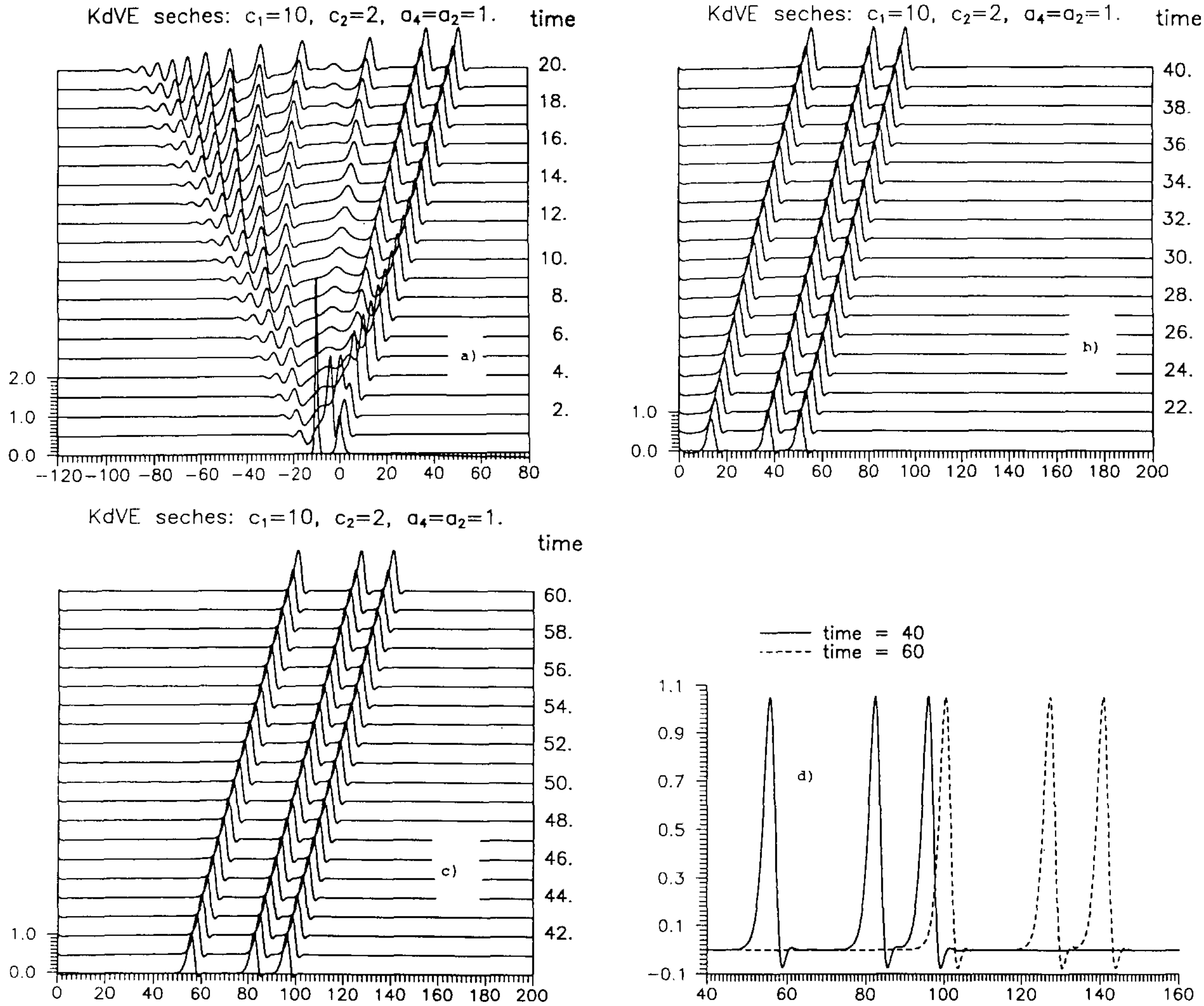

Fig. 6. Overtaking interactions of a sech with $c_1 = 10$ and $c_2 = 2$ in KdV-KSVE for $\varepsilon = \alpha_2 = \alpha_4 = 1$ and small initial separation. (a) $0 < t < 20$. (b) $20 < t < 40$. (c) $40 < t < 60$. (d) Comparison of the solution for $t = 40$ and $t = 60$.

