Int. Journal of Math. Analysis, Vol. 6, 2012, no. 12, 555 - 562

On Generalizations of Pseudo-Injectivity

S. Baupradist[1], T. Sitthiwirattham[2,3] and S. Asawasamrit[2]

[1]Department of Mathematics, Faculty of Science
Chulalongkorn University, Bangkok, 10330, Thailand
samruam.b@chula.ac.th

[2]Department of Mathematics, Faculty of Applied Science
King Mongkut 's University of Technology North Bangkok, Bangkok 10800, Thailand
tst@kmutnb.ac.th, suphawata@kmutnb.ac.th

[3]Centre of Excellence in Mathematics
CHE, Si Ayuthaya Road
Bangkok, 10400, Thailand

Abstract

Let M be a right R-module. A right R-module N is called *pseudo-M-c-injective* if for any monomorphism from a closed submodule of M to N can be extended to homomorphism from M to N. In this paper, we give some properties of pseudo-M-c-injective modules and characterization of commutative semi-simple rings is given in terms of quasi-pseudo-c-injective modules. Sufficient conditions are given for a quasi-pseudo-c-injective module to become co-Hopfian.

Mathematics Subject Classification: 16D50, 16D70, 16D80

Keywords: pseudo-M-c-injective modules, quasi-pseudo-c-injective modules, right self-pseudo-c-injective modules

1 Introduction and Preliminaries

Throughtout this paper, R is an associative ring with identity and Mod-R is the category of unitary right R-modules. Let M be a right R-module and $S = End_R(M)$, its endomorphism ring. A submodule N of a right R-module M is called an *essential submodule* in M (denoted by $N \subset^e_> M$) if $N \cap A \neq 0$ for any non-zero submodule A of M. For a submodule N of a right R-module M is called *closed* in M (denoted by $N \subset^c_> M$) if N has no proper essential extension inside M. Clearly, every direct summand of a right R-module M is closed in M. For the property of closed submodules, the reader is referred to [13].

Consider the following conditions for a right R-module M:

(C_1) : Every submodule of M is an essential in a direct summand of M.

(C_2) : If a submodule of M is isomorphic to a direct summand of M then it is itself a direct summand of M.

(C_3) : If A and B are direct summand of M with $A \cap B = 0$ then $A \oplus B$ is also a direct summand of M.

A right R-module M is called *CS-module* (or *extending module*) if M satisfies (C_1). We have a right R-module M, which is CS-module if and only if every closed submodule in M is direct summand of M. For a right R-module M is called *continuous* if it satisfies (C_1), (C_2) and *quasi-continuous* if it satisfies (C_1),(C_3). For more details of CS-modules, continuous modules and quasi-continuous modules, we can refer to [8].

In [6], let M and N be right R-modules, N is called *M-closed-injective* (briefly M-c-injective) if every homomorphism $\alpha : K \to N$ where K is a closed submodule of M can be extended to a homomorphism $\bar{\alpha} : M \to N$. A right R-module M is called *quasi-c-injective* (the original terminology is self-c-injective) if M is a M-c-injective. Under M-closed-injective condition, Chaturvedi A. K., Pandeya B. M., Tripathi A. M. and Mishra O. P. (2010) [5] gave some new characterizations of self-c-injective modules in terms endomorphism ring of an R-module satisfying the CM-property. For the more details of M-c-injective module, we can refer to [5], [6] and [12].

A generalization of injective module is studied that is pseudo-injective module which have been studied in [10], [14], [15]. In 2005, H. Q. Dinh [7] introduced the notion of pseudo-M-injective modules (the original terminology is M-pseudo-injective) following which a right R-module N is called *pseudo-M-injective* if for any monomorphism from submodule of M to N can be extended to a homomorphism from M to N. Lately, S. Baupradist and S. Asawasamrit (2011) [1] introduced the notion of pseudo-M-c-injective modules and studied their properties.

