DIRECT-SUM DECOMPOSITIONS OVER LOCAL RINGS

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Abstract. Let $(R, \mathbf{m})$ be a local ring (commutative and Noetherian). If R is complete (or, more generally, Henselian), one has the Krull-Schmidt uniqueness theorem for direct sums of indecomposable finitely generated R-modules. By passing to the $\mathbf{m}$-adic completion $\widehat{R}$, we can get a measure of how badly the Krull-Schmidt theorem can fail for a more general local ring. We assign to each finitely generated R-module M a full submonoid $\Lambda(M)$ of $\mathbb{N}^n$, where n is the number of distinct indecomposable direct summands of $\widehat{R} \otimes_R M$. This monoid is naturally isomorphic to the monoid $+(M)$ of isomorphism classes of modules that are direct summands of direct sums of finitely many copies of M. The main theorem of this paper states that every full submonoid of $\mathbb{N}^n$ arises in this fashion. Moreover, the local ring R realizing a given full submonoid can always be taken to be a two-dimensional unique factorization domain. The theorem has two non-commutative consequences: (1) a new proof of a recent theorem of Facchini and Herbera characterizing the monoid of isomorphism classes of finitely generated projective right modules over a non-commutative semilocal ring, and (2) a characterization of the monoids $+(N)$, where N is an Artinian right module over an arbitrary ring.

Suppose we wish to determine how badly the Krull-Schmidt uniqueness theorem fails for direct-sum decompositions of finitely generated modules over a given ring. One approach—the one we will follow here—is to associate to a given finitely generated module M the monoid $+(M)$ consisting of isomorphism classes of modules that are direct summands of direct sums of finitely many copies of M. (For example, M might be the direct sum of a bunch of indecomposable modules demonstrating failure of the Krull-Schmidt theorem.) The structure of these monoids tells us exactly how badly the Krull-Schmidt theorem can fail. Suppose, for example that we determine that $+(M) \cong \Lambda := \{(a,b,c) \in \mathbb{N}^3 \mid a + 4b = 5c\}$, with M corresponding to $(1,1,1)$. The minimal elements of Λ, namely, $(1,1,1)$, $(0,5,4)$ and $(5,0,1)$ correspond to the indecomposable modules in $+(M)$. In particular, M is indecomposable. Moreover, writing aM for the direct sum of a copies of M, we see that $2M$, $3M$ and $4M$ all have unique representations as direct sums of indecomposables. However, $5M = P \oplus Q$, where P and Q are the indecomposable modules corresponding to the other two minimal elements of Λ. In this way we can completely describe the decompositions of direct sums of copies of M.

In §1 of this paper we review basic terminology concerning monoids and describe a natural isomorphism between $+(M)$ and a certain full submonoid $\Lambda(M)$ of $\mathbb{N}^n$ (where n is the

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number of distinct indecomposable factors of the completion of M). Also in §1, we develop some machinery to use in §2 and prove that $\Lambda(M)$ is an expanded submonoid of $\mathbb{N}^n$ if M is a torsion-free module over a one-dimensional analytically unramified local domain.

The main results of this paper characterize the submonoids of $\mathbb{N}^n$ that arise in this fashion. We give two separate constructions to realize these monoids: In §2 we show that every *expanded* submonoid of $\mathbb{N}^n$ can be realized by a torsion-free module over a suitable one-dimensional ring, and in §3 we treat the (more general) *full* submonoids. The first construction is straightforward: All we need is an irreducible curve singularity with lots of analytic branches, each of which has high multiplicity. The second construction is less explicit: We start with a two-dimensional complete normal domain A with large divisor class group and then employ a remarkable theorem of Heitmann [He] to get a local unique factorization domain whose completion is A. Both of these constructions depend on the fact that we know exactly which torsion-free, respectively, reflexive modules over the completion $\widehat{R}$ are extended from R-modules. In §4 we introduce a little non-commutativity and apply our results to give a new proof (and somewhat stronger statement) of a recent result of Facchini and Herbera [FH]. Also in §4, we obtain a characterization of the monoids $+(N)$, where N is an Artinian module over an arbitrary ring.

Except in the last section §4, all rings are commutative and Noetherian.

The one-dimensional case of this construction (which works only for expanded submonoids) was reported in [W2], though details of the construction were omitted. This construction is an outgrowth of an *ad hoc* construction reported in [W1]. In working out the more general two-dimensional construction, I have discussed this work with many colleagues, among them Dale Cutkosky, Frank DeMeyer and Ray Heitmann. I thank them for their input. I am grateful also to Winfried Bruns for an enlightening comment on affine monoids, to Alberto Facchini for an extremely careful reading of an earlier version of this manuscript, and to Dolors Herbera for pointing out the application to Artinian modules at the end of the paper. Finally, I thank Anna Guerrieri for her hospitality at Università di L'Aquila, where part of this research was carried out.