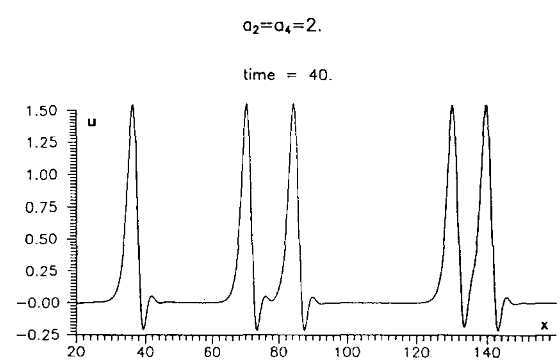

Fig. 7. The bound state for $\varepsilon = \alpha_2 = \alpha_4 = 2$ and $t = 40$.

3.3. Collisions of essentially dissipative solitons

In this subsection we turn to true permanent solutions of KdV–KSVE, i.e., we take infinity or steadiness in the strict mathematical sense. While the case with dissipation being relatively small disturbance is pertinent to the specific physical condition in thin liquid layers, the quest for permanent solutions and investigation of their collisions are of primary importance for extending the soliton paradigm into the realms of dissipation-dominated systems.

Permanent solitary waves of KSE are possible when the contributions of different terms (production, dissipation, nonlinearity) are in balance. If the dissipation enters the picture multiplied by a small parameter, then the permanent solution is necessarily a *short-wave* one, i.e., the permanent solutions of the intrinsically dissipative system under consideration are never long ones. Or vice versa – on the class of true permanent solutions, the KdV–KSVE (6) is intrinsically dissipative, rather than a dissipation modified essentially elastic system.

In a previous work [8] we had treated in detail the impact of Marangoni nonlinearity (proportional to α_5) on the shape of the solitary waves of KSVE. We concentrate in the present paper mostly on the effects contained in the original KS model and in the additional dispersion (proportional to α_3). For this reason we concern ourselves in what follows with the following values of the parameters:

$$\alpha_1 = 3\,, \quad \alpha_2 = \alpha_4 = 1\,, \quad \alpha_5 = 0\,. \tag{29}$$

We begin with the (α_3) interaction of two humps for the KSE case. A hump of permanent form is necessarily a homoclinic solution of the ODE to which the evolution equation reduces in the moving frame.

The problem of identification of homoclinics is an inverse one and needs special treatment. It turned out that a hump shape propagating to the right exists in KSE only for a single value of phase velocity $c = 1.216$ [18,73] although in certain papers [9,16,37], the possibility that the spectrum could be continuous was not excluded. Following [13] we provided here in Fig. 8 an explanation of this purely numerical effect. As it turns out, heteroclinics does exist for continuous spectrum of phase velocities and has exactly the same undulate forerunner as the homoclinics. Then if one matches a tail exponentially decaying towards $-\infty$ to arbitrary heteroclinics, one obtains a homoclinic shape which satisfies the equation everywhere, save the point of matching where only the second derivative is discontinuous. Then it is clear, that such a shape would give a very small residual if introduced into the equation (for the case $c = 1$ in Fig. 8 the integral of the square of the residual is approximately $4 \cdot 10^{-3}$). There is no surprise then, that if calculations are conducted with single precision, the artificial humps could be mistaken for a real homoclinic solution to the KSE. It took us special effort using refined calculations with double precision to rule out the homoclinics, except for the value $c = 1.216$ when the residual goes down to 10^{-9} [18].

Due to the symmetry of the problem the hump shape (homoclinics) exists also for $c = -1.216$, but with negative amplitude. The shape with positive phase velocity is very similar to one presented in Fig. 8 for $c = 1$ (see for exact picture [18]). Unlike the sech solitons of KdVE it exhibits a wavy forerunner. Since the magnitude of phase velocity is unique, then two coherent structures with phase velocities of the same sign could never collide. A possible interaction is the *head-on* collision of two shapes with opposite phase velocities.

