# Discrete-Time Recurrent Neural Networks With Complex-Valued Linear Threshold Neurons

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***Abstract*—This brief discusses a class of discrete-time recurrent neural networks with complex-valued linear threshold neurons. It addresses the boundedness, global attractivity, and complete stability of such networks. Some conditions for those properties are also derived. Examples and simulation results are used to illustrate the theory.**

***Index Terms*—Complex-valued neural networks (NNs), discrete-time recurrent neural networks (RNNs), linear threshold (LT).**

## I. Introduction

COMPLEX number calculus has been found useful in such areas as electrical engineering, informatics, control engineering, bioengineering, and other related fields. It is therefore not surprising to see that complex-valued neural networks (NNs), which deal with complex-valued data, complex-valued weights, and neuron activation functions, have also been widely studied in recent years [1], [2].

There exist two main approaches to the analysis and application of such networks: The first one takes advantage of the error-correction algorithm [3] that attempts to minimize the cumulative cycle error between the real network output and the target output by adjusting the network parameters in the hidden and output layers. Some applications include nonlinear filters [4]–[7], classifiers [8], [9], and time series prediction [10].

The second approach is to treat a complex-valued NN as a recurrent dynamic system that is able to acquire some useful properties without supervised training. This approach is often related to associative memory design [2], [11]–[13]. While stability analysis and qualitative analysis are popular and well developed in real-valued recurrent dynamic NNs, research into complex-valued networks as dynamic systems has not progressed equally fast.

In this brief, we propose a class of discrete-time recurrent NNs (RNNs) with complex-valued linear threshold (CLT) neurons that is described by the following equation:

$$z(k+1) = W\sigma\left(z(k)\right) + H \tag{1}$$

or its equivalent component form

$$z_i(k+1) = \sum_{j=1}^{N} w_{ij}\sigma\left(z_j(k)\right) + h_i \tag{2}$$

for $k \geq 0$, where each $z_i$ denotes the activity of neuron $i$ $(1 \leq i \leq N)$; $z(k)$ and $H$ are $n \times 1$ vectors, i.e., $z(k) = [z_1(k), \ldots, z_N(k)]^T$ and $H = [h_1, \ldots, h_N]^T$; and CLT function $\sigma$ is defined as

$$\sigma(z) = \max\left(0, \mathrm{Re}(z)\right) + i \cdot \max\left(0, \mathrm{Im}(z)\right), \qquad z \in \mathbb{C}$$

which is continuous, unbounded, and nondifferentiable.

Many researchers have recently studied RNNs with linear threshold (LT) neurons [14]–[20]. The dynamical properties of RNNs play primarily important roles in their applications. Compared with other NNs with nonlinear activation functions, one advantage of the LT NNs is that the network can be looked at as a linear system if each neuron's output is always greater than or less than zero, and LT NNs will change one linear system to another similar linear system only when some neurons' outputs switch on the zero boundary. Therefore, the LT NNs can accurately be described indeed by a linear system by using the effective recurrence matrix, instead of the original network weight [15], [16], which also offers the possibility of using some linear system dynamical analysis for LT NNs. Some related work can be seen in [14]–[19].

This brief focuses on the convergence analysis of discrete-time CLT RNNs. Some related work on discrete-time RNNs can be found in [21]–[26]. Our work can be looked at as extended work from the previous real-valued LT RNNs [19] to the complex-valued ones. Since a complex number exhibits strong component correlation between its real and imaginary parts, we cannot analyze the whole RNNs just as two separate systems by simply splitting a complex value into real and imaginary parts. For example, if we want to use the first approach previously noted, split-complex neuron activation functions are not analytic, and hence, the Cauchy–Riemann equations do not apply. In our research, we keep this fully complex nature by building an energy function, which not only integrates the influence on the whole dynamic system from both parts but can also be used to perfectly estimate system properties. Based on the energy minimization performed by the proposed network, we present some stability criteria, which show that our discrete-time CLT networks have similar properties as the real-valued ones. Simulation results illustrate these properties.

