Applied Mathematics and Computation 218 (2011) 3733–3745

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Stability analysis of diffusive predator–prey model with modified Leslie–Gower and Holling-type III schemes ☆

Yanling Tian *, Peixuan Weng

School of Mathematics, South China Normal University, Guangzhou 510631, PR China

ARTICLE INFO

Keywords:
Diffusive predator–prey model
Modified Leslie–Gower and Holling-type II schemes
Comparison method
Persistence
Stability

ABSTRACT

The stability of a diffusive predator–prey model with modified Leslie–Gower and Holling-type III schemes is investigated. A threshold property of the local stability is obtained for a boundary steady state, and sufficient conditions of local stability and un-stability for the positive steady state are also obtained. Furthermore, the global asymptotic stability of these two steady states are discussed. Our results reveal the dynamics of this model system.

1. Introduction

One of dominant themes in both ecology and mathematical ecology is the dynamic relationship between predators and their prey due to its universal existence and importance in population dynamics. The investigations on predator–prey models are developed during these thirty years, and more realistic models are derived in view of laboratory experiments and observations. In these models, more factors such age-structure, seasonal effects, radio dependence, etc. are included into consideration (see [1,4–7,10] and the references therein). In [1,7], Nindjin et al. considered a predator–prey model incorporating a modified version of Leslie–Gower functional response as well as the Holling-type II functional response:

$$\begin{aligned} \dot{x} &= x\left(a_1 - bx - \frac{c_1 y}{x+k_1}\right), \\ \dot{y} &= y\left(a_2 - \frac{c_2 y}{x+k_2}\right), \end{aligned} \tag{1.1}$$

where (1.1) is considered associated with initial conditions $x(0) > 0$, $y(0) > 0$.

Model (1.1) describes a prey population x which serves as food for a predator with population y. The parameters a_1, a_2, b, c_1, c_2, k_1 are assumed to be only positive values: a_1 and a_2 are the growth rate of prey x and predator y respectively, b measures the strength of competition among individuals of species x, c_1 is the maximum value of the per capita reduction rate of x due to y, k_1 and k_2 measure the extent to which environment provides protection to prey x and to predator y respectively, and c_2 has a similar meaning as c_1.

The first equation of (1.1) is a standard equation with Holling-type II response function, but the second is not standard. The intactness of the equation contains a modified Leslie–Gower term $\frac{y}{lx}$. If the growth of the predator population is of logistic form, then:

☆ Supported by the Natural Science Foundation of China (11171120), Research Fund for the Doctoral Program of Higher Education of China (20094407110001) and Natural Science Foundation of Guangdong Province (10151063101000003).

* Corresponding author.
E-mail addresses: tianyl@scnu.edu.cn (Y. Tian), wengpx@scnu.edu.cn (P. Weng).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2011.09.018

$$\frac{dy}{dt} = y\left(a_2 - \frac{y}{C}\right),$$

here C measures the carry capacity set by the environmental resources. Let $C = lx$, where l is the conversion factor of prey into predators, then the above equation becomes:

$$\frac{dy}{dt} = y\left(a_2 - \frac{y}{lx}\right),$$

and the term $\frac{y}{lx}$ measures the loss in the predator population due to the rarity (per capita y/x) of its favorite food. If this favorite food is lacking severely, the predator y will switch to other population, but its growth will be limited. By adding a positive constant to the denominator, the equation becomes:

$$\frac{dy}{dt} = y\left(a_2 - \frac{y}{lx + d}\right),$$

and thus the second equation of (1.1) follows, where $c_2 = \frac{1}{l}, k_2 = \frac{d}{l}$.

As well known, Holling-type III response exists universally in population dynamics. Hence it is natural to consider the system incorporating a modified version of Leslie–Gower functional response as well as that of the Holling-type III:

$$\begin{aligned}\dot{x} &= x\left(a_1 - bx - \frac{c_1 y^2}{x^2 + k_1}\right),\\ \dot{y} &= y\left(a_2 - \frac{c_2 y}{x + k_2}\right).\end{aligned} \tag{1.2}$$

Taking the diffusion of the species into account as in [9,10,12,13], one obtains the following reaction–diffusion model:

$$\begin{aligned}\frac{\partial U}{\partial t} &= D_1 \Delta U + U\left(a_1 - bU - \frac{c_1 W^2}{U^2 + k_1}\right),\\ \frac{\partial W}{\partial t} &= D_2 \Delta W + W\left(a_2 - \frac{c_2 W}{U + k_2}\right).\end{aligned} \tag{1.3}$$

to describe the interaction of spatially distributed populations of predator W and prey U, here $U = U(t,x)$, $W = W(t,x),\ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Let:

$$u = \frac{b}{a_1}U, \quad w = \frac{c_1}{a_1}W, \quad t' = a_1 t, \quad x' = \frac{x}{\sqrt{\frac{D_2}{a_1}}}, \quad D = \frac{D_1}{D_2}, \tag{1.4}$$

then:

$$\frac{\partial w}{\partial t'} = \frac{c_1}{a_1^2}\frac{\partial W}{\partial t}, \quad \Delta w = \frac{D_2 c_1}{a_1^2}\Delta W,$$

where $\Delta w = \sum_{i=1}^n \frac{\partial^2 w}{\partial (x_i')^2}, \Delta W = \sum_{i=1}^n \frac{\partial^2 W}{\partial x_i^2}$. Substituting (1.4) in (1.3), we have:

$$\frac{\partial w}{\partial t'} = \Delta w + \alpha w\left(1 - \frac{\beta_2 w}{u + \bar{k}_2}\right),$$

where $\alpha = \frac{a_2}{a_1}, \beta_2 = \frac{c_2 b}{c_1 a_2}, \bar{k}_2 = \frac{b}{a_1}k_2$. Similarly, we obtain:

$$\frac{\partial u}{\partial t'} = \frac{b}{a_1^2}\frac{\partial U}{\partial t}, \quad \Delta u = D_2 \frac{b}{a_1^2}\Delta U = \frac{D_1 b}{D a_1^2}\Delta U,$$

and then:

$$\frac{\partial u}{\partial t'} = D\Delta u + u\left(1 - u - \frac{\beta_1 w^2}{u^2 + \bar{k}_1}\right),$$

where $\beta_1 = \frac{b^2}{c_1}, \bar{k}_1 = \frac{b^2 k_1}{a_1^2}$. By rewriting $t', x', \bar{k}_1, \bar{k}_2$ as t, x, k_1, k_2, (1.3) becomes:

$$\begin{aligned}\frac{\partial u}{\partial t} &= D\Delta u + u\left(1 - u - \frac{\beta_1 w^2}{u^2 + k_1}\right),\\ \frac{\partial w}{\partial t} &= \Delta w + \alpha w\left(1 - \frac{\beta_2 w}{u + k_2}\right),\end{aligned}$$

where D, α, β_1, β_2, k_1, k_2 are positive constants.

Assume that the predator and prey are confined to a bounded domain Ω in $\mathbb{R}^n$ with a smooth boundary. In this article, we consider the following reaction–diffusion system:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = D\Delta u + u\left(1 - u - \frac{\beta_1 w^2}{u^2 + k_1}\right), \quad t > 0, \quad x \in \Omega,\\
&\frac{\partial w}{\partial t} = \Delta w + \alpha w\left(1 - \frac{\beta_2 w}{u + k_2}\right), \quad t > 0, \quad x \in \Omega,\\
&\frac{\partial u}{\partial \boldsymbol{n}} = \frac{\partial w}{\partial \boldsymbol{n}} = 0, \quad t > 0, \quad x \in \partial\Omega,\\
&u(0,x) = u_0(x) \geqslant 0, \quad w(0,x) = w_0(x) \geqslant 0, \quad x \in \Omega,
\end{aligned} \tag{1.5}$$

where $\boldsymbol{n}$ is the outward unit normal vector of the boundary $\partial\Omega$, u_0, w_0 are continuous functions of x. We mention here that [8, Theorem 2.1] guarantees that (1.5) has a unique nonnegative solution (u,w) defined on $[0,\infty) \times \Omega$. In addition, from the maximum principle, we can see that the solution is positive, i.e., $u(t,x) > 0$, $w(t,x) > 0$ on $\overline{\Omega}$ for all $t > 0$, provided $u_0 \not\equiv 0$, $w_0 \not\equiv 0$.

