ON (GORENSTEIN) INJECTIVE AND (GORENSTEIN) FLAT DIMENSIONS OF RIGHT DERIVED SECTION FUNCTOR OF COMPLEXES

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ABSTRACT. Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, $\mathfrak{a}$ be a proper ideal of R and M be an R-complex in $\mathrm{D}(R)$. We prove that if $M \in \mathrm{D}^f_{\sqsubset}(R)$ (respectively, $M \in \mathrm{D}^f_{\sqsupset}(R)$), then $\mathrm{id}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{id}_R M$ (respectively, $\mathrm{fd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{fd}_R M$). Next, it is proved that the right derived section functor of a complex $M \in \mathrm{D}_{\sqsubset}(R)$ (R is not necessarily local) can be computed via a genuine left-bounded complex $G \simeq M$ of Gorenstein injective modules. We show that if R has a dualizing complex and M is an R-complex in $\mathrm{D}^f_{\square}(R)$, then $\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M$ and $\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gid}_R M$. Also, we show that if M is a relative Cohen-Macaulay R-module with respect to $\mathfrak{a}$ (respectively, Cohen-Macaulay R-module of dimension n), then $\mathrm{Gfd}_R \mathbf{H}^{\mathrm{ht}_M \mathfrak{a}}_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M + n$ (respectively, $\mathrm{Gid}_R \mathbf{H}^n_{\mathfrak{m}}(M) = \mathrm{Gid}_R M - n$). The above results generalize some known results and provide characterizations of Gorenstein rings.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring, $\mathfrak{a}$ is a proper ideal of R and M is an R-complex. The category of R-complexes is denoted $\mathrm{C}(R)$, and we use subscripts $\sqsubset$, $\sqsupset$ and $\square$ to denote genuine boundedness conditions. Also, the derived category is denoted $\mathrm{D}(R)$, and we use subscripts $\sqsubset$, $\sqsupset$ and $\square$ to denote homological boundedness conditions. The symbol $\simeq$ is the sign for isomorphism in $\mathrm{D}(R)$ and quasi-isomorphisms in $\mathrm{C}(R)$. We also use superscript f to signify that the homology modules are degreewise finitely generated. An R-complex I is semiinjective if the functor $\mathrm{Hom}_R(-, I)$ converts injective quasiisomorphisms into surjective quasiisomorphisms. A semiinjective resolution of an R-complex M is a semiinjective complex I and a quasiisomorphism $M \xrightarrow{\simeq} I$. For an R-complex M the injective dimension $\mathrm{id}_R M$ is defined as

$$\mathrm{id}_R M = \inf \left\{ \ell \in \mathbb{Z} \mid {\exists \text{ semiinjective } R\text{-complex } I \text{ such that} \atop M \simeq I \text{ in } \mathrm{D}(R) \text{ and } I_v = 0 \text{ for all } v < -\ell} \right\}.$$

Several of the main results of this paper involve the hypothesis that R has a dualizing complex. A complex $D \in \mathrm{D}^f_{\square}(R)$ is dualizing for R if it has finite injective dimension and the canonical morphism $\chi^R_M : R \to \mathbf{R}\mathrm{Hom}_R(D, D)$ is an isomorphism in $\mathrm{D}(R)$. If R has a dualizing complex D, we may consider the functor $-^{\dagger} = \mathbf{R}\mathrm{Hom}_R(-, D)$. The notion of Gorenstein injective module was introduced by E.E. Enochs and O.M.G. Jenda in [6]. An R-module M is said to be Gorenstein injective, if there exists a $\mathrm{Hom}_R(\mathcal{I}, -)$ exact acyclic complex E of injective R-modules such that $M = \mathrm{Ker}(E_0 \to E_{-1})$. The Gorenstein injective dimension, $\mathrm{Gid}_R M$, of $M \in \mathrm{D}_{\sqsubset}(R)$ is defined to be the infimum of the set of integers n such that there exists a complex $G \in \mathrm{C}_{\sqsubset}(R)$ consisting of Gorenstein injective modules satisfying $M \simeq G$ and $G_n = 0$ for $n < -\ell$. Also, an

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R-complex F is semiflat if the functor $-\otimes_R F$ preserves injective quasiisomorphisms. For an R-complex M the flat dimension $\mathrm{fd}_R M$ is defined as

$$\mathrm{fd}_R M = \inf\left\{ n\in\mathbb{Z} \mid {\exists \textit{ semiflat } R\textit{-complex } F \textit{ such that} \atop F\simeq M \textit{ in } \mathrm{D}(R) \textit{ and } F_v=0 \textit{ for all } v>n} \right\}.$$

An R-module M is said to be Gorenstein flat, if there exists an $\mathcal{I}\otimes_R -$ exact acyclic complex F of of flat R-modules such that $M = \mathrm{Ker}(F_0 \to F_{-1})$. The Gorenstein flat dimension, $\mathrm{Gfd}_R M$, of $M \in \mathrm{D}_{\sqsupset}(R)$ is defined to be the infimum of the set of integers n such that there exists a complex $F \in \mathrm{C}_{\sqsupset}(R)$ consisting of Gorenstein flat modules satisfying $M \simeq F$ and $G_n = 0$ for $n < \ell$. Let M be an R-complex in $\mathrm{D}(R)$. The right derived section functor of the complex M is defined as $\mathbf{R}\Gamma_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(E)$, where E is a semiinjective resolution of M (see [8] and [16]).

It has been shown in [7, Theorem 6.5] that the right derived section functor (with support in any ideal $\mathfrak{a}$) sends complexes of finite flat dimension (respectively, finite injective dimension) to complexes of finite flat dimension (respectively, finite injective dimension). In section 2, as a main result, we prove that if $(R,\mathfrak{m})$ is a local ring and $M \in \mathrm{D}^f_{\sqsubset}(R)$, then $\mathrm{id}_R M$ and $\mathrm{id}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M)$ are simultaneously finite, and that $\mathrm{id}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{id}_R M$ (see Theorem 2.6). Then, as a corollary, we provide a characterization of Gorenstein rings which recovers [20, Corollary 2.7]. Also, in 2.10, we prove that if $(R,\mathfrak{m})$ is a local ring and M is an R-complex in $\mathrm{D}^f_{\sqsubset}(R)$ with $\sup \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \inf \mathbf{R}\Gamma_{\mathfrak{a}}(M) = -n$, then $\mathrm{id}_R M$ and $\mathrm{id}_R \mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))$ ($\mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))$ is considered as a module) are simultaneously finite and there is an equality $\mathrm{id}_R \mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M)) = \mathrm{id}_R M - n$. Notice that, this result is a generalization of [20, Theorem 2.5].