The result is presented in Fig. 9 where the evolution of instability is clearly seen. The two humps collide and eventually form a single structure (see $t = 18{,}20$). On this new shape, the conservation of energy is not maintained. Rather, the balance law provides for the evolution of the total energy,

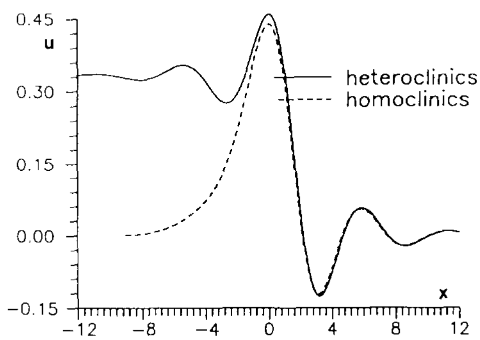

Fig. 8. Comparison of the real heteroclinics and the pseudo-homoclinics of KSE for $c = 1$.

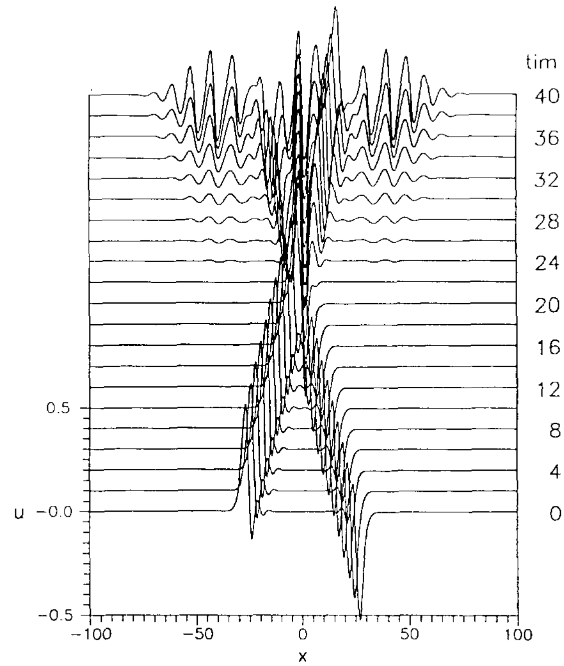

Fig. 9. The head-on collision of two hump-shape solutions of KSE, Eq. (3).

because the production and dissipation are no more in equilibrium, although they were equilibrated for each of the initial hump shapes and for their superposition when separated enough. After being kicked out of equilibrium the system cannot return and the two humps do not re-emerge after the collision. Rather a new solution is formed (see Fig. 9, $t \geq 28$) for which the production of energy slightly exceeds the dissipation and hence the energy increases with time. We have already mentioned that the production (connected to the second derivative of the shape) could only exceed the dissipation (related to the fourth derivative) if the wave length is long enough. That is exactly what happens in the sequence of Fig. 9. It is clearly seen that the said figure represents a slowly evolving signal which is not a spurious numerical artifact.

The fact that initial shapes disappear after the interaction is enough to discard the humps of KSE as true solitons, not to speak about the energy not being conserved in the course of collision. This does not, however, mean that a solitonic (in the sense of "particle-like") behaviour is strictly impossible in intrinsically dissipative systems. The problem with homoclinics of KSE is that its tail is so unstable that the shape is easily destroyed. An appropriate extension of KSE may very well cope with this difficulty. The other candidates for soliton title are the heteroclinic solutions (kinks) for which both the tails and forerunners are undulate and stable (at least – metastable). One can expect then, that the distortion of a heteroclinic shape due to the interaction with another one, should not be so dangerous and would not switch the instability mechanism. That is why, it is expected that the heteroclinic shapes should persist even after a collision.

The analytical kink solution (a combination of hyperbolic tangents) was obtained by Kuramoto and Tsuzuki [43]. The analytical solution was found to exist only for a single value of the phase velocity. For the more general equation KdV–KSVE the same kind of solution was found in [42]. Our previous numerical work [13] based on the Method of Variational Imbedding had shown that various kink solutions do exist for all phase velocities $c > 0.3$. It is interesting to note that for intermediate values of phase velocity even more than one kink solution appears. When $c < 0.3$, the limit of linear waves is approached and no localized solutions of kink type are numerically found. In Fig. 8 is presented one kink as found in [13]. This is the kink that persists for largest interval of phase velocity. The numerical results suggests that this solution is present for phase velocities up to $c = \infty$. In Fig. 10 are shown the shapes of kinks and their derivatives for different phase velocities. The dotted line presents the solution of an equation obtained by rescaling the variables $x_1 = c^{1/2}$, $u_1 = cu$, $u_1' = c^{3/4}u'$ of (7) and taking the limit $c \to \infty$.

