

Induced Modules by an Endomorphism of Finitely Generated Modules

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Abstract

Let R be a commutative ring. A finitely generated R -module M can be converted into an $R[X]$ -module by an R -endomorphism of M (see for example [4]). In this work, we first give a structure Theorem for finitely generated modules over local rings in term of Fitting ideals. And then we consider an $R[X]/(f(X))$ -finitely generated module $M_{u,f}$ induced on M by an endomorphism u which annihilate a monic polynomial $f(X)$. We establish a structure Theorem for $M_{u,f}$ which shall have interesting applications in linear algebra.

Keywords: Fitting ideals, structure theorem, induced modules

1 Introduction

In this paper, and except supplementary indication, R will be a commutative ring with unit. Let M be a finitely generated R -module, and u an R -endomorphism of M . The endomorphism u converts M into an $R[X]$ finitely generated module by $X.m = u(m)$ for all $m \in M$. We denote M as $R[X]$ -module via u by M_u .

In the classical case, i.e. when $R = K$ is a field and $E = K^n$, E_u is a $K[X]$ torsion module. The structure Theorem for finitely generated modules over principal ideal domains says that E_u decompose into a direct sum of

cyclic torsion modules (i.e. there exists an integer r and elements p_1, \dots, p_m in $K[X]$ with $p_1|p_2\dots|p_m$ such that $E_u \cong \sum_{i=1}^{i=r} R/(p_i)$). It is well known that this structure Theorem is used to recover the possible canonical forms for matrices of endomorphisms over K -vector spaces.

In the general case, i.e. when R is a commutative ring, the $R[X]$ -module structure induced by an endomorphism was used to study the classical problem of classification of M -endomorphisms (see for example [4]). But the structure of $R[X]$ still an obstacle since there is no structure theorems over this ring.

In this paper we are concerned with the M -endomorphisms which annihilate monic polynomials in $R[X]$. So, if $f(X)$ is a monic polynomial in $R[X]$ cancelled by an R -endomorphism u of M , we consider the ring $\Lambda = R[X]/(f(X))$ and define a Λ -module structure on M via u in the same way as above. Our goal is to give a structure theorem for these modules. A structure theorem for a finitely generated modules M over a commutative ring R is a cyclic decomposition $M \cong \bigoplus_{i=1}^{i=n} R/(a_i)$, with a_i divides a_{i+1} . The uniqueness of such a decomposition is always assured (see for example [6, Theorem 2.4]).

This paper is organized in the following way. In section 2, we exhibit some properties of Fitting ideals of direct sum of finitely generated modules. This properties allows us to show Theorem 2.6 , a structure theorem for finitely generated modules over local rings.

In section 2, we suppose that u annihilates a monic polynomial $f(X)$ in $R[X]$, we define the $R[X]/(f(X))$ -module $M_{u,f}$ associated to M_u and $f(X)$. Then we investigate a case in which $M_{u,f}$ shall have a structure theorem. We give Theorem 3.5 which can be used to characterize, under assumptions, invariant factors and canonical forms for endomorphisms and matrices over some local rings.

2 A structure Theorem for finitely generated modules over local rings

Let R be a commutative ring and M a finitely generated R -module. Let $\underline{x} = (x_1, \dots, x_n)$ be a set of generators of M . A relation of M is a vector (a_1, \dots, a_n) in R^n such that $\sum_{i=1}^n a_i x_i = 0$. For a positive integer $k = 0, \dots, n-1$, the k^{th} Fitting ideal of M is defined to be the ideal $F_k(M)$ generated by the determinants of all $(n-k) \times (n-k)$ -submatrices of the matrix

$$K(M) = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & & \vdots \end{pmatrix},$$

where the vectors (a_{i1}, \dots, a_{in}) are the relations of M . If $k \geq n$, we define $F_k(M) = R$. These ideals form an ascending sequence of invariant ideals for M , independently of the choice of \underline{x} , and have nice properties(see [7]).

In the following, we will denote by $\mu(M)$ the minimal number of generators of M . We define $\omega(M) := \min\{k \mid F_k(M) \neq 0\}$.