In this paper, we give more characterizations of pseudo-M-c-injective module. After this we have provided a characterization of commutative semi-simple ring by using pseudo-c-injectivity have been provided (S. Baupradist and S. Asawasamrit [1]). A sufficient condition for a quasi-pseudo-c-injective module to be co-Hopfian is given.

2 Pseudo-C-Injectivity

Definition 2.1. Let M be a right R-module. A right R-module N is called *pseudo-M-closed-injective* (briefly pseudo-M-c-injective) if for any monomorphism from a closed submodule of M to N can be extended to homomorphism from M to N. The right R-module N is called *pseudo-c-injective* if N is a pseudo-A-c-injective for all $A \in$ Mod-R. A right R-module N is called *quasi-pseudo-c-injective* if N is a pseudo-N-c-injective. A ring R is called *right self-pseudo-c-injective* if it is a pseudo-R_R-c-injective.

Remark.

(1) Clearly, every M-c-injective module is a pseudo-M-c-injective module.

(2) The following example (see [3]), shows that there exists a right R-module M with is a quasi-pseudo-c-injective. For the right $\mathbf{Z}$-module $M = (\mathbf{Z}/p\mathbf{Z}) \oplus \mathbf{Q}$ where p is a prime number, $\mathbf{Z}$ is the set of all integers and $\mathbf{Q}$ is the set of all rational numbers. By [3], we have M is a quasi-c-injective module. Hence M is a quasi-pseudo-c-injective module.

Proposition 2.2. *Let M and N be right R-modules. If N is a pseudo-M-c-injective module, A is a direct summand of N and B is a closed submodule of M then*

1. *A is a pseudo-B-c-injective module.*
2. *A is a pseudo-M-c-injective module.*
3. *N is a pseudo-B-c-injective module.*

Proof. 1. Let K be a closed submodule of B and φ a monomorphism from K to A. Since A is a direct summand of M, there exists a submodule $\acute{A}$ of N such that $N = A \oplus \acute{A}$. Let i_K be an inclusion map from K to B, i_B an inclusion map from B to M and i_A an injection map from A to $N = A \oplus \acute{A}$. Since N is a pseudo-M-c-injective module and $i_A\varphi$ is a monomorphism, there exists a homomorphism α from M to N such that $\alpha i_B i_K = i_A\varphi$. Choose $\overline{\varphi} = \pi_A \alpha i_B$ where π_A is a projection map from M to A. Clearly, $\overline{\varphi}$ is a homomorphism from B to A and $\overline{\varphi} i_K = \pi_A \alpha i_B i_K = \pi_A i_A \varphi = \varphi$. Hence A is a pseudo-B-c-injective module.

2. By (1), we have A is a pseudo-M-c-injective module.

3. Let K be a closed submodule of B and φ a monomorphism from K to N. Since N is a pseudo-M-c-injective module, there exists α a homomorphism from M to N such that $\alpha i_K i_B = \varphi$ where $i_K : K \to B$ and $i_B : B \to M$ are inclusion map. Choose $\overline{\varphi} = \alpha i_B$. Clearly, $\overline{\varphi}$ is a homomorphism from B to N and $\overline{\varphi} i_K = \alpha i_B i_K = \varphi$. Hence N is a pseudo-B-c-injective module. □

Corollary 2.3. *Let M and N be right R-modules. Then N is a pseudo-M-c-injective module if and only if N is a pseudo-X-c-injective module for very closed submodule X of M.*

Proof. Suppose that N is a pseudo-M-c-injective module. By proposition 2.2. (3), we have N is a pseudo-X-c-injective module for very closed submodule X of M. Conversely, Since M is a closed submodule of M and by assumption, we have N is a pseudo-M-c-injective module. □

Proposition 2.4. *Let X, Y and M be right R modules such that $X \cong Y$. Then the following statements hold:*

1. If X is a pseudo-M-c-injective module then Y is a pseudo-M-c-injective module.

2. If M is a pseudo-X-c-injective module then M is a pseudo-Y-c-injective module.

Proposition 2.5. *Let M be a right R-module and X a closed submodule of M. If X is a pseudo-M-c-injective module then X is a direct summand of M.*