In what follows we investigate the collisions of kinks (Fig. 10). We compose the initial condition of two kinks: the first (the left) between the levels $u_{\text{left}} = u_{-\infty}$ and $u_{\text{right}} = u_m$; and the second between $u_{\text{left}} = u_m$ and $u_{\text{right}} = u_{+\infty}$. We start with the case without dispersion. In all figures related to these results, both the shapes of kinks and the shapes of the first spatial derivative are illustrated. This means that for KdV–KSVE we can investigate the dynamics and interaction of kinks in a similar manner as it is usually done for the humps (seches of KdV). This is also physically clear because in a dissipative system to have a sustained structure one needs to pump energy. For the kinks the energy comes from the difference between the levels behind and ahead of the wave. The energy dissipated in the shock is compensated by the difference of the levels of the solution at left and right infinities ($u_{-\infty}$ and $u_{+\infty}$, respectively) and for each value of the difference between levels a respective permanent kink-shape is formed. Hence the KdV–KSVE is similar to the Burgers equation where shocks are formed between two stationary states [80]. Some comments on this point are given further down.

In Fig. 11a is shown the evolution of a system of two kinks both moving to the right (the spatial derivative is presented in Fig. 11b). The kink of higher amplitude is superimposed over the smaller kink. The faster kink catches up with the slower one and after a short collision time they form a single kink proceeding with slower phase velocity but with the same *momentum* (12) and a *mass* equal to the sum of masses of the two original kinks. This scenario is repeated when the two kinks are of equal amplitude. Once again the left kink is faster since it is between two higher levels (see Eq. (10) for the expression of the phase velocity). Even when the left kink has a smaller amplitude it still goes faster, because it is superimposed over the level of the right kink.

In Fig. 11c are shown the trajectories of the "centers" of the coherent structures (kinks). The center of a kink is defined here as the point where $u = (u_{\text{left}} + u_{\text{right}})/2$. If the center is defined as the maximum of slope (maximum of first spatial derivative), the results differ insignificantly. According to this definition after the two kinks fuse to form a single one, then two different points are referred to as centers. Although very close, the trajectories of these two points are different and this is seen in the figures as two parallel lines that are very close to each other.

Although the interaction is completely inelastic, one can see that what can be conserved is indeed conserved and the trajectories exhibit no phase shift. The notion of phase shift, however, needs some clarification in this case. In the lowermost figure are presented the trajectories of the kinks. In addition, the trajectories (dashed lines) of non-interacting kinks are also added. The dashed line in the middle of corresponding figure represents the estimate for the trajectory of a single kink with *mass* that is the sum of two initial *masses* and phase velocity defined from the total *momentum* when that said "compound" particle commences its motion from the position of the "mass center" of the system of two initial kinks. One sees that after the composite coherent structure (the composite *particle*) is formed in the numerical simulations, it proceeds exactly (within the error of the scheme) alongside the projected trajectory of the single *particle*-kink containing the total *mass* of the system. In this sense there is no phase shift in

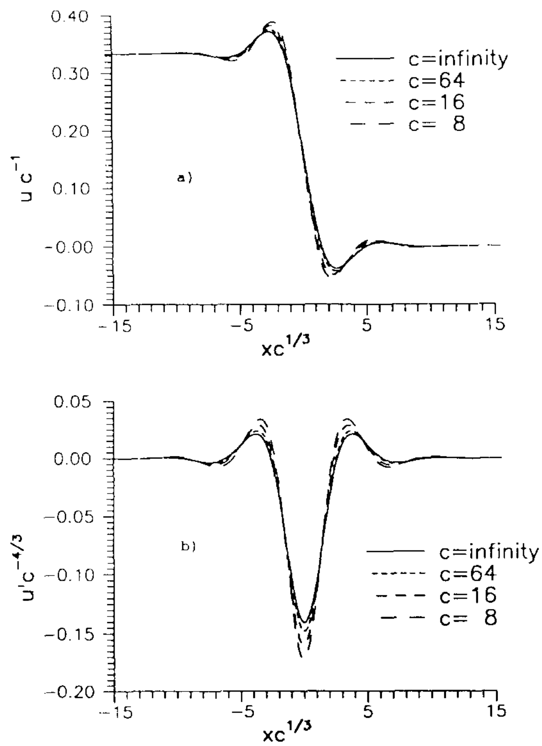

Fig. 10. The kink-shape solution of KSE Eq. (3) for various values of phase velocity. The function, derivative and argument are scaled by $c, c^{4/3}$ and $c^{-1/3}$. (a) The kink shape (b) The shape of the derivative.