The rest of this brief is organized as follows: Preliminaries are given in Section II. In Section III, a theoretical analysis of the network is given, which includes boundedness, global attractivity, and complete stability of the proposed network. Simulations and illustrative examples are presented in Section IV. Conclusions are given in Section V.

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## II. Preliminaries

In this section, we provide preliminaries, which will be used here to establish the theory.

For any $c \in \mathbb{C}^n$, we also denote

$$c^* = (\bar{c})^T$$

where $\bar{c}$ is the conjugate of $c$.

*Lemma 1:* Given any $z \in \mathbb{C}$, the following holds:

$$\sigma^*(z) = \sigma^*(\mathrm{Re}(z)) - i\sigma^*(\mathrm{Im}(z)).$$

*Proof:* We assume that $z = a + ib$, where $a, b \in \mathbb{R}$.

Case 1) $a, b \le 0$. Then

$$\sigma^*(z) = 0 = \sigma^*(\mathrm{Re}(z)) - i\sigma^*(\mathrm{Im}(z)).$$

Case 2) $a > 0, b \le 0$. Then

$$\sigma^*(z) = a = \sigma^*(\mathrm{Re}(z)) - i\sigma^*(\mathrm{Im}(z)).$$

The remaining two conditions, i.e., Case 3 ($a \le 0, b > 0$) and Case 4 ($a, b > 0$), can also be easily proven.

The proof is completed.

*Lemma 2:* The following holds:

$$\mathrm{Re}\left(\int_{z_1}^{z_2} [\sigma^*(s) - \sigma^*(z_1)]\, ds\right) \ge \frac{1}{2}(\sigma^*(z_2) - \sigma^*(z_1)) \cdot (\sigma(z_2) - \sigma(z_1))$$

for all $z_1, z_2 \in \mathbb{C}$.

*Proof:* We assume that $z_1 = a + ib$ and $z_2 = c + id$, where $a, b, c, d \in \mathbb{R}$. According to the signs of $a$, $b$, $c$, and $d$, we need to consider 16 conditions. Since four conditions of $z_1, z_2 \in \mathbb{R}$ have been proven in [19], we just need to consider the remaining 12 conditions.

1) $a \le 0, b > 0, c \le 0$, and $d \le 0$. Then, by Lemma 1

$$\frac{1}{2}(\sigma^*(z_2) - \sigma^*(z_1))(\sigma(z_2) - \sigma(z_1)) = \frac{1}{2}b^2,$$

$$\begin{aligned} \mathrm{Re}&\left(\int_{z_1}^{z_2} [\sigma^*(s) - \sigma^* z_1)]\, ds\right) \\ &= \mathrm{Re}\left(\int_{z_1}^{z_2} \sigma^*(s)ds - \sigma^*(z_1)(z_2 - z_1)\right) \\ &= -\frac{1}{2}b^2 + b^2 - bd \ge \frac{1}{2}b^2. \end{aligned}$$

2) $a > 0, b > 0, c \le 0$, and $d \le 0$. Then, we have

$$\frac{1}{2}(\sigma^*(z_2) - \sigma^*(z_1))(\sigma(z_2) - \sigma(z_1)) = \frac{1}{2}(a^2 + b^2),$$

$$\begin{aligned} \mathrm{Re}&\left(\int_{z_1}^{z_2} [\sigma^*(s) - \sigma^*(z_1)]\, ds\right) \\ &= -\frac{1}{2}a^2 - \frac{1}{2}b^2 + a^2 - ac + b^2 - bd \ge \frac{1}{2}(a^2 + b^2). \end{aligned}$$

For simplicity, we only prove the preceding two conditions; the other ten conditions can also be easily proven.

This proof is completed.

*Definition 1:* Network (2) is said to be bounded if each trajectory is bounded.

*Definition 2:* Let $S$ be a compact subset of network (2). We denote the $\epsilon$ neighborhood of $S$ by $S_\epsilon$. Compact set $S$ is said to globally attract network (2) if, for any $\epsilon \ge 0$, all trajectories of network (2) ultimately enter and remain in $S_\epsilon$. Set $S$ is called an attractive set.