§1. The monoid associated to a module

Conventions and notation. Throughout this section $(R, \mathbf{m})$ is a commutative, Noetherian local ring with $\mathbf{m}$-adic completion $\widehat{R}$, and M is a non-zero finitely generated R-module with completion $\widehat{M} = \widehat{R} \otimes_R M$. We denote the isomorphism class of an R-module N by $[N]$. The direct sum of a copies of a module N is denoted by aN. The notation $P \mid Q$ means that the module P is isomorphic to a direct summand of the module Q. The monoid of non-negative integers is denoted by $\mathbb{N}$, and, as usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of integers, rationals, reals and complex numbers, respectively. We denote by $+(M)$ the monoid $\{[N] : \ N \mid tM \text{ for some } t \in \mathbb{N}\}$, where the monoid operation is given by $[N_1] + [N_2] = [N_1 \oplus N_2]$.

Write $\widehat{M} = a_1V_1 \oplus \cdots \oplus a_nV_n$, where the V_i are pairwise non-isomorphic indecomposable $\widehat{R}$-modules and the a_i are *positive* integers. The positive integers n, a_i and the modules V_i are determined up to isomorphism and reordering. We will keep this notation and the ordering of the indecomposables V_i fixed for the rest of this section. Let $\Lambda(M)$ denote the submonoid of $\mathbb{N}^n$ consisting of those n-tuples $\beta := (b_1, \dots b_n)$ such that $b_1V_1 \oplus \cdots \oplus b_nV_n \cong \widehat{N}$ for some finitely generated R-module N. We recall the following definitions from [BH]:

1.1. Definition. Let $\Lambda \subseteq \mathbb{N}^n$. Then Λ is an *expanded* submonoid of $\mathbb{N}^n$ provided $\Lambda = L \cap \mathbb{N}^n$ for some $\mathbb{Q}$-subspace L of $\mathbb{Q}^n$. More generally, Λ is a *full* submonoid of $\mathbb{N}^n$ provided $\Lambda = G \cap \mathbb{N}$ for some subgroup G of the additive group $\mathbb{Z}^n$. Note that Λ is expanded if and only if it is of the form $\mathrm{Ker}(\Phi) \cap \mathbb{N}^n$ for some $m \times n$ matrix Φ with entries in $\mathbb{Z}$ (or $\mathbb{Q}$). These monoids are studied extensively in the first chapter of Stanley's book [S].

Suppose now that $[N] \in +(M)$. Since $\widehat{N} \mid t\widehat{M}$ for some $t \in \mathbb{N}$, we can write $\widehat{N} \cong b_1V_1 \oplus \cdots \oplus b_nV_n$ with $b_i \in \mathbb{N}$. Again, the n-tuple $\beta := (b_1, \dots, b_n)$ is uniquely determined by the isomorphism class $[N]$. We let $\lambda : +(M) \to \Lambda(M)$ be the map taking $[N]$ to the n-tuple β determined in this way. We will show that λ is a monoid isomorphism from $+(M)$ onto $\Lambda(M)$.

It is well known that two finitely generated R-modules with isomorphic completions are already isomorphic over R. We need a refinement of this fact, which follows easily from the results in [G]. A direct proof, essentially the same as that of a special case in [RR], appears in [W1]:

1.2. Proposition. *Let N_1 and N_2 be finitely generated R-modules. If $\widehat{N_1} \mid \widehat{N_2}$, then $N_1 \mid N_2$. In particular, if $\widehat{N_1} \cong \widehat{N_2}$, then $N_1 \cong N_2$.* □

1.3. Propostion.

(1) *The map $\lambda : +(M) \to \Lambda(M)$ defined above is an isomorphism of monoids.*
(2) *$\Lambda(M)$ is a full submonoid of $\mathbb{N}^n$.*

Proof. (1) The map λ is clearly a homomorphism of monoids, and it is injective by the last sentence of (1.2). To prove that λ is surjective, let $\beta = (b_1, \dots, b_n) \in \Lambda(M)$. There is a finitely generated R-module N such that $\widehat{N} \cong b_1V_1 \oplus \cdots \oplus b_nV_n$. Choosing t so large that $t\alpha \geq \beta$, we see that $\widehat{N} \mid t\widehat{M}$. Then $[N] \in +(M)$ by (1.2), and $\lambda([N]) = \beta$. (2) Let G be the subgroup of $\mathbb{Z}^n$ generated by $\Lambda(M)$. Given $\gamma \in G \cap \mathbb{N}^n$, write $\gamma = \gamma_2 - \gamma_1$, with $\gamma_i \in \Lambda(M)$. Choose finitely generated modules N_i such that $\lambda([N_i]) = \gamma_i$. Since $\gamma_2 \geq \gamma_1$, $\widehat{N_1} \mid \widehat{N_2}$, whence $N_1 \mid N_2$ by (1.2). Writing $N_1 \oplus Q \cong N_2$, we see that $\gamma = \lambda([Q]) \in \Lambda(M)$. □