In section 3, as a main result, a Gorenstein flat version of 2.6 is demonstrated. Indeed, it is shown, in Theorem 3.5, that if $(R,\mathfrak{m})$ is a local ring and $M \in \mathrm{D}^f_{\sqsupset}(R)$, then $\mathrm{fd}_R M$ and $\mathrm{fd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M)$ are simultaneously finite, and that $\mathrm{fd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{fd}_R M$. Next, as an application of Theorem 3.5, we prove a Gorenstein flat version of 2.10 (see Theorem 3.7). It has been proved in [4, Theorem 5.9] that if $M \in \mathrm{D}_{\square}(R)$, then

$$\mathrm{Gid}_R M < \infty \Rightarrow \mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty.$$

Moreover, if R has a dualizing complex, the above implication may be reversed if $\mathfrak{a}$ is in the Jacobson radical of R and $M \in \mathrm{D}^f_{\square}(R)$. In Theorem 3.10, as a main result, we show that if $(R,\mathfrak{m})$ is a local ring admitting a dualizing complex and M is an R-complex in $\mathrm{D}^f_{\square}(R)$, then $\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M$. Then, as a corollary, we prove that if M is a relative Cohen-Macaulay R-module with respect to $\mathfrak{a}$, where is defined as in [20], and that $n = \mathrm{grade}(\mathfrak{a}, M)$, then $\mathrm{Gfd}_R \mathbf{H}^n_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M + n$.

In section 4, we prove that if C is an $\Gamma_{\mathfrak{a}}$-acyclic R-complex in $\mathrm{C}_{\sqsubset}(R)$, then $\mathbf{R}\Gamma_{\mathfrak{a}}(C) \simeq \Gamma_{\mathfrak{a}}(C)$ (see Theorem 4.5). As an application of this Theorem, in Corollary 4.7, we show that, the right derived section functor of a complex $M \in \mathrm{D}_{\sqsubset}(R)$ can be computed via a genuine left-bounded complex $G \simeq M$ of Gorenstein injective modules. Also, as a main result, we show that if $(R,\mathfrak{m})$ is a local ring admitting a dualizing complex and M is an R-complex in $\mathrm{D}^f_{\square}(R)$, then $\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gid}_R M$ (see Theorem 4.10). As a corollary, we provide a characterization of Gorenstein rings which recovers [20, Corollary 3.10] and [19, Theorem 2.6]. Next, in Theorem 4.12, we prove a complex version of 2.6 which improves [20, Theorem 3.8]. As a corollary, in 4.13, we deduce that $\mathrm{Gid}_R \mathbf{H}^n_{\mathfrak{m}}(M) = \mathrm{Gid}_R M - n$, wherever $(R,\mathfrak{m})$ is a local ring and M is a Cohen-Macaulay R-module with $\dim_R M = n$ (without assuming that R is Cohen-Macaulay). This corollary improves [20, Corollary 3.9].

2. RIGHT DERIVED SECTION FUNCTOR AND INJECTIVE DIMENSION

We recall the following lemma which shows that the right derived section functor can be computed via Čech complexes.

Lemma 2.1. *Let $\mathfrak{a}$ be a proper ideal of R, and let M be an R-complex in $\mathrm{D}(R)$. If $\underline{x} = x_1, \cdots, x_r$ is a generating set for the ideal $\mathfrak{a}$ and $\check{C}_{\underline{x}}$ the corresponding Čech complex, then*

$$\mathbf{R}\Gamma_{\mathfrak{a}}(M) \simeq M \otimes^{\mathbf{L}}_R \check{C}_{\underline{x}}.$$

Proof. see [15, Theorem 1.1(iv)]. □

The following proposition is one of the main results of this section which has led to some interesting results.

Proposition 2.2. *Let $(R, \mathfrak{m}, k)$ be a local ring, and let M be an R-complex in $\mathrm{D}_{\sqsubset}(R)$. Then*

$$\mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\Gamma_{\mathfrak{a}}(M)) \simeq \mathbf{R}\mathrm{Hom}_R(k, M).$$

Proof. Let F be a semifree resolution of the residue field k. Let $\underline{x} = x_1, \cdots, x_r$ be a generating set for the ideal $\mathfrak{a}$ and $\check{C}_{\underline{x}}$ be the Čech complex with respect to $\underline{x}$. Then, in view of [2, Theorem 4.4.5(a)], there are isomorphisms

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(k, M \otimes^{\mathbf{L}}_R \check{C}_{\underline{x}}) &\simeq \mathbf{R}\mathrm{Hom}_R(k, M) \otimes^{\mathbf{L}}_R \check{C}_{\underline{x}} \\ &\simeq \mathbf{R}\mathrm{Hom}_R(k, M) \otimes^{\mathbf{L}}_k (k \otimes^{\mathbf{L}}_R \check{C}_{\underline{x}})\end{aligned}$$

in $\mathrm{D}(R)$. But, as in the proof of [11, Lemma 2.4], there exists a quasiisomorphism $F \otimes_R \check{C}_{\underline{x}} \simeq F$ in $\mathrm{C}(R)$. Hence $k \otimes^{\mathbf{L}}_R \check{C}_{\underline{x}} \simeq k$, and so

$$\mathbf{R}\mathrm{Hom}_R(k, M \otimes^{\mathbf{L}}_R \check{C}_{\underline{x}}) \simeq \mathbf{R}\mathrm{Hom}_R(k, M).$$

The result therefore follows from the fact that $M \otimes^{\mathbf{L}}_R \check{C}_{\underline{x}} \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(M)$. □

Corollary 2.3. *Let $(R, \mathfrak{m}, k)$ be a local ring, and let M be an R-complex in $\mathrm{D}_{\sqsubset}(R)$. Then the following statements hold.*

(i) $\mathrm{depth}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{depth}_R M$.
(ii) *If R admits a dualizing complex D, then*

$$\mathrm{width}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M)^{\dagger} = \mathrm{width}_R M^{\dagger}.$$

Proof. (i) Immediate from Proposition 2.2. (ii) By [2, Theorem 4.4.6(a)] and Proposition 2.2, there are isomorphisms

$$\begin{aligned}k \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(M), D) &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\Gamma_{\mathfrak{a}}(M)), D) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(k, M), D) \\ &\simeq k \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, D)\end{aligned}$$

in $\mathrm{D}(R)$. The result now follows by definition of width. □

The following corollary, which is an immediate consequence of [10, Corollary 3.4.4] and 2.2, determines the i-th Bass number $\mu^i(\mathfrak{p}, \mathbf{R}\Gamma_{\mathfrak{a}}(M)) := \mu^i_{R_{\mathfrak{p}}}(\mathbf{R}\Gamma_{\mathfrak{a}}(M)_{\mathfrak{p}})$ of $\mathbf{R}\Gamma_{\mathfrak{a}}(M)$, where is defined as in [2].