Proposition 2.1 *Suppose R is a local ring. Let M_1 and M_2 be two finitely generated R -modules. Then $\mu(M_1 \oplus M_2) = \mu(M_1) + \mu(M_2)$.*

Proof. Suppose R is a local ring with maximal ideal m . Let $M = M_1 \oplus M_2$ be the direct sum of two finitely generated R -modules. So $M/mM \cong M_1/mM_1 \oplus M_2/mM_2$. Let (x_1, \dots, x_{n_1}) and (y_1, \dots, y_{n_2}) be respective bases of the R/m -vector spaces M_1/mM_1 and M_2/mM_2 . It is known that $n_1 = \mu(M_1)$ and $n_2 = \mu(M_2)$. Since $(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2})$ is a R/m -basis of M/mM , we have $\mu(M) = n_1 + n_2 = \mu(M_1) + \mu(M_2)$.

Proposition 2.2 *Let $M = M_1 \oplus M_2$ be the direct sum of two finitely generated R -modules M_1 and M_2 . Then :*

- (i) $F_k(M_1) \subseteq F_{n_2+k}(M)$ and $F_k(M_2) \subseteq F_{n_1+k}(M)$, $\forall k \geq 0$, with n_1 and n_2 the sizes of two sets of generators of M_1 and M_2 respectively.
- (ii) $F_k(M_i) \subseteq F_{k+1}(M_i).F_{n-1}(M)$ for any positive integer $k \leq n_i - 1$, where $i = 1, 2$ and $n = n_1 + n_2$.

Proof. (i) Let $\underline{x}_1 = (x_1, \dots, x_{n_1})$ be a set of generators of M_1 and $\underline{x}_2 = (y_1, \dots, y_{n_2})$ a set of generators of M_2 . Then M is generated by $(\underline{x}_1, \underline{x}_2)$. Consequently, any relation of M_i can be completed to a relation of M , for $i = 1, 2$. Let k be a positive integer $< n_1$ and $n = n_1 + n_2$. Then any $(n_1 - k) \times (n_1 - k)$ -submatrix of a matrix whose rows are relations of \underline{x} is an $(n - (n_2 + k)) \times (n - (n_2 + k))$ -submatrix of a matrix whose rows are relations of $(\underline{x}_1, \underline{x}_2)$. So, $\forall k \leq n_1$, $F_k(M_1) \subseteq F_{n_2+k}(M)$. This inclusion is trivial if $k \geq n_1$. Then we have (i).

(ii) Let k be a positive integer $\leq n_i - 1$ ($i=1,2$). The first step in computing, by the Laplace method, the determinant of any $(n_i - k) \times (n_i - k)$ -submatrix of a matrix whose rows are relations of \underline{x}_i gives an element $\sum_j \alpha_j \beta_j$, where α_j is a coefficient of some row of $K(M)$ and β_j is the determinant of some $(n_i - (k + 1)) \times (n_i - (k + 1))$ -submatrix of a matrix whose rows are relations of $\underline{x}_i, \forall i$. Since the coefficients of $K(M)$ are generators of $F_{n-1}(M)$ and $F_{k+1}(M_i)$ is generated by the determinants of the $(n_i - (k + 1)) \times (n_i - (k + 1))$ -submatrices from $K(M_i)$ (by Definition). This is true for $i = 1, 2$, and then we have the result.

Proposition 2.3 *Let M and M' be two finitely generated R -modules such that $M=M' \oplus R/F_{r-1}(M)$, where $r = \mu(M)$. Then $F_k(M) = F_k(M')F_{r-1}(M)$ for any positive integer $k \leq r - 1$.*

Proof. Let k be a positive integer $\leq r - 1$. If $M = M' \oplus R/F_{r-1}(M)$, then $F_k(M) = F_k(M')F_{r-1}(M) + F_{k-1}(M') + \dots + F_0(M')$ (by Proposition 10.8 page 487 in [5]). So $F_k(M) = F_k(M')F_{r-1}(M) + F_{k-1}(M')$ (the Fitting ideals form an ascending sequence). So, by (ii) in Proposition 2.2, $F_k(M) = F_k(M')F_{r-1}(M)$.

Recall that an element in R is said to be regular if it is not a zero divisor in R . We say that an ideal is regular if it contains a regular element. We also say that an ideal I of R can be simplified if for any ideals J and J' in R we have $IJ = IJ'$ implies $J = J'$.