Obviously, if every trajectory $z(k)$ of network (2) satisfies

$$\begin{cases} \lim\limits_{k\to+\infty} \sup z(k) \in S \\ \lim\limits_{k\to+\infty} \inf z(k) \in S, \end{cases}$$

then $S$ globally attracts network (2).

*Definition 3:* Vector $z^\star$ is called an equilibrium point (fixed point) of (2) if it satisfies

$$z^\star = W\sigma(z^\star) + H.$$

We denote the set of equilibrium points of network (2) as $\Omega$.

*Definition 4:* An equilibrium point $z^\star$ is said to be stable (in the sense of Lyapunov) if the following statement is true:

For every $\epsilon > 0$, there exists $\delta > 0$ such that every solution $z(k)$ with $\|z(0) - z^\star\| < \delta$ exists for all $k \ge 0$ and satisfies the inequality $\|z(k) - z^\star\| < \epsilon$ for $k \ge 0$. The norm $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^n$ or $\mathbb{C}^n$.

*Definition 5:* Network (2) is said to be completely convergent (completely stable) if each trajectory $z(k)$ satisfies

$$dist\,(z(k), \Omega) \triangleq \min_{z^\star \in \Omega} \|z(k) - z^\star\| \to 0$$

as $k \to +\infty$.

Throughout this brief, for all constant $c \in \mathbb{C}$, denote

$$\begin{cases} \mathrm{Re}^+(c) = \max(0, \mathrm{Re}(c)) \\ \mathrm{Re}^-(c) = \min(0, \mathrm{Re}(c)) \\ \mathrm{Im}^+(c) = \max(0, \mathrm{Im}(c)) \\ \mathrm{Im}^-(c) = \min(0, \mathrm{Im}(c)). \end{cases}$$

Obviously, $\mathrm{Re}^+(c), \mathrm{Im}^+(c) \ge 0$, $\mathrm{Re}^-(c)$, and $\mathrm{Im}^-(c) \le 0$.

## III. Properties of Discrete-Time Complex RNNs With LT Neurons

In this section, conditions guaranteeing the boundedness of network (2) are derived. Moreover, a compact set that globally attracts all the trajectories of network (2) is represented in an explicit expression. Afterward, complete stability conditions of network (2) are given to allow network (2) to possess multiequilibrium points.

*Theorem 1:* If there exist constants $\alpha_i > 0 (i = 1, \ldots, N)$ such that

$$\frac{1}{\alpha_i}\sum_{j=1}^{N} \alpha_j \left(\mathrm{Re}^+(w_{ij}) + |\mathrm{Im}(w_{ij})|\right) < 1, \qquad (i = 1, \ldots, N)$$

then network (2) is bounded, and the compact set

$$S = \{z | a_i \le \mathrm{Re}(z_i) + \mathrm{Im}(z_i) \le b_i, 1 \le i \le N\}$$

globally attracts the network, where

$$\begin{cases} b_i = \frac{\alpha_i}{1-\gamma} \cdot \max\limits_{1\le j\le N} \left\{\frac{\mathrm{Re}^+(h_j) + \mathrm{Im}^+(h_j)}{\alpha_j}\right\} \\ a_i = \sum\limits_{j=1}^{N} \left(\mathrm{Re}^-(w_{ij}) - |\mathrm{Im}(w_{ij})|\right) b_j + \left(\mathrm{Re}^-(h_i) + \mathrm{Im}^-(h_i)\right) \\ \gamma = \max\limits_{1\le i\le N} \left\{\frac{1}{\alpha_i}\sum\limits_{j=1}^{N} \alpha_j \left(\mathrm{Re}^+(w_{ij}) + |\mathrm{Im}(w_{ij})|\right)\right\}. \end{cases}$$

*Proof:* Clearly, we have the expressions for $\mathrm{Re}\,(z_i(k+1))$ and $\mathrm{Im}\,(z_i(k+1))$, shown at the bottom of the page. Then, we have the expressions for $\sigma(\mathrm{Re}\,(z_i(k+1)))$ and $\sigma\,(\mathrm{Im}\,(z_i(k+1)))$, shown at the bottom of the page, for $i = 1,\ldots,N$.