Corollary 2.4. *Let $(R, \mathfrak{m}, k)$ be a local ring, and let M be an R-complex in $\mathrm{D}_{\sqsubset}(R)$. Then $\mu^i_R(\mathbf{R}\Gamma_{\mathfrak{a}}(M)) = \mu^i_R(M)$ for all $i \in \mathbb{Z}$; In particular, for every $\mathfrak{p} \in \mathbf{V}(\mathfrak{a})$ there is an equality $\mu^i(\mathfrak{p}, \mathbf{R}\Gamma_{\mathfrak{a}}(M)) = \mu^i(\mathfrak{p}, M)$ for all $i \in \mathbb{Z}$.*

Next, we recall the following lemma which is needed in the proof of Theorem 2.6.

Lemma 2.5. *Let M be a complex in $\mathrm{D}(R)$. Then*

(i) $\mathrm{id}_R M = \sup\{-\inf \mathbf{R}\mathrm{Hom}_R(T,M) \mid$ *T is a cyclic R-module*$\}$; *and*

(ii) *If $M \in \mathrm{D}_{\sqsubset}(R)$, then*

$$\mathrm{id}_R M = \sup\{m \in \mathbb{Z} \mid \exists \mathfrak{p} \in \mathrm{Spec} R : \mu^m_{R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0\}.$$

Proof. Parts (i) and (ii) have been proved in [2, Theorem 5.1.6] and [2, Lemma 6.1.19], respectively. □

The following theorem, which is one of the main results of this section, provides a comparison between the injective dimensions of a complex and its right derived section functor.

Theorem 2.6. *Let $(R,\mathfrak{m},k)$ be a local ring, and let M be an R-complex in $\mathrm{D}_{\sqsubset}(R)$.*

(i) *If $\mathrm{id}_R M < \infty$, then $\mathrm{id}_R \mathbf{R}\Gamma_\mathfrak{a}(M) < \infty$.*

(ii) *The converse holds whenever $M \in \mathrm{D}^f_{\sqsubset}(R)$.*

Furthermore, if $M \in \mathrm{D}^f_{\sqsubset}(R)$, then $\mathrm{id}_R \mathbf{R}\Gamma_\mathfrak{a}(M) = \mathrm{id}_R M$.

Proof. (i) Let $s := \mathrm{id}_R M < \infty$. Then, in view of Lemma 2.5(ii) and Corollary 2.4, $\mu^{i+s}(\mathfrak{p}, \mathbf{R}\Gamma_\mathfrak{a}(M)) = 0$ for all $\mathfrak{p} \in \mathrm{Spec} R$ and for all $i > 0$. Therefore, it follows from Lemma 2.5(ii) that $\mathrm{id}_R \mathbf{R}\Gamma_\mathfrak{a}(M) \leq s$.

(ii) Let $t := \mathrm{id}_R \mathbf{R}\Gamma_\mathfrak{a}(M) < \infty$. Then, by Lemma 2.5(i), $\inf \mathbf{R}\mathrm{Hom}_R(T, \mathbf{R}\Gamma_\mathfrak{a}(M)) \geq -t$ for all cyclic R-module T. Hence, in view of Proposition 2.2, there are isomorphisms

$$\mathbf{H}_{-t-i}(\mathbf{R}\mathrm{Hom}_R(k,M)) \cong \mathbf{H}_{-t-i}(\mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\Gamma_\mathfrak{a}(M))) \cong 0$$

for all $i > 0$. Therefore $-\inf \mathbf{R}\mathrm{Hom}_R(k,M) \leq t$, and so $\mathrm{id}_R M \leq t < \infty$ by [2, Theorem 6.1.13]. The final assertion is a consequence of (i) and (ii). □

The following corollary, which recovers [20, Corollary 2.7], is an immediate consequence of the previous Theorem.

Corollary 2.7. *Let $(R,\mathfrak{m})$ be a local ring. Then the following statements are equivalent:*

(i) *R is Gorenstein;*

(ii) *$\mathrm{id}_R \mathbf{R}\Gamma_\mathfrak{a}(R) < \infty$ for any ideal $\mathfrak{a}$ of R;*

(iii) *$\mathrm{id}_R \mathbf{R}\Gamma_\mathfrak{a}(R) < \infty$ for some ideal $\mathfrak{a}$ of R.*

Remark 2.8. Note that if M is an R-complex in $\mathrm{D}(R)$ such that $\sup \mathbf{R}\Gamma_\mathfrak{a}(M) = \inf \mathbf{R}\Gamma_\mathfrak{a}(M) = -n$, then there is an isomorphism $\mathbf{R}\Gamma_\mathfrak{a}(M) \simeq \mathbf{H}_{-n}(\mathbf{R}\Gamma_\mathfrak{a}(M))$ in $\mathrm{D}(R)$. Hence $\Sigma^n \mathbf{R}\Gamma_\mathfrak{a}(M)$ is equivalent to the module $\Sigma^n \mathbf{H}_{-n}(\mathbf{R}\Gamma_\mathfrak{a}(M))$ in the category of R-modules. We will abbreviate $\Sigma^n \mathbf{H}_{-n}(\mathbf{R}\Gamma_\mathfrak{a}(M))$ to $\mathbf{H}^n_\mathfrak{a}(M)$ when $\sup \mathbf{R}\Gamma_\mathfrak{a}(M) = \inf \mathbf{R}\Gamma_\mathfrak{a}(M) = -n$.