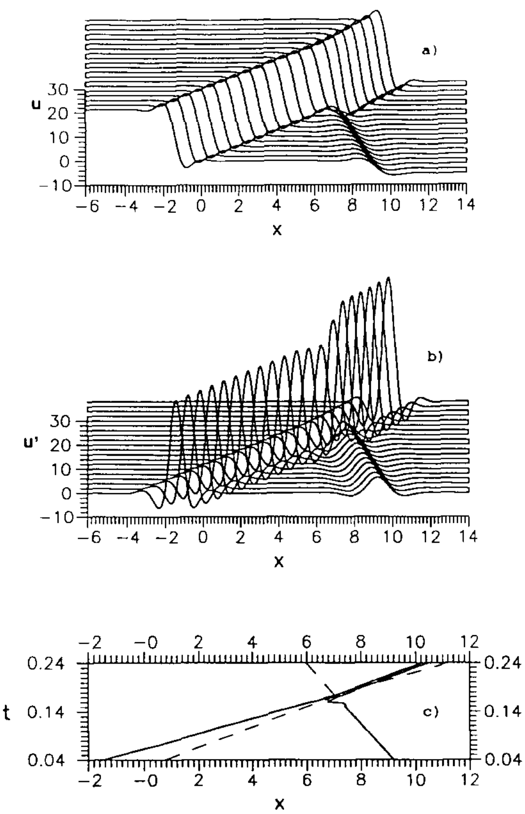

Fig. 11. Overtaking collision of two kinks in KSE Eq. (3): $u_{-\infty} = \frac{96}{3}$, $u_m = \frac{16}{3}$, $u_{+\infty} = 0$. (a) The shape of the solution u. (b) The shape of the derivative u_x. (c) Trajectories of the "centers" of solitons.

KSE. The total (*pseudo*)*energy* is not conserved during the collision and the final state has the same *momentum* but different (*pseudo*)*energy*. In fact part of the initial (*pseudo*)*energy* of the system of two kinks has been transformed into "internal energy" of the deformation suffered by the wave profile.

The picture is the same if we consider the head-on collision of kinks. In Figs. 12a, b is shown the case without dispersion for two kinks moving with opposite but equal speeds. Once again the interaction is inelastic and there is no phase shift. In fact the motion stops completely because of the conservation of momentum as shown in the figure. We may speak of "completely inelastic interaction". Note that incorporating dispersion ($\alpha_3 \neq 0$) does not change the character of the interaction leaving it inelastic (see Fig. 13). Only the shape of the kink becomes more wavy. For an essentially dissipative evolution the dispersion is a regular small perturbation that changes the quantitative results but does not interfere with the main qualitative traits of the behaviour of solitary waves. To stress the completely inelastic nature of the interactions among the coherent structures of the intrinsically dissipative system as KdV–KSVE we prefer to use the coinage clayons stemming from the notion that after the collision two solitary waves of KdV–KSVE stick to each other as two clay balls [38,80].

The results concerning the shape of the bore (Fig. 13) are of intrinsic interest irrespective of their behaviour in interactions. Localized solutions like stationary propagating bores (hydraulic jumps) do not exist for the KDVE [82], because the undulate tail does not decay at $-\infty$. As shown numerically by