1.4. Dimension one. For the remainder of this section we concentrate on the following special case: R is a one-dimensional local ring whose completion $\widehat{R}$ is reduced, and M is torsion-free. We will show that in this case $\Lambda(M)$ is an expanded submonoid of $\mathbb{N}^n$. The machinery developed here in order to prove this will be used also in the next section. We assume throughout that R is not a discrete valuation ring. (Without this assumption, (2) of (1.5) below would be false.)

Let $\widetilde{R}$ denote the integral closure of R in its total quotient ring, and let $\mathfrak{c}$ denote the conductor, that is, the largest ideal of $\widetilde{R}$ that is contained in R. Since $\widehat{R}$ is reduced, $\widetilde{R}$ is finitely generated as an R-module, and we have a pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & \widetilde{R} \\ \downarrow & & \downarrow \\ R/\mathfrak{c} & \longrightarrow & \widetilde{R}/\mathfrak{c} \end{array} \qquad (1.4.1)$$

The bottom line of (1.4.1) is an *Artinian pair*, [CWW], that is, a module-finite extension $A \to B$ of Artinian rings. (Our assumption that R is not a discrete valuation ring means that $R/\mathfrak{c}$ is not the zero ring.) We denote the Artinian pair $R/\mathfrak{c} \to \widetilde{R}/\mathfrak{c}$ by R_{art}. A *module* over the Artinian pair $A \to B$ is a pair $V \to W$, where W is a finitely generated projective B-module and V is an A-submodule of W satisfying $BV = W$. Every finitely generated torsion-free R-module N arises as a pullback:

(1.4.2)
$$\begin{array}{ccc} N & \longrightarrow & \widetilde{R}N \\ \downarrow & & \downarrow \\ N/\mathfrak{c}N & \longrightarrow & \widetilde{R}N/\mathfrak{c}N \end{array},$$

where $\widetilde{R}N$ is the free $\widetilde{R}$-module $(\widehat{R} \otimes_R N)/\text{torsion}$. ($\widetilde{R}N$ is free because $\widetilde{R}$ is a semilocal Dedekind domain.) The bottom line of (1.4.2) is an R_{art}-module; we denote it by N_{art}. We record the following facts (see [CWW, (1.6)]):

1.5. Proposition. *Let R be a one-dimensional local ring with reduced completion $\widehat{R}$.*

(1) *Given two finitely generated torsion-free R-modules N_1, N_2, we have $N_1 \cong N_2$ if and only if $N_{1\text{art}} \cong N_{2\text{art}}$.*
(2) *Assume R is an integral domain. Given an R_{art}-module $V \to W$, there exists a finitely generated torsion-free R-module N such that $N_{\text{art}} \cong (V \to W)$ if and only if W is free as an $\widetilde{R}/\mathfrak{c}$-module.* □

1.6. Notation. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the minimal prime ideals of the completion $\widehat{R}$, and let κ_i be the field $\widehat{R}_{\mathfrak{p}_i}$. If N is a finitely generated, torsion-free $\widehat{R}$-module, the *rank* $\text{rk}(N)$ of N is the s-tuple $(\dim_{\kappa_1} N_{\mathfrak{p}_1}, \dots, \dim_{\kappa_s} N_{\mathfrak{p}_s})$. We say N has *constant rank* r if $\text{rk}(N) = (r, \dots, r)$. Note that if $(A \to B) = \widehat{R}_{\text{art}}$ and $(V \to W) = N_{\text{art}}$, then N has constant rank r if and only if W is a free B-module of rank r.

The following theorem (a very special case of [LO, (6.15)]) was shown to me by L. S. Levy:

1.7. Proposition. *Let R be a one-dimensional local ring whose completion $\widehat{R}$ is reduced.*

(1) *If N is a finitely generated torsion-free R-module, then $\widehat{N}$ is a torsion-free $\widehat{R}$-module.*
(2) *Assume R is an integral domain. Let N be a finitely generated torsion-free $\widehat{R}$-module. Then $N \cong \widehat{Q}$ for some finitely generated torsion-free R-module Q if and only if N has constant rank.*

Proof. (1) Since N is torsion-free, N can be embedded in a finitely generated free R-module F. Then $\widehat{N}$ is embedded in the free $\widehat{R}$-module $\widehat{F}$ and therefore is torsion-free.