Example 2.9. Let $(R,\mathfrak{m})$ be a Gorenstein local ring with $\dim(R) = d$, and let M be the R-complex

$$0 \longrightarrow \bigoplus_{\substack{\mathfrak{p} \in \mathrm{Spec}(R) \\ \mathrm{ht}\mathfrak{p}=0}} E(R/\mathfrak{p}) \to \bigoplus_{\substack{\mathfrak{p} \in \mathrm{Spec}(R) \\ \mathrm{ht}\mathfrak{p}=1}} E(R/\mathfrak{p}) \to \cdots \to E(R/\mathfrak{m}) \longrightarrow 0.$$

Since $M \simeq R$, $\mathrm{id}_R M = d$. On the other hand, $\mathbf{R}\Gamma_\mathfrak{m}(M) \simeq \mathbf{H}_{-d}(\mathbf{R}\Gamma_\mathfrak{m}(M))$, and so $\Sigma^d \mathbf{R}\Gamma_\mathfrak{m}(M) \cong E(R/\mathfrak{m})$. Therefore

$$\mathrm{id}_R \mathbf{R}\Gamma_\mathfrak{m}(M) = d = \mathrm{id}_R M.$$

The following theorem, which is an immediate consequence of Theorem 2.10, is a generalization of [20, Theorem 2.5].

Theorem 2.10. *Let $(R,\mathfrak{m})$ be a local ring, and let n be an integer. Suppose that M is an R-complex in $\mathrm{D}_{\sqsubset}(R)$ such that* $\sup\mathbf{R}\Gamma_{\mathfrak{a}}(M)=\inf\mathbf{R}\Gamma_{\mathfrak{a}}(M)=-n$.

(i) *If* $\mathrm{id}_R M<\infty$, *then* $\mathrm{id}_R\mathbf{H}^n_{\mathfrak{a}}(M)<\infty$.
(ii) *The converse holds whenever* $M\in\mathrm{D}^f_{\sqsubset}(R)$.

Furthermore, if $M\in\mathrm{D}^f_{\sqsubset}(R)$, *then* $\mathrm{id}_R\mathbf{H}^n_{\mathfrak{a}}(M)=\mathrm{id}_R M-n$.

Proof. Parts (i) and (ii) follow from Theorem 2.6 in view of Remark 2.8.

For the final assertion, we notice that, since $\mathbf{R}\Gamma_{\mathfrak{a}}(M)\simeq\mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))$, there are equalities

$$\begin{aligned}\mathrm{id}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M)&=\mathrm{id}_R\Sigma^{-n}(\Sigma^n\mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M)))\\&=\mathrm{id}_R\Sigma^n\mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))+n.\end{aligned}$$

Then, in view of Remark 2.8, $\mathrm{id}_R\mathbf{H}^n_{\mathfrak{a}}(M)=\mathrm{id}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M)-n$. The desired equality now follows from Theorem 2.6. □

3. Right derived section functor and (Gorenstein) flat dimension

The following proposition is one of the main results of this section which has led to some interesting results.

Proposition 3.1. *Let $(R,\mathfrak{m},k)$ be a local ring, and let M be an R-complex in $\mathrm{D}(R)$. Then*

$$k\otimes^{\mathbf{L}}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M)\simeq k\otimes^{\mathbf{L}}_R M.$$

Proof. Let $\underline{x}=x_1,\cdots,x_r$ be a generating set for the ideal $\mathfrak{a}$ and $\check{C}_{\underline{x}}$ be the Čech complex with respect to $\underline{x}$. Then, there exists an isomorphism $k\otimes^{\mathbf{L}}_R\check{C}_{\underline{x}}\simeq k$ in $\mathrm{D}(R)$ (see 2.2). Hence, we have

$$k\otimes^{\mathbf{L}}_R\check{C}_{\underline{x}}\otimes^{\mathbf{L}}_R M\simeq k\otimes^{\mathbf{L}}_R M.$$

But $\check{C}_{\underline{x}}\otimes^{\mathbf{L}}_R M\simeq\mathbf{R}\Gamma_{\mathfrak{a}}(M)$. Therefore $k\otimes^{\mathbf{L}}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M)\simeq k\otimes^{\mathbf{L}}_R M$ as desired. □

Corollary 3.2. *Let $(R,\mathfrak{m},k)$ be a local ring. Suppose that M and N are R-complexes in $\mathrm{D}(R)$. Then the following statements hold.*

(i) $\mathrm{width}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M)=\mathrm{width}_R M$.
(ii) *If N is an R-complex in $\mathrm{D}_{\sqsubset}(R)$ and $M\in\mathrm{D}_{\sqsupset}(R)$, then*

$$\mathrm{depth}_R\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(M),N)=\mathrm{depth}_R\mathbf{R}\mathrm{Hom}_R(M,N).$$

In particular, If R admits a dualizing complex D, then

$$\mathrm{depth}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M)^{\dagger}=\mathrm{depth}_R M^{\dagger}.$$

Proof. (i) Immediate from Proposition 3.1. (ii) A straightforward application of [2, Proposition 5.2.6], part (i) and Corollary 2.3(i). □

The following corollary, which is an immediate consequence of [10, Corollary 3.4.4] and 3.1, determines the i-th Betti number $\beta_i(\mathfrak{p},\mathbf{R}\Gamma_{\mathfrak{a}}(M)):=\beta_i^{R_{\mathfrak{p}}}(\mathbf{R}\Gamma_{\mathfrak{a}}(M)_{\mathfrak{p}})$ of $\mathbf{R}\Gamma_{\mathfrak{a}}(M)$, where is defined as in [2].

Corollary 3.3. *Let $(R,\mathfrak{m},k)$ be a local ring, and let M be an R-complex in $\mathrm{D}(R)$. Then $\beta_i^R(\mathbf{R}\Gamma_{\mathfrak{a}}(M))=\beta_i^R(M)$ for all $i\in\mathbb{Z}$; In particular, for every $\mathfrak{p}\in\mathbf{V}(\mathfrak{a})$ there is an equality $\beta_i(\mathfrak{p},\mathbf{R}\Gamma_{\mathfrak{a}}(M))=\beta_i(\mathfrak{p},M)$ for all $i\in\mathbb{Z}$.*

Next, we recall the following lemma which is needed in the proof of Theorem 3.5.

Lemma 3.4. *Let M be a complex in $\mathrm{D}(R)$. Then*

(i) $\mathrm{fd}_R\ M = \sup\{\sup T \otimes^{\mathbf{L}}_R M \mid\ T$ *is a cyclic R-module*$\}$; *and*
(ii) *There is an equality*

$$\mathrm{fd}_R\ M = \sup\{m \in \mathbb{Z} \mid \exists \mathfrak{p} \in \mathrm{Spec}R : \beta^{R_\mathfrak{p}}_m(M_\mathfrak{p}) \neq 0\}.$$

Proof. Parts (i) and (ii) have been proved in [2, Theorem 5.1.9] and [2, Lemma 6.1.15], respectively. □

Theorem 3.5, which is a flat version of 2.6, is one of the main results of this section. This theorem recovers some interesting results that have currently been appeared in the literature.