Obviously, $(0,0), (1,0), \left(0, \frac{k_2}{\beta_2}\right)$ are the three constant steady states of (1.5). Moreover, the following proposition guarantees the existence and uniqueness of the positive constant steady state.

Proposition 1.1. *System (1.5) has a unique interior equilibrium $E_1 = (u^*, w^*)$ (i.e., $u^* > 0, w^* > 0$) if the following condition holds:*

$$\left(1 - \frac{\beta_1}{\beta_2^2}\right)^2 > 3\left(k_1 + \frac{2\beta_1 k_2}{\beta_2^2}\right), \quad k_1 - \frac{\beta_1 k_2^2}{\beta_2^2} > 0. \tag{1.6}$$

Proof. If $E_1 = (u^*, w^*)$ is the interior equilibrium of (1.5), then:

$$(1 - u^*)\left((u^*)^2 + k_1\right) - \beta_1\left(\frac{u^* + k_2}{\beta_2}\right)^2 = 0.$$

Consider the function:

$$s(h) = (1 - h)(h^2 + k_1) - \frac{\beta_1}{\beta_2^2}(h + k_2)^2,$$

and calculate its derivative, we have:

$$s'(h) = -(h^2 + k_1) + 2h(1 - h) - \frac{2\beta_1}{\beta_2^2}(h + k_2).$$

Obviously:

$$\begin{aligned}
&\left(1 - \frac{\beta_1}{\beta_2^2}\right)^2 > 3\left(k_1 + \frac{2\beta_1 k_2}{\beta_2^2}\right) \Rightarrow s'(h) < 0,\\
&k_1 - \frac{\beta_1 k_2^2}{\beta_2^2} > 0 \Rightarrow s(0) > 0,
\end{aligned}$$

then the curve of the function $s(h)$ intersects x axis only once, hence the existence and the uniqueness of the interior equilibrium is guaranteed. The proof is completed. □

As well known, the persistence and the stability are two important topics in the study of dynamics for differential equations and population models (e.g., see [4,5,9,11,12]). In the present article, we shall also concern with these two topics. This article is organized as follows. Section 2 is for the preliminaries. In Section 3, the persistence of the system is discussed. The stability of the boundary steady state $\left(0, \frac{k_2}{\beta_2}\right)$ and the positive steady state (u^*, w^*) are investigated in Section 4. A threshold property or sufficient conditions of the local stability is obtained for $\left(0, \frac{k_2}{\beta_2}\right)$ and (u^*, w^*) by the linearized method and eigenvalue method, respectively (see Theorems 4.1 and 4.4 in Section 4). Sufficient conditions for globally asymptotic stability of these two steady states are also obtained. The techniques used for globally asymptotic stability are the Lyapunov function method and the comparison principle.

2. Preliminaries

In order to discuss the dynamics of the steady states, we introduce some lemmas in this section for the use of convenience. Consider the following equation:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1\Delta u + F_1(u,w), \quad t>0, \quad x\in\Omega,\\ \frac{\partial w}{\partial t} = d_2\Delta u + F_2(u,w), \quad t>0, \quad x\in\Omega,\\ \frac{\partial u}{\partial \boldsymbol{n}} = \frac{\partial w}{\partial \boldsymbol{n}} = 0, \quad t>0, \quad x\in\Omega,\\ u(0,x)\geqslant 0, \quad w(0,x)\geqslant 0, \quad x\in\Omega. \end{cases} \tag{2.1}$$

Let $(\check{u},\check{w})$ be a steady state of the equation, i.e., $F_1(\check{u},\check{w}) = F_2(\check{u},\check{w}) = 0$.

Define

$$\mathbb{X} = \left\{(u,w)^T \in [\mathcal{C}^1(\overline{\Omega})]\times[\mathcal{C}^1(\overline{\Omega})] \mid \frac{\partial u}{\partial \boldsymbol{n}} = \frac{\partial w}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad \partial\Omega\right\}.$$

Assume that $0 = \mu_0 < \mu_1 < \mu_2 < \cdots < \cdots$ denote the eigenvalues of the operator $-\Delta$ on $\mathbb{X}$ with the homogeneous Neumann boundary condition. Hence there is a decomposition $\mathbb{X} = \oplus_{i=0}^{\infty}\mathbb{X}_i$ such that $\mathbb{X}_i$ is the eigenspace corresponding to μ_i.

Define an operator $\mathcal{L}$ in the form:

$$\mathcal{L} = \begin{pmatrix} d_1\Delta + \theta_1 & -\theta_2 \\ \theta_3 & d_2\Delta - \theta_4 \end{pmatrix},$$

here:

$$\theta_1 = \frac{\partial F_1}{\partial u}|_{u=\check{u},w=\check{w}}, \quad \theta_2 = -\frac{\partial F_1}{\partial w}|_{u=\check{u},w=\check{w}},$$
$$\theta_3 = \frac{\partial F_2}{\partial u}|_{u=\check{u},w=\check{w}}, \quad \theta_4 = -\frac{\partial F_2}{\partial w}|_{u=\check{u},w=\check{w}}.$$

As usual, we identify an abstract vector function $(u(t), w(t)) = (u(t,\cdot), w(t,\cdot))$. Therefore, we can rewrite (2.1) as:

$$\frac{d}{dt}\begin{pmatrix} u-\check{u} \\ w-\check{w} \end{pmatrix} = \mathcal{L}\begin{pmatrix} u-\check{u} \\ w-\check{w} \end{pmatrix} + \begin{pmatrix} f_1(u-\check{u}, w-\check{w}) \\ f_2(u-\check{u}, w-\check{w}) \end{pmatrix},$$

where $f_i(z_1,z_2) = o\left(\sqrt{z_1^2+z_2^2}\right), i = 1,2$.

Motivated from [11], we have the following lemma:

Lemma 2.1. *If:*

$$\theta_1 - \theta_4 < 0, \quad \theta_2 \geqslant 0, \quad \theta_3 \geqslant 0, \quad \theta_2\theta_3 - \theta_1\theta_4 > 0, \tag{2.2}$$

then $(\check{u},\check{w})$ is uniformly asymptotically stable. Furthermore, if $\theta_2\theta_3 - \theta_1\theta_4 < 0$, then $(\check{u},\check{w})$ is unstable.