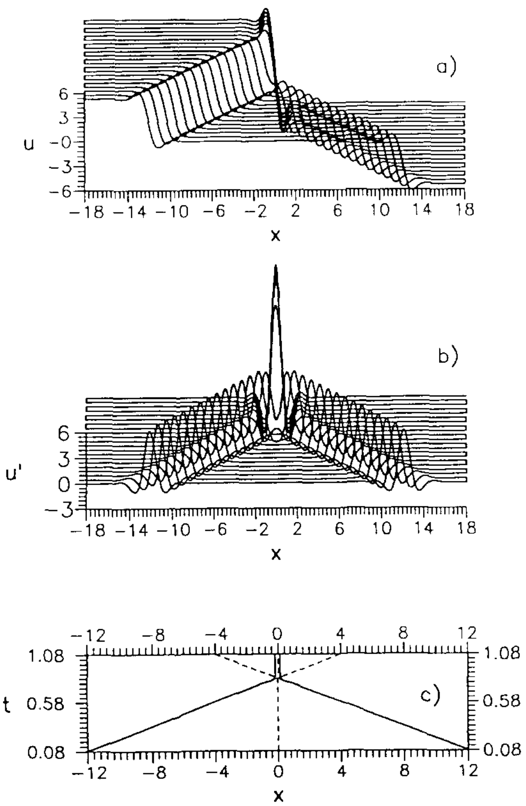

Fig. 12. Head-on collision of two kinks in KSE Eq. (3): $u_{-\infty}=\frac{16}{3}$, $u_m=0$, $U_{+\infty}=-\frac{16}{3}$. (a) The shape of the solution u. (b) The shape of the derivative u_x. (c) Trajectories of the "centers" of solitons.

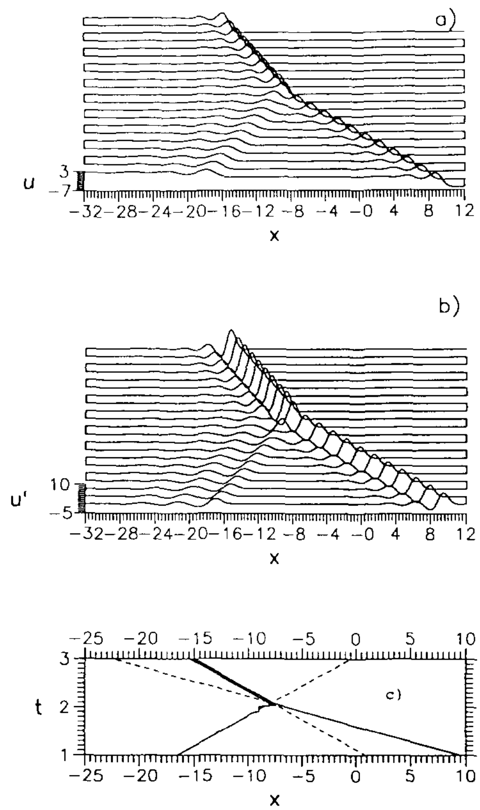

Fig. 13. Head-on collision of two kinks in KdV-KSVE Eq. (6): $\alpha_1=3$, $\alpha_3=3$, $u_{-\infty}=\frac{8}{3}$, $u_m=0$, $u_{+\infty}=-\frac{16}{3}$. (a) The shape of the solutions u. (b) The shape of the derivative u_x. (c) Trajectories of the "centers" of solitons.

Peregrine [56], the undular bore is not a stationary solution, rather its wavy tail spans ever larger intervals with time. On can "localize" the bore by adding dissipation. It was done by Johnson [31] who discussed the solutions of a Korteweg–de Vries–Burgers equation KdVBE in which dissipation is represented by the usual (Burgers) second-order spatial derivative. He had shown, however, that there was a threshold for the amplitude of longitudinal velocity beyond which the sign of the second derivative changes to the improper one (which is in fact the sign in KSE). Hence our bores are of radically different nature than the bores of [31], the former being damped by the fourth derivative and excited by the term which played a damping role in [31]. However, from the point of view of the universal, paradigmatic approach taken here the two cases are rather similar. Indeed, on the one hand the presence of dissipation is what makes a stationary bore possible and on the other hand an increase of dissipation decreases the undulations of the bore (according to our view, reduces the time allowed for the practical permanence of the bore). Vice versa, increasing the dispersion parameter increases the undulations of the bore. The only difference here is that due to the production term, the bore is never monotonic in the KdV–KSVE even for vanishingly small values of $\alpha_3=0$.

