Jitter in ATM networks and its impact on peak rate enforcement

James ROBERTS
FRANCE TELECOM CNET/PAA/ATR
38, Rue du Général Leclerc
92131 ISSY-LES-MOULINEAUX

Fabrice GUILLEMIN
FRANCE TELECOM CNET/LAA/SLC
Route de Trégastel
22300 LANNION

Abstract

Cells arriving to an ATM network experience random delays due to queueing in upstream multiplexing stages, notably in customer premises. This is the phenomenon of jitter and the aim of the present paper is to study its influence on peak rate enforcement. We first introduce some general characterizations of jitter and then, describe two models of jittered flows based on simple queueing systems. We discuss the objectives of peak rate enforcement and study the impact of jitter on the dimensioning of Jumping Window and Leaky Bucket mechanisms. A useful synthetic characterization of jitter appears to be a remote quantile of the cell delay distribution expressed in units of the initial inter-cell interval.

1 Introduction

The use of the Asynchronous Transfer Mode (ATM) in the Broadband-Integrated Services Digital Network (B-ISDN) allows users to establish connections of widely different bit rates over an access link of around 150 Mbit/s. The price of this flexibility is the danger that a user having contracted a communication at, say, 10 Mbit/s, in fact uses his access to emit data at a much greater rate (up to 150 Mbit/s) bringing the risk of network performance degradation. It is essential to be able to enforce the contracted peak rate by means of a policing device at the user-network interface. (Network operators may also wish to enforce bit rates coming from the network of another operator at the network-node interface). This enforcement is complicated by the fact that an initially periodic cell stream is altered by the random delays affecting cells in multiplexing stages (notably in customer premises equipment) between the source and the policing device. This is the phenomenon of jitter and requires that, instead of simply measuring the interval between two cells, we make a statistical estimation of the

In a recent study [Nie 90], Niestegge has proposed a method for dimensioning a leaky bucket for peak rate policing based on a remote quantile of the delay distribution of an arbitrary cell. While the simplicity of this approach is appealing for practical applications, we have preferred to establish a more complete description of the jitter phenomenon taking account, notably, of the correlations between successive cell arrival epochs.

Different characterizations of jitter are discussed in Section 2 and applied to models of the jitter affecting a multiplexed periodic source in Section 3. The results are used in Section 4 to dimension the parameters of jumping window and leaky bucket policing mechanisms.

2 Characterizing jitter

2.1 Quantifying jitter

Jitter is an expression of the random delay affecting cells at multiplexing stages in the network. A first quantification is simply the probability distribution of the delay of an arbitrary cell and, notably, its remote quantiles (e.g. δ such that $\Pr\{\text{delay} > \delta\} < 10^{-10}$). The alteration of the periodic nature of the cell arrival process is partially characterized by the inter-cell distribution and, in particular, by its moments; the squared coefficient of variation is a convenient dimensionless measure of jitter. However, since the periodic nature of the original stream is preserved over relatively long intervals, additional quantifications are necessary to account for correlations between successive interarrivals.

2.2 A Markovian jittered process

To simplify the discussion, we consider the jitter in a discrete time process where the time unit is, however, arbitrary. We consider an initially periodic stream whose inter-cell interval is d (Fig. 1). The i^{th} cell has a sojourn time in system (multiplex or network) of $D+W_i$ where D is a constant (propagation time, etc.) and W_i is a non-negative random delay component. We assume the W_i constitute a stationary ergodic process with probability distribution :

$$w_k = \Pr\{W_i = k\}, \quad \text{for} \quad k \geq 0.$$

We further assume that the dependence between successive delays is first order Markovian characterized by the transition probabilities:

$$q_{jk} = \Pr\{W_i = k \,|W_{i-1} = j\,\}, \quad \text{for } j \geq 0,\; k \geq 0.$$

The w_k then satisfy the equations:

$$w_k = \sum_{j \geq 0} w_j q_{jk}. \tag{1}$$

The constant delay D which depends only on the route followed by the cells of the connection considered is usually ignored.