(2) We first show that the completion T of $\widetilde{R}$ is the integral closure of $\widehat{R}$. To see this, we note that we can compute both completions ($R \to \widehat{R}$ and $\widetilde{R} \to T$) with respect to powers of the conductor $\mathfrak{c}$. Since $\widetilde{R}$ is a finite birational extension of R, T is a finite birational extension of $\widehat{R}$. Further, T is a direct product of finitely many discrete valuation rings, being the completion of the semilocal Dedekind domain $\widetilde{R}$. Therefore T is integrally closed and is therefore the integral closure of $\widehat{R}$.

Next note that $\widehat{\mathfrak{c}}$ is the conductor of $\widehat{R}$ in its integral closure T. It follows that $R_{\text{art}} = \widehat{R}_{\text{art}}$, and that $Q_{\text{art}} = \widehat{Q}_{\text{art}}$ for every finitely generated torsion-free R-module (since the completion of the pullback diagram (1.4.2) for Q is the pullback diagram for the $\widehat{R}$-module $\widehat{Q}$).

Now let Q be any finitely generated torsion-free R-module, and let $(V \to W) = Q_{\text{art}}$. Since R is a domain, W is free as an $\widetilde{R}$-module by (1.5). By the last sentence in (1.6), $\widehat{Q}$ has constant rank. This proves the "only if" part of (2).

To prove the converse, let N be an $\widehat{R}$-module of constant rank, and let $(V \to W) = N_{\text{art}}$. Then W is a free $R/\mathfrak{c}$-module, so by (1.5) there is a finitely generated torsion-free R-module Q such that $Q_{\text{art}} \cong (V \to W)$. Then $\widehat{Q}_{\text{art}} \cong (V \to W) = N_{\text{art}}$. By (1) of (1.5), $\widehat{Q} \cong N$. □

1.8. Corollary. *Let R be a one-dimensional local domain with reduced completion, and let M be a finitely generated torsion-free R-module. Then $\Lambda(M)$ is an expanded submonoid of $\mathbb{N}^n$.*

Proof. Write $\Lambda(M) = G \cap \mathbb{N}^n$, where G is the subgroup of $\mathbb{Z}^n$ generated by $\Lambda(M)$. Let $L = \mathbb{Q}G \subseteq \mathbb{Q}^n$. We have to show that $L \cap \mathbb{N}^n = \Lambda(M)$. If $\gamma = (c_1, \dots, c_n) \in L \cap \mathbb{N}^n$, then $t\gamma \in G$ for some positive integer t. But then $t\gamma \in G \cap \mathbb{N}^n = \Lambda(M)$. Now let $N = c_1V_1 \oplus \cdots \oplus c_nV_n$ (where the V_i are as in the second paragraph of this section). Since $t\gamma \in \Lambda(M)$, tN has constant rank by (2) of (1.7). Then N also has constant rank, and, by (2) of (1.7), $\gamma \in \Lambda(M)$. □

§2. Realization of expanded submonoids of $\mathbb{N}^n$

The goal in this section is to prove the following companion to Corollary 1.8:

2.1. Theorem. *Let n be a positive integer, and let Λ be an expanded submonoid of $\mathbb{N}^n$. Assume that Λ contains an element $\alpha = (a_1, \dots, a_n)$ in which each a_i is positive. Then there exist a one-dimensional local domain R with reduced completion and a finitely generated torsion-free R-module M such that $\Lambda(M) = \Lambda$ and the isomorphism $\lambda : +(M) \to \Lambda(M)$ takes $[M]$ to α.*

The proof depends on a technical lemma, which is based on a construction due to Drozd and Roĭter [DR].

2.2. Lemma. *Let $\mathbb{F}$ be a field, and let $D_1, \dots, D_s$ be finite-dimensional $\mathbb{F}$-algebras. Assume that for each $i \le s$ there exist elements $a_i, b_i \in D_i$ such that $\{1, a_i, a_i^2, b_i\}$ is linearly independent over $\mathbb{F}$. Put $D = D_1 \times \cdots \times D_s$. Let $r_1, \dots, r_s$ be non-negative integers, and let $W_i = r_iD_i$, the free D_i-module of rank r_i. Then there exists an indecomposable $(\mathbb{F} \to D)$-module $V \to W$, where $W = W_1 \times \cdots \times W_s$.*

Proof. We assume $r_1 \ge \cdots \ge r_s$ for convenience. Let $L = r_1\mathbb{F}$ be the $\mathbb{F}$-vector space of $r_1 \times 1$ column vectors. We regard L as a subspace of W by sending the column $[c_1, \dots, c_{r_1}]^{\text{t}}$ to the element of W whose i^{th} coordinate is $[c_1, \dots, c_{r_i}]^{\text{t}}$.