Proof. The proof is routine. □

Proposition 2.6. *Let M and N be right R-modules. If N is a pseudo-M-c-injective module, A is a direct summand of N and B is a direct summand of M then A is a pseudo-B-c-injective module.*

Proof. Let K be a closed submodule of B and φ a monomorphism from K to A. Since A is a direct summand of N and B is a direct summand of M, there exist submodules A' of N and B' of M such that $N = A \oplus A'$ and $M = B \oplus B'$. Let i_A be an injective map from A to $N = A \oplus A'$, i_B an injective map from B to $M = B \oplus B'$ and i_K an inclusion map from K to B. Since $B \subset^{\oplus}_{>} M$, B is a closed submodule of M and B is not an essential in M. Then K is a closed submodule of M. But N is a pseudo-M-c-injective module, so $i_A\varphi$ can be extended to a homomorphism α from M to N such that $\alpha i_B i_K = i_A\varphi$. Choose $\bar{\varphi} = \pi_A \alpha i_B$ where π_A is the canonical projection from N to A. We have $\bar{\varphi}$ is an extension of φ. Therefore A is a pseudo-B-c-injective module. □

Proposition 2.7. *A right R-module M satisfies C_1. (CS-module) if and only if every right R-module N is a pseudo-M-c-injective.*

Proof. A similar proving to ([3], Proposition 3.1.1.). □

Proposition 2.8. *Every quasi-pseudo-c-injective module satisfies C_2.*

Proof. Assume that M is a quasi-pseudo-c-injective module. Let A and B be submodules of M such that $A \cong B$ and $A \subset^{\oplus}_{>} M$. We have A is a closed submodule of M. But $A \cong B$ then B is a closed submodule of M. Since M is a quasi-pseudo-c-injective module and $A \subset^{\oplus}_{>} M$, then A is a pseudo-M-c-injective module. But $A \cong B$ thus B is a pseudo-M-c-injective module. By proposition 2.5, we have $B \subset^{\oplus}_{>} M$. Hence M satisfies C_2. □

Corollary 2.9. *Every quasi-c-injective module satisfies C_2.*

In the following, we give a weaker conditions of the C_1 and C_2 conditions.

Proposition 2.10. *Let M be a quasi-pseudo-c-injective module.*

(1) (CC_1): If A is a direct summand of M and B is a closed submodule of M such that $A \cong B$ then B is a direct summand of M.

(2) (CC_2): Let A and B be two direct summand of M with $A \cap B = 0$. If $A \oplus B$ is a closed submodule of M then $A \oplus B$ is a direct summand of M.

Proof. By proposition 2.8, we have (1) and (2). □

Theorem 2.11. *Let M be a right R-module. If M is a quasi-pseudo-c-injective which is a uniform module then M is a quasi-c-injective module.*

Proof. Assume that M is a quasi-pseudo-c-injective which is a uniform module. Let X be a closed submodule of M and $\varphi : X \to M$ a homomorphism. Since M is a uniform module, either φ or $I_X - \varphi$ is a monomorphism where I_X is an inclusion map from X to M.
Case I. φ is a monomorphism. Since M is a quasi-pseudo-c-injective module, there exists a homomorphism $\overline{\varphi} : M \to M$.
Case II. $I_X - \varphi$ is a monomorphism. Since M is a quasi-pseudo-c-injective module, there exists a homomorphism $g : M \to M$ such that $gI_X = I_X - \varphi$. Hence $I_X - g$ is an extension of φ.
Therefore M is a quasi-c-injective module. □

Proposition 2.12. *Suppose that R is a commutative domain and let c be a non-zero non-unit element of R. Then the right R-module $R \oplus (R/cR)$ is not a quasi-pseudo-c-injective.*

Proof. A similar proving to ([3], Proposition 3.2.3.). □

All right R-modules $M_i, i \in I$ are called *relatively pseudo-c-injective modules* if M_i is a pseudo-M_j-c-injective for all distinct $i, j \in I$, where I is the index set.