Proposition 2.4 *Suppose R is a local ring. Let M and M' be two finitely generated R -modules such that $M = M' \oplus R/F_{r-1}(M)$, $r = \mu(M)$. If the Fitting ideals of M are principal regular ideals that can be simplified then*

- (i) $\omega(M) = \omega(M')$.
- (ii) *The Fitting ideals of M' are principal regular ideals that can be simplified.*

Proof. (i) Set $\omega(M) = s$ and $\omega(M') = s'$. Then $s, s' \leq r - 1$ and, by Proposition 2.3, $F_{s-1}(M) = F_{s-1}(M')F_{r-1}(M)$. So $F_{s-1}(M')F_{r-1}(M) = 0$. Hence $F_{s-1}(M') = 0$ and $s \leq s'$. So $F_{s'-1}(M) = F_{s'-1}(M')F_{r-1}(M)$. So $F_{s'-1}(M) = 0$ ($F_{s'-1}(M') = 0$). Hence $s' \leq s$. Thereby $s = s'$.

(ii) One has $\mu(M') = r - 1$ (by Proposition 2.1). So if k is a positive integer $\geq r - 1$ then $F_k(M') = R$. Suppose $k < r - 1$. Proposition 2.3 implies that $F_k(M) = F_k(M')F_{r-1}(M)$. Let's put $F_k(M) = (\alpha_k)$ for $k = 0, \dots, r - 1$. Since $F_k(M) \subseteq F_{r-1}(M)$, there exists an element β_k in R such that $\alpha_k = \beta_k \cdot \alpha_{r-1}$ for $k = 0, \dots, r - 2$. So $(\beta_k)F_{r-1}(M) = F_k(M')F_{r-1}(M)$. Consequently $F_k(M') = (\beta_k)$ (since $F_{r-1}(M)$) can be simplified). Therefore it remains to show that all the $F_k(M')$ can be simplified. Let I and J be two ideals in R and k a positive integer such that $F_k(M')I = F_k(M')J$. Hence $F_k(M')F_{r-1}(M)I = F_k(M')F_{r-1}(M)J$ ($F_{r-1}(M)$ is not nil). So $F_k(M)I = F_k(M)J$. Consequently, by hypothesis, $I = J$. Hence the Fitting ideals of M' can be simplified.

Proposition 2.5 *Let $I_1 \subseteq I_2 \subseteq \dots \subseteq I_n$ be an increasing sequence of ideals in R . Let k be a positive integer. Set $M = \bigoplus_{i=1}^{i=n} R/I_i$. Then*

$$F_k(M) = \begin{cases} I_{k+1} \dots I_n & \text{if } k = 0, 1, \dots, n - 1. \\ R & \text{if } k \geq n. \end{cases}$$

In particular, if a_1, \dots, a_n are elements of R such that a_i divides a_{i+1} for $i = 1, \dots, n - 1$ then $F_k(\bigoplus_{i=1}^{i=n} R/(a_i)) = (a_1 \dots a_{n-k})$, for $k \leq n - 1$.

Proof. We have $F_k(\bigoplus_{i=1}^n R/I_i) = \sum_{i_1 + \dots + i_n = k} F_{i_1}(R/I_1) \dots F_{i_n}(R/I_n)$ (by [5, Proposition 10.8]). So, by [5, Corollary 10.6], $F_k(\bigoplus_{i=1}^{i=n} R/I_i)$ is sum of ideals of the form $I_{i_1} \dots I_{i_{n-k}}$ where (i_1, \dots, i_{n-k}) are the $(n - k)$ -tuples of $\{1, \dots, n\}$.

Since $I_i \subseteq I_{i+1}$ for $1 \leq i \leq n$, $I_{i_1} \dots I_{i_{n-k}} \subseteq I_{k+1} \dots I_n$ for any $n - k$ -tuple (i_1, \dots, i_{n-k}) . Consequently $F_k(\bigoplus_{i=1}^{i=n} R/I_i) = I_{k+1} \dots I_n$.

Recall that a structure theorem for a finitely generated module M over a commutative ring R is a cyclic decomposition $M \cong \bigoplus_{i=1}^{i=n} R/(a_i)$ with a_i divides a_{i+1} . The following theorem gives a structure theorem for a finitely generated R -module over a local ring.