Proof. For each $i, i = 0,1,2,\ldots,\mathbb{X}_i$ is invariant under the operator $\mathcal{L}$ and ξ_i is an eigenvalue of $\mathcal{L}$ on $\mathbb{X}_i$ if and only if ξ_i is an eigenvalue of the matrix:

$$M_i = \begin{pmatrix} -d_1\mu_i + \theta_1 & -\theta_2 \\ \theta_3 & -d_2\mu_i - \theta_4 \end{pmatrix}.$$

By direct calculation and (2.2), we obtain:

$$Re\xi_0^- = \frac{1}{2}\left[\theta_1 - \theta_4 - \sqrt{(\theta_1-\theta_4)^2 - 4(\theta_2\theta_3 - \theta_1\theta_4)}\right] < 0,$$
$$Re\xi_0^+ = \frac{1}{2}\left[\theta_1 - \theta_4 + \sqrt{(\theta_1-\theta_4)^2 - 4(\theta_2\theta_3 - \theta_1\theta_4)}\right] < 0.$$

We also have:

$$\det M_i = d_1d_2\mu_i^2 + (\theta_4 d_1 - \theta_1 d_2)\mu_i + \theta_2\theta_3 - \theta_1\theta_4 > 0,$$
$$trM_i = -(d_1+d_2)\mu_i + \theta_1 - \theta_4 < 0,$$

hence:

$$Re\xi_i^- = \frac{1}{2}\left(trM_i - \sqrt{(trM_i)^2 - 4\det M_i}\right) \leqslant \frac{1}{2}trM_i < \frac{1}{2}trM_1 < 0,$$

$$Re\xi_i^+ = \frac{1}{2}\left(trM_i + \sqrt{(trM_i)^2 - 4\det M_i}\right) \leqslant \frac{\det M_i}{trM_i} < -\frac{d_1d_2\mu_i^2}{(d_1+d_2)\mu_i + (\theta_4-\theta_1)\mu_i} < \begin{cases} -\frac{d_1d_2}{(d_1+d_2)+(\theta_4-\theta_1)} = -\delta < 0, & \mu_i > 1, \\ -\frac{d_1d_2\mu_1^2}{(d_1+d_2)+(\theta_4-\theta_1)} = -\delta < 0, & \mu_i \leqslant 1. \end{cases}$$

From the above discussion, we know that there is:

$$\tilde{\delta} = \min\left\{\frac{d_1 d_2}{(d_1+d_2)+(\theta_4-\theta_1)}, \frac{d_1 d_2 \mu_1^2}{(d_1+d_2)+(\theta_4-\theta_1)}\right\} > 0,$$

such that $Re\xi_i^+ < -\tilde{\delta}$, for all i. Consequently, the spectrum of $\mathcal{L}$, which consists of eigenvalues, lies in $\{Re\xi < -\tilde{\delta}\}$. An application of [3, Theorem 5.1.1] leads to the first conclusion of the theorem.

Note that if $\theta_2\theta_3 - \theta_1\theta_4 < 0$, then:

$$Re\xi_0^+ = \frac{1}{2}\left[\theta_1 - \theta_4 + \sqrt{(\theta_1-\theta_4)^2 - 4(\theta_2\theta_3 - \theta_1\theta_4)}\right] > 0.$$

Hence the spectrum $\sigma(\mathcal{L})\bigcap\{Re\lambda > 0\} \neq \emptyset$, and therefore $(\check{u}, \check{w})$ is unstable, which yields to the second conclusion of the theorem. □

The following lemma is also from [11]:

Lemma 2.2. *Let a and b be positive constants. Assume that* $\phi, \varphi \in C^1([a, +\infty)), \varphi \geqslant 0$ *and* ϕ *is bounded from below. If* $\phi'(t) \leqslant -b\varphi(t)$ *and* $\varphi'(t) \leqslant k$ *in* $[a, +\infty)$ *for some positive constant k, then* $\lim_{t\to+\infty}\varphi(t) = 0$.

Poincare inequality is usually used in order to give the estimation of the solutions, we introduce it as follows:

Lemma 2.3 (*Poincare inequality*). *Assume that* $1 \leqslant p \leqslant \infty$ *and that* Ω *is a bounded open subset of n-dimensional Euclidean space* $\mathbb{R}^n$ *having Lipschitz boundary (i.e.,* Ω *is an open bounded Lipschitz domain.) Then there exists a constant C, depending only on* Ω *and p such that, for every function u in the sobolov space* $W^{1,p}(\Omega)$:

$$\|u - \bar{u}\|_{L^p(\Omega)} \leqslant C\|\nabla u\|_{L^p(\Omega)}$$

where $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u(y)dy$ *is the average value of u over* Ω, *with* $|\Omega|$ *standing for the Lebesgue measures of the domain* $\overline{\Omega}$.

The last lemma is from [2, Theorem A2].

Lemma 2.4. *Consider the following equation:*

$$\begin{aligned}
&\frac{\partial u_i}{\partial t} = \Delta u_i + f_i(u_1, \ldots, u_n), \quad 1 \leqslant i \leqslant m,\\
&u_i(0, x) = u_{i_0}(x),\\
&\frac{\partial u_i}{\partial \boldsymbol{n}} = 0, \quad x \in \partial\Omega,
\end{aligned}$$

suppose that $|u_i(t,x)| < \overline{K}$ *for* $(t,x) \in \mathbb{R}^+ \times \Omega, 1 \leqslant i \leqslant n$, *and* f_i *is of class* C^1 *on* $\Sigma = \prod_{i=1}^n[-\overline{K}, \overline{K}], 1 \leqslant i \leqslant n$, *where* $u = \{u_1, u_2, \ldots, u_n\}$ *is the solution of the above system. Finally, suppose that* $f_i(u) = 0$ *if* $u_i = 0$. *Then there exists* $N > 0$ *depending only on* $\Omega, \alpha, \overline{K}$ *and* df_i *(the total differential of* f_i*) such that* $\|\boldsymbol{u}\|_{C^{2,\alpha}(\overline{\Omega})} \leqslant N$.

3. Persistence

In this section, we shall discuss the persistence of (1.5). Firstly, we consider the upper boundedness of the solutions for (1.5).

Theorem 3.1. *Any solution* (u, w) *of* (1.5) *satisfies:*

$$\limsup_{t\to\infty}\max_{x\in\overline{\Omega}} u(t,\cdot) \leqslant 1, \quad \limsup_{t\to\infty}\max_{x\in\overline{\Omega}} w(t,\cdot) \leqslant \frac{1+k_2}{\beta_2}. \tag{3.1}$$

Proof. Assume that (u, w) is any solution of (1.5). The first element u satisfies:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - D\Delta u \leqslant u(1-u), \quad t > 0, \quad x \in \Omega,\\
&\frac{\partial u}{\partial \boldsymbol{n}} = 0, \quad t > 0, \quad x \in \partial\Omega,\\
&u(0,x) = u_0(x) \geqslant 0, \quad x \in \Omega.
\end{aligned}$$

Consider the following initial value problem of ordinary differential equation:

$$\begin{aligned}
&\frac{dv}{dt} = v(1-v), \quad t > 0,\\
&v(0) = \max_{\overline{\Omega}} u_0(\cdot).
\end{aligned}$$

Since $\lim_{t\to+\infty} v(t) \leqslant 1$, the first inequality of (3.1) is followed by the standard comparison principle.

As a result, for any $\varepsilon > 0$, there exists $T > 0$ such that $u(t,\cdot) \leqslant 1 + \varepsilon$ for $t \geqslant T$. Then w satisfies:

$$\begin{aligned}
&\frac{\partial w}{\partial t} - \Delta w \leqslant \alpha w\left(1 - \frac{\beta_2 w}{1+\varepsilon+k_2}\right), \quad t \geqslant T, \quad x \in \Omega,\\
&\frac{\partial w}{\partial \boldsymbol{n}} = 0, \quad t > T, \quad x \in \partial\Omega,\\
&w(T,x) \geqslant 0, \quad x \in \Omega.
\end{aligned}$$

Consider the following initial value problem of ordinary differential equation:

$$\begin{aligned}
&\frac{dz}{dt} = \alpha z\left(1 - \frac{\beta_2 z}{1+\varepsilon+k_2}\right), \quad t > T,\\
&z(T) = \max_{\overline{\Omega}} w_0(\cdot).
\end{aligned}$$

Since $\lim_{t\to+\infty} z(t) \leqslant \frac{1+k_2+\varepsilon}{\beta_2}$, we have:

$$\limsup_{t\to\infty} \max_{\overline{\Omega}} w(t,\cdot) \leqslant \frac{1+k_2+\varepsilon}{\beta_2}.$$

Let $\varepsilon \to 0$, then the second inequality of (3.1) is valid. □

The above theorem implies that for any $\varepsilon > 0$, the rectangle $[0, 1+\varepsilon) \times [0, \frac{1+k_2+\varepsilon}{\beta_2})$ is a global attractor of (1.5) in $\mathbb{R}^2_+$.