Define

$$v_i(k) = \frac{1}{\alpha_i}\left(\sigma\left(\mathrm{Re}\,(z_i(k))\right) + \sigma\left(\mathrm{Im}\,(z_i(k))\right)\right), \qquad (i = 1,\ldots,N)$$

for all $k \geq 0$.

Then, we have

$$\begin{aligned} v_i(k+1) &\leq \sigma\left\{\frac{1}{\alpha_i}\sum_{j=1}^{N}\alpha_j\cdot\left(\mathrm{Re}^+(w_{ij}) + |\mathrm{Im}(w_{ij})|\right)\cdot v_j(k) + \frac{\mathrm{Re}^+(h_i)+\mathrm{Im}^+(h_i)}{\alpha_i}\right\} \\ &= \frac{1}{\alpha_i}\sum_{j=1}^{N}\alpha_j\cdot\left(\mathrm{Re}^+(w_{ij}) + |\mathrm{Im}(w_{ij})|\right)\cdot v_j(k) + \frac{\mathrm{Re}^+(h_i)+\mathrm{Im}^+(h_i)}{\alpha_i} \end{aligned}$$

for $k \geq 0$.

Define

$$V(k) = \max_{1\leq i\leq N}\{v_i(k)\}$$

for $k \geq 0$.

Using the similar method referred to in [19], we can get

$$\begin{aligned} V(k+1) &\leq \gamma\cdot V(k) + \max_{1\leq j\leq N}\left\{\frac{\mathrm{Re}^+(h_j)+\mathrm{Im}^+(h_j)}{\alpha_j}\right\} \\ &\leq \gamma^{k+1}V(0) + \frac{1}{1-\gamma}\max_{1\leq j\leq N}\left\{\frac{\mathrm{Re}^+(h_j)+\mathrm{Im}^+(h_j)}{\alpha_j}\right\}, \\ \lim_{k\to+\infty}&\sup\left(\mathrm{Re}\,(z_i(k+1)) + \mathrm{Im}\,(z_i\,(k+1))\right)) \\ &\leq \lim_{k\to+\infty}\sup\left(\sigma\left(\mathrm{Re}\,(z_i(k+1))\right) + \sigma\left(\mathrm{Im}\,(z_i(k+1))\right)\right)) \\ &\leq \frac{\alpha_i}{1-\gamma}\cdot\max_{1\leq j\leq N}\left\{\frac{\mathrm{Re}^+(h_j)+\mathrm{Im}^+(h_j)}{\alpha_j}\right\} = b_i. \end{aligned}$$

On the other hand, it can easily be proven that

$$\begin{aligned} \lim_{k\to+\infty}&\inf\left(\mathrm{Re}\,(z_i(k+1)) + \mathrm{Im}\,(z_i(k+1))\right) \\ &\geq \sum_{j=1}^{N}\left(\mathrm{Re}^-(w_{ij}) - |\mathrm{Im}(w_{ij})|\right)\cdot b_j \\ &\quad + \mathrm{Re}^-(h_i) + \mathrm{Im}^-(h_i) \\ &= a_i. \end{aligned}$$

This shows that compact set $S$ globally attracts any trajectory of network (2).

The proof is completed.

*Theorem 2:* Suppose the network is bounded. If there exists a diagonal positive-definite matrix $D$ such that $DW$ is a Hermitian matrix and $D(I+W)$ is a symmetric positive-definite matrix, then network (2) is completely convergent.

*Proof:* Suppose that $D = \mathrm{diag}(d_1,\ldots,d_N)$. Let us construct the following energy function:

$$\begin{aligned} E(k) = &-\frac{1}{2}\sigma^*\left(z(k)\right)DW\sigma\left(z(k)\right) \\ &-\frac{1}{2}\sigma^*\left(z(k)\right)DH + \frac{1}{2}\sigma^*\left(z(k)\right)Dz(k) \\ &-\frac{1}{2}z^{-1/2}\sum_{i=1}^{N}d_i\int_0^{z_i(k)}\sigma^*(s)ds \\ &+\left(-\frac{1}{2}\sigma^*\left(z(k)\right)DH + \frac{1}{2}\sigma^*\left(z(k)\right)Dz(k)\right. \\ &\left.\quad -\frac{1}{2}z^{-1/2}\sum_{i=1}^{N}d_i\int_0^{z_i(k)}\sigma^*(s)ds\right)^*. \end{aligned}$$

Clearly, energy functional $E(k)$ is real valued as long as the synapic matrix $DW$ is a Hermitian matrix.