Theorem 3.5. *Let $(R, \mathfrak{m}, k)$ be a local ring, and let $\mathfrak{a}$ be a proper ideal of R. Suppose that M is an R-complex in $\mathrm{D}(R)$.*
(i) *If $\mathrm{fd}_R M < \infty$, then $\mathrm{fd}_R \mathbf{R}\Gamma_\mathfrak{a}(M) < \infty$.*
(ii) *The converse holds whenever $M \in \mathrm{D}^f_\sqsupset(R)$.*

Furthermore, if $M \in \mathrm{D}^f_\sqsupset(R)$, then $\mathrm{fd}_R \mathbf{R}\Gamma_\mathfrak{a}(M) = \mathrm{fd}_R M$.

Proof. (i) Let $s := \mathrm{fd}_R M < \infty$. Then, in view of Lemma 3.4(ii) and Corollary 3.3, $\beta_{i+s}(\mathfrak{p}, \mathbf{R}\Gamma_\mathfrak{a}(M)) = 0$ for all $\mathfrak{p} \in \mathrm{Spec}R$ and for all $i > 0$. Therefore, it follows from Lemma 3.4(ii) that $\mathrm{fd}_R \mathbf{R}\Gamma_\mathfrak{a}(M) \leq s$.

(ii) Let $t := \mathrm{fd}_R \mathbf{R}\Gamma_\mathfrak{a}(M) < \infty$. Then, by Lemma 3.4(i), $\sup T \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_\mathfrak{a}(M)) \geq t$ for all cyclic R-module T. Hence, in view of Proposition 3.1, there are isomorphisms

$$\mathbf{H}_{t+i}(k \otimes^{\mathbf{L}}_R M) \cong \mathbf{H}_{t+i}(k \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_\mathfrak{a}(M)) \cong 0$$

for all $i > 0$. Therefore $\sup k \otimes^{\mathbf{L}}_R M \leq t$, and so $\mathrm{fd}_R M \leq t < \infty$ by [2, Theorem 6.1.13]. The final assertion is a consequence of (i) and (ii). □

Example 3.6. Let $(R, \mathfrak{m})$ be a Gorenstein local ring with $\dim(R) = d$, and let M be as in 2.9. Since $M \simeq R$, $\mathrm{fd}_R M = 0$. On the other hand, $\mathbf{R}\Gamma_\mathfrak{m}(M) \simeq \mathbf{H}_{-d}(\mathbf{R}\Gamma_\mathfrak{m}(M))$, and so $\Sigma^d \mathbf{R}\Gamma_\mathfrak{m}(M) \cong \mathbf{H}^n_\mathfrak{m}(R)$. Hence, by [21, Corollary 3.4], $\mathrm{fd}_R \mathbf{R}\Gamma_\mathfrak{m}(M) + d = d$. Therefore

$$\mathrm{fd}_R \mathbf{R}\Gamma_\mathfrak{m}(M) = 0 = \mathrm{fd}_R M.$$

The following theorem, which is an immediate consequence of Theorem 3.5, is a flat version of 2.10.

Theorem 3.7. *Let $(R, \mathfrak{m})$ be a local ring, and let n be an integer. Suppose that M is an R-complex in $\mathrm{D}(R)$ such that $\sup \mathbf{R}\Gamma_\mathfrak{a}(M) = \inf \mathbf{R}\Gamma_\mathfrak{a}(M) = -n$.*
(i) *If $\mathrm{fd}_R M < \infty$, then $\mathrm{fd}_R \mathbf{H}^n_\mathfrak{a}(M) < \infty$.*
(ii) *The converse holds whenever $M \in \mathrm{D}^f_\sqsupset(R)$.*

Furthermore, if $M \in \mathrm{D}^f_\sqsupset(R)$, then $\mathrm{fd}_R \mathbf{H}^n_\mathfrak{a}(M) = \mathrm{fd}_R M + n$.

Proof. Parts (i) and (ii) follow from Theorem 3.5 in view of Remark 2.8.

For the final assertion, we notice that, since $\mathbf{R}\Gamma_\mathfrak{a}(M) \simeq \mathbf{H}_{-n}(\mathbf{R}\Gamma_\mathfrak{a}(M))$, there are equalities

$$\begin{aligned}\mathrm{fd}_R \mathbf{R}\Gamma_\mathfrak{a}(M) &= \mathrm{fd}_R \Sigma^{-n}(\Sigma^n \mathbf{H}_{-n}(\mathbf{R}\Gamma_\mathfrak{a}(M))) \\ &= \mathrm{fd}_R \Sigma^n \mathbf{H}_{-n}(\mathbf{R}\Gamma_\mathfrak{a}(M)) - n.\end{aligned}$$

Then, in view of Remark 2.8, $\mathrm{fd}_R \mathbf{H}^n_\mathfrak{a}(M) = \mathrm{fd}_R \mathbf{R}\Gamma_\mathfrak{a}(M) + n$. The desired equality now follows from Theorem 3.5. □

In the rest of this section, we make a comparison between the Gorenstein flat dimensions of a complex and its right derived section functor.

Proposition 3.8. *Suppose that M is an R-complex in $\mathrm{D}_{\square}(R)$. Then*

$$\mathrm{Gfd}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) \leq \mathrm{Gfd}_R M.$$

Proof. Notice that if $\mathrm{Gfd}_R M = \infty$, then there is nothing to prove. So, we may assume that $\mathrm{Gfd}_R M < \infty$. Hence, it follows from [4, Theorem 5.9] that $\mathrm{Gfd}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty$. Now, by [9, Theorem 8.8], there exists $\mathfrak{p} \in \mathbf{V}(\mathfrak{a})$ such that

$$\mathrm{Gfd}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{depth} R_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}}.$$

But, by [10, Corollary 3.4.4] and Corollary 2.3(i), $\mathrm{depth}_{R_{\mathfrak{p}}}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}} = \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$. It follows, again by [9, Theorem 8.8], that

$$\mathrm{Gfd}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{depth} R_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \mathrm{Gfd}_R M$$

as desired. □

Proposition 3.9. *Let $(R, \mathfrak{m}, k)$ be a local ring, and let M be an R-complex in $\mathrm{D}^f_{\square}(R)$ such that $\mathrm{Gfd}_R M < \infty$. Then*

$$\mathrm{Gfd}_R M \leq \mathrm{Gfd}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M).$$

