UNIFORM EXPONENTIAL ATTRACTORS FOR A SINGULARLY PERTURBED DAMPED WAVE EQUATION

Pierre Fabrie, Cedric Galusinski

Université Bordeaux-I
Mathématiques Appliquées
351 Cours de la Libération, 33405 Talence Cedex, France

Alain Miranville, Sergey Zelik

Université de Poitiers
Laboratoire d'Applications des Mathématiques, SP2MI
Boulevard Marie et Pierre Curie, Téléport 2
86962 Chasseneuil Futuroscope Cedex, France

Dedicated to Professor Mark Iosifovich Vishik on the occasion of his eightieth birthday.

Abstract. Our aim in this article is to construct exponential attractors for singularly perturbed damped wave equations that are continuous with respect to the perturbation parameter. The main difficulty comes from the fact that the phase spaces for the perturbed and unperturbed equations are not the same; indeed, the limit equation is a (parabolic) reaction-diffusion equation. Therefore, previous constructions obtained for parabolic systems cannot be applied and have to be adapted. In particular, this necessitates a study of the time boundary layer in order to estimate the difference of solutions between the perturbed and unperturbed equations. We note that the continuity is obtained without time shifts that have been used in previous results.

1. Introduction. The study of the long time behavior of equations arising from mechanics and physics is very important, as it is essential, for practical purposes, to understand and predict the asymptotic behavior of the system. Several objects have been introduced for this study.

A first object is the global attractor. It is a compact set, invariant by the flow, which attracts all the trajectories as time goes to infinity. Since it is the smallest (with respect to inclusion) set enjoying these properties, it is a suitable object for the study of the long time behavior of the system. Furthermore, in many cases, one can prove that it has finite dimension (in the sense of the Hausdorff or the fractal dimension). We refer the reader to [BV], [CV], [H], [L] and [T] for extensive reviews on this subject. Now, the global attractor has two drawbacks. Indeed, it can attract the trajectories slowly (see for instance [Ko]) and (consequently) it can be sensitive to perturbations.

In order to overcome these difficulties, Foias, Sell and Temam have introduced in [FoST] the notion of an inertial manifold. An inertial manifold is a smooth, finite dimensional, hyperbolic (and thus robust), semi-invariant manifold which contains the global attractor and attracts the trajectories exponentially. Unfortunately, all the known constructions of inertial manifolds are based on a very restrictive property, the so-called spectral gap condition. Consequently, the existence of inertial manifolds is not known for many physically important equations such as the Navier-Stokes equations (even in two space dimensions) and reaction-diffusion and damped

wave equations in three space dimensions. A non-existence result has even been obtained for reaction-diffusion equations in higher space dimensions.

So, Eden, Foias, Nicolaenko and Temam have introduced in [EFNT] the notion of an exponential attractor, which can be seen as an intermediate object between the two ideal objects that the global attractor and an inertial manifold are. An exponential attractor is a compact, semi-invariant set which contains the global attractor, attracts the trajectories exponentially and has finite fractal dimension. So, compared with an inertial manifold, an exponential attractor is not necessarily regular and, compared with the global attractor, it is expected to be more stable (since it attracts the trajectories exponentially). We shall come back to this last point below. We note finally that, since it is not unique, the actual choice of an exponential attractor may, in a sense, be artificial.

Exponential attractors have been constructed for a large class of equations (see [BN], [EFNT], [EfM1], [EfM2], [EfMZ1], [EfMZ2], [EfMZ3], [FG1], [FG2], [FM], [FN], [M1], [M2] and the references therein). In particular, in [EfMZ1] (see also [EfMZ2] and [EfMZ3]), a construction that is valid in Banach spaces is given (all the previous constructions made an essential use of orthogonal projectors with finite ranks and were thus valid in Hilbert spaces only); another construction, due to Le Dung and Nicolaenko and valid in Banach spaces, is given in [LdN]. So, exponential attractors are as general as global attractors.