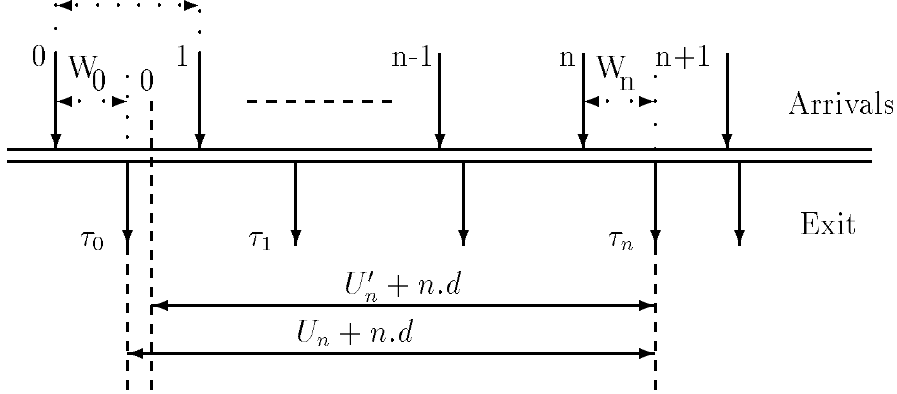

Figure 1: Periodic stream passing through a jittering element

2.3 Inter-cell time distributions. .

Let 0 be an arbitrary instant and let τ_0 be the cell exit instant immediately preceding 0. Let $\tau_i,\ i \geq 1$, be the exit instants of subsequent cells. We introduce the random variables U_n and U'_n defined by:

$$\begin{aligned} U_n &= \tau_n - \tau_0 - nd, \\ U'_n &= \tau_n - nd. \end{aligned}$$

U_n is the variation of the n^{th} order inter-exit time with respect to the interarrival interval nd; U'_n has a similar interpretation when the exit time is measured from an arbitrary instant. Jitter is characterized by the distributions of the random variables $U_n, n \geq 1$, and especially that of $U_1 = W_1 - W_0$. The distribution of U_1 allows a comparison between the interarrival in the jittered process with that in the initial flow which is constant and exactly equal to d.

For further characterization we introduce some notation. Let $f_n(k)$ and $f'_n(k)$ be the distributions of U_n and U'_n respectively. Define:

$$g_n(j,k) = \Pr\{\tau_2 - \tau_1 = d + j, \tau_{n+1} - \tau_2 = (n-1)d + k\}$$

i.e., g_n is the joint distribution of an arbitrary interval (not the interval containing the arbitrary instant 0) and the sum of the $n-1$ subsequent intervals. Then

$$f_n(k) = \sum_{i \geq 0} w_i q^{(n)}_{i,i+k}, \tag{2}$$

$$g_n(j,k) = \sum_{i \geq 0} w_i q_{i,i+j} q^{(n)}_{i+j,i+j+k}, \tag{3}$$

$$f'_n(k) = \frac{1}{d} \sum_{t=-d+1}^{k+(n-1)(d-1)} \sum_{t=0}^{k} g_n(r, k-t), \tag{4}$$

where $q^{(n)}_{ij}$ is the $i-j$ component of the n^{th} power of the transition matrix. The relation for $f'_n(k)$ is derived as follows.

both $\tau_1 - \tau_0$ (the arbitrary instant has more chance of falling in a long interval) and to the distribution $g_n(j,k)$ (the arbitrary interval has more chance of falling in a frequently occuring interval length); on normalizing, we deduce:

$$\Pr\{\tau_1 - \tau_0 = d + r, \tau_n - \tau_1 = (n-1)d + s\} = \frac{(d+r)g_n(r,s)}{d},$$

d being the expected inter-exit time. Now, given $\tau_1 - \tau_0$, the distribution of τ_1 is uniform on $(0, \tau_1 - \tau_0)$:

$$\Pr\{\tau_1 = d + t \,|\tau_1 - \tau_0 = d + r\} = \frac{1}{d+r}, \quad \text{for } -d < t \leq r,$$

so that

$$\Pr\{\tau_1 = d + t, \tau_n - \tau_1 = (n-1)d + s\} = \frac{1}{d}\sum_{r \geq t} g_n(r,s).$$

We obtain (4) on setting $s = k - t$ and summing over t in the range $-d < t \leq k$.

This result can be established more rigourously by using the theory of point processes as follows. We have assumed that the cell process is stationary and therefore, we can consider that the points τ_k are defined for $k \in \mathcal{Z}$ with $\ldots \tau_{-1} < \tau_0 \leq 0 < \tau_1 < \ldots$ and the process $\{\tau_k\}$ is a stationary point process. For $n \geq 1$, write $\sigma_k^n = \tau_{n+k} - \tau_{k+1}$. The process $\{\tau_k, \sigma_k^n\}$ is a stationary marked point process. Adapting formula (4.1.3) in [Bac 90] to the discrete time case, we get:

$$P\{\tau_1 = k, \sigma_k^n = r\} = \frac{1}{d}\sum_{l=k}^{\infty} P^0\{\tau_1 = l, \sigma_k^n = r\}$$

where P^0 is the Palm probability associated with the point process $\{\tau_k\}$. Consequently, we have

$$\begin{aligned} P\{\tau_1 = d + t, \tau_n - \tau_1 = (n-1)d + s\} &= \frac{1}{d}\sum_{l=d+t}^{\infty} P^0\{\tau_1 = l, \sigma_0^n = (n-1)d + s\} \\ &= \frac{1}{d}\sum_{r \geq t} g_n(r,s). \end{aligned}$$

The remainder of the proof is as above.