Let $a = (a_1, \dots, a_s)$, $b = (b_1, \dots, b_s)$; and let ν be the nilpotent $r_1 \times r_1$ matrix with 1's on the superdiagonal and 0's elsewhere. Let V be the $\mathbb{F}$-subspace of W consisting of elements of the form $u + av + b\nu v$, where u and v range over L. Then $V \to W$ is a $(\mathbb{F} \to D)$-module, and exactly as in the proof of [CCW, (2.6)] one checks that every D-endomorphism of W carrying V into V is actually in $\mathbb{F}[\nu]$. Since $\mathbb{F}[\nu]$ is local, it follows that the $(\mathbb{F} \to D)$-module $V \to W$ is indecomposable. □

We will apply this lemma to the following specific ring:

2.3. The ring. Let s be a positive integer, and let $\mathbb{F}$ be any field with at least s distinct elements $t_1, \dots, t_s$. Let S be the complement of the union of the maximal ideals $(X - t_i)\mathbb{F}[X], i = 1, \dots, s$. We define $R = R_s$ by pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & S^{-1}\mathbb{F}[X] \\ \downarrow & & \downarrow \pi \\ \mathbb{F} & \longrightarrow & S^{-1}\mathbb{F}[X]/(X - t_1)^4 \cdot \dots \cdot (X - t_s)^4 \ , \end{array} \tag{2.3.1}$$

where π is the natural map. Then R is a one-dimensional local domain with reduced completion, and (2.3.1) is the conductor square for R. We can rewrite the bottom line R_{art} as $\mathbb{F} \to D_1 \times \dots \times D_s$ where $D_i = \mathbb{F}[X]/(X^4)$. The conductor square for the completion is

$$\begin{array}{ccc} \widehat{R} & \longrightarrow & T_1 \times \dots \times T_s \\ \downarrow & & \downarrow \\ \mathbb{F} & \longrightarrow & D_1 \times \dots \times D_s \ , \end{array} \tag{2.3.1}$$

where each T_i is isomorphic to $\mathbb{F}[[X]]$.

2.4. Proposition. *For any sequence of non-negative integers $r_1, \dots, r_s$, the complete ring $\widehat{R} = \widehat{R}_s$ in (2.3) has a finitely generated torsion-free module N with $\operatorname{rk}(N) = (r_1, \dots, r_s)$.*

Proof. Let $F = r_1 T_1 \times \dots \times r_s T_s$, a projective module over $T := T_1 \times \dots \times T_s$. Let $V \to W$ be the $\widehat{R}_{\text{art}}$-module supplied by (2.2). Since $F/\mathfrak{c}F \cong W$, there is a torsion-free $\widehat{R}$-module N (unique up to isomorphism) such that $N_{\text{art}} \cong (V \to W)$. (See (3) of [CWW, (1.6)]. In fact, N is just the pullback of F and V over W.)

Proof of Theorem 2.1. Write $\Lambda = \mathbb{N}^n \cap \operatorname{Ker}(\Phi)$, where $\Phi : \mathbb{Q}^n \to \mathbb{Q}^m$ is a linear transformation. We regard Φ as an $m \times n$ matrix $[q_{ij}]$, and we can assume without loss of generality that the entries of Φ are in $\mathbb{Z}$. Select a positive integer h such that $q_{ij} + h \geq 0$ for all i, j. Let R be the ring of (2.3), with $s = m + 1$.

For $j = 1, \dots, n$, choose, using (2.4), an indecomposable, finitely generated, torsion-free $\widehat{R}$-module N_j such that

$$\operatorname{rk}(N_j) = (q_{1j} + h, \dots, q_{mj} + h, h), \ j = 1, \dots, n.$$

Suppose now that $\beta = (b_1, \dots, b_n) \in \mathbb{N}^n$. If $N = b_1 N_1 \oplus \dots \oplus b_n N_n$ then

$$\operatorname{rk}(N) = (\sum_{j=1}^{n} (q_{1j} + h) b_j, \dots, \sum_{j=1}^{n} (q_{mj} + h) b_j, (\sum_{j=1}^{n} b_j) h).$$

Therefore, by (1.7), N is extended from an R-module if and only if $\sum_{j=1}^{n} (q_{ij} + h) b_j = (\sum_{j=1}^{n} b_j) h$ for $1 \leq i \leq m$, that is, if and only if $\beta \in \mathbb{N}^n \cap \operatorname{Ker}(\Phi) = \Lambda$.