Theorem 2.6 *Suppose R is a local ring. Let M be a finitely generated R -module. Let $\mu(M) = r$. Suppose that the Fitting ideals of M are principal regular ideals. Then $M \cong \bigoplus_{i=1}^{i=r-s} R/(a_i) \oplus R^s$, where a_i divides a_{i+1} for $1 \leq i \leq r - s - 1$, $s = \omega(M)$ and $a_1 R \neq R$.*

Proof. To prove that (i) implies (ii) we proceed by induction on $r = \mu(M)$. For $r=1$, we have $M \cong R/F_0(M)$ (by [1, Proposition 4], since $F_0(M)$ is principal). The assertion is then true for $r=1$. For $r > 1$, [1, Proposition 4] implies $M \cong R/F_{r-1}(M) \oplus M'$, M' is a finitely generated R -module and $\mu(M') = r - 1$ (by Proposition 2.1). Furthermore, according to Proposition 2.4, the Fitting ideals of M' are principal and $I(M')$ is regular (since the Fitting ideals of M are principal and regular). So, by induction hypothesis, $M' \cong (\bigoplus_{i=1}^{i=r-1-s} R/b_i R) \oplus R^s$ where b_i divides b_{i+1} for $1 \leq i \leq r - s - 1$, $s = \omega(M')$. Hence $M \cong R/F_{r-1}(M) \oplus (\bigoplus_{i=1}^{i=r-1-s} R/b_i R) \oplus R^s$. Let $a_1 R = F_{r-1}(M)$ and $a_{i+1} = b_i$ for $i = 1, \dots, r - 1 - s$. We have $F_{r-2}(M') = b_1 R = a_1 R$ and $F_{r-2}(M') \subseteq F_{r-1}(M)$ (see Proposition 2.2 and 2.5). Then a_1 divides a_2 . So by Proposition 2.4, $s = \omega(M') = \omega(M)$. So $M \cong (\bigoplus_{i=1}^{i=r-s} R/a_i R) \oplus R^s$ where $a_i | a_{i+1}$ for $1 \leq i \leq r - s - 1$ and $s = \omega(M)$. Furthermore, Proposition 2.5 implies $a_1 R = F_{r-1}(M) \neq R$.

3 Module induced by an endomorphism which annihilate a monic polynomial

Let M be a finitely generated R -module, and u an R -endomorphism of M . Let $f(X)$ be a monic polynomial in $R[X]$ cancelled by the endomorphism u . Then, an $R[X]$ -module associated to u can be defined on M by $X.m = u(m)$ for all $m \in M$. We denote M as $R[X]$ -module via u by M_u . Furthermore, since $f(X)R[X] \subseteq \text{Ann}_{R[X]}(M_u)$, we can convert the $R[X]$ -module M_u into a $R[X]/(f(X))$ -module by $\mathbf{g}(\mathbf{X}).m = g(u)(m)$, for all $\mathbf{g}(\mathbf{x}) \in \Lambda$ and all $m \in M$, where $\mathbf{g}(\mathbf{X})$ is the image of an element $g(X)$ of $R[X]$ in $R[X]/(f(X))$. We put $\Lambda = R[X]/(f(X))$ and we will denote M_u as $R[X]/(f(X))$ -module by $M_{u,f}$. $M_{u,f}$ is an $R[X]/(f(X))$ -lattice, i.e., a $R[X]/(f(X))$ -finitely generated module, R -torsion free module.

Recall that the k^{th} determinantal ideal of an $n \times n$ square matrix A is the ideal $\mathcal{D}_k(A)$ generated by all the $(k \times k)$ -minors of A . We put $\mathcal{D}_0(A) = R$ and

$\mathcal{D}_k(A) = 0$ if $k \geq n$.

Recall also that if M is a finitely presented module with finite presentation $R^m \xrightarrow{v} R^n \xrightarrow{v'} M \rightarrow 0$, the k^{th} Fitting ideal of M is the ideal $\mathcal{D}_k(A)$, where A is a matrix of the homomorphism v (see [7]),

In the following proposition, we suppose M to be free of finite rank n and we establish the link between the Fitting ideals of $M_{u,f}$ and the ideals $\mathcal{D}_k(\mathbf{X}\mathbf{I}_n - \mathbf{A})$, where \mathbf{A} is the image in $M_n(\Lambda)$ of a matrix A of u and \mathbf{I}_n the unit matrix of size n .