Let us come back to the problem of the robustness of the global attractor. Generally, global attractors are only upper semicontinuous with respect to perturbations. The lower semicontinuity property is much more delicate to obtain and can be established only for some particular cases (see e.g. [SH], [R] and [BV]); for instance, it is true when the semigroup possesses a global Lyapunov function and all equilibria are hyperbolic. In this particular case, the corresponding attractor (the so-called regular attractor in the terminology of Babin and Vishik) is exponential and is robust under perturbations (i.e. it is upper and lower semicontinuous with respect to perturbations, see [BV]). Moreover, if $\mathcal{A}_\varepsilon$ is the global attractor of a perturbed system and $\mathcal{A}_0$ is that of the unperturbed one, then, under natural assumptions on the perturbations, we have $\mathrm{dist}_{sym}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \leq c\varepsilon^\kappa$, where dist_{sym} denotes the symmetric distance between sets, $\kappa \in (0, 1)$ and $\varepsilon > 0$ is the perturbation parameter.

As already mentioned, exponential attractors are more robust objects. In particular, one can prove the continuity of exponential attractors under perturbations in many cases (see [EFNT] for the continuity for Galerkin approximations and [FGM], [FG1] and [FG2] for examples of (singular) perturbations of partial differential equations), even when this property is not known or is violated for the global attractor. The drawback of these results is that the continuity is obtained up to a time shift. To be more precise, the continuity can be expressed as follows:

$$\lim_{\varepsilon \to 0} \limsup_{t \to \infty} \left[\mathrm{dist}(S_t(\varepsilon)\mathcal{M}_\varepsilon, \mathcal{M}_0) + \mathrm{dist}(S_t(0)\mathcal{M}_0, \mathcal{M}_\varepsilon) \right] = 0,$$

where $S_t(\varepsilon)$ and $S_t(0)$ are the perturbed and unperturbed semigroups respectively, $\mathcal{M}_\varepsilon$ and $\mathcal{M}_0$ are the corresponding exponential attractors and dist denotes the nonsymmetric Hausdorff distance.

In [EfMZ3], by adapting the construction of [EfMZ1], a construction of continuous exponential attractors, without such time shifts, is given. Moreover, we obtain analogous (to the case of regular attractors) estimates for the symmetric distance between the perturbed ($\mathcal{M}_\varepsilon$) and unperturbed ($\mathcal{M}_0$) exponential attractors, namely

$\text{dist}_{sym}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \leq c_1 \varepsilon^{\kappa_1}$, without assuming that the system possess a global Lyapunov function and that all equilibria be hyperbolic. We also note that, in contrast to the case of regular attractors, the constants c_1 and κ_1 can be computed explicitly in terms of the physical parameters of the problem in specific examples. This construction was then applied to a (singularly) perturbed viscous Cahn-Hilliard system.

Our aim in this article is to extend this construction to singularly perturbed damped wave equations (see e.g. [R] for the study of the (upper semi)continuity of the global attractor and [FGM] for the continuity (up to a time shift) of exponential attractors). The difficulty here comes from the fact that the phase spaces for the perturbed and unperturbed equations are not the same (the limit equation being a (parabolic) reaction-diffusion equation). Therefore, the abstract construction of [EfMZ3] cannot be applied and must be deeply reworked (we can also note that the construction of [EfMZ3] is valid for (strongly) dissipative systems only).

This article is dedicated to Professor Mark Iosifovich Vishik on the occasion of his eightieth birthday in recognition of the impact he had on the development of the theory of infinite dimensional dynamical systems in mathematical physics.

2. Setting of the problem. We consider the following singularly perturbed damped wave equation in a smooth bounded domain $\Omega \subset \mathbb{R}^3$:

$$\varepsilon \partial_t^2 u + \gamma \partial_t u - \Delta_x u + f(u) = g, \quad u\big|_{t=0} = u_0, \quad \partial_t u\big|_{t=0} = u'_0, \quad u\big|_{\partial\Omega} = 0, \quad (2.1)$$

where $\varepsilon > 0$ is a small parameter and $\gamma > 0$. We assume that the nonlinear interaction function f satisfies the following assumptions:

$$\begin{cases} 1. \ f \in C^2(\mathbb{R}), \ \ f(0) = 0, \ \ 2. \ f'(u) \geq -K, \\ 3. \ f(u).u \geq 0 \text{ if } |u| \geq L, \ \ 4. \ |f''(u)| \leq C(1 + |u|), \end{cases} \quad (2.2)$$

where C, K and L are fixed positive constants. We also assume that the external force g belongs to the Sobolev space $H^1(\Omega)$.