Then, we have

$$\begin{aligned} \Delta E &= E(k+1) - E(k) \\ &= -\frac{1}{2}Z^*(k+1)DWZ(k+1) \\ &\quad + M(k+1) + M^*(k+1) \\ &= -\frac{1}{2}Z^*(k+1)DWZ(k+1) + 2\mathrm{Re}\left(M(k+1)\right) \qquad (3) \end{aligned}$$

$$\begin{cases} \mathrm{Re}\,(z_i(k+1)) \leq \sum_{j=1}^{N}\left[\mathrm{Re}^+(w_{ij})\mathrm{Re}\left(\sigma\left(z_j(k)\right)\right) - \mathrm{Im}^-(w_{ij})\mathrm{Im}\left(\sigma\left(z_j(k)\right)\right)\right] + \mathrm{Re}^+(h_i) \\ \mathrm{Im}\,(z_i(k+1)) \leq \sum_{j=1}^{N}\left[\mathrm{Re}^+(w_{ij})\mathrm{Im}\left(\sigma\left(z_j(k)\right)\right) + \mathrm{Im}^+(w_{ij})\mathrm{Re}\left(\sigma\left(z_j(k)\right)\right)\right] + \mathrm{Im}^+(h_i) \end{cases}$$

$$\begin{cases} \sigma(\mathrm{Re}\,(z_i(k+1))) \leq \sum_{j=1}^{N}\left[\mathrm{Re}^+(w_{ij})\mathrm{Re}\left(\sigma\left(z_j(k)\right)\right) + |\mathrm{Im}(w_{ij})|\,\sigma\left(\mathrm{Im}\left(\sigma\left(z_j(k)\right)\right)\right)\right] + \mathrm{Re}^+(h_i) \\ \sigma\left(\mathrm{Im}\,(z_i(k+1))\right) \leq \sum_{j=1}^{N}\left[\mathrm{Re}^+(w_{ij})\sigma\left(\mathrm{Im}\left(\sigma\left(z_j(k)\right)\right)\right) + \left|\mathrm{Im}^+(w_{ij})\right|\sigma\left(\mathrm{Re}\left(\sigma\left(z_j(k)\right)\right)\right)\right] + \mathrm{Im}^+(h_i) \end{cases}$$

for all $k \geq 0$, where

$$\begin{aligned}
Z(k+1) =& \sigma\left(z(k+1)\right) - \sigma\left(z(k)\right),\\
M(k+1) =& -\frac{1}{2}\sigma^*\left(z(k+1)\right) DW\left(\sigma\left(z(k)\right)\right)\\
&+\frac{1}{2}\sigma^*\left(z(k)\right) DW\sigma\left(z(k)\right)\\
&-\frac{1}{2}\left(\sigma\left(z(k+1)\right)-\sigma\left(z(k)\right)\right)^* DH\\
&+\frac{1}{2}\sigma^*\left(z(k+1)\right) Dz(k+1)\\
&-\frac{1}{2}\sigma^*\left(z(k)\right) Dz(k)\\
&-\frac{1}{2}z^{-1/2}\sum_{i=1}^{N} d_i \int\limits_{z_i(k)}^{z_i(k+1)} \sigma^*(s)ds.
\end{aligned}$$

For the term $M(k+1)$, we have

$$\begin{aligned}
M(k+1) =& -\frac{1}{2}z^{-1/2}\sum_{i=1}^{N} d_i \int\limits_{z_i(k)}^{z_i(k+1)} \left[\sigma^*(s) - \sigma^*\left(z_i(k)\right)\right] ds\\
=& -\frac{1}{2}\sum_{i=1}^{N} d_i \Delta M_i(k+1)
\end{aligned}$$

for all $k \geq 0$.