Proposition 3.1 *Suppose M is a free R -module of rank n . Then*

(i) $\Lambda \otimes_R M \xrightarrow{\Psi_u} \Lambda \otimes_R M \xrightarrow{\Phi_u} M_{u,f} \rightarrow 0$ is an exact sequence, where $\Psi_u(\mathbf{g}(\mathbf{X}) \otimes m) = \mathbf{X}\mathbf{g}(\mathbf{X}) \otimes m - \mathbf{g}(\mathbf{X}) \otimes u(m)$ and $\Phi_u(\mathbf{g}(\mathbf{X}) \otimes m) = \mathbf{g}(\mathbf{X}).m$, for all $\mathbf{g}(\mathbf{X}) \in \Lambda$ and all $m \in M$.

(ii) $F_k(M_{u,f}) = \mathcal{D}_{n-k}(\mathbf{X}\mathbf{I}_n - \mathbf{A})$, for all $k \geq 0$, \mathbf{A} is the image in $M_n(\Lambda)$ of a matrix A of u .

Proof. (i) Let $g(X)$ in $R[X]$ and m in M . Then $\Phi_u.\Psi_u(\mathbf{g}(\mathbf{X}) \otimes m) = \Phi_u(\mathbf{X}\mathbf{g}(\mathbf{X}) \otimes m - \mathbf{g}(\mathbf{X}) \otimes u(m)) = Xg(X).m - g(X).u(m) = g(X).u(m) - g(X).u(m) = 0$. So, $Im(\Psi_u) \subseteq Ker(\Phi_u)$. Conversely, Let $z \in Ker(\Phi_u)$ and $z = \sum_{i=0}^{i=r} \mathbf{X}^i \otimes m_i$. Then we have, $\Phi_u(z) = \sum_{i=0}^{i=r} u^i(m_i) = 0$. So, $z = \sum_{i=0}^{i=r} (\mathbf{X}^i \otimes m_i - 1 \otimes u^i(m_i)) = \sum_{i=0}^{i=r} (\tilde{\mathbf{X}}^i - u^i)(1 \otimes m_i) = \sum_{i=0}^{i=r} (\tilde{X}^i - u^i)(1 \otimes m_i)$. Since u and \tilde{X} commute, $\tilde{X}^i - u^i = (\tilde{X} - u) \sum_{j=0}^{j=i-1} \tilde{X}^j.u^{i-j-1}$. Then $z = \sum_{i=0}^{i=r} \Psi_u(\sum_{j=0}^{j=i-1} \tilde{X}^j.u^{i-j-1})(1 \otimes m_i) \in Im(\Psi_u)$. Consequently $Ker(\Phi_u) \subseteq Im(\Psi_u)$. So $Ker(\Phi_u) = Im(\Psi_u)$. The sequence is then exact.

(ii) According to (i), $F_k(M_u) = F_k(\Psi_u)$, for any integer $k \geq 0$. Let $(x_i)_{1 \leq i \leq n}$ be an R -basis of M . Then $(1 \otimes x_i)_{1 \leq i \leq n}$ is a basis of the Λ -free module $\Lambda \otimes_R M$. So, if $A = (a_{ij})_{1 \leq i, j \leq n}$, $\Psi_u(1 \otimes x_j) = \mathbf{x} \otimes x_j - 1 \otimes x_j = \mathbf{X} \otimes x_j - 1_\Lambda \otimes \sum_{i=1}^{i=n} a_{ij}x_i = \mathbf{X} \otimes x_j - \sum_{i=1}^{i=n} \mathbf{a}_{ij}(1 \otimes x_i) = \mathbf{X}(1_\Lambda \otimes x_j) - \sum_{i=1}^{i=n} \mathbf{a}_{ij}(1 \otimes x_i) = \sum_{i=1}^{i=n} \mathbf{x}(\delta_{ij}(1_\Lambda \otimes x_i) - \mathbf{a}_{ij}(1 \otimes x_i))$, where δ_{ij} is the kronecker symbol. Hence $\mathbf{X}\mathbf{I}_n - \mathbf{A}$ is a matrix of Ψ_u . The Fitting ideals of $M_{u,f}$ are then the determinantal ideals of $\mathbf{X}\mathbf{I}_n - \mathbf{A}$.