Let M be the finitely generated R-module (unique up to isomorphism) such that $\widehat{M} \cong a_1 V_1 \oplus \cdots \oplus a_n V_n$ (where $\alpha = (a_1, \ldots, a_n) \in \Lambda$ is the given element with each $a_i > 0$). Then $\Lambda(M) = \Lambda$ and $\lambda([M]) = \mathbf{a}$, as desired. □

It is interesting to note that every full submonoid of $\mathbb{N}^n$ is *isomorphic* (as an abstract monoid) to $\Lambda(M)$ for some finitely generated torsion-free module M over a one-dimensional local ring R. Recall [BH] that a *positive normal* monoid is a monoid Γ such that

(1) Γ is finitely generated.
(2) Γ is isomorphic to a submonoid of $\mathbb{N}^n$ for some n, and
(3) if $n\alpha \in \Gamma$ for some positive integer n and some $\alpha \in \mathbb{Z}\Gamma$ (the group universally associated to Γ), then $\alpha \in \Gamma$.

The following characterization of positive normal monoids is the subject of Exercise 6.4.16 in [BH]:

2.5. Proposition. *These conditions on a monoid Γ are equivalent:*

(1) Γ *is a positive normal monoid.*
(2) Γ *is isomorphic to a full submonoid of* $\mathbb{N}^m$ *for some* m.
(3) Γ *is isomorphic to an expanded submonoid of* $\mathbb{N}^n$ *for some* n. □

Obviously (3) ⇒ (2) ⇒ (1). The implication (2) implies (3) is due to Hochster and is the crux of the proof. For a simple example illustrating the procedure, consider the full (but not expanded) submonoid of $\mathbb{N}^2$ consisting of all pairs (a, b) satisfying $a \equiv b \pmod 3$. This is isomorphic to the expanded submonoid of $\mathbb{N}^3$ consisting of all (a, b, c) satisfying the equation $a + 2b = 3c$.

2.6. Corollary. *The following conditions on a monoid Γ are equivalent:*

(1) Γ *is a positive normal monoid.*
(2) Γ *is isomorphic to* $+(M)$ *for some finitely generated module* M *over a local ring* R.
(3) Γ *is isomorphic to* $+(M)$ *for some finitely generated torsion-free module over a one-dimensional local domain with reduced completion.*

Proof. Obviously (3) implies (2), and (2) implies (1) by (1.3) and (2.5). To prove that (1) implies (3), we may assume, by (2.5), that $\Gamma = L \cap \mathbb{N}^n$ for some $\mathbb{Q}$-subspace L of $\mathbb{Q}^n$. The problem is that L might not contain a strictly positive element. Among all elements of Γ, choose an element $\beta = (b_1, \ldots, b_n)$ with the greatest number of positive entries. By renumbering, we may assume that $b_i > 0$ for $i = 1, \ldots, t$ and $b_{t+1} = \cdots = b_n = 0$. Then for *every* element $\alpha = (a_1, \ldots, a_n) \in \Gamma$ we have $a_{t+1} = \cdots = a_n = 0$ (else $\alpha + \beta$ would have more positive entries than $\mathbf{b}$). Therefore the projection $\pi : \mathbb{Q}^n \twoheadrightarrow \mathbb{Q}^t$ (onto the first t coordinates) carries Γ isomorphically onto $\Delta := \mathbb{N}^t \cap \pi(L)$. Since Δ contains the strictly positive element $\pi(\beta)$, we can use (2.1) to get a suitable ring and a module M such that $\Lambda(M) = \Delta$. Since $+(M) \cong \Lambda(M)$ and $\Delta \cong \Gamma$, we are done. □

§3. Realization of full monoids of $\mathbb{N}^n$

Here we prove an refinement of (2.6), where we realize a given full submonoid of $\mathbb{N}^n$ *exactly*, rather than just up to isomorphism.

3.1. Main Theorem. *Let Λ be a full submonoid of $\mathbb{N}^n$ containing an element $\alpha = (a_1, \dots, a_n)$ with each $a_i > 0$. Then there exist a two-dimensional local unique factorization domain R and a finitely generated reflexive R-module M such that $\Lambda(M) = \Lambda$ and $\lambda([M]) = \alpha$.*

Proof. Let $G = \mathbb{Z}^n/\mathbb{Z}\Lambda$, and write G as a direct sum of cyclic groups: $G \cong C_1 \oplus \cdots \oplus C_t$. Choose a positive integer d such that $(d-1)(d-2) \geq t$, and let V be a smooth plane projective curve of degree d over $\mathbb{C}$. Let A be the homogeneous coordinate ring of V for some embedding of V in $\mathbf{P}^2_{\mathbb{C}}$. Our first task is to compute the divisor class group $\mathrm{Cl}(A)$. (Note that A is a two-dimensional normal domain by [H, Chap. II, Exer. 8.4 (b)].)