2.4 Arrivals in a window

Consider a time window of width x: let $\theta_x(n)$ be the probability this window contains n cell exits assuming the window starts just after a cell exit and let $\theta'_x(n)$ be the same probability assuming the window starts at an arbitrary instant. These probabilities are derived as follows:

$$\begin{aligned} \theta_x(n) &= \Pr\{\text{at least } n \text{ cells}\} - \Pr\{\text{at least } n+1 \text{ cells}\} \\ &= \Pr\{U_n + nd \leq x\} - \Pr\{U_{n+1} + (n+1)d \leq x\} \\ &= \sum_{j \leq -nd+x} f_n(j) - \sum_{j \leq -(n+1)d+x} f_{n+1}(j). \end{aligned} \quad (5)$$

$j \le -nd+x \qquad j \le -(n+1)d+x$

2.5 Limit distributions

In the limit $n \to \infty$, $q_{ij}^{(n)} \to w_j$, for all i, and we obtain limit distributions:

$$\begin{aligned} f_\infty(k) &= \sum_i w_i w_{i+k}, \\ g_\infty(j,k) &= \sum_i w_i q_{i,i+j} w_{i+j+k}, \qquad (7) \\ f'_\infty(k) &= \frac{1}{d} \sum_{t \ge -d+1} \sum_{r \ge t} g_\infty(r, k-t). \end{aligned}$$

When n is large we can thus approximate $\theta_x(n)$ and $\theta'_x(n)$ by $\phi_x(n)$ and $\phi'_x(n)$ defined by:

$$\begin{aligned} \phi_x(n) &= \sum_{j=-(n+1)d+x+1}^{-nd+x} f_\infty(j), \qquad (8) \\ \phi'_x(n) &= \sum_{j=-(n+1)d+x+1}^{-nd+x} f'_\infty(j). \end{aligned}$$

Note that the probabilities $\phi_x(n)$, in particular, are fairly simple to obtain and depend only on the stationary waiting time distribution.

3 Models of jitter

3.1 Multiplexing periodic sources

If all sources superposed in an ATM multiplexer have the same bit rate there is no jitter since, for any given periodic stream, the delay is identical for every cell. When sources have different bit rates, the degree of jitter depends on their relative rates. The greatest effect will be felt by the highest rate streams since the number of cells from other streams arriving between successive periodic cells will then be most variable; for a low rate stream, on the other hand, roughly the same number of cells will arrive between any two successive cells in roughly the same relative positions and the delay of each cell will therefore be almost identical. The following models for calculating the jitter of a periodic stream assume the number of cells from other streams arriving in successive intervals are independent and should lead to an overestimation of the induced delay variability.

3.2 Jitter due to a multiplexing stage

We assume a multiplex receives the superposition of a periodic stream of cells of period d and a Poisson stream of rate λ where the unit of time is the cell transmission time [Rob 89]. We assume the multiplex can only start to transmit cells at specific instants ..., -2, -1, 0, arrival is a Markov process. Assuming FIFO service, the queue length at the i^{th} periodic cell arrival is identical to the waiting time W_i introduced in Section 2. To calculate the transition probabilities q_{ij} we introduce the conditional probabilities:

$$\begin{aligned} Q(j,k) &= \Pr\{W_i > k \,|W_{i-1} = j\}, \\ P_n(j,k) = \Pr\{W_i > k \,|W_{i-1} &= j \text{ and } n \text{ Poisson arrivals in } ((i-1)d, id)\}. \end{aligned} \tag{9}$$