Definition 3.1. The problem (1.5) is said to have the persistence property if for any nonnegative initial data $(u_0(x), w_0(x))$, with $u_0(x) \not\equiv 0$, $w_0(x) \not\equiv 0$, there exists a positive constant $\eta = \eta(u_0, w_0)$, such that the corresponding solution, (u, w) of (1.5) satisfies:

$$\liminf_{t\to\infty} \min_{\overline{\Omega}} u(t,\cdot) \geqslant \eta, \quad \liminf_{t\to\infty} \min_{\overline{\Omega}} w(t,\cdot) \geqslant \eta.$$

Theorem 3.2. *Suppose* $1 - \frac{\beta_1}{k_1}\left(\frac{1+k_2}{\beta_2}\right)^2 > 0$, *then* (1.5) *is persistent.*

Proof. Suppose that (u, w) is the solution of (1.5) with $u_0(x) \geqslant 0$, $w_0(x) \geqslant 0$ and $u_0(x) \not\equiv 0$, $w_0(x) \not\equiv 0$. Since $u_0(x) \not\equiv 0$, we have $u(t,\cdot) \geqslant 0$ for all $t \geqslant 0$ and $u(t,\cdot) > 0$ for all $t > 0$. Hence:

$$1 - \frac{\beta_2 w}{u+k_2} \geqslant 1 - \frac{\beta_2 w}{k_2} \quad \text{forall } t \geqslant 0, \quad x \in \overline{\Omega}.$$

Let $T_1 > 0$ and $w(T_1,\cdot) > 0$. Consider the following initial value problem:

$$\begin{aligned}
&\dot{z} = \alpha z\left(1 - \frac{\beta_2 z}{k_2}\right), \quad t > T_1,\\
&z(T_1) = \min_{\overline{\Omega}} w(T_1,\cdot) > 0.
\end{aligned}$$

It is easy to have $\lim_{t\to+\infty} z(t) = \frac{k_2}{\beta_2}$, it then follows by the comparison principle that:

$$\liminf_{t\to+\infty} \min_{\overline{\Omega}} w(t,\cdot) \geqslant \frac{k_2}{\beta_2}. \tag{3.2}$$

From the second inequality of (3.1) we obtain $T_2 > 0$ such that:

$$1 - u - \frac{\beta_1 w^2}{u^2+k_1} \geqslant 1 - u - \frac{\beta_1\left(\frac{1+k_2}{\beta_2}\right)^2}{k_1}, \quad t \geqslant T_2, \quad x \in \overline{\Omega}.$$

Thus there is:

$$\eta_1 = 1 - \frac{\beta_1\left(\frac{1+k_2}{\beta_2}\right)^2}{k_1},$$

being the limit of the solutions of the following initial value problem:

$$\begin{aligned}
&\dot{v} = \alpha v\left(1 - v - \frac{\beta_1\left(\frac{1+k_2}{\beta_2}\right)^2}{k_1}\right), \quad t > T_2,\\
&v(T_2) = \min_{\overline{\Omega}} u(T_2,\cdot) > 0.
\end{aligned}$$

We then obtain from the comparison theorem that:

$$\liminf_{t\to+\infty} \min_{\overline{\Omega}} u(t,\cdot) \geqslant \eta_1. \tag{3.3}$$

(3.2) together with (3.3) yields the conclusion of this theorem. □

4. Stability

Since $(0,0)$ and $(1,0)$ are unstable, then stability of the steady states $\left(0,\frac{k_2}{\beta_2}\right)$ and (u^*,w^*) are considered. Theorems 4.1, 4.2, 4.3 describe the stability of $\left(0,\frac{k_2}{\beta_2}\right)$, while Theorems 4.4, 4.5 describe the stability of (u^*,w^*). Multiple methods and techniques are used in this section. By the analysis of eigenvalues of the linearized matrixes, we obtain Theorems 4.1, 4.4, which are the results on local stability. Lemma 2.2 and the analysis on the Lyapunov function yield Theorems 4.2, 4.5, as well as the comparison method leads to Theorem 4.3. They are the results of globally asymptotic stability.

Theorem 4.1. *The following conclusions hold.*

(1) *If* $\frac{\beta_1}{k_1}\left(\frac{k_2}{\beta_2}\right)^2 > 1$, *then the constant solution* $\left(0,\frac{k_2}{\beta_2}\right)$ *of* (1.5) *is uniformly asymptotically stable.*

(2) *If* $\frac{\beta_1}{k_1}\left(\frac{k_2}{\beta_2}\right)^2 < 1$, *then the solution* $\left(0,\frac{k_2}{\beta_2}\right)$ *is unstable.*

Proof. Define $\hat{w} = \frac{k_2}{\beta_2}$, Let:

$$\theta_1 = 1 - \frac{\beta_1}{k_1}(\hat{w})^2 = 1 - \frac{\beta_1 k_2^2}{k_1\beta_2^2}, \quad \theta_2 = 0,$$

$$\theta_3 = \frac{\alpha\beta_2(\hat{w})^2}{(k_2)^2} = \frac{\alpha}{\beta_2}, \quad \theta_4 = -\alpha + \frac{2\alpha\beta_2\hat{w}}{k_2} = \alpha,$$

then the linearization of (1.5) at $(0,\hat{w})$ can be written as:

$$\frac{d}{dt}\begin{pmatrix} u \\ w-\hat{w}\end{pmatrix} = \mathcal{L}\begin{pmatrix} u \\ w-\hat{w}\end{pmatrix} + \begin{pmatrix} f_1(u,w-\hat{w}) \\ f_2(u,w-\hat{w})\end{pmatrix},$$

where

$$\mathcal{L} = \begin{pmatrix} D\Delta+\theta_1 & -\theta_2 \\ \theta_3 & \Delta-\theta_4\end{pmatrix}, \quad f_i(z_1,z_2) = o\left(\sqrt{z_1^2+z_2^2}\right), \quad i=1,2.$$

If $\frac{\beta_1}{k_1}\left(\frac{k_2}{\beta_2}\right)^2 > 1$, then a straightforward calculation yields:

$$\theta_1 < 0, \quad \theta_2 = 0, \quad \theta_3 > 0, \quad \theta_4 > 0, \quad \theta_2\theta_3 - \theta_1\theta_4 > 0,$$

then $\left(0,\frac{k_2}{\beta_2}\right)$ is uniformly asymptotically stable by Lemma 2.1.