Corollary 2.13. *Let M_1 and M_2 be right R-modules and $M = M_1 \oplus M_2$. If M is a quasi-pseudo-c-injective module then M_1 and M_2 are both quasi-pseudo-c-injectives and are relatively pseudo-c-injective module.*

Proof. By Proposition 2.2.(2) and Corollary 2.3., we have M_1 and M_2 are quasi-pseudo-c-injective modules. It is also obvious that M_1 and M_2 are relatively pseudo-c-injective module. □

The converse of Corollary 2.13. is not true. We can consider the example ([3]), let $M_1 = \mathbf{Z}$ and $M_2 = \mathbf{Z}/p\mathbf{Z}$, where p is a prime number and $\mathbf{Z}$ is the set of all integers, be right $\mathbf{Z}$-modules. Since M_1 and M_2 are uniform, M_1 and M_2 are quasi-pseudo-c-injectives and relatively pseudo-c-injective. By Proposition 2.12., we have $M_1 \oplus M_2$ is not quasi-pseudo-c-injective module.

Now we consider the sufficient conditions for a direct sum of two quasi-pseudo-c-injective modules to be quasi-pseudo-c-injective. We have the following result.

Theorem 2.14. *For a commutative ring R. Then the following conditions are equivalent:*

(1) The direct sum of two quasi-pseudo-c-injective module is quasi-pseudo-c-injective module;

(2) Every quasi-pseudo-c-injective is injective;

(3) R is a semi-simple artinian.

Proof. (1) $\Rightarrow$ (2) Let M be a quasi-pseudo-c-injective module and $E(M)$ be an injective hull of M. So $E(M)$ is a quasi-pseudo-c-injective module and by assumtion, we have $M \oplus E(M)$ is a quasi-pseudo-c-injective module. Let $i_M : M \to M \oplus E(M)$ be a nature injective map. There exits a homomorphism $f : M \oplus E(M) \to M \oplus E(M)$ such that $i_M = f i_E i$ where i is an inclusion map from M to $E(M)$ and i_E is an nature injective map from E to $M \oplus E(M)$. Thus $I = \pi_M i_M = \pi_M f i_E i$ where I is an identity map from M to M and π_M is a nature projection map from $M \oplus E(M)$ to M. Then $I = gi$ where $g = \pi_M f i_E$. We have M is a direct summand of $E(M)$. Hence M is an injective module.
(2) $\Rightarrow$ (3) Assume that every quasi-pseudo-c-injective is injective. Since every simple R-module is quasi-pseudo-c-injective, its is injective. Hence R is a V-ring and thus a Von-Neuman regular ring due to commutativity. Let M be a completely-reducible R-module. So we have very closed submodule is a direct summand. Then M satisfies CS-module. By ([1], Proposition 2.7), we have M is a quasi-pseudo-c-injective module. By assumption, it is injective module. By [11], it follows that if the countable direct sum of injective hull of simple module is injective then R is a Noetherian ring. Thus R is a Noetherian and regular, so it is semi-simple artenian.
(3) $\Rightarrow$ (1) It is obvious. □

Remark. In the above theorm, commutativity of ring is needed to prove only the part (2) $\Rightarrow$ (3).

Corollary 2.15. *For a commutative ring R. Then the following conditions are equivalent:*

(1) The direct sum of two quasi-c-injective module is quasi-c-injective module;

(2) Every quasi-c-injective is injective;
(3) R is a semi-simple artinian.

By Haghany A. and Vedadi M. R. [9], a right R-module M is called *co-Hopfian (Hopfian)* if every injective (surjective) endomorphism $f : M \to M$ is an automorphism. According to [13], a right R-module M is called *directly finite* if it is not isomorphic to a proper direct summand of M.

Lemma 2.16. *([13], Proposition 1.25) An right R-module M is directly finite if and only if $fg = I$ implies that $gf = I$ for all $f, g \in S = End_R(M)$* where I is an identity map from M to M.