We then have,

$$\begin{aligned} q_{jk} &= Q(j,k-1) - Q(j,k), \\ Q(j,k) &= \sum_n P_n(j,k) \frac{(\lambda d)^n}{n!} e^{-\lambda d}, \end{aligned} \tag{10}$$

and it remains to calculate $P_n(j,k)$. It is shown in appendix A that these probabilities are given by:

$$P_n(j,k) = \begin{cases} 0, & \text{for } j+1 \geq d \text{ and } n \leq d+k-j-1 \\ & \text{or } j+1 < d \text{ and } n \leq k, \\ \sum_{s=1}^{n-k} \binom{n}{s+k} \left(\frac{s}{d}\right)^{s+k} \left(1-\frac{s}{d}\right)^{n-s-k} \frac{d-n+k}{d-s}, & \\ & \text{for } j+1 < d \text{ and } k < n \leq d+k-j-1, \\ 1 & \text{for } n > d+k-j-1. \end{cases} \tag{11}$$

The complicated expression in the middle is the probability the queue length is k at the end of an interval of length d in which there occur n arrivals at uniformly distributed instants [Vir 89].

The delay distribution w_k is deduced on solving the state equations. Figure 2 plots $\Pr\{W_i \geq k\}$ for a multiplex load of 0.7 and different values of the period d. Also plotted is the M/D/1 waiting time distribution which provides an upper bound on the waiting time distribution quantiles and constitutes a good approximation for reasonably large d.

The variability of the inter-exit time distribution $f_1(k)$ depends on d. Figure 3 gives the squared coefficient of variation for different values of d and two multiplex loads, 0.7 and 0.85. Table 1 gives the 10^{-10} quantile δ of the random delay W_i. Jitter is clearly most significant for higher loads and smaller periods.

The convergence of the n^{th} order inter-exit time distribution to the limit $f_\infty(k)$ is illustrated in Table 2 for load 0.7 and period 15 (corresponding, for example, to a 10 Mbit/s connection on a 150 Mbit/s link). For all practical purposes, convergence is attained after 8 intervals in this example. Convergence is slower for higher loads but, in all cases, the f_∞ distribution constitutes a conservative approximation for the effects of jitter.

3.3 Jitter due to traffic on a bus

In the following alternative model of the jitter affecting a multiplexed periodic stream, the considered stream is supposed to be inserted on a bus already partially occupied by upstream

Figure 2: Waiting time distribution – FIFO multiplex.

Figure 3: Squared coefficient of variation of inter-exit time –FIFO multiplex.

$\rho = 0.7$	16	28	32	34	35
$\rho = 0.85$	36	59	66	71	72

Table 1: Quantile δ such that $\Pr\{W_i > \delta\} \leq 10^{-10}$ – FIFO multiplex.

k	-30	-15	0	15	30
$f_1(k)$	0	0	.22	.22e-06	.37e-15
$f_4(k)$	.47e-11	.50e-05	.22	.54e-05	.30e-10
$f_8(k)$	.87e-10	.60e-05	.22	.60e-05	.96e-10
$f_\infty(k)$	.10e-09	.60e-05	.22	.60e-05	.10e-09

Table 2: Inter-exit time distributions – FIFO multiplex (d=15, load=0.7)

stations. We assume each slot is already occupied with probability p, independently of the state of occupation of the other slots. This situation is encountered when modelling a Local Area Network based upon the DQDB access protocol [Tra 89]. The cells of the periodic stream are served by the first unoccupied slot occuring after their arrival. The waiting time of the periodic cells is then Markovian with transition probabilities:

$$q_{jk} = \begin{cases} p^k(1-p), & \text{for} j \leq d-1 \\ p^{d+k-j}(1-p), & \text{for } d \leq j \leq d+k \\ 0, & \text{for } d+k < j \end{cases}$$

Rather than directly solving the state equations, we can derive the distribution w_k by arguing as follows. The sojourn time of a periodic cell in this system, $W_i + 1$, is exactly the same as the sojourn time of a customer in a $D/Geo/1$ queueing system (i.e. a single server with periodic arrivals and shifted geometrically distributed service time) with interarrival time d and service time distribution

$$\Pr\{\text{service time} = k \text{ slots}\} = p^{k-1}(1-p), \qquad \text{for } k \geq 1.$$

This statement may readily be verified on writing down transition probablities for the latter process. Now, the sojourn time in the $D/Geo/1$ queue is the sum of two independent random variables: the service time and the waiting time. The distribution of the latter is derived in appendix B. We have :

$$\Pr\{\text{waiting time } = k \text{ slots}\} = \begin{cases} 1-\sigma, & \text{for } k = 0, \\ \sigma(1-\sigma)(1-p)\left(1-(1-\sigma)(1-p)\right)^{k-1}, & \text{for } k \geq 1, \end{cases}$$

where σ is the real root between 0 and 1 of the equation

$$z = (p + (1-p)z)^d .$$

$$w_k = (1-p)(1-\sigma)(1-(1-p)(1-\sigma))^k, \quad \text{for } k \geq 0. \qquad (12)$$

Figure 4 gives the squared coefficient of variation for the same parameter values as in Figure 3, and Table 3 gives the 10^{-10} quantile δ of the random delay for different values of the multiplex load $\rho = p + 1/d$, namely $\rho = 0.7$ and $\rho = 0.85$.