Remark 2.1. We note that assumptions (2.2) are satisfied for cubic nonlinearities ($f(u) = u^3 - \alpha u, \ \alpha \in \mathbb{R}$). We also note that, for simplicity, we require the regularity $g \in H^1(\Omega)$ for the external force g, although our approach allows us (after minor changes) to obtain the main results for the case $g \in L^2(\Omega)$ as well. We finally note that assumption $(2.2)_3$ is not satisfied for the Sine-Gordon equation ($f(u) = \sin u$). However, the calculations are simpler in that case (since f and its derivatives are bounded) and the results obtained below are valid for this equation as well.

In order to simplify the notations, we set $\xi_u(t) := [u(t), \partial_t u(t)]$ and introduce the following energy norms depending on the small parameter $\varepsilon \geq 0$ on the pairs of functions $\xi := [u, v]$:

$$\|\xi\|_{\mathcal{E}^\kappa(\varepsilon)}^2 := \|u\|_{H^{\kappa+1}}^2 + \varepsilon \|v\|_{H^\kappa}^2 + \|v\|_{H^{\kappa-1}}^2. \quad (2.3)$$

In view of these norms, we introduce the energy spaces $\mathcal{E}^\kappa(\varepsilon)$ as follows: for $\varepsilon \neq 0$, we set

$$\mathcal{E}^\kappa(\varepsilon) := \left(H^{\kappa+1}(\Omega) \cap \{u\big|_{\partial\Omega} = 0\}\right) \times \left(H^\kappa(\Omega) \cap \{v\big|_{\partial\Omega} = 0\}\right), \quad (2.4)$$

and, for $\varepsilon = 0$, we set

$$\mathcal{E}^\kappa(0) := \left(H^{\kappa+1}(\Omega) \cap \{u\big|_{\partial\Omega} = 0\}\right) \times \left(H^{\kappa-1}(\Omega) \cap \{v\big|_{\partial\Omega} = 0\}\right), \quad (2.5)$$

where $H^{\kappa}(\Omega)$ denotes the standard Sobolev space; we agree that the boundary conditions are added only for the κ for which they have a sense (in the sense of the trace). We will write in the sequel $\mathcal{E}(\varepsilon)$ instead of $\mathcal{E}^0(\varepsilon)$.

We note that the space (2.4) is a natural phase space for the hyperbolic equation (2.1) (see e.g. [BV]). On the other hand, in the case $\varepsilon = 0$, the space (2.5) is naturally associated with the limit parabolic problem (2.1) (with $\varepsilon = 0$). As we will show below, the norms (2.3) reflect in a right way the dependence on ε of the norms of the solutions of the singularly perturbed hyperbolic equations (2.1) as $\varepsilon \to 0$.

The rest of the article is organized as follows.

Section 3 is devoted to the study of singular perturbations of exponential attractors in an abstract setting.

In Section 4, we give uniform (with respect to $\varepsilon \to 0$) estimates on the solutions of (2.1) in the phase spaces $\mathcal{E}^{\kappa}(\varepsilon)$, $\kappa = 0, 1, 2$, which are necessary in order to apply the abstract scheme of Section 3 to study the hyperbolic equation (2.1) and estimates on the difference of solutions of problem (2.1), which are crucial for our study of exponential attractors (for the reader's convenience, the proofs of the most classical estimates are given in Section 6). In particular, we need to study the time boundary layer near $t = 0$ as $\varepsilon \to 0$ in order to obtain the proper estimates on the difference of solutions of the perturbed ($\varepsilon \neq 0$) and unperturbed ($\varepsilon = 0$) equations (2.1). We then prove that the dynamical systems associated with problems (2.1) possess uniform exponential attractors $\mathcal{M}_{\varepsilon}$ in the spaces $\mathcal{E}^2(\varepsilon)$ such that

$$\text{dist}_{sym, \mathcal{E}(\varepsilon)} \left(\mathcal{M}_{\varepsilon}, \mathcal{M}_0 \right) \leq c_1 \varepsilon^{\kappa_1}, \qquad (2.6)$$

where $c_1 > 0$ and $0 < \kappa_1 < 1$ are independent of ε.