Proposition 2.17. *A quasi-pseudo-c-injective module M is a directly finite if and only if it is co-Hopfian.*

Proof. Let f be an injective endomorphism of M and I an identity homomorphism from M to M. Since M is a quasi-pseudo-c-injective module, there exits a homomorphism $g : M \to M$ such that $gf = I$. By Lemma 2.16, we have $fg = I$ which implies that f is an automorphism. Hence M is co-Hopfian. Conversely, assume that M is a co-Hopfian. Let $f, g \in S = End_R(M)$ such that $fg = I$. Then g is an injective homomorphism and g^{-1} exists. Thus $f = fgg^{-1} = Ig^{-1} = g^{-1}$. So $gf = gg^{-1} = I$. By Lemma 2.16, we have M is a directly finite. □

Corollary 2.18. *If M is an indecomposable quasi-pseudo-c-injective module then M is a co-Hofian.*

Proof. Since every indecomposable module is directly finite and by proposition 2.17, M is a co-Hofian. □

Corollary 2.19. *If M is a quasi-pseudo-c-injective and Hofian module then M is co-Hopfian.*

Proof. Since every Hopfian module is a directly finite and by proposition 2.17, M is co-Hopfian. □

Acknowledgment. The first and second authors were supported from Centre of Excellence in Mathematics, CHE, Si Ayuthaya Road, Bangkok 10400, THAILAND.

References

[1] S. Baupradist and S. Asawasamrit, A weaker form of pseudo-injective modules, to appear in *JP Journal of Algebra, Number Theory and Applications.*

[2] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, 1956.

[3] Catarina Arajo de Santa Clara Gomes, *Some generalizations of Injectivity*, Thesis submitted for the Degree of Doctor of Philosophy at the University of Glasgow, 1998.

[4] A.K. Chaturvedi, B.M. Pandeya and A.J. Gupta, Quasi-c-principally injective modules and self-c-principally injective rings, *Southeast Asian Bulletin of Mathematics*, **33** (2009), 685-702.

[5] A.K. Chaturvedi, B.M. Pandeya, A.M. Tripathi and O.P. Mishra, On M-C-Injective and Self-C-Injective Modules, *Asian-European Journal of Mathematics*, **3** No.3 (2010), 387-393.

[6] C.S. Clara and P.F. Smith, Modules which are self-injective relative to closed submodules, *Contemporary of Mathematics 259 Amer. Math. Soc. Providence* (2000), 487-499.

[7] H.Q. Dinh, A note on pseudo-injective modules, *Communication in Algebra*, **33** (2005), 361-369.

[8] N.V. Dung, D.V. Huynh, P.F. Smith and R. Wisbauer, *Extending Modules, Research Notes in Mathematics Series*, 313, Pitman London 1994.

[9] A. Haghany and M.R. Vedadi, Modules whose injective endomorphisms are essential, *J. Algebra*, **243** (2001), 765-779.

[10] S.K. Jain and S. Singh, Quasi-injective and pseudo-injective modules, *Canad. Math. Bull.*, **18** No.3 (1975), 134-141.

[11] R.P. Kurshan, Ring whose cyclic modules have finitely generated socle, *J. Algebra*, **15** (1970), 376-386.

[12] E. Mermut, C.S. Clara and P.F. Smith, Injectivity relative to closed submodules, *Journal of Algebra*, **321** (2009), 548-557.

[13] S.H. Mohamed and B.J. Müller, *Continuous and Discrete Modules*, Cambridge Univ. Press, Cambridge, UK, 1990.

[14] S. Singh, On pseudo-injective modules, *Riv. Mat. Univ. Parma*, **2** No.9 (1968), 59-65.

[15] S. Singh and S.K. Jain, On pseudo injective modules and self pseudo-injective rings, *The Journal of Mathematical Sciences*, **2** No.1 (1967), 125-133.

Received: October, 2011