Proof. By [4, Theorem 5.9], $\mathrm{Gfd}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty$. Hence, by [9, Theorem 8.7] and Corollary 2.3(i), there are equalities

$$\begin{aligned}\sup(E(k) \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M)) &= \mathrm{depth} R - \mathrm{depth}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M)\\ &= \mathrm{depth} R - \mathrm{depth}_R M = \sup(E(k) \otimes^{\mathbf{L}}_R M).\end{aligned}$$

Since $M \in \mathrm{D}^f_{\square}(R)$, $\sup(E(k) \otimes^{\mathbf{L}}_R M) = \mathrm{Gfd}_R M$. The result now follows from the fact that $\mathrm{Gfd}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) \geq \sup(E(k) \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M))$ (see [4, Corollary 3.6]). □

The following theorem, which is a Gorenstein flat version of Theorem 3.5, is one of the main results of this section.

Theorem 3.10. *Let $(R, \mathfrak{m})$ be a local ring, and let M be an R-complex in $\mathrm{D}^f_{\square}(R)$.*
- (i) *If $\mathrm{Gfd}_R M < \infty$, then $\mathrm{Gfd}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M$.*
- (ii) *If R admits a dualizing complex, then $\mathrm{Gfd}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M$.*

Proof. (i) A straightforward application of Proposition 3.8 and Proposition 3.9.

(ii) In view of part (i), we may assume that $\mathrm{Gfd}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty$. Hence, by [4, Theorem 5.9], $\mathrm{Gfd}_R M < \infty$. The desired equality now follows from Proposition 3.8 and Proposition 3.9. □

The next theorem is a Gorenstein flat version of Theorem 3.7.

Theorem 3.11. *Let $(R, \mathfrak{m})$ be a local ring admitting a dualizing complex, and let n be an integer. Suppose that M is an R-complex in $\mathrm{D}^f_{\square}(R)$ such that $\sup\mathbf{R}\Gamma_{\mathfrak{a}}(M) = \inf\mathbf{R}\Gamma_{\mathfrak{a}}(M) = -n$.*
- (i) *If $\mathrm{Gfd}_R M < \infty$, then $\mathrm{Gfd}_R\mathbf{H}^n_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M + n$.*
- (ii) *If R admits a dualizing complex, then $\mathrm{Gfd}_R\mathbf{H}^n_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M + n$.*

Proof. Follows from Theorem 3.10 in view of Remark 2.8. □

Corollary 3.12. *Let $(R, \mathfrak{m})$ be a local ring. Suppose that M is relative Cohen-Macaulay with respect to $\mathfrak{a}$ and that $n = \mathrm{grade}(\mathfrak{a}, M)$. Then $\mathrm{Gfd}_R\mathbf{H}^n_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M + n$.*

Proof. Notice that $(\widehat{R}, \widehat{\mathfrak{m}})$ is a local ring admitting a dualizing complex and $M \otimes_R \widehat{R}$ is a relative Cohen-Macaulay $\widehat{R}$-module with respect to $\mathfrak{a}\widehat{R}$ and that $\mathrm{grade}(\mathfrak{a}\widehat{R}, M \otimes_R \widehat{R}) = n$. Hence, in view of [3, Theorem 4.27], we may assume that R is complete; and so it has a dualizing complex. The result therefore follows from Theorem 3.11. □

4. Right derived section functor and Gorenstein injective dimension

In this section the category of R-modules is denoted $\mathcal{C}(R)$. Recall from [1, Exercise 4.1.2] that the local cohomology modules of R-module M with respect to $\mathfrak{a}$ can be calculated by an $\Gamma_{\mathfrak{a}}$-acyclic resolution of M. First, we prove the complex version of this exercise.

Definition 4.1. (see [17]) Let $\mathrm{F} : \mathcal{C}(R) \to \mathcal{C}(R)$ be a left exact functor, and assume that M is an R-complex in $\mathrm{C}_{\sqsubset}(R)$. If $0 \to M \to C_{*,0} \to C_{*,1} \to \cdots \to C_{*,q} \to$ is a Cartan-Eilenberg injective resolutions of M, where is defined as in [13], define $\mathbb{R}^i(\mathrm{F}M)$ to be $\mathbf{H}_i(\mathrm{Tot}(\mathrm{F}C))$.

Lemma 4.2. *Let M and $\acute{M}$ be two R-complexes in $\mathrm{C}_{\sqsubset}(R)$, and let $\zeta : M \to \acute{M}$ be a morphism of R-complexes. Suppose that $0 \to M \to C_{*,0} \to C_{*,1} \to \cdots \to C_{*,q} \to$ and $0 \to \acute{M} \to \acute{C}_{*,0} \to \acute{C}_{*,1} \to \cdots \to \acute{C}_{*,q} \to$ are Cartan-Eilenberg injective resolutions of M and $\acute{M}$, respectively. Then there exists a sequence $\{\zeta_{*,q}\}_{q\in\mathbb{N}_0}$ of morphisms $\zeta_{*,q} : C_{*,q} \to \acute{C}_{*,q}$ of R-complexes over ζ.*

Proof. A straightforward application of [12, Theorem 19]. □

Lemma 4.3. *Let $\mathrm{F} : \mathcal{C}(R) \to \mathcal{C}(R)$ be a left exact functor, and let M and $\acute{M}$ be two R-complexes. Then*

(i) *Any quasiisomorphism $\zeta : M \to \acute{M}$ induces isomorphism*

$$\mathbb{R}^i(\mathrm{F}M) \cong \mathbb{R}^i(\mathrm{F}\acute{M})$$

for all $i \in \mathbb{Z}$; and

(ii) *If M is F-acyclic R-complex in $\mathrm{C}_{\sqsubset}(R)$, that is, M_i is F-acyclic for all $i \in \mathbb{Z}$, then*

$$\mathbb{R}^i(\mathrm{F}M) = \mathbf{H}_i(\mathrm{F}M)$$

for all $i \in \mathbb{Z}$.

Proof. Straightforward verification similar to the proof of [17, Corollary 5.7.7]. □

The following theorem, which is one of the main results of this section, provide us to prove some interesting results.