We note that, in Section 4, we construct the exponential attractors in the spaces $\mathcal{E}^2(\varepsilon)$ only, although the convergence (2.6) is established in a weaker space (namely, $\mathcal{E}(\varepsilon)$). In order to overcome this disadvantage, we prove, in Section 5, an abstract result on the transitivity of exponential attraction and, based on this result, we verify that $\mathcal{M}_{\varepsilon}$ attracts exponentially the bounded subsets of $\mathcal{E}(\varepsilon)$ as well. Thus, we finally prove that the uniform exponential attraction property for $\mathcal{M}_{\varepsilon}$ and the convergence (2.6) take place in the *same* phase space $\mathcal{E}(\varepsilon)$. We can note that this result gives a positive answer to a question raised in [EFNT], Chapter 6.

3. Perturbations of exponential attractors: the abstract setting. In this section, we formulate and prove an abstract result on the construction of uniform exponential attractors for a singularly perturbed family of maps, which generalizes that given in [EfMZ3] and will be applied to our problem in Section 4 below (see also [MZ] for the application to phase-field type equations). In order to do so, we will use the concept of Kolmogorov ε-entropy (see e.g. [KT] for details).

Definition 3.1. Let K be a (pre)compact set in a metric space V and let $N_{\mu}(K, V)$ be the minimal number of balls of radius $\mu > 0$ in V that are necessary to cover K. Then, the Kolmogorov μ-entropy of K is the number $\mathbb{H}_{\mu}(K, V) := \ln N_{\mu}(K, V)$. We recall that the fractal dimension of the set K can be expressed in terms of the μ-entropy, namely $\dim_F (K, V) := \limsup_{\mu \to 0^+} \frac{\mathbb{H}_{\mu}(K,V)}{\ln \frac{1}{\mu}}$.

We are now in a position to formulate our abstract scheme.

Let $\mathcal{E}(\varepsilon)$ and $\mathcal{E}^1(\varepsilon)$, $\varepsilon \in [0, 1]$, be two families of Banach spaces (which are embedded into a larger topological space $\mathcal{V}$) such that $\mathcal{E}^1(\varepsilon) \subset\subset \mathcal{E}(\varepsilon)$, for every

$\varepsilon \in [0, 1]$. We also assume that these compact embeddings are in some sense uniform with respect to ε, which we express in the following way:

$$\mathbb{H}_\mu(B_{\mathcal{E}^1(\varepsilon)}(0, 1), \mathcal{E}(\varepsilon)) \leq \mathbb{M}(\mu), \quad \forall \mu > 0, \tag{3.1}$$

where $\mathbb{M}(\mu)$ is some monotonic function that is independent of ε (here and below, $B_V(v, R)$ denotes the R-ball in V centered at v).

We further assume that we are given a family of closed sets $B_\varepsilon \subset \mathcal{E}(\varepsilon)$ (with B_0 bounded in $\mathcal{E}(0)$) and a family of maps $S_\varepsilon : B_\varepsilon \to B_\varepsilon$ such that

1. For every $\varepsilon \in [0, 1]$, $B_0 \subset \mathcal{E}(\varepsilon)$ and

$$\|b_0\|_{\mathcal{E}(\varepsilon)} \leq C_1 \|b_0\|_{\mathcal{E}(0)} + C_2 \varepsilon, \quad \forall b_0 \in B_0. \tag{3.2}$$

2. There exist maps $\mathcal{C}_\varepsilon$ and $\mathcal{K}_\varepsilon$ (which map B_ε into $\mathcal{E}(\varepsilon)$) such that $S_\varepsilon = \mathcal{C}_\varepsilon + \mathcal{K}_\varepsilon$ and, for every b_ε^1, $b_\varepsilon^2 \in B_\varepsilon$, the following estimates hold:

$$\|\mathcal{C}_\varepsilon b_\varepsilon^1 - \mathcal{C}_\varepsilon b_\varepsilon^2\|_{\mathcal{E}(\varepsilon)} \leq \kappa \|b_\varepsilon^1 - b_\varepsilon^2\|_{\mathcal{E}(\varepsilon)}, \quad \|\mathcal{K}_\varepsilon b_\varepsilon^1 - \mathcal{K}_\varepsilon b_\varepsilon^2\|_{\mathcal{E}^1(\varepsilon)} \leq K \|b_\varepsilon^1 - b_\varepsilon^2\|_{\mathcal{E}(\varepsilon)}, \tag{3.3}$$

where $\kappa < \frac{1}{2}$ and K are independent of ε.