On the other hand, the assumption $\frac{\beta_1}{k_1}\left(\frac{k_2}{\beta_2}\right)^2 < 1$ yields $\theta_2\theta_3 - \theta_1\theta_4 < 0$, then $\left(0,\frac{k_2}{\beta_2}\right)$ is unstable by Lemma 2.1. □

Next, the globally asymptotic stability of $(0,\frac{k_2}{\beta_2})$ is considered. Let (u,w) be the unique positive solution of system (1.5). Using Theorem 3.1, there exists a constant C, which does not depend on x and t, such that $\|u(t,\cdot)\|_\infty \leqslant \overline{C}, \|w(t,\cdot)\|_\infty \leqslant \overline{C}$ for $t \geqslant 0$. By Lemma 2.4, there is $A > 0$ such that:

$$\|u(t,\cdot)\|_{C^{2,\alpha}(\overline{\Omega})} \leqslant A, \|w(t,\cdot)\|_{C^{2,\alpha}(\overline{\Omega})} \leqslant A. \tag{4.1}$$

Theorem 4.2. *Suppose that* $\frac{k_1}{\beta_1} < \frac{k_2}{\beta_2^2}$, *if* $k_2 > 1$ *and* $\beta_1 > \beta_2^2$, *then the positive constant solution* $\left(0,\frac{k_2}{\beta_2}\right)$ *of* (1.5) *is globally asymptotically stable.*

Proof. Define a Lyapunov function:

$$V(u,w)(t) = \int_\Omega \frac{\alpha u}{\beta_2^2}dx + \int_\Omega \left[\frac{w^2}{2} - \hat{w}^2\left(1+\ln\frac{w}{\hat{w}}\right)\right]dx,$$

here $\hat{w} = \frac{k_2}{\beta_2}$. Obviously V is bounded below for all $t > 0$. We calculate the derivative of V along the solutions of (1.5):

$$\frac{dV}{dt}\Big|_{(1.5)} = \int_\Omega \frac{\alpha}{\beta_2^2}\frac{\partial u}{\partial t}dx + \int_\Omega \left(1-\frac{\hat{w}}{w}\right)(w+\hat{w})\frac{\partial w}{\partial t}dx$$
$$= \int_\Omega \frac{\alpha u}{\beta_2}\left(1-u-\frac{\beta_1 w^2}{u^2+k_1}\right)dx + \int_\Omega \left(1-\frac{\hat{w}}{w}\right)(w+\hat{w})\left[\Delta w + \alpha w\left(1-\frac{\beta_2 w}{u+k_2}\right)\right]dx. \tag{4.2}$$

Integration by parts, we have:

$$\int_\Omega \left(1-\frac{\hat{w}}{w}\right)(w+\hat{w})\Delta w dx = -\int_\Omega \left(1+\frac{\hat{w}^2}{w^2}\right)(|\nabla w|)^2 dx. \tag{4.3}$$

Direct calculation leads to:

$$1-\frac{\beta_2 w}{u+k_2} = \frac{u}{u+k_2} + \frac{\beta_2(\hat{w}-w)}{(u+k_2)}. \tag{4.4}$$

If $\beta_1 > \beta_2^2, \quad \frac{k_1}{\beta_1} < \frac{k_2}{\beta_2^2}$, then:

$$\frac{\beta_1 w^2}{u^2+k_1} > \frac{\beta_2^2 w^2}{u+k_2}. \tag{4.5}$$

Substitute (4.3)–(4.5) into (4.2) and note that $k_2 > 1$, we obtain:

$$\frac{dV}{dt}|_{(1.5)} \leqslant \int_\Omega \frac{\alpha u}{\beta_2^2}\left(1-u-\frac{\beta_2^2 w^2}{u+k_2}\right)dx - \int_\Omega \left(1+\frac{\hat{w}^2}{w^2}\right)(|\nabla w|)^2 dx - \int_\Omega (w+\hat{w})\frac{\alpha\beta_2(w-\hat{w})^2}{u+k_2}dx$$
$$+ \int_\Omega (w+\hat{w})\frac{\alpha u(w-\hat{w})}{u+k_2}dx$$
$$\leqslant -\int_\Omega \frac{\alpha}{\beta_2^2}u^2 dx + \int_\Omega \frac{\alpha u}{\beta_2^2}\left(\frac{u+k_2^2-\beta_2^2 w^2}{u+k_2}\right)dx - \int_\Omega \left(1+\frac{\hat{w}^2}{w^2}\right)(|\nabla w|)^2 dx - \int_\Omega (w+\hat{w})\frac{\alpha\beta_2(w-\hat{w})^2}{u+k_2}dx$$
$$+ \int_\Omega (w+\hat{w})\frac{\alpha u(w-\hat{w})}{u+k_2}dx = -\int_\Omega \frac{\alpha}{\beta_2^2}u^2 dx + \int_\Omega \frac{\alpha u}{\beta_2^2}\left(\frac{u}{u+k_2}\right)dx + \int_\Omega \frac{\alpha u}{\beta_2^2}\left(\frac{\beta_2^2(\hat{w}-w)(\hat{w}+w)}{u+k_2}\right)dx$$
$$- \int_\Omega \left(1+\frac{\hat{w}^2}{w^2}\right)(|\nabla w|)^2 dx - \int_\Omega (w+\hat{w})\frac{\alpha\beta_2(w-\hat{w})^2}{u+k_2}dx + \int_\Omega (w+\hat{w})\frac{\alpha u(w-\hat{w})}{u+k_2}dx$$
$$\leqslant -\int_\Omega \frac{\alpha}{\beta_2^2}u^2\left(1-\frac{1}{u+k_2}\right)dx - \int_\Omega (w+\hat{w})\frac{\alpha\beta_2(w-\hat{w})^2}{u+k_2}dx - \int_\Omega \left(1+\frac{\hat{w}^2}{w^2}\right)(|\nabla w|)^2 dx. \tag{4.6}$$

Also by $k_2 > 1$, we have:

$$\frac{dV}{dt}|_{(1.5)} \leqslant -\int_\Omega \frac{\alpha}{\beta_2}u^2\left(1-\frac{1}{u+k_2}\right)dx - \int_\Omega (w+\hat{w})\frac{\alpha\beta_2(w-\hat{w})^2}{u+k_2}dx \leqslant -\int_\Omega \frac{\alpha}{\beta_2}\left(1-\frac{1}{k_2}\right)u^2 dx - \int_\Omega \frac{\alpha k_2(w-\hat{w})^2}{1+k_2}dx \leqslant 0.$$

By Theorem 3.1, u, w are bounded, so there are positive constants B_1, B_2 such that:

$$\frac{d\int_\Omega u^2 dx}{dt} = \int_\Omega 2u\frac{\partial u}{\partial t}dx = \int_\Omega 2u\left[D\Delta u + u\left(1-u-\frac{\beta_1 w^2}{u^2+k_1}\right)\right]dx \leqslant B_1,$$

$$\frac{d\int_\Omega (w-\hat{w})^2 dx}{dt} = \int_\Omega 2(w-\hat{w})\frac{\partial w}{\partial t}dx = \int_\Omega 2(w-\hat{w})\left[\Delta w + \alpha w\left(1-\frac{\beta_2 w}{u+k_2}\right)\right]dx \leqslant B_2.$$

Apply Lemma 2.2, we obtain:

$$\lim_{t\to\infty}\int_\Omega u^2 dx = \lim_{t\to\infty}\int_\Omega (w-\hat{w})^2 dx = 0. \tag{4.7}$$

On the other hand, we have from (4.6) that there is $B_3 > 0$ such that:

$$\frac{dV}{dt}\Big|_{(1.5)} \leqslant -\int_\Omega \left(1+\frac{\hat{w}^2}{w^2}\right)(|\nabla w|)^2 dx \leqslant -B_3\int_\Omega (|\nabla w|)^2 dx \triangleq -g(t).$$

Calculating the derivative of $g(t)$, we have:

$$g'(t) = B_3\int_\Omega 2\sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial(\frac{\partial u}{\partial x_i})}{\partial t}dx = B_3\int_\Omega 2\sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial(\frac{\partial u}{\partial t})}{\partial x_i}dx.$$

By using inequality (4.1) and system (1.5), $g'(t)$ is bounded in $[1,+\infty)$. Therefore Lemma 2.2 also yields:

$$\lim_{t\to+\infty}\int_\Omega (|\nabla w|^2)dx = 0.$$

The Poincare inequality yields:

$$\lim_{t\to+\infty}\int_\Omega (w-\bar{w})^2 dx = 0, \tag{4.8}$$

where $\bar{w} = \frac{1}{|\Omega|}\int_\Omega w dx$ and

$$|\Omega|(\bar{w}-\hat{w})^2 = \int_\Omega (\bar{w}-\hat{w})^2 dx = \int_\Omega (\bar{w}-w+w-\hat{w})^2 dx \leqslant 2\int_\Omega (\bar{w}-w)^2 dx + 2\int_\Omega (w-\hat{w})^2 dx.$$