By [H, Appendix B, §5], $\mathrm{Pic}^0(V) \cong D := 2g(\mathbb{R}/\mathbb{Z})$, where $g = \frac{1}{2}(d-1)(d-2)$ is the genus of V. Now $\mathrm{Pic}^0(V)$ is the kernel of the degree map $\mathrm{Pic}(V) \twoheadrightarrow \mathbb{Z}$, so $\mathrm{Cl}(V) = \mathrm{Pic}(V) \cong D \oplus \mathbb{Z}\sigma$, where σ is the class of a divisor of degree 1. There is a short exact sequence

$$0 \to \mathbb{Z} \to \mathrm{Cl}(V) \to \mathrm{Cl}(A) \to 0,$$

in which $1 \in \mathbb{Z}$ maps to the divisor class $\tau = [H.V]$, where H is a line in $\mathbf{P}^2_{\mathbb{C}}$. (See [H, Chap. II, Exer. 6.3]. Thus $\mathrm{Cl}(A) \cong \mathrm{Cl}(V)/\mathbb{Z}\tau$. Since τ has degree d, $\tau - d\sigma \in D$. Since D is a divisible group there is an element $\delta \in D$ such that $d\delta = \tau - d\sigma$. We define a homomorphism $\mathrm{Cl}(V) \twoheadrightarrow D \oplus \mathbb{Z}/d\mathbb{Z}$ by $x \mapsto (x, 0)$ if $x \in D$ and $\sigma \mapsto (-\delta, \bar{1})$. It is easy to see that the kernel of this map is $\mathbb{Z}\tau$. This shows that $\mathrm{Cl}(A) \cong D \oplus \mathbb{Z}/d\mathbb{Z}$.

Let $\mathfrak{p}$ be the irrelevant maximal ideal of A. By [H, Chap. II, Exer. 6.3 (d)], $\mathrm{Cl}(A_{\mathfrak{p}}) \cong D \oplus \mathbb{Z}/dZ$. Let B be the $\mathfrak{p}$-adic completion of A. Since $\mathrm{Cl}(A_{\mathfrak{p}}) \to \mathrm{Cl}(B)$ is injective (by faithfully flat descent), $\mathrm{Cl}(B)$ contains a copy of $D = (d-1)(d-2)\mathbb{R}/\mathbb{Z}$, which in turn contains a copy of G. Therefore there is a homomorphism $\phi : \mathbb{Z}^n \to \mathrm{Cl}(B)$ such that $\mathrm{Ker}(\phi) = \mathbb{Z}\Lambda$.

Now B is a complete local normal domain [ZS, Chap. VIII, §13]. By Heitmann's theorem [He], there is a local unique factorization domain R such that $\widehat{R} = B$.

Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{Z}^n$ (row vectors), and choose, for each i, a divisorial ideal J_i of $\widehat{R}$ such that $\phi(e_i) = [J_i]$. If $(b_1, \dots, b_n) \in \mathbb{N}^n$, the divisor class of the $\widehat{R}$-module $b_1 J_1 \oplus \cdots \oplus b_n J_n$ is $b_1[J_1] + \dots b_n[J_n] = \phi(b_1, \dots, b_n)$. (See [B, Chap.VII, §4.7].) By [We, (1.5)] or [RWW, Prop. 3], the module $b_1 J_1 \oplus \cdots \oplus b_n J_n$ is extended from an R-module if and only if its divisor class is trivial, that is, if and only if $(b_1, \dots, b_n) \in \mathbb{Z}\Lambda \cap \mathbb{N}^n = \Lambda$. We let M be the finitely generated R-module such that $\widehat{M} \cong a_1 J_1 \oplus \cdots \oplus a_n J_n$, and the proof is complete.

4. A non-commutative excursion

Suppose E is a not necessarily commutative semilocal ring, that is, E/J is Artinian (where J is the Jacobson radical of E). Then E/J, being semisimple Artinian, has only finitely many non-isomorphic simple right modules, say $X_1, \dots, X_n$, and every finitely generated right E/J-module is uniquely a direct sum of copies of these. Given a finitely generated projective right E-module P, write $P/PJ \cong a_1 X_1 \oplus \cdots \oplus a_n X_n$. The map $\xi : [P] \mapsto (a_1, \dots, a_n)$ is an isomorphism between the monoid $\mathcal{P}_E$ of isomorphism classes of finitely generated projective right E-modules (with $\oplus$ as the operation) and its image $\Xi(E)$, a full submonoid of $\mathbb{N}^n$. (This depends on the fact—a consequence of Nakayama's lemma—that

$P \mid Q$ if and only if $P/PJ \mid Q/QJ$, for finitely generated projective right E-modules P and Q. See [FS].)