3. There exist nonlinear "projectors" $\Pi_\varepsilon : B_\varepsilon \to B_0$ such that $\Pi_\varepsilon B_\varepsilon = B_0$ and

$$\|S_\varepsilon^{(k)} b_\varepsilon - S_0^{(k)} \Pi_\varepsilon b_\varepsilon\|_{\mathcal{E}(\varepsilon)} \leq C \varepsilon L^k, \tag{3.4}$$

for every $b_\varepsilon \in B_\varepsilon$, where the constants C and L are independent of ε (here and below, $S_\varepsilon^{(k)}$ denotes the kth iteration of the map S_ε).

The main result of this section is the following theorem.

Theorem 3.1. *Let assumptions (3.1)–(3.4) hold. Then, there exists a family of exponential attractors $\mathcal{M}_\varepsilon \subset B_\varepsilon$ for the maps S_ε such that $S_\varepsilon \mathcal{M}_\varepsilon \subset \mathcal{M}_\varepsilon$ and the following conditions are satisfied:*

1. The rate of exponential attraction is uniform with respect to ε:

$$\mathrm{dist}_{\mathcal{E}(\varepsilon)}(S_\varepsilon^{(k)} B_\varepsilon, \mathcal{M}_\varepsilon) \leq C_3 e^{-\nu k}, \tag{3.5}$$

where the positive constants C_3 and ν are independent of ε.

2. The sets $\mathcal{M}_\varepsilon$ are compact in $\mathcal{E}(\varepsilon)$, for $\varepsilon \in [0, 1]$, and their fractal dimensions are uniformly bounded with respect to ε:

$$\dim_F(\mathcal{M}_\varepsilon, \mathcal{E}(\varepsilon)) \leq C_4, \tag{3.6}$$

where the positive constant C_4 is independent of ε.

3. The symmetric distance between $\mathcal{M}_\varepsilon$ and $\mathcal{M}_0$ satisfies

$$\mathrm{dist}_{sym, \mathcal{E}(\varepsilon)}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \leq C_5 \varepsilon^\tau. \tag{3.7}$$

Moreover, the constants C_i, $i = 3, 4, 5$, and $0 < \tau < 1$ can be calculated explicitly.

Proof. We first construct the exponential attractor $\mathcal{M}_0$. To this end, we construct a family of sets $V_i \subset S_0^{(i)} B_0$, $i = 1, 2, \cdots$, by the following inductive procedure. We recall that, due to our assumptions, the set B_0 is bounded in $\mathcal{E}(0)$. Consequently, there exists a ball $B_{\mathcal{E}(0)}(b_0, R)$ such that $b_0 \in B_0$ and $B_0 \subset B_{\mathcal{E}(0)}(b_0, R)$. We

set $V_0 := \{b_0\}$. We now assume that the set V_k is already constructed such that $V_k \subset S_0^{(k)} B_0$ and V_k is an $R_k := R\left(\kappa + \frac{1}{2}\right)^k$-net of $S_0^{(k)} B_0$ (we recall that, due to our assumptions, $\kappa < \frac{1}{2}$). We then construct the next set V_{k+1} preserving these properties. To this end, we use estimates (3.3). Let b belong to V_k. We consider a ball $B_{\mathcal{E}(0)}(b, R_k)$ and its image under the map S_0. It follows from (3.3) that

$$\mathcal{K}_0 B_{\mathcal{E}(0)}(b, R_k) \subset B_{\mathcal{E}^1(0)}(\mathcal{K}_0 b, K R_k). \tag{3.8}$$

We now recall that $\mathcal{E}^1(0) \subset\subset \mathcal{E}(0)$. Consequently, there exists a covering of the right-hand side of (3.8) by a finite number of $R_k \frac{1-2\kappa}{4}$-balls in $\mathcal{E}(0)$. We fix a covering with a minimal number of balls and denote by $W_k(b)$ the set of all the centers of this covering. Furthermore, we note that

$$\#W_k(b) = N_{R_k \frac{1-2\kappa}{4}}\left(B_{\mathcal{E}^1(0)}(\mathcal{K}_0 b, K R_k), \mathcal{E}(0)\right) \leq$$
$$\leq N_{\frac{1-2\kappa}{4K}}\left(B_{\mathcal{E}^1(0)}(0, 1), \mathcal{E}(0)\right) \leq \exp\left(\mathbb{M}(\frac{1-2\kappa}{4K})\right) := N. \tag{3.9}$$

It is essential for our construction that the number N defined in (3.9) be independent of k (and of $b \in V_k$; moreover, thanks to (3.1), this number will be independent of ε as well if we replace the spaces $\mathcal{E}(0)$ and $\mathcal{E}^1(0)$ by $\mathcal{E}(\varepsilon)$ and $\mathcal{E}^1(\varepsilon)$).