By Lemma 2, we have

$$\begin{aligned}
\mathrm{Re}\left(M(k+1)\right) =& \mathrm{Re}\left(-\frac{1}{2}\sum_{i=1}^{N} d_i \Delta M_i(k+1)\right)\\
\leq& -\frac{1}{4}Z^*(k+1)DZ(k+1) \leq 0 \qquad (4)
\end{aligned}$$

for all $k \geq 0$, which means that the term $M(k+1)$ is always located in the right half-plane of the complex plane.

From (3) and (4), we have

$$\Delta E = -\frac{1}{2}Z^*(k+1)D(W+I)Z(k+1) \leq 0.$$

Therefore, $E(k)$ is monotonously decreasing. Especially, if $\Delta E = 0$, we have $\sigma(z(k+1)) = \sigma(z(k))$, which means that

$$z_i^\star = z_i(k+1) = \sigma\left(z_i(k)\right)$$

for all $i = 1, \ldots, N$. The network is then convergent.

This completes the proof.

## IV. Simulation

In this section, we provide simulation results to illustrate and verify the theory developed.

*Example 1:* Consider a CLT NN with two neurons

$$z(k+1) = W\sigma\left(z(k)\right) + H \qquad (5)$$

where

$$W = \begin{bmatrix} -0.6356 & -1.6725 - 0.5000i \\ -1.6725 + 0.5000i & -0.7405 \end{bmatrix}$$
$$H = \left[1.7515 + 0.2730i \quad 1.4746 + 0.0235i\right]^T.$$

Clearly, network (5) satisfies Theorem 1; thus, it is bounded, and there exists a compact set to attract all the trajectories of the network. Simple calculations yield $a_1 = -10.1098$, $a_2 = -9.8603$, and $b_1 = b_2 = 3.5030$. Thus,

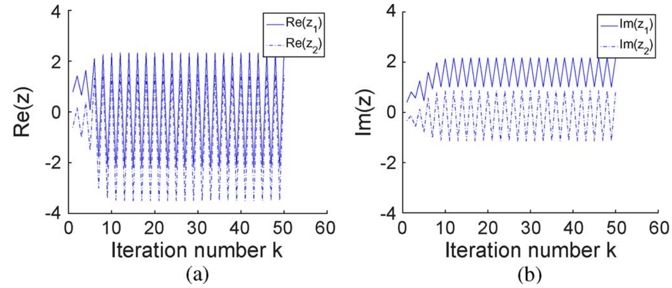


Fig. 1. Periodic trajectory of network (5). (a) Real part activities of $z_1$ and $z_2$. (b) Imaginary part activities of $z_1$ and $z_2$.

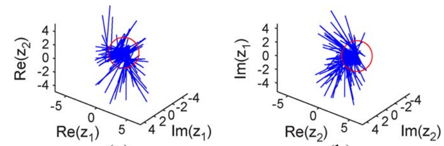


Fig. 2. Global attractivity and complete stability of network (6). The attractive regions are marked with a circle according to the activities of $z_1$ and $z_2$. (a) Trajectories projected on the $\mathrm{Re}(z_1) - \mathrm{Im}(z_1) - \mathrm{Re}(z_2)$ subspace. (b) Trajectories projected on the $\mathrm{Re}(z_2) - \mathrm{Im}(z_2) - \mathrm{Im}(z_1)$ subspace. Here, network (6) is a monostable network.

we have an attracting set $S = \{z| -10.1098 \leq \mathrm{Re}(z_1) + \mathrm{Im}(z_1) \leq 3.5030, -9.8603 \leq \mathrm{Re}(z_2) + \mathrm{Im}(z_2) \leq 3.5030\}$.

We choose the initial value $z(0) = [0.7878 + 0.40255i - 0.60172 - 0.32289i]^T$. By simulation, we can find that this network has two period-2 fixed points (Fig. 1): $z_1^\star = [2.1891 + 1.0103i, -3.497 - 1.1363i]^T$ and $z_2^\star = [2.3197 + 2.1735i, 1.4746 + 0.86495i]^T$.