Since every simple right E/J-module is a direct summand of E/J, $\Xi(E)$ contains an element (namely $\xi([E])$) all of whose coordinates are positive. Facchini and Herbera [FH] show conversely that every full submonoid Ξ of $\mathbb{N}^n$ containing such an element actually arises in this fashion. We will give a new proof of this fact using our Main Theorem 3.1. The new twist is that the semilocal ring realizing a given monoid can be taken to be module-finite over its center. In the following result, only the last statement is new; the rest is due to Facchini and Herbera [FH]:

4.1. Theorem. *Let Ξ be a full submonoid of $\mathbb{N}^n$, and assume Ξ contains an element $\alpha = (a_1, \dots, a_n)$ with each $a_i > 0$. Then there exists a semilocal ring E such that E/J has exactly n simple modules, $\Xi(E) = \Xi$, and $\xi([E]) = \alpha$. Moreover, E can be taken to be the endomorphism ring of a finitely generated reflexive module over a commutative Noetherian local unique factorization domain of dimension* 2.

Proof. Choose, by (3.1), a two-dimensional local unique factorization domain R and a finitely generated reflexive module M such that $\Lambda(M) = \Xi$ and $\lambda([M]) = \alpha$. Let E be the endomorphism ring of M. There is an isomorphism $\eta : +(M) \to \mathcal{P}_E$ taking the isomorphism class $[N]$ to $[\operatorname{Hom}_R(M, N)]$. The inverse map takes $[P]$ to $[P \otimes_E M]$. (Andreas Dress used this isomorphism in a 1969 paper [D].) Let $\widehat{E} = \widehat{R} \otimes_R E = \operatorname{End}_{\widehat{R}}(\widehat{M})$. Then $\widehat{J} := \widehat{R} \otimes_R J$ is the Jacobson radical of $\widehat{E}$, and since E/J has finite length as an R-module the natural map $E/J \to \widehat{E}/\widehat{J}$ is an isomorphism. Since $\widehat{E}$ is module-finite over the Henselian local ring $\widehat{R}$, the natural map $\mathcal{P}_{\widehat{E}} \to \mathcal{P}_{\widehat{E}/\widehat{J}}$ is an isomorphism.

Let $V_1, \dots, V_n$ be the distinct indecomposable summands of $\widehat{M}$ (as in §1), and let $P_i = \operatorname{Hom}_{\widehat{R}}(\widehat{M}, V_i)$. The P_i are the distinct indecomposables in $\mathcal{P}_{\widehat{E}}$. If we let $Q_i \in \mathcal{P}_{E/J}$ correspond to P_i via the natural isomorphisms $\mathcal{P}_{E/J} \xrightarrow{\cong} \mathcal{P}_{\widehat{E}/\widehat{J}} \xleftarrow{\cong} \mathcal{P}_{\widehat{E}}$, we see that $Q_1, \dots, Q_n$ are the distinct simple modules in $\mathcal{P}_{E/J}$. The following commutative diagram completes the proof:

$$\begin{array}{ccc} +(M) & \xrightarrow[\cong]{\lambda} & \Lambda(M) \\ {\scriptstyle \eta}\big\downarrow{\scriptstyle \cong} & & \big\downarrow{\scriptstyle =} \\ \mathcal{P}_E & \xrightarrow{\xi} & \Xi(E) \end{array}$$

Artinian modules. Suppose now that M is an Artinian right R-module, where R is an arbitrary ring (not necessarily commutative). We can define the monoid $+(M)$ exactly as in the beginning of this paper. A result of Camps and Dicks [CD] says that $E := \operatorname{End}_R(M)$ is semilocal. As in the proof of (4.1), $+(M) \cong \mathcal{P}_E$, which is a positive normal monoid, by the first paragraph of this section. In fact, every positive normal monoid arises in this way. The key result is due to Facchini, Herbera, Levy and Vámos [FHLV, Corollary 1.3]: Given any module-finite algebra A over a commutative semilocal ring R, there exists a cyclic Artinian module N over some ring with $\operatorname{End}(N) \cong A$. Letting A be the endomorphism ring of the module M constructed in the proof of the Main Theorem (3.1), we obtain the following extension of Corollary 2.6 and Theorem 4.1:

4.2. Corollary. *The following conditions on a monoid* Γ *are equivalent:*

(1) Γ *is a positive normal monoid.*
(2) $\Gamma \cong +(N)$ *for some (cyclic) Artinian module* N *over some ring.*
(3) $\Gamma \cong \mathcal{P}_E$ *for some semilocal ring* E.
(4) $\Gamma \cong +(M)$ *for some finitely generated module over a commutative Noetherian semilocal ring.* □

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