Figure 4: Squared coefficient of variation of inter-exit time – bus multiplex.

Note that the variation attains a maximum between $d = 2$ and $d = 5$: when d is very small, the interferring traffic is too low in volume to affect the periodic stream; as d increases, the cell delay also increases but becomes insignificant with respect to the period.

For large d, it is clear that the value of σ is very small (e.g. for d= 15, we find $\sigma = 0.001$). Consequently, we have $w_k \approx q_{jk}$ and the inter-exit time distributions $f_n(k)$ converge rapidly to $f_\infty(k)$.

In table 4, we compare functions $\phi_x(n)$ and $\phi'_x(n)$ by their particular values $\phi_{nd}(n+y)$ and $\phi'_{nd}(n+y)$ for load 0.7 and period 15. These functions are independent of n and the argument y expresses the deviation from the expected number of arrivals in an arbitrary interval. $\phi'_x(n)$ differs from $\phi_x(n)$ by a horizontal translation of between 0 and 1. The same is true in general for θ'_x and θ_x and, in particular, we have:

$$\theta_{nd}(n+y-1) > \theta'_{nd}(n+y) > \theta_{nd}(n+y)$$

for $y \geq 1$.

4 Peak rate enforcement

4.1 Policing mechanisms

We consider two alternative mechanisms for peak rate enforcement: the jumping window and the leaky bucket [Rat 90]. The former is conceptually the simplest and allows a direct

$\rho = 0.7$	16	27	31	33	34
$\rho = 0.85$	35	56	63	67	68

Table 3: Quantile δ such that $\Pr\{W_i > \delta\} \leq 10^{-10}$ –Bus multiplex.

y	-4	-2	0	2	4
$\phi'_{nd}(n+y)$	.63e-09	.24e-03	.81	.12e-03	.13e-09
$\phi_{nd}(n+y)$	.86e-10	.76e-04	.86	.75e-04	.86e-10

Table 4: Distribution of excess cells in a window - bus multiplex.

appreciation of the effects of jitter. It consists in a device which counts the number of cell arrivals in a window of length Nd (we assume an integer multiple of the policed period for simplicity) and discards any cells over an allowed limit $N + V$. The window is re-initialized every Nd time units. The leaky bucket is more complex consisting in a counter which is incremented on each cell arrival and decremented at a fixed rate a. Cells arriving when the counter attains a certain threshold M are discarded. Our objective is to dimension N and V and a and M to effectively police the jittered streams described by the above models.

4.2 Policing objectives

It is essential that the network can be sure that a user, having obtained a resource allocation for a certain peak rate, does not then emit data at a higher rate, thus provoking congestion and cell losses for all contending communications. It has been recognized that any bandwidth enforcement mechanism must realize a compromise between three factors [Eck 90]:

- responsiveness – the mechanism should react rapidly to control a non-conforming stream;
- allowed margin – the difference between the contracted parameter and the maximum rate which can pass the mechanism should not be too large;
- cell discard rate – a conforming source should suffer minimal cell loss (e.g. $< 10^{-10}$).

Fixing the maximum cell loss rate of a conforming source at 10^{-10}, we consider two possibilities for the responsiveness – allowed margin trade-off:

a) minimize response time while limiting peak rate excess (to 10% or 20%, for example);

b) minimize excess peak rate while limiting response time (e.g. reject cells occuring within a certain interval of a preceding cell).

	N	V	N	V	N	V
FIFO	30	3	30	3	20	2
bus	60	6	50	5	40	4

Table 5: Jumping window dimensioning - load=0.70.

4.3 Dimensioning a jumping window

Suppose we allow a peak rate margin of r (i.e. the policing mechanism will allow a peak rate up to $(1+r)$ times the policed rate). The required window size Nd and the number of excess cells allowed V must then satisfy:

$$\sum_{n>V} \theta'_{Nd}(N+n) < 10^{-10} \text{ and } V/N < r.$$

In the results presented in Table 5 we have used the approximation $\phi'_x(n)$ for $\theta'_x(n)$. This table gives the required values of N and V for a multiplex load of 0.7 in the two considered models.