*Example 2:* Consider a CLT NN with two neurons

$$z(k+1) = W\sigma\left(z(k)\right) + H \qquad (6)$$

where

$$W = \begin{bmatrix} -0.1589 & -0.4181 - 0.3500i \\ -0.4181 + 0.3500i & -0.1851 \end{bmatrix}$$
$$H = \left[0.7519 + 1.2049i \quad 1.8203 + 1.6188i\right]^T.$$

Clearly, network (6) satisfies Theorems 1 and 2, so it is completely stable. By simple calculations, we have $a_1 = a_2 = -4.9047$ and $b_1 = b_2 = 5.2909$. Thus, we have an attracting set $S = \{z| -4.9047 \leq \mathrm{Re}(z_j) + \mathrm{Im}(z_j) \leq 5.2909, j = 1, 2.\}$. Fig. 2 shows the complete convergence of network (6) for 100 trajectories originating from randomly selected initial points in $\mathrm{Re}(s(0)), \mathrm{Im}(s(0)), \in S$, and we find one fixed point from the simulations

$$z^\star = \left[0.64732 - 0.10594i \quad 1.2763 - 1.5198i\right]^T.$$

*Example 3:* Consider a CLT NN with four neurons

$$z(k+1) = W\sigma\left(z(k)\right) + H \qquad (7)$$

where

$$W = \begin{bmatrix} 0.41 & -0.01i & -1 & 0 \\ 0.01i & -0.41 & 0 & -1 \\ -1 & 0 & 0.41 & -0.01i \\ 0 & -1 & -0.01i & 0.41 \end{bmatrix}$$
$$H = [1 \quad 1 \quad 1 \quad 1]^T.$$

Network (7) satisfies Theorems 1 and 2, so it is completely stable. We thus have an attracting set $S = \{z| -1.7414 \leq$

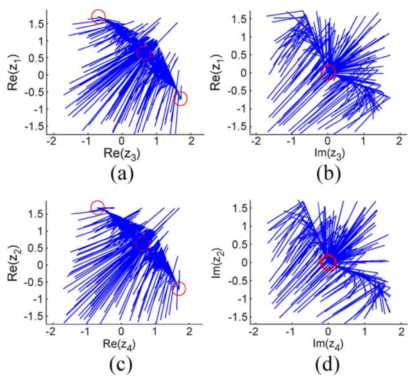


Fig. 3. Global attractivity and complete stability of network (7). The attractive regions are marked with a circle according to the activities of $z_1 - z_4$. (a) Real part activities of $z_1$ and $z_3$. (b) Imaginary part activities of $z_1$ and $z_3$. (c) Real part activities of $z_2$ and $z_4$. (d) Imaginary part activities of $z_2$ and $z_4$. Clearly, network (7) is a multistable network.

$\mathrm{Re}(z_j) + \mathrm{Im}(z_j) \leq 1.7241, j = 1, 2, 3, 4.\}$. Fig. 3 shows the complete convergence of network (7) for 200 trajectories originating from randomly selected initial points in $\mathrm{Re}(s(0)), \mathrm{Im}(s(0)) \in S$, and we find at least eight fixed points from the simulations. Clearly, our network is also a multistable network [19], [21].

## V. Conclusion

In this brief, we have investigated a class of discrete-time RNNs with complex-valued LT neurons. We have discussed three basic dynamical problems for this class of networks: 1) boundedness; 2) global attractivity; and 3) complete stability. The simulations carried out have validated theoretical findings.

The motivation for this study includes two main objectives: The first one is to extend the convergence study on the real-valued LT RNNs [19] to the complex-valued ones, and the second one is to develop some analysis method for complex-valued NNs from the dynamic system view. We hope that the method developed here may also be beneficial to the analysis of other complex-valued RNN properties.

This brief may lead to the design of a complex-valued associative memory based on such networks. Further research is necessary to investigate the dynamical properties, such as to find stable or unstable fixed points and to store some complex-valued patterns into those stable fixed points. These complex-valued patterns depend on real applications, such as image features [12], [13] or complex-valued signals.

Another possible further research is to find a supervised training algorithm for such networks and use them in engineering application. Some research methods used in other complex-valued NNs could be practicable.

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