Hence from (4.7) and (4.8) we have:

$$\bar{w}(t) \to \hat{w} \quad \text{as} \quad t\to+\infty. \tag{4.9}$$

Define $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u dx$, then by Schwarz inequality, we have:

$$\bar{u}^2 = \left(\frac{1}{|\Omega|}\int_\Omega u dx\right)^2 \leqslant \int_\Omega u^2 dx.$$

Hence by (4.7), we also have:

$$\bar{u}(t) \to 0 \quad \text{as} \quad t\to+\infty. \tag{4.10}$$

According to (4.1), there exists a subsequence of $\{t_m\}$, denoted still by $\{t_m\}$, and nonnegative functions $\tilde{u},\tilde{w}\in\mathcal{C}^2(\overline{\Omega})$ such that:

$$\lim_{m\to+\infty}\|u(t_m,\cdot)-\tilde{u}(\cdot)\|_{\mathcal{C}^2(\overline{\Omega})} = \lim_{m\to+\infty}\|w(t_m,\cdot)-\tilde{w}(\cdot)\|_{\mathcal{C}^2(\overline{\Omega})} = 0. \tag{4.11}$$

Note that:

$$\int_\Omega (\tilde{w}(x)-\hat{w})^2 dx \leqslant 3\int_\Omega (\tilde{w}(x)-w(t_m,x))^2 dx + 3\int_\Omega (w(t_m,x)-\bar{w}(t_m))^2 dx + 3\int_\Omega (\bar{w}(t_m)-\hat{w})^2 dx.$$

Let $m\to+\infty$, (4.8), (4.9) and (4.11) imply the fact $\tilde{w}(\cdot)\equiv\hat{w}$. Similarly, we can obtain $\tilde{u}\equiv 0$. Therefore we have:

$$\lim_{m\to+\infty}\|u(t_m,\cdot)-0\|_{\mathcal{C}^2(\overline{\Omega})} = \lim_{m\to+\infty}\|w(t_m,\cdot)-\hat{w}\|_{\mathcal{C}^2(\overline{\Omega})} = 0.$$

This equality and Theorem 4.1 yield that $\left(0,\frac{k_2}{\beta_2}\right)$ is global asymptotic stable. □

In the following, we use the comparison method to obtain Theorem 4.3, which is also the result on the global asymptotic stability of $(0,\frac{k_2}{\beta_2})$.

Theorem 4.3. *Suppose that:*

$$\frac{1}{3} < k_1 < \beta_1\left(\frac{k_2}{\beta_2}\right)^2, \tag{4.12}$$

then:

$$\lim_{t\to+\infty}(u(t,\cdot),w(t,\cdot)) = \left(0,\frac{k_2}{\beta_2}\right) \text{ uniformly on } \overline{\Omega}$$

provided $w_0(x)\not\equiv 0$. Thus $\left(0,\frac{k_2}{\beta_2}\right)$ is globally asymptotically stable.

Proof. Since $u(t,\cdot)\geqslant 0$, $w(t,\cdot)\geqslant 0$ for $t\geqslant 0$ and $w_0(x)\not\equiv 0$, from the second equation of the system, there is $t_1>0$ such that:

$$\begin{aligned}
&\frac{\partial w}{\partial t} - \Delta w \geqslant \alpha w\left(1-\frac{\beta_2 w}{k_2}\right), \quad t>t_1, \quad x\in\Omega,\\
&\frac{\partial w}{\partial \boldsymbol{n}} = 0, \quad t>t_1, \quad x\in\partial\Omega,\\
&w(t_1,x)>0, \quad x\in\Omega.
\end{aligned}$$

Consider the corresponding initial value problem:

$$\begin{aligned}
&\dot{z}(t) = \alpha z\left(1-\frac{\beta_2 z}{k_2}\right), \quad t>t_1,\\
&z(t_1) = \min_{\overline{\Omega}} w(t_1,\cdot) > 0.
\end{aligned}$$

An application of the comparison principle gives:

$$\liminf_{t\to\infty} \min_{\overline{\Omega}} w(t,\cdot) \geqslant \frac{k_2}{\beta_2}. \tag{4.13}$$

Hence for any $\varepsilon > 0$, there is $T > 0$ such that $w(t,\cdot) > \frac{k_2}{\beta_2} - \varepsilon$ for $t > T$. By (4.12), we know that $(1-u)(u^2+k_1) < k_1$ for $u > 0$ since $\frac{d[(1-u)(u^2+k_1)]}{du} < 0$, moreover we can choose ε small enough that $k_1 - \beta_1\left(\frac{k_2}{\beta_2} - \varepsilon\right)^2 < 0$, it follows that:

$$\frac{\partial u}{\partial t} - D\Delta u \leqslant u\left(1 - u - \frac{\beta_1\left(\frac{k_2}{\beta_2}-\varepsilon\right)^2}{u^2+k_1}\right) < 0,$$

then:

$$\limsup_{t\to+\infty} \max_{\overline{\Omega}} u(t,\cdot) = 0.$$

Hence there is $T_1 > 0$ such that $u(t,\cdot) < \varepsilon$ for all $t > T_1$ and it follows that:

$$\frac{\partial w}{\partial t} - \Delta w \leqslant \alpha w\left(1 - \frac{\beta_2 w}{\varepsilon + k_2}\right).$$

The standard comparison argument leads to:

$$\limsup_{t\to+\infty} \max_{\overline{\Omega}} w(t,\cdot) \leqslant \frac{\varepsilon + k_2}{\beta_2}. \tag{4.14}$$

Let $\varepsilon \to 0$, we have $w \to \frac{k_2}{\beta_2}$ uniformly on $\overline{\Omega}$ as $t \to +\infty$ by (4.13) and (4.14). □

From the above three theorems, we can see that the condition $\frac{k_1}{\beta_1} < \left(\frac{k_2}{\beta_2}\right)^2$ play an important role in the stability of $(0, \frac{k_2}{\beta_2})$. In what follows, we want to investigate the stability of (u^*, w^*). Note form Proposition 1.1 that the assumption $\frac{k_1}{\beta_1} > \left(\frac{k_2}{\beta_2}\right)^2$ guarantees the uniqueness and existence of the positive steady state (u^*, w^*). Hence we give the assumption that $\frac{k_1}{\beta_1} > \left(\frac{k_2}{\beta_2}\right)^2$ for all the rest theorems.

Theorem 4.4. *The following conclusions hold:*

(1) *Suppose $2u^* - k_1 - 3(u^*)^2 < 0$, then the positive constant solution (u^*, w^*) of (1.5) is uniformly asymptotically stable.*
(2) *If $2u^* - k_1 - 3(u^*)^2 > \frac{2\beta_1 w^*}{\beta_2}$, then the solution (u^*, w^*) is unstable.*