Proposition 3.2 *Suppose M is a free R -module of rank n . Then*

(i) $M_{u,f} \stackrel{\Lambda}{\cong} \Lambda \otimes_{R[X]} M_u$.

(ii) $F_k(M_{u,f}) = F_k(M_u).\Lambda$, for any integer $k \geq 0$.

Proof. The assertion (ii) is a direct consequence of (i) (by [3, Corollary 20.5]). For the assertion (i), we consider the bilinear map :

$$\begin{aligned} \varphi : \Lambda \times M_u &\longrightarrow M_{u,f} \\ (\lambda, x) &\longmapsto \lambda.x. \end{aligned}$$

The universal property of the tensor product ensure the existence of an homomorphism of $R[X]$ -module $L_\varphi : \Lambda \otimes_{R[X]} M_u \longrightarrow M_{u,f}$ such that $L_\varphi(\lambda \otimes x) =$

$\lambda.x, \quad \forall \lambda \in \Lambda, \quad \forall x \in M_u.$ L_φ is also an homomorphism of Λ -module : $L_\varphi(\lambda'(\lambda \otimes x)) = L_\varphi((\lambda'\lambda) \otimes x) = (\lambda'\lambda).x = \lambda'(\lambda.x) = \lambda' L_\varphi(\lambda \otimes x).$ We will show that L_φ has an inverse. Consider the application

$$\begin{aligned} \psi : M_{u,f} &\longrightarrow \Lambda \otimes_{R[X]} M_u \\ x &\longmapsto 1 \otimes x. \end{aligned}$$

ψ is an homomorphism of Λ -module. Let $\lambda \otimes x \in \Lambda \otimes_{R[X]} M_u.$ So $\psi L_\varphi(\lambda \otimes x) = v(\lambda.x) = 1 \otimes \lambda x.$ Put $\lambda = \lambda(X) + f(X)R[X].$ So $\psi L_\varphi(\lambda \otimes x) = \lambda(X) \otimes x.$ Hence $L_\varphi = Id_{\Lambda \otimes M_u}.$ This on the one hand, and on the other hand one has, $L_\varphi \psi(x) = L_\varphi(1 \otimes x) = x,$ for any $x \in M.$ Then $L_\varphi \psi = Id_{M_u}.$ So ψ is the inverse of $L_\varphi,$ and hence L_φ is an isomorphism of Λ -module.

If R is local with maximal ideal \mathfrak{m} we denote by $\bar{g}(X)$ the image of a polynomial $g(X)$ of R in $\mathbf{k}[X],$ where $\mathbf{k} = R/\mathfrak{m}$ is the residue field or $R.$

Proposition 3.3 *Suppose R is a local ring with maximal ideal \mathfrak{m} and residue field $\mathbf{k}.$ If $\bar{f}(X) = \bar{g}(X)^\alpha,$ where $\bar{g}(X)$ is irreducible in $\mathbf{k}[X]$ and α a positive integer, then $R[X]/(f(X))$ is a local ring with maximal ideal $(\mathfrak{m}, \mathbf{g}(\mathbf{x})),$ generated by \mathfrak{m} and $\mathbf{g}(\mathbf{x}).$*

Proof. This result comes directly from [8, Lemma 4].

Proposition 3.4 *Suppose R is a local ring with maximal ideal \mathfrak{m} and residue field $\mathbf{k}.$ Suppose $\bar{f}(X) = \bar{g}(X)^\alpha,$ where $\bar{g}(X)$ is irreducible in $\mathbf{k}[X]$ and α is a positive integer. If $F_{r-1}(M_{u,f})$ is principal, then $\Lambda/F_{r-1}(M_{u,f})$ is a direct summand of $M_{u,f},$ $r = \mu(M_{u,f}).$*

Proof. By the previous Proposition, Λ is a local ring. So the result comes from [1, Proposition 4].