We note that the first estimate of (3.3) implies that the system of $R_k \frac{1-2\kappa}{4} + \kappa R_k = R_k \frac{1+2\kappa}{4}$-balls centered at the points of the set $\mathcal{C}_0 b + W_k(b)$ covers the set $S_0 B_{\mathcal{E}(0)}(b, R_k)$ and that the number of balls in this system does not exceed N. Therefore, the system of $R_k \frac{1-2\kappa}{4}$-balls centered at the points of $\cup_{b \in V_k}(\mathcal{C}_0 b + W_k(b))$ covers $S_0^{(k+1)} B_0$ and the number of balls in this system is not greater than $N \# V_k$. Increasing the radius of every ball in this covering by a factor of two, we can assume that the centers of the covering belong to $S_0^{(k+1)} B_0$. We now denote by V_{k+1} the (new) centers of this covering and we note that $R_{k+1} = R_k \frac{1+2\kappa}{2}$.

Thus, we have constructed by induction the family of sets $V_k \subset S_0^{(k)} B_0$, $k \in \mathbb{N}$, which satisfy the following properties:

$$1. \ \#V_k \leq N^k, \quad 2. \ \text{dist}_{\mathcal{E}(0)}\left(S_0^{(k)} B_0, V_k\right) \leq R(\kappa + \frac{1}{2})^k. \tag{3.10}$$

(We recall that, due to our assumptions, $\kappa + \frac{1}{2} < 1$.)

We now define a new family of sets E_k, $k \in \mathbb{N}$, by the following inductive formula:

$$E_1 := V_1, \quad E_{k+1} := S_0 E_k \cup V_{k+1}.$$

Then, obviously

$$\begin{cases} 1. \ E_k \subset S_0^{(k)} B_0, \quad S_0 E_k \subset E_{k+1}, \quad 2. \ \#E_k \leq N^{k+1}, \\ 3. \ \text{dist}_{\mathcal{E}(0)}\left(S_0^{(k)} B_0, E_k\right) \leq R(\kappa + \frac{1}{2})^k. \end{cases} \tag{3.11}$$

We finally set

$$\mathcal{M}_0' := \cup_{k \in \mathbb{N}} E_k, \quad \mathcal{M}_0 := [\mathcal{M}_0']_{\mathcal{E}(0)}, \tag{3.12}$$

where $[\cdot]_V$ denotes the closure in V. It is not difficult to verify that the set $\mathcal{M}_0$ is an exponential attractor for the map S_0 on B_0. Indeed, $(3.11)_1$ and (3.12) imply

that $S_0 \mathcal{M}_0 \subset \mathcal{M}_0$ and the exponential attraction (3.5) is an immediate corollary of $(3.11)_3$. Furthermore, it follows from $(3.11)_2$, together with $(3.11)_1$, that the fractal dimension of $\mathcal{M}_0$ satisfies (see [EfMZ1] for details)

$$\dim_F (\mathcal{M}_0, \mathcal{E}(0)) \leq \frac{\mathbb{M}(\frac{1-2\kappa}{4K})}{\ln \frac{2}{2\kappa+1}}. \tag{3.12'}$$

We then construct the exponential attractors $\mathcal{M}_\varepsilon$, for $\varepsilon \neq 0$, using the construction of the attractor $\mathcal{M}_0$. To this end, we fix inverse images of the sets E_k under the maps $S_0^{(k)}$ (it is possible to do so thanks to $(3.11)_1$). To be more precise, we assume that the family of sets $\widehat{E}_k \subset B_0$ is such that

$$1. \ \ S_0^{(k)} \widehat{E}_k = E_k, \quad 2. \ \ \#\widehat{E}_k = \#E_k \leq N^{k+1}. \tag{3.13}$$

We fix an arbitrary $\varepsilon \in (0, 1]$ and arbitrary liftings of the sets $\widehat{E}_k \subset B_0$ to B_ε (with respect to the projection Π_ε), i.e. the $\widehat{E}_k(\varepsilon) \subset B_\varepsilon$ are such that

$$1. \ \ \Pi_\varepsilon \widehat{E}_k(\varepsilon) = \widehat{E}_k, \quad 2. \ \ \#\widehat{E}_k(\varepsilon) = \#E_k \leq N^{k+1}. \tag{3.14}$$