Proof. Let:

$$\theta_1 = 1 - 2u^* - \beta_1 (w^*)^2\left(\frac{-(u^*)^2 + k_1}{\left[(u^*)^2 + k_1\right]^2}\right) = 1 - 2u^* - \frac{(1-u^*)\left[(u^*)^2 + k_1\right]\left[k_1 - (u^*)^2\right]}{\left[(u^*)^2 + k_1\right]^2}$$
$$= \frac{u^*}{\left[(u^*)^2 + k_1\right]}\left[2u^* - k_1 - 3(u^*)^2\right],$$

$$\theta_2 = \frac{2\beta_1 u^* w^*}{(u^*)^2 + k_1}, \quad \theta_3 = \frac{\alpha\beta_2 (w^*)^2}{(u^* + k_2)^2} = \frac{\alpha}{\beta_2}, \quad \theta_4 = -\alpha + 2\frac{\alpha\beta_2 w^*}{u^* + k_2} = \alpha,$$

and

$$\mathcal{L} = \begin{pmatrix} D\Delta + \theta_1 & -\theta_2 \\ \theta_3 & \Delta - \theta_4 \end{pmatrix},$$

then the abstract form of (1.5) at (u^*, w^*) can be written as:

$$\frac{d}{dt}\begin{pmatrix} u - u^* \\ w - w^* \end{pmatrix} = \mathcal{L}\begin{pmatrix} u - u^* \\ w - w^* \end{pmatrix} + \begin{pmatrix} f_1(u-u^*, w-w^*) \\ f_2(u-u^*, w-w^*) \end{pmatrix},$$

where $f_i(z_1, z_2) = o(\sqrt{z_1^2 + z_2^2}), i = 1, 2$. If $2u^* - k_1 - 3(u^*)^2 < 0$, a straightforward calculation yields:

$$\theta_1 < 0, \quad \theta_2 > 0, \quad \theta_3 > 0, \quad \theta_4 > 0, \quad \theta_2\theta_3 - \theta_1\theta_4 > 0.$$

An application of Lemma 2.1 leads to the first conclusion of the theorem.

On the other hand, the assumption $2u^* - k_1 - 3(u^*)^2 > \frac{2\beta_1 w^*}{\beta_2}$ yields $\theta_2\theta_3 - \theta_1\theta_4 < 0$, hence (u^*, w^*) is unstable by Lemma 2.1. □

Now we discuss the global stability of (u^*, w^*) by using Lemma 2.2 and Lyapunov function.

Theorem 4.5. *Suppose that $(u^*)^2 + k_1 - 1 > 0$ and $k_1k_2u^* - k_2^2 - k_2u^* - k_1 > 0$ hold, then the positive constant solution (u^*, w^*) of (1.5) is globally asymptotically stable.*

Proof. Define a Lyapunov function:

$$V(u,w)(t) = \int_\Omega \frac{\alpha}{\beta_1}\left[\frac{u^2}{2} - u(k_2+u^*) + \left(k_2(k_2+u^*) + k_1 + \frac{k_1u^*}{k_2}\right)\ln\frac{u+k_2}{u^*+k_2} - \frac{k_1u^*}{k_2}\ln\frac{u}{u^*}\right]dx + \int_\Omega\left(\frac{w^2}{2} - (w^*)^2\left(1+\ln\frac{w}{w^*}\right)\right)dx.$$

Obviously V is bounded below for all $t > 0$. Define:

$$\widehat{B} = k_2(k_2+u^*) + k_1 + \frac{k_1u^*}{k_2}, \quad \widehat{C} = \frac{k_1u^*}{k_2}.$$

Calculating the derivative of V along the solutions of (1.5), we obtain:

$$\begin{aligned}
\left.\frac{dV}{dt}\right|_{(1.5)} &= \int_\Omega \frac{\alpha}{\beta_1}\left[u - (k_2+u^*) + \widehat{B}\frac{1}{u+k_2} - \widehat{C}\frac{1}{u}\right]\frac{\partial u}{\partial t}dx + \int_\Omega (w+w^*)\left(1-\frac{w^*}{w}\right)\frac{\partial w}{\partial t}dx \\
&= \int_\Omega \frac{\alpha}{\beta_1}\left[u - (k_2+u^*) + \widehat{B}\frac{1}{u+k_2} - \widehat{C}\frac{1}{u}\right]\left[D\Delta u + u\left(1-u-\frac{\beta_1w^2}{u^2+k_1}\right)\right]dx \\
&\quad + \int_\Omega (w+w^*)\left(1-\frac{w^*}{w}\right)\left[\Delta w + \alpha w\left(1-\frac{\beta_2 w}{u+k_2}\right)\right]dx \\
&= -\int_\Omega \frac{D\alpha}{\beta_1}\left(1 - \frac{\widehat{B}}{(u+k_2)^2} + \frac{\widehat{C}}{u^2}\right)(|\nabla u|)^2dx - \int_\Omega\left(1+\frac{(w^*)^2}{w^2}\right)(|\nabla w|)^2dx + \int_\Omega\frac{\alpha}{\beta_1} \\
&\quad \times \frac{(u-u^*)(u^2+k_1)}{u+k_2}\left(1-u-\frac{\beta_1w^2}{u^2+k_1}\right)dx + \int_\Omega \alpha(w+w^*)(w-w^*)\left(1-\frac{\beta_2w}{u+k_2}\right)dx \\
&\leqslant -\int_\Omega \frac{D\alpha}{\beta_1}\left(\frac{\widehat{C}(u+k_2)^2 - \widehat{B}u^2}{u^2(u+k_2)^2}\right)(|\nabla u|)^2dx - \int_\Omega\left(1+\frac{(w^*)^2}{w^2}\right)(|\nabla w|)^2dx + \int_\Omega\frac{\alpha}{\beta_1} \\
&\quad \times \frac{(u-u^*)(u^2+k_1)}{u+k_2}\left(1-u-\frac{\beta_1w^2}{u^2+k_1}\right)dx + \int_\Omega \alpha(w+w^*)(w-w^*)\left(1-\frac{\beta_2w}{u+k_2}\right)dx.
\end{aligned}$$

Since:

$$\begin{aligned}
1-u-\frac{\beta_1w^2}{u^2+k_1} &= 1-u-\frac{\beta_1w^2}{u^2+k_1} - \left(1-u^*-\frac{\beta_1(w^*)^2}{(u^*)^2+k_1}\right) = -(u-u^*) - \left(\frac{\beta_1w^2}{u^2+k_1} - \frac{\beta_1(w^*)^2}{(u^*)^2+k_1}\right) \\
&= -(u-u^*) - \left(\frac{\beta_1w^2}{u^2+k_1} - \frac{\beta_1(w^*)^2}{u^2+k_1} + \frac{\beta_1(w^*)^2}{u^2+k_1} - \frac{\beta_1(w^*)^2}{(u^*)^2+k_1}\right) \\
&= -(u-u^*) - \frac{\beta_1(w+w^*)}{u^2+k_1}(w-w^*) + \frac{\beta_1(w^*)^2(u-u^*)(u^*+u)}{((u^*)^2+k_1)(u^2+k_1)} \\
&= -(u-u^*) - \frac{\beta_1(w+w^*)}{u^2+k_1}(w-w^*) + \frac{u-u^*}{u^2+k_1}(u^*+u)(1-u^*) \\
&= -(u-u^*)\left(\frac{u^2+k_1-(u^*+u)(1-u^*)}{u^2+k_1}\right) - \frac{\beta_1(w+w^*)}{u^2+k_1}(w-w^*)
\end{aligned}$$

and

$$1-\frac{\beta_2w}{u+k_2} = \frac{-\beta_2}{u+k_2}(w-w^*) + \frac{u-u^*}{u+k_2},$$

then together with $\widehat{C} - \widehat{B} < 0$, we have:

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{(1.5)} &\leqslant -\int_\Omega \frac{D\alpha}{\beta_1}\left(\frac{\widehat{C}-\widehat{B}+\widehat{C}k_2^2}{u^2(u+k_2)^2}\right)(|\nabla u|)^2dx - \int_\Omega\left(1+\frac{(w^*)^2}{w^2}\right)(|\nabla w|)^2dx \\ &\quad - \int_\Omega \frac{\alpha}{\beta_1}\frac{(u-u^*)(u^2+k_1)}{u+k_2}\left[(u-u^*)\left(\frac{u^2+k_1-1+(u^*)^2}{u^2+k_1}\right)+\frac{\beta_1(w+w^*)}{u^2+k_1}(w-w^*)\right] \\ &\quad + \int_\Omega \alpha(w-w^*)(w+w^*)\left(\frac{-\beta_2}{u+k_2}(w-w^*)+\frac{u-u^*}{u+k_2}\right) \\ &\leqslant \int_\Omega\left[-\frac{\alpha}{\beta_1}\frac{u^2+k_1-1+(u^*)^2}{u+k_2}(u-u^*)^2-\frac{\alpha\beta_2 w^*}{u+k_2}(w-w^*)^2\right]dx - \int_\Omega\frac{D\alpha}{\beta_1}\left(\frac{k_1k_2u^*-k_2^2-k_2u^*-k_1}{u^2(u+k_2)^2}\right)(|\nabla u|)^2dx \\ &\quad - \int_\Omega\left(1+\frac{(w^*)^2}{w^2}\right)(|\nabla w|)^2dx.\end{aligned}$$

Hence if $(u^*)^2+k_1-1>0$ and $k_1k_2u^*-k_2^2-k_2u^*-k_1>0$, then:

$$\left.\frac{dV}{dt}\right|_{(1.5)} \leqslant \int_\Omega\left[-\frac{\alpha}{\beta_1}\frac{u+k_1-1+(u^*)^2}{u+k_2}(u-u^*)^2-\frac{\alpha\beta_2w^*}{u+k_2}(w-w^*)^2\right]dx \leqslant 0. \tag{4.15}$$

By Theorem 3.1, u, w are bounded, so there are positive constants C_1, C_2 such that:

$$\frac{d\int_\Omega(u-u^*)^2dx}{dt} = \int_\Omega 2(u-u^*)\frac{\partial u}{\partial t}dx = \int_\Omega 2(u-u^*)\left[D\Delta u + u\left(1-u-\frac{\beta_1 w^2}{u^2+k_1}\right)\right]dx \leqslant C_1,$$

$$\frac{d\int_\Omega(w-w^*)^2dx}{dt} = \int_\Omega 2(w-w^*)\frac{\partial w}{\partial t}dx = \int_\Omega 2(w-w^*)\left[D\Delta w + w\left(1-\frac{\beta_2 w}{u+k_2}\right)\right]dx \leqslant C_2.$$

Apply Lemma 2.2, we obtain:

$$\lim_{t\to\infty}\int_\Omega(u-u^*)^2dx = \lim_{t\to\infty}\int_\Omega(w-w^*)^2dx = 0. \tag{4.16}$$

On the other hand, we find from (4.15) that there is $C_3>0$ such that:

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{(1.5)} &\leqslant -\int_\Omega\frac{D\alpha}{\beta_1}\left(\frac{k_1k_2u^*-k_2^2-k_2u^*-k_1}{u^2(u+k_2)^2}\right)(|\nabla u|)^2dx - \int_\Omega\left(1+\frac{(w^*)^2}{w^2}\right)(|\nabla w|)^2dx \\ &\leqslant -C_3\int_\Omega(|\nabla u|^2+|\nabla w|^2)dx \triangleq -g(t).\end{aligned}$$

Using inequality (4.1) and system (1.5), $g'(t)$ is bounded in $[1,+\infty)$. Therefore:

$$\lim_{t\to+\infty}\int_\Omega(|\nabla u|^2+|\nabla w|^2)dx = 0.$$

The Poincare inequality yields:

$$\lim_{t\to+\infty}\int_\Omega(u-\bar{u})^2dx = \lim_{t\to+\infty}\int_\Omega(w-\bar{w})^2dx = 0, \tag{4.17}$$

where $\bar{u}=\frac{1}{|\Omega|}\int_\Omega u\,dx, \bar{w}=\frac{1}{|\Omega|}\int_\Omega w\,dx$ and

$$|\Omega|(\bar{u}-u^*)^2 = \int_\Omega(\bar{u}-u^*)^2dx = \int_\Omega(\bar{u}-u+u-u^*)^2dx \leqslant 2\int_\Omega(\bar{u}-u)^2dx + 2\int_\Omega(u-u^*)^2dx,$$

$$|\Omega|(\bar{w}-w^*)^2 \leqslant 2\int_\Omega(\bar{w}-w)^2dx + 2\int_\Omega(w-w^*)^2dx.$$

Hence from (4.16) and (4.17):

$$\bar{u}(t)\to u^*, \quad \bar{w}(t)\to w^* \quad \text{as} \quad t\to+\infty. \tag{4.18}$$

According to (4.1), there exists a subsequence of $\{t_m\}$, denoted still by $\{t_m\}$, and nonnegative functions $\tilde{u},\tilde{w}\in C^2(\overline{\Omega})$ such that:

$$\lim_{m\to+\infty}\|u(t_m,\cdot)-\tilde{u}(\cdot)\|_{C^2(\overline{\Omega})} = \lim_{m\to+\infty}\|w(t_m,\cdot)-\tilde{w}(\cdot)\|_{C^2(\overline{\Omega})} = 0.$$

Combined with (4.16)–(4.18), we have:

$$\lim_{m\to+\infty}\|u(t_m,\cdot)-u^*\|_{C^2(\overline{\Omega})} = \lim_{m\to+\infty}\|w(t_m,\cdot)-w^*\|_{C^2(\overline{\Omega})} = 0.$$

Since $(u^*)^2 + k_1 - 1 > 0$ implies $2u^* - k_1 - 3(u^*)^2 < 0$, then the above equality and Theorem 4.4 yield the conclusion which (u^*, w^*) is globally asymptotically stable. □

References

[1] M.A. Aziz-Alaoui, M. Daher Okiye, Boundeness and global stability for a predator-prey model with modified Leslie–Gower and Holling-type II schemes, Appl. Math. Lett. 16 (2003) 1069–1075.
[2] J. Brown, P.C. Dunne, R.A. Gardner, A semilinear parabolic system arising in the theory of superconductivity, J. Differ. Equations 40 (1981) 232–252.
[3] D. Henry, Geometric theory of semilinear parabolic equations, Lecture Notes in Mathematics***, vol. 840, Springer-Verlag, New York, 1993.
[4] W. Ko, K. Ryu, Qualitative analysis of a predator-prey model with Holling type II functional response incorporating a prey refuge, J. Differ. Equations 231 (2006) 534–550.
[5] T. Lindstrom, Global stability of a model for competing predators: an extension of the Aradito and Ricciardi Lyapunov function, Nonlinear Anal. 39 (2000) 793–805.
[6] J.D. Murray, Mathematical Biology, I & II, Springer-Verlag, New York, 2002.
[7] A.F. Nindjin, M.A. Aziz-Alaoui, M. Cadivel, Analysis of a predator-prey model with modified Leslie–Gower and Holling-type II schemes with delay, Nonlinear Anal. Real World Appl. 7 (2006) 1104–1118.
[8] C.V. Pao, Dynamics of nonlinear parabolic systems with time delays, J. Math. Anal. Appl. 205 (1996) 751–779.
[9] R. Peng, M.X. Wang, Note on a ratio-dependent predator-prey system with diffusion, Nonlinear Anal. Real World Appl. 7 (2006) 1–11.
[10] R.K. Upadhyay, S.R.K. Iyengar, Effect of seasonality on the dynamics of 2 and 3 species prey-prator systems, Nonlinear Anal. Real World Appl. 6 (2005) 509–530.
[11] M.X. Wang, Nonlinear Parabolic Equation of Parabolic Type (in Chinese), Science Press, Beijing, 1993.
[12] Y. Yamada, Global solution for quasilinear parabolic systems with cross-diffusion effects, Nonlinear Anal. TMA 24 (1995) 1395–1412.
[13] X.Z. Zeng, A ratio-dependent predator-prey model with diffusion, Nonlinear Anal. Real World Appl. 8 (2007) 1062–1078.