Concerning possibility b) in Section 4.2, it proves impossible to determine a minimal interval between cells since the probability two cells arrive back to back is greater than the objective discard probability of 10^{-10}.

4.4 Dimensioning a leaky bucket

The value of the leaky bucket counter behaves like the number of customers in a $G/D/1/M$ queue with service rate a (leak rate) and arrival process that of the actual cell stream [Rat 90]. Note that the leak rate may be greater than the policed rate in order to decrease the value of M and consequently, to increase the responsiveness of the mechanism. Dimensioning a leaky bucket consists in estimating the required value of M to achieve a 10^{-10} cell loss rate for given leak rate and arrival process. In this section we dimension M in the case of the jittered stream described in Section 3.3. We apply the Beneš result [Nor 91] to derive an upper bound on the queue length quantiles of the infinite capacity queue. These are used as a conservative estimate on the required bucket capacity.

Let $D = 1/a$ be the service time of the '(jittered stream)/D/1' queue. The Beneš result yields the following bound:

$$\begin{aligned}
\Pr\{X > m\} &\leq \sum_{n\geq 1} \Pr\{n+m \text{ arrivals in an interval of length } nD\} \\
&= \sum_{n\geq 1} \theta'_{nD}(n+m) \\
&< \sum_{n\geq 1} \theta_{nD}(n+m-1).
\end{aligned}$$

Section 3.3. We have the following expression for $\phi_{nD}(n+m-1)$:

$$\phi_{nD}(n+m-1) = \frac{1}{1+\alpha}\left[\alpha^{(n+m-1)d-nD} - \alpha^{(n+m)d-nD}\right]$$

and the above bound yields :

$$\Pr\{X > m\} \leq \frac{\alpha^{-D}(1-\alpha^{d})}{(1+\alpha)(1-\alpha^{d-D})}\alpha^{md}$$

where $\alpha = (1-(1-p)(1-\sigma))$.

Determining the leaky bucket capacity M from the 10^{-10} quantile of this bound, we deduce:

$$M \approx \frac{-10}{\log_{10}\alpha}\frac{1}{d} + \frac{1}{ad}. \tag{13}$$

Thus M is almost independent of the leak rate a $(0 < a < 1/d)$. The first term in (13) is the 10^{-10} quantile of the cell waiting time distribution w_k divided by d. This result supports the heuristic approach in [Nie 90] and suggests that the remote delay quantile expressed in units of the source period is a useful synthetic characterization of jitter. For a multiplex load of 0.85 and $d = 15$, we would require a value of $M = 8$ for a 10% margin of the leak rate over the policed rate. Comparison with the results of Table 5 (for a lower load) confirms that the leaky bucket is much more responsive than the jumping window. However, this mechanism would still allow significant bursts to enter the network provoking congestion or requiring multiplex buffer over-dimensioning [Gui 91].

5 Conclusion

An assumption of Markovian dependence between the delays of successive cells allows us to calculate various quantities of interest for characterizing jitter. This assumption is appropriate for two ATM multiplex models and we have been able to investigate the effects of different source and multiplex parameters on the degree of jitter introduced. Jumping window and leaky bucket mechanisms have been dimensioned to enforce the peak rate of jittered streams. The leaky bucket is seen to be considerably more responsive than the jumping window. A synthetic characterization of jitter consists in the remote delay quantile divided by the period of the considered stream. This is sufficient to dimension the leaky bucket in some cases and supports the heuristic approach of [Nie 90].

A Transition probabilities in the $M+D/D/1$ queue

The $M+D/D/1$ queue with FIFO service has been introduced in Section 3.2. We wish here to determine the transition probabilities q_{jk} for the cells of the periodic stream. Since the service discipline is FIFO, W_i is identical to the queue length L_i seen by the i^{th} arriving periodic cell. With the notation of Section 3.2, we have the following theorem:

Theorem 1 *The conditional probabilities $P_n(j,k)$ are given by:*

$$P_n(j,k) = \begin{cases} 0, & \text{for } j+1 \geq d \text{ and } n \leq d+k-j-1 \\ & \text{or } j+1 < d \text{ and } n \leq k, \\ \sum_{s=1}^{n-k} \binom{n}{s+k} \left(\frac{s}{d}\right)^{s+k} \left(1-\frac{s}{d}\right)^{n-s-k} \frac{d-n+k}{d-s}, & \\ & \text{for } j+1 < d \text{ and } k < n \leq d+k-j-1, \\ 1 & \text{for } n > d+k-j-1. \end{cases}$$

Proof :

It is clearly impossible to have $L_i > k$ if $j+1 \geq d$ (i.e. the server is always busy in the interval) and at most $k+d-j-1$ cells arrive or, if $j+1 < d$ and at most k cells arrive. Similary, L_i is certainly greater than k if more than $k+d-j-1$ cells arrive. There remains the intermediate case $j+1 < d$ and $k < n \leq k+d-j-1$.