We finally set $\tilde{E}_k(\varepsilon) := S_\varepsilon^{(k)} \widehat{E}_k(\varepsilon)$. We claim that

$$\text{dist}_{\mathcal{E}(\varepsilon)} \left(S_\varepsilon^{(k)} B_\varepsilon, \tilde{E}_k(\varepsilon) \right) \leq 2C\varepsilon L^k + C_1 R(\kappa + \tfrac{1}{2})^k + C_2 \varepsilon, \tag{3.15}$$

where the constants C, C_1, C_2 and L are defined in (3.2) and (3.4). Indeed, let $b \in B_\varepsilon$ be an arbitrary point. We set $b_0 := S_0^{(k)} \Pi_\varepsilon b \in S_0^{(k)} B_0$. We recall that E_k is an $R(\kappa + \frac{1}{2})^k$-net of the set $S_0^{(k)} B_0$. Consequently, there exists a point $b_k \in E_k$ such that

$$\|b_0 - b_k\|_{\mathcal{E}(\varepsilon)} \leq C_1 \|b_0 - b_k\|_{\mathcal{E}(0)} + C_2 \varepsilon \leq C_1 R(\kappa + \tfrac{1}{2})^k + C_2 \varepsilon. \tag{3.16}$$

Let $\widehat{b}_k$, $\widehat{b}_k(\varepsilon)$ and $\tilde{b}_k(\varepsilon)$ be the images of the point b_k in $\widehat{E}_k$, $\widehat{E}_k(\varepsilon)$ and $\tilde{E}_k(\varepsilon)$ respectively (i.e. $b_k = S_0^{(k)} \widehat{b}_k$, $\Pi_\varepsilon \widehat{b}_k(\varepsilon) = \widehat{b}_k$ and $\tilde{b}_k(\varepsilon) = S_\varepsilon^{(k)} \widehat{b}_k(\varepsilon)$). Then, estimates (3.4) imply that

$$\|S_\varepsilon^{(k)} b - b_0\|_{\mathcal{E}(\varepsilon)} \leq C\varepsilon L^k, \ \ \|\tilde{b}_k(\varepsilon) - b_k\|_{\mathcal{E}(\varepsilon)} \leq C\varepsilon L^k. \tag{3.17}$$

Combining estimates (3.16) and (3.17), we find $\|S_\varepsilon^{(k)} b - \tilde{b}_k(\varepsilon)\|_{\mathcal{E}(\varepsilon)} \leq \|S_\varepsilon^{(k)} b - b_0\|_{\mathcal{E}(\varepsilon)} + \|b_0 - b_k\|_{\mathcal{E}(\varepsilon)} + \|b_k - \tilde{b}_k(\varepsilon)\|_{\mathcal{E}(\varepsilon)} \leq 2C\varepsilon L^k + C_1 R(\kappa + \frac{1}{2})^k + C_2 \varepsilon$, which proves (3.15).

Let now $k(\varepsilon)$ and $1 > \tau > 0$ be the solutions of

$$\varepsilon L^k = (\tfrac{1}{2} + \kappa)^k = \varepsilon^\tau, \tag{3.18}$$

i.e. $k(\varepsilon) = \frac{\log_L \varepsilon}{\log_L(\frac{1}{2}+\kappa)-1}$, $\tau = \frac{-\log_L(\frac{1}{2}+\kappa)}{1-\log_L(\frac{1}{2}+\kappa)}$. Then, it follows from (3.4), (3.15) and (3.18) that

$$\begin{cases} 1. \ \ \text{dist}_{sym,\mathcal{E}(\varepsilon)} \left(\tilde{E}_k(\varepsilon), E_k \right) \leq C'\varepsilon^\tau, \\ 2. \ \ \text{dist}_{\mathcal{E}(\varepsilon)} \left(S_\varepsilon^{(k)} B_\varepsilon, \tilde{E}_k(\varepsilon) \right) \leq C'' R(\tfrac{1}{2} + \kappa)^k, \end{cases} \tag{3.19}$$

for every $1 \leq k \leq k(\varepsilon)$, where the constants C' and C'' are independent of k and ε.