Theorem 4.4. *Let $\mathrm{F} : \mathcal{C}(R) \to \mathcal{C}(R)$ be a left exact functor, and let M be an F-acyclic R-complex in $\mathrm{C}_{\sqsubset}(R)$. Assume that I is F-acyclic and $\mathrm{F}(I)$ is injective for every injective R-module I. Then $\mathrm{F}(M) \simeq \mathrm{F}(E)$, for every semiinjective resolution $E \in \mathrm{C}_{\sqsubset}(R)$ of M.*

Proof. Let E be a semiinjective resolution of M with $E_v = 0$ for $v > \sup M$, and let $\zeta : M \xrightarrow{\simeq} E$ be an quasiisomorphism. By [13, Theorem 10.45], there exist Cartan-Eilenberg injective resolutions $0 \to M \to C_{*,0} \to C_{*,1} \to \cdots \to C_{*,q} \to$ and $0 \to E \to \acute{C}_{*,0} \to \acute{C}_{*,1} \to \cdots \to \acute{C}_{*,q} \to$. Hence, in view of Lemma 4.2, there is a sequence $\{\zeta_{*,q}\}_{q\in\mathbb{N}_0}$ of morphisms of R-complexes such that the diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & E & \longrightarrow & \acute{C}_{*,0} & \longrightarrow & \acute{C}_{*,1} & \longrightarrow & \cdots & \longrightarrow & \acute{C}_{*,q} & \longrightarrow \\
 & & \uparrow{\scriptstyle\zeta} & & \uparrow{\scriptstyle\zeta_{*,0}} & & \uparrow{\scriptstyle\zeta_{*,1}} & & & & \uparrow{\scriptstyle\zeta_{*,q}} & \\
0 & \longrightarrow & M & \longrightarrow & C_{*,0} & \longrightarrow & C_{*,1} & \longrightarrow & \cdots & \longrightarrow & C_{*,q} & \longrightarrow
\end{array}$$

commutes in $\mathrm{C}(R)$. By Lemma 4.3(ii), $\mathbb{R}^p(\mathrm{F}(M)) = \mathbf{H}_p(\mathrm{F}(M))$ for all $p \in \mathbb{Z}$, so that the natural morphism $\mathrm{F}(M) \to \mathrm{Tot}(\mathrm{F}(C))$ is a quasiisomorphism. Similarly, $\mathbb{R}^p(\mathrm{F}(E)) = \mathbf{H}_p(\mathrm{F}(E))$ for all $p \in \mathbb{Z}$, so that the natural morphism $\mathrm{F}(E) \to \mathrm{Tot}(\mathrm{F}(\acute{C}))$

is a quasiisomorphism. Thus, by [2, Proposition 3.3.5(a)], there exists a quasiisomorphism $\mathrm{Tot}(\mathrm{F}(\acute{C})) \to \mathrm{F}(E)$, since $\mathrm{F}(E)$ is injective.

But, by Lemma 4.3(i), there are isomorphisms $\mathbb{R}^p(\mathrm{F}(M)) \cong \mathbb{R}^p(\mathrm{F}(E))$ for all $p \in \mathbb{Z}$. Hence the morphism $\zeta_* : \mathrm{Tot}(\mathrm{F}(C)) \longrightarrow \mathrm{Tot}(\mathrm{F}(\acute{C}))$ is a quasiisomorphism, where

$$\zeta_n = \sum_{p+q=n} \mathrm{F}(\zeta_{p,q}) : \mathrm{Tot}(\mathrm{F}(C))_n \to \mathrm{Tot}(\mathrm{F}(\acute{C}))_n$$

for all $n \in \mathbb{Z}$. Therefore, there are quasiisomorphisms

$$\mathrm{F}(M) \xrightarrow{\simeq} \mathrm{Tot}(\mathrm{F}(C)) \xrightarrow{\simeq} \mathrm{Tot}(\mathrm{F}(\acute{C})) \xrightarrow{\simeq} \mathrm{F}(E).$$

Now $\mathrm{F}(M) \simeq \mathrm{F}(E)$ as desired. □

The next theorem, which offers an application of the previous theorem, is a complex version of [1, Exercise 4.1.2].

Theorem 4.5. *Let C be an $\Gamma_{\mathfrak{a}}$-acyclic R-complex in $\mathrm{C}_{\sqsubset}(R)$. Then $\mathbf{R}\Gamma_{\mathfrak{a}}(C) \simeq \Gamma_{\mathfrak{a}}(C)$.*

It should be noted that the following corollary, as an application of Theorem 4.5, follows from the result of J. Lipman in [10, Corollary 3.4.3]; however, the proof below is different from his.

Corollary 4.6. *Let $f : R \to R'$ be a ring homomorphism of commutative Noetherian rings, and let $\lceil_R : \mathcal{C}(R') \to \mathcal{C}(R)$ to denote the functor obtained from restriction of scalars (using f). Suppose that M' is an R'-complex in $\mathrm{D}_{\sqsubset}(R')$. Then there is an isomorphism*

$$\mathbf{R}\Gamma_{\mathfrak{a}}(M'\lceil_R) \simeq \mathbf{R}\Gamma_{\mathfrak{a}R'}(M')\lceil_R$$

in $\mathrm{D}(R)$. In particular, $\mathbf{H}_{-i}(\mathbf{R}\Gamma_{\mathfrak{a}}(M'\lceil_R)) \cong \mathbf{H}_{-i}(\mathbf{R}\Gamma_{\mathfrak{a}R'}(M'))\lceil_R$ for all $i \in \mathbb{Z}$.

Proof. Let $I' \in \mathrm{C}_{\sqsubset}(R')$ be an injective resolutions of M'. Note that $M'\lceil_R \simeq I'\lceil_R$. But, by [1, Theorem 4.1.6], $I'\lceil_R$ is $\Gamma_{\mathfrak{a}}$-acyclic R-complexes in $\mathrm{C}_{\sqsubset}(R)$, and so, by Theorem 4.5, there is an isomorphism

$$\mathbf{R}\Gamma_{\mathfrak{a}}(M'\lceil_R) \simeq \Gamma_{\mathfrak{a}}(I'\lceil_R)$$

in $\mathrm{D}(R)$. The result now follows, since $\Gamma_{\mathfrak{a}}(I'\lceil_R) = \Gamma_{\mathfrak{a}R'}(I')\lceil_R$. □

It has been proved in [14, Corollary 3.3] that if R admits a dualizing complex and $M \in \mathrm{D}_{\sqsubset}(R)$, then $\mathrm{Gid}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) \leq \mathrm{Gid}_R M$. The next corollary together with [14, Theorem 3.2] recover this result. Also, notice that [14, Theorem 3.4] is an immediate consequence of 4.7.