Let $\nu_n(k)$ be the complementary distribution of the queue length at service instant id due uniquely to the n Poisson arrivals (i.e. discounting the $j+1$ cells already present at $(i-1)d$). $P_n(j,k)$ is exactly equal to $\nu_n(k)$ if, and only if, the queue is empty somewhere in the interval (by *empty* in the present case of a synchronous server, we mean the queue is empty at a service instant).

To show the conditions defining the considered intermediate case do indeed imply that the queue must empty at some point in $((i-1)d, id)$, first, suppose the contrary : the system is always busy. Then, $L_i = n+1+j-d$ and $L_i > k$. But this contradicts the condition on n and we can conclude that the queue must empty. It remains to show that $\nu_n(k)$ is given by the expression in Theorem 1.

Using Beneš result [Nor 91], we have:

$$\nu_n(k) = \sum_{s=1}^{n-k} \Pr \quad \{k+s \text{ arrivals in } (id-s, id)\} \times$$
$$\Pr \quad \{\text{ queue empty at } id-s \mid k+s \text{ arrivals in } (id-s, id)\}.$$

Poisson arrivals being uniformly distributed over any finite interval, the probability of exactly $k+s$ arrivals in an interval of length s is clearly :

$$\binom{n}{k+s} \left(\frac{s}{d}\right)^{k+s} \left(1-\frac{s}{d}\right)^{n-k-s}.$$

It remains to calculate the second probability. If there are $r+s$ arrivals in $(id-s, id)$, there are $n-r-s$ arrivals in $((i-1)d, id-s)$. Let n_l be the number of Poisson arrivals distributed, and therefore exchangeable. Consequently, we can write :

$$
\begin{aligned}
\Pr & \{ \text{ queue empty at } id-s \mid r+s \text{ arrivals in } (id-s, id)\} \\
= & \Pr\{ \text{ queue empty at } id-s \mid n-r-s \text{ arrivals in } ((i-1)d, id-s)\} \\
= & \Pr\{N_l < l, l=1,\cdots,d-s \mid N_{d-s} = n-r-s\} = \frac{d-n+r}{d-s}.
\end{aligned}
$$

from [Tak 77], Theorem 1, p.10. This completes the proof.

B The $D/Geo/1$ queue

We consider a discrete-time queueing system where events can only occur at specific instants $k, k \in \mathcal{Z}$. Customers arrive to a single-server queue with infinite capacity, their interarrival time is deterministic and equal to d and they require service times which are independent and shifted geometrically distributed with parameter p. More precisely, if S_i denotes the service time of the i^{th} customer, we have :

$$k \geq 1, \Pr\{S_i = k\} = (1-p)p^{k-1}$$

We consider a sequence $\{X_i\}_{i\geq 1}$ of independent and geometrically distributed random variables with parameter p. Let $r \geq 1$. The generating function of the random variable $X_1 + \cdots + X_r$ is given by:

$$E\left[z^{X_1+\cdots+X_r}\right] = \frac{[(1-p)z]^r}{(1-pz)^r} = (1-p)^r z^r \sum_{n=0}^{\infty} \binom{n+r-1}{r-1} z^n p^n \text{ for } \mid z \mid \leq 1,$$

It follows that:

$$\Pr\{X_1 + \cdots + X_r = n\} = (1-p)^r p^{n-r} \binom{n-1}{r-1} \text{ for } n \geq r.$$

Consider a discrete-time renewal process in which the interrarival times X_i, $i \geq 1$, are shifted geometrically distributed with parameter p. Let N denote the number of renewals in an interval of length d and let $b_r = \Pr\{N = r\}$. We have:

$$
\begin{aligned}
b_r &= \Pr\{X_1 + \cdots + X_r \leq d, X_1 + \cdots + X_{r+1} > d\} \\
&= (1-p)^r p^{d-r} \sum_{k=r}^{d} \binom{k-1}{r-1} = \binom{d}{r} (1-p)^r p^{d-r}
\end{aligned}
$$

by using the Vandermonde convolution formula ([Rio 82], p.148). Finally, we have:

$$b_r = \begin{cases} \binom{d}{r} (1-p)^r p^{d-r} & \text{if } 0 \leq r \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

matrix is given by:

$$\Pi = ((\pi_{i,j})) \text{ where } \begin{cases} \pi_{i,0} = 1 - \sum_{j=0}^{i} b_j, & i \geq 0, \\ \pi_{i,j} = b_{i-j+1}, & i \geq 0, j = 1, \cdots, i+1, \\ \pi_{i,j} = 0, & \text{otherwise.} \end{cases}$$

