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Lattices of two-sided ideals of locally matricial algebras and the Γ-invariant problem. (English summary)
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An algebra R over a field F is full matricial if it is isomorphic to the algebra of $n \times n$ matrices with entries in F, for some positive integer n. We say that R is matricial if it is isomorphic to a finite product of full matricial algebras, and locally matricial if it is isomorphic to a direct limit of matricial algebras (all over the same field F). Of course, every locally matricial algebra is von Neumann regular, that is, it satisfies the axiom $(\forall x)(\exists y)(xyx = x)$. As the $(\vee, 0)$-semilattice $\mathrm{Id}^{\mathrm{c}} R$ of a von Neumann regular ring R is distributive, this holds a fortiori for R locally matricial.

The present paper is concerned with the converse problem. It has been proved by the reviewer in [Israel J. Math. **103** (1998), 177–206; MR1613568 (99g:06023)] and [Proc. Amer. Math. Soc. **127** (1999), no. 2, 363–370; MR1468207 (99c:06007)] that there are distributive semilattices of size $\aleph_2$ (or any larger cardinal number) that are not isomorphic to $\mathrm{Id}^{\mathrm{c}} R$, for any von Neumann regular ring R. A well-known unpublished note by G. M. Bergman from 1986 shows that every distributive semilattice which is either countable or strongly distributive (i.e., every element is a finite join of join-irreducibles) is isomorphic to $\mathrm{Id}^{\mathrm{c}} R$, for some locally matricial algebra R (over any given field). The main result of the present paper states that every bounded distributive lattice (so not only those semilattices) is isomorphic to $\mathrm{Id}^{\mathrm{c}} R$, for some locally matricial algebra R (over any given field). The construction is very sophisticated, and it is given by an explicit formula. It also gives a new proof of Bergman's above-cited results.

In the last section of the paper, the author uses his main result to solve the so-called Γ-invariant problem, stated by P. C. Eklof and J. Trlifaj in 1998. So he proves, in particular, the following (Theorem 5.1): For any uncountable regular cardinal κ, every subset of κ (modulo the closed unbounded filter) appears as the Γ-invariant of some strongly uniform module over a unit-regular ring.

Reviewed by *Friedrich Wehrung*

References

1. G. M. Bergman, *Von Neumann regular rings with tailor-made ideal lattices*, unpublished notes (1986).
2. P. C. Eklof and J. Trlifaj, Γ-*invariants for dense lattices*, Algebra Universalis **40** (1998), 427–445. MR1681834 (2000f:06011)
3. K. R. Goodearl, *Von Neumann Regular Rings*, Robert E. Krieger Publishing Co., Inc., Malabar, FL, 1991, pp. xviii+412. MR1150975 (93m:16006)
4. G. Grätzer, *General Lattice Theory*, 2nd edition, Birkhäuser-Verlag, Basel, 1998. MR1670580 (2000b:06001)

5. K. R. Goodearl and F. Wehrung, *Representations of distributive lattices in ideal lattices of various algebraic structures*, Algebra Universalis **45** (2001), 71–102. MR1809858 (2002g:06008)
6. R. N. McKenzie, G. F. McNulty and W. F. Taylor, *Algebras, Lattices, Varieties. Volume I*, The Wadsworth & Brooks/Cole Mathematics Series, Monterey, California: Wadsworth & Brooks/Cole Advanced Books & Software, 1987, pp. xii + 361. MR0883644 (88e:08001)
7. P. Pudlák, *On congruence lattices of lattices*, Algebra Universalis **20** (1985), 96–114. MR0790908 (87d:06024)
8. S. Shelah and J. Trlifaj, *Spectra of the* Γ*-invariants of uniform modules*, Journal of Pure and Applied Algebra **162** (2001), 367–379. MR1843814 (2002i:16005)
9. J. Trlifaj, *Modules over non-perfect rings*, in *Advances in Algebra and Model Theory*, Gordon and Breach, Philadelphia, 1996, pp. 471–492. MR1687740 (2000m:16009)
10. J. Trlifaj, *Uniform modules,* Γ*-invariants, and Ziegler spectra for regular rings*, Proc. ICAGM'98, Birkhäuser Verlag, Basel, 1999, pp. 327–340. MR1735580 (2001a:16018)
11. F. Wehrung, *Non-measurability properties of interpolation vector spaces*, Israel Journal of Mathematics **103** (1998), 177–203. MR1613568 (99g:06023)
12. F. Wehrung, *A uniform refinement property of certain congruence lattices*, Proceedings of the American Mathematical Society **127** (1999), 363–370. MR1468207 (99c:06007)
13. F. Wehrung, *Representation of algebraic distributive lattices with* $\aleph_1$ *compact elements as ideal lattices of regular rings*, Publicacions Matemàtiques (Barcelona) **44** (2000), 419–435. MR1800815 (2001k:16015)
14. F. Wehrung, *Semilattices of finitely generated ideals of exchange rings with finite stable rank*, Transactions of the American Mathematical Society **356** (2004), 1957–1970. MR2031048 (2004j:06006)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.