4. Concluding remarks

Three decades ago Zabusky and Kruskal [84,83] pioneered the work on solitons by exploring the behaviour of solutions of KdVE, which is a mathematical object containing an appropriate (local) balance between nonlinearity and dispersion. This balance allows for traveling permanent localized structures (solitary waves and crests of periodic wave trains/series of solitons) that behave upon collisions as stable particles. Hence the coinage *soliton* for designating a solitary wave-particle. The Zabusky–Kruskal's and FPU's precursor computer experiments were of those cases where numerical computations revealed new and unexpected results. The KdVE was later found not to be an isolated curiosity and a new scientific field was uncovered. Their physical problem involved no dissipation as it referred to a collisionless plasma kinetic model.

In the present paper we have taken up Zabusky and Kruskal's problem, acknowledging dissipation as it arises in the description of some physical problems, e.g., surface waves or internal waves in driven systems which obey the KdV–KSV equation (6).

First we have treated the case of physical relevance when the coefficient of KS part of the equation is small, i.e., when dissipation slightly modifies the KdVE with a smallness parameter ε. For this case we have shown that:

(1) The (nearly) solitonic behaviour is preserved for times smaller than the characteristic time defined by the inverse of the small parameter ε. There is a difference: the KdVE seches start "aging" when introduced into KdV–KSVE and eventually assume the permanent shapes that are solutions of KdV–KSVE.

(2) Selection of the phase velocity and a discrete spectrum is observed due to the production–dissipation mechanism which although an ε-small disturbance, destroys the integrability of the system. Each initial shape eventually ends up in one of the permanent shapes propagating with one of the phase velocities (terminal velocity) from the discrete spectrum.

(3) If the KdVE-seches are initially close enough, they interact (almost) as solitons. After collision they transform into permanent shapes propagating with the terminal velocity and form a bound state which propagates as a single structure. If initially they are not close enough, they evolve into the permanent shape before they reach each other and form the bound state without passing through each other (note the similarity with van der Waals bound states formed with imperfect particles due to infinitely long ranged weakly attractive forces [33]. Hard spheres, perfect particles cannot form bound states and hence do not allow gas-liquid condensation).

(4) Increasing the value of the smallness parameter ε controlling the KS part exaggerates the described effects and a threshold is reached past which the solution goes chaotic.

We have also considered the dissipation-modified KdVE (6) as a mathematical object worth exploring in itself, i.e., as a prototype of a non-integrable model problem with no conservation laws, but rather with input-dissipation energy balance. In doing so we depart from underlying specific physics, but try to uncover possibly new phenomena of some universality of hopefully genuine dissipative nonlinear wave equations.

We have discovered that:

(5) The production–dissipation energy balance is capable of sustaining permanent shapes just as the nonlinearity-dispersion balance does in integrable systems like KdVE. The permanent hump shapes of KSE are represented by a single homoclinic solution with dimensionless phase velocity $c = \pm$

1.216. The more interesting is the class of heteroclinic permanent shapes (bores/kinks/hydraulic jumps) which are found to exist for $c > 0.3$.

(6) Upon head-on collision the hump shapes do not recover and yield to an ever-evolving wave pattern which is not of permanent shape. In this sense the hump does not seem to qualify for a soliton.

(7) The interaction of kinks produces a permanent shape of the same class (kinks), thus exhibiting particle-like behaviour. However, the interaction here is *completely inelastic* in the sense that after the collision a larger kink is formed from the two initial kinks and this new wave-particle proceeds with phase velocity defined by the conservation of (*pseudo*)*momentum*. This justifies calling the kinks "inelastic solitons" (or "clayons" since they behave as two clay balls [80]).

(8) The soliton/clayon properties of kinks are not disproved by the presence of dispersion. Simply, the shapes become more undulate, but the interaction remains completely inelastic even for dispersion coefficients 20 times larger than the production–dissipation coefficient ε. This means that in the class of permanent kink solutions the KdV–KSVE is essentially dissipative (parabolic).

(9) Items (5)–(8) allow us to conclude that the notion of soliton (wave-particle) can be extended to dissipative systems, but the wave-particles collide in a completely inelastic manner forming upon collision a larger particle that carries the total momentum of the system.

We used fully implicit four-stage-in-time difference scheme with Newton's quasi-linearization of the nonlinear terms. The practical stability of the scheme is virtually unlimited which allowed us to go after the very-long time evolution of the solution.

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Note added in proof

Since the manuscript was finished we became aware of the following references that with theory, experiment or both, bear upon similar issues of our text.

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