The Markov chain $\{Q_n\}$ has a stationary distribution if and only if the equation $\pi.\Pi = \pi$ has a unique solution which satisfies: $\pi_i \geq 0$, for $i \geq 0$, and $\sum_{i=0}^{\infty} \pi_i = 1$. It can be shown that this is equivalent to the following condition: the equation $z = [p + (1-p)z]^d$ has a unique solution between 0 and 1. Using Rouché's theorem, the condition is satisfied when $\rho = p + 1/d < 1$. In this case, let σ denote that solution. The stationary distribution is given by: $k \geq 0, \pi_k = \sigma^k(1-\sigma)$.

The distribution of the waiting time W_∞ can be easily derived:

$$\begin{aligned} \Pr\{W_\infty = 0\} &= 1 - \sigma, \\ \Pr\{W_\infty = k\} &= \sigma(1-\sigma)(1-p)[1-(1-p)(1-\sigma)]^{k-1} \text{ for } k \geq 1. \end{aligned}$$

The distribution of the inter-exit time I is given by:

$$\begin{aligned} \Pr\{I = k\} &= \frac{1-p}{1+p(1-\sigma)}(1-\sigma)[1-(1-p)(1-\sigma)]^{d-k} \\ &+ \frac{1-p}{1+p(1-\sigma)}p^{k-1}[1-(1-p)(1-\sigma)]^d \text{ for } 1 \leq k \leq d, \\ \Pr\{I = k\} &= \frac{1-p}{1+p(1-\sigma)}\left[(1-\sigma)p^{k-d} + p^{k-1}[1-(1-p)(1-\sigma)]^d\right] \text{ for } k \geq d. \end{aligned}$$

The squared coefficient of variation Cv^2 of the inter-exit time I is given after some algebra by:

$$Cv^2 = \frac{\mathrm{Var}[I]}{\mathrm{E}[I]^2} = \frac{-2\sigma}{d(1-p)(1-\sigma)} + \frac{2(p+(1-p)\sigma)}{d^2(1-p)^2(1-\sigma)}.$$

References

[Bac 90] Baccelli F., Brémaud, P. : *Elements of queueing theory.* Springer Verlag, to appear.

[Eck 90] Eckberg A.E., Luan D.T., Lucantoni D.M. *An approach to controlling congestion in ATM networks.* Intl J. of Digital and Analogue Comm Systems. Vol 3, pp 127-135, 1990.

[Gui 91] Guillemin F., Boyer P., Dupuis A., Romoeuf L. : *Peak rate enforcement in ATM networks.* Submitted to Infocom'92.

[Kle 76] Kleinrock L. : *Queueing theory, Vol 1.* Wiley, New York 1976.

[Nie 90] Niestegge G. : *The 'leaky bucket' policing method in the ATM network.* Intl J. of Digital and Analogue Comm Systems. Vol 3, pp. 187-197, 1990.

[Nor 91] Norros I., Roberts J., Simonian A. Virtamo J. : *The superposition of VBR sources in an ATM multiplexer.* IEEE JSAC, Vol. 9, n^o 3, pp. 378-387, April 1991.

[Rio 82] Riordan, J. : *Combinatorial identities.* R. Kriger Publishing Company, 1979.

[Rob 89] Roberts J. : *Jitter due to an ATM multiplex.* COST 224 document TD(89)035.

[Tak 77] Takacs, L. : *Combinatorial methods in the theory of stochastic processes.* R. Kriger Publishing Company, 1977.

[Tra 89] Tran Gia, P., Stock, T. : *Approximate performance analysis of the DQDB access protocol.* ITC Specialist Seminar, Adelaide 1989, paper 16.1.

[Vir 89] Virtamo J. Roberts J. W. : *Evaluating buffer requirements in an ATM multiplex.* Globecom '89, Dallas, Nov 1989. Statistical multiplexing of on-off sources. COST 224 document TD(90)034, 1990.