Corollary 4.7. *Let M be an R-complex in $\mathrm{D}_{\sqsubset}(R)$, and let $G \in \mathrm{C}_{\sqsubset}(R)$ be an R-complex of Gorenstein injective modules such that $M \simeq G$. Then $\mathbf{R}\Gamma_{\mathfrak{a}}(M) \simeq \Gamma_{\mathfrak{a}}(G)$.*

Proof. Since by [18, Lemma 1.1] every Gorenstein injective module is $\Gamma_{\mathfrak{a}}$-acyclic, the result follows from Theorem 4.5. □

The following proposition together with [4, Theorem 5.9] recover [14, Corollary 3.3].

Proposition 4.8. *Suppose that M is an R-complex in $\mathrm{D}_{\sqsubset}(R)$ such that $\mathrm{Gid}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty$. Then*

$$\mathrm{Gid}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) \leq \mathrm{Gid}_R M.$$

Proof. Notice that if $\mathrm{Gid}_R M = \infty$, then there is nothing to prove. So, we may assume that $\mathrm{Gid}_R M < \infty$. By [5, Theorem 2.2], there exists $\mathfrak{p} \in \mathbf{V}(\mathfrak{a})$ such that

$$\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{depth} R_{\mathfrak{p}} - \mathrm{width}_{R_{\mathfrak{p}}} (\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}}.$$

But, by [10, Corollary 3.4.4] and Corollary 3.2(i), $\mathrm{width}_{R_{\mathfrak{p}}} (\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}} = \mathrm{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$. It follows, again by [5, Theorem 2.2], that

$$\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{depth} R_{\mathfrak{p}} - \mathrm{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \mathrm{Gid}_R M$$

as desired. □

Proposition 4.9. *Let $(R, \mathfrak{m})$ be a local ring, and let M be an R-complex in $\mathrm{D}^f_{\square}(R)$ such that $\mathrm{Gid}_R M < \infty$. Then*

$$\mathrm{Gid}_R M \leq \mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M).$$

Proof. By [4, Proposition 6.3], there is an inequality

$$\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) \geq \mathrm{depth} R - \mathrm{width}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M).$$

But, by Corollary 3.2(i), $\mathrm{width}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{width}_R M$. Thus, we have

$$\begin{aligned} \mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) &\geq \mathrm{depth} R - \mathrm{width}_R M \\ &= \mathrm{depth} R - \inf M. \end{aligned}$$

The result therefore follows from [5, Corollary 2.3]. □

The following theorem, which is a Gorenstein injective version of Theorem 2.6, is one of the main results of this section.

Theorem 4.10. *Let $(R, \mathfrak{m})$ be a local ring admitting a dualizing complex, and let M be an R-complex in $\mathrm{D}^f_{\square}(R)$. Then $\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gid}_R M$.*

Proof. A straightforward application of [4, Theorem 5.9], Proposition 4.8 and Proposition 4.9. □

In [20, Corollary 3.10] a characterization of a Gorenstein local ring R is given. The next corollary recovers that characterization. Also, notice that Corollary 4.11 together with Corollary 2.7 recover [19, Theorem 2.6].

Corollary 4.11. *Let $(R, \mathfrak{m})$ be a local ring admitting a dualizing complex. Then the following statements are equivalent:*

(i) *R is Gorenstein;*
(ii) *$\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(R) < \infty$ for any ideal $\mathfrak{a}$ of R;*
(iii) *$\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(R) < \infty$ for some ideal $\mathfrak{a}$ of R.*

Proof. A straightforward application of Theorem 4.10 and [3, Proposition 3.11]. □

The next theorem, which is a Gorenstein injective version of Theorem 2.10, recovers [20, Theorem 3.8].

Theorem 4.12. *Let $(R, \mathfrak{m})$ be a local ring admitting a dualizing complex, and let n be an integer. Suppose that M is an R-complex in $\mathrm{D}^f_{\square}(R)$ such that $\sup \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \inf \mathbf{R}\Gamma_{\mathfrak{a}}(M) = -n$. Then $\mathrm{Gid}_R \mathbf{H}^n_{\mathfrak{a}}(M) = \mathrm{Gid}_R M - n$.*

Proof. Follows from Theorem 4.10 in view of Remark 2.8. □

Let $(R,\mathfrak{m})$ be a local ring, and let M be an R-module. As a result, it has been shown in [20, Corollary 3.9] that if R and M are Cohen-Macaulay with $\dim_R M = n$, then $\mathrm{Gid}_R \mathbf{H}^n_{\mathfrak{m}}(M) = \mathrm{Gid}_R M - n$. We show that [20, Corollary 3.9] holds without assuming that R is Cohen-Macaulay.

Corollary 4.13. *Let $(R,\mathfrak{m})$ be a local ring, and let M be a Cohen-Macaulay R-module with $\dim_R M = n$. Then the following statements hold.*

(i) $\mathrm{Gid}_{\widehat{R}} \mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}}(M \otimes_R \widehat{R}) = \mathrm{Gid}_R M$.
(ii) $\mathrm{Gid}_R \mathbf{H}^n_{\mathfrak{m}}(M) = \mathrm{Gid}_R M - n$.

Proof. Notice that $(\widehat{R},\widehat{\mathfrak{m}})$ is a local ring admitting a dualizing complex and $M \otimes_R \widehat{R}$ is a Cohen-Macaulay $\widehat{R}$-module of dimension n.

(i) A straightforward application of Theorem 4.10 and [3, Theorem 3.24].

(ii) It follows, by Remark 2.8, that

$$\mathrm{Gid}_{\widehat{R}} \mathbf{R}\Gamma_{\widehat{\mathfrak{m}}}(M \otimes_R \widehat{R}) = \mathrm{Gid}_{\widehat{R}} \mathbf{H}^n_{\widehat{\mathfrak{m}}}(M \otimes_R \widehat{R}) + n$$

But, in view of [14, Lemma 3.6], $\mathrm{Gid}_{\widehat{R}} \mathbf{H}^n_{\widehat{\mathfrak{m}}}(M \otimes_R \widehat{R}) = \mathrm{Gid}_R \mathbf{H}^n_{\mathfrak{m}}(M)$. The result now follows from part (i). □

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