Theorem 3.5 *Suppose R is a local ring with residue field $\mathbf{k}.$ Suppose $\bar{f}(X) = \bar{g}(X)^\alpha,$ where $\bar{g}(X)$ irreducible in $\mathbf{k}[X]$ and α a positive integer. Suppose $F_{\omega(M_{u,f})}(M_{u,f})$ is regular and $F_k(M_{u,f})$ is principal for any $k \geq \omega(M_{u,f}).$ Then $M_{u,f} \cong \bigoplus_{i=1}^{i=r-s} \Lambda/g_i(X)\Lambda \oplus \Lambda^s,$ where $r = \mu(M_{u,f}),$ $g_i(X)|g_{i+1}(X)$ for $i = 1, \dots, r - s - 1$ and $s = \omega(M_{u,f}).$*

Proof. According to Proposition 3.3, Λ is a local ring. So we apply Theorem 2.6 to get the result.

Proposition 3.6 *Suppose R is a local ring with only one prime ideal $\mathfrak{m}.$ Let $\mathbf{k} = R/\mathfrak{m}$ be residue field of $R.$ Suppose $f(X)$ is a monic polynomial. If $\bar{f}(X) = \bar{g}(X)^\alpha,$ where $\bar{g}(X)$ is irreducible in $\mathbf{k}[X]$ and α is a positive integer. If $h(X) \in R[X]$ divide $f(X)$ then $h(X)R[X]$ can be generated by a monic polynomial in $R[X].$*

Proof. Since $\bar{h}(X)$ divides $\bar{f}(X)$ in $\mathbf{k}[X]$, there exists a positive integer β such that $\bar{h}(X) = \bar{g}(X)^\beta$. Let s be the degree of $\bar{h}(X)$. Set $h(X) = a_n X^n + \dots + a_0$. Then $n \geq s$ and a_s is the coefficient of smaller index not belonging to \mathbf{m} . So a_s is a unit in R . Besides, for any $i > s$, if $a_i \neq 0$ then $a_i \in \mathbf{m}$. Since \mathbf{m} is the unique prime ideal of R , \mathbf{m} is the nil radical of R . Then a_i is nilpotent, for $i > s$. Consequently, according to [1, Corollary 10.1], $h(X)R[X]$ can be generated by a monic polynomial in $R[X]$.

Corollary 3.7 *Suppose R is a local ring with only one prime ideal \mathbf{m} and residue field \mathbf{k}/\mathbf{m} . Let M be a free R -module of rang n . Suppose $f(X)$ is a monic polynomial and $\bar{f}(X) = \bar{g}(X)^\alpha$, where $\bar{g}(X)$ is irreducible in $\mathbf{k}[X]$ and α is a positive integer. If $\mathcal{D}_k(X.I_n - A)$ is principal, $\forall k \geq 0$, and $\mathcal{D}_{n-\omega(M_{u,f})}(X.I_n - A)$ is regular then $M_u \cong_{R[X]} \bigoplus_{i=1}^{i=r-s} R[X]/(g_i(X)) \oplus (R[X]/(f(X)))^s$, where the $g_i(X)$ are monic polynomials such that $g_i(X)$ divides $g_{i+1}(X)$ and s a positive integer.*

Proof. Let $s = \omega(M_{u,f})$. We have $M_{u,f} \cong \bigoplus_{i=1}^{i=r-s} \Lambda/g_i(X)\Lambda \oplus \Lambda^s$, with $r = \mu(M_{u,f})$ and $g_i(X)$ divides $g_{i-1}(X)$ for $i = 2, \dots, r-s$ (by (ii) in the previous theorem). Since $\Lambda/g_i(X)\Lambda \cong R[X]/(g_i(X))$ ($g_i(X)$ divides $f(X)$, for $i = 1, \dots, r-s$, since $f(X) \in \text{Ann}_{R[X]}(M_u)$), $M_u \cong_{R[X]} \bigoplus_{i=1}^{i=r-s} R[X]/(g_i(X)) \oplus (R[X]/(f(X)))^s$, where $f(X)$ is monic, $g_i(X)$ divides $g_{i+1}(X)$ and the $g_i(X)$ can be chosen monic (by Proposition 3.6).

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Received: December, 2008