Thus, we can take $E_k(\varepsilon) := \tilde{E}_k(\varepsilon)$, if $k \leq k(\varepsilon)$. In order to construct the sets $E_k(\varepsilon)$ for $k > k(\varepsilon)$, we forget the exponential attractor $\mathcal{M}_0$ and the sets $\tilde{E}_k(\varepsilon)$ and construct them by the inductive procedure described above based on (3.3) and starting from $E_{k(\varepsilon)}(\varepsilon)$ rather. Thus, we have a family of sets $E_k(\varepsilon)$ which satisfy the following conditions:

$$\begin{cases} 1. & E_k(\varepsilon) \subset S_\varepsilon^{(k)} B_\varepsilon, \quad S_\varepsilon E_k(\varepsilon) \subset E_{k+1}(\varepsilon), \quad \#E_k(\varepsilon) \leq N^{k+1}, \\ 2. & \text{dist}_{\mathcal{E}(\varepsilon)}\left(S_\varepsilon^{(k)} B_\varepsilon, E_k(\varepsilon)\right) \leq C'R(\tfrac{1}{2} + \kappa)^k. \end{cases} \tag{3.20}$$

Moreover, for $k \leq k(\varepsilon)$, we have

$$\text{dist}_{sym,\mathcal{E}(\varepsilon)}(E_k(\varepsilon), E_k) \leq C''\varepsilon^\tau. \tag{3.21}$$

We finally define the exponential attractor $\mathcal{M}_\varepsilon$ as follows:

$$\mathcal{M}'_\varepsilon := \cup_{k \in \mathbb{N}} E_k(\varepsilon), \quad \mathcal{M}_\varepsilon := [\mathcal{M}'_\varepsilon]_{\mathcal{E}(\varepsilon)}. \tag{3.22}$$

Arguing as in [EfMZ3], we can verify that the family of exponential attractors $\mathcal{M}_\varepsilon$ satisfies all the assertions of the theorem. Indeed, the semi-invariance property is straightforward. Furthermore, the uniform exponential attraction property (3.5) is an immediate corollary of $(3.20)_2$. Estimate (3.6) for the fractal dimension of $\mathcal{M}_\varepsilon$ can be derived from (3.20) as in the case of $\mathcal{M}_0$ (see [EfMZ1]). Moreover, since the quantities N, κ and C'' in (3.20) are independent of ε, the estimate that we will obtain will be uniform with respect to ε (actually, the dimension of $\mathcal{M}_\varepsilon$ for $\varepsilon \neq 0$ has the same upper bound (3.12') as in the case $\varepsilon = 0$).

So, there remains to verify estimate (3.7). We shall actually only prove the estimate

$$\text{dist}_{\mathcal{E}(\varepsilon)}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \leq C_5\varepsilon^\tau. \tag{3.23}$$

The other inequality (giving the symmetric distance) can be proved analogously. Moreover, it is sufficient to check (3.23) for $\mathcal{M}'_\varepsilon$ instead of $\mathcal{M}_\varepsilon$. So, let b_ε belong to $\mathcal{M}'_\varepsilon$. Then, there exists $k \in \mathbb{N}$ such that $b_\varepsilon \in E_k(\varepsilon)$. If $k \leq k(\varepsilon)$, the result follows immediately from (3.21). Let us therefore assume that $k > k(\varepsilon)$. There exists $\widehat{b}_\varepsilon \in S_\varepsilon^{(k-[k(\varepsilon)])} B_\varepsilon \subset B_\varepsilon$ such that $S_\varepsilon^{([k(\varepsilon)])}\widehat{b}_\varepsilon = b_\varepsilon$. We set $\widehat{b}_0 := \Pi_\varepsilon\widehat{b}_\varepsilon$. Then, on the one hand, (3.4) implies that

$$\|b_\varepsilon - S_0^{([k(\varepsilon)])}\widehat{b}_0\|_{\mathcal{E}(\varepsilon)} \leq C\varepsilon L^{k(\varepsilon)}, \tag{3.24}$$

and, on the other hand, it follows from (3.11) that

$$\text{dist}_{\mathcal{E}(\varepsilon)}\left(S_0^{([k(\varepsilon)])}\widehat{b}_0, \mathcal{M}_0\right) \leq R(\kappa + \frac{1}{2})^{[k(\varepsilon)]}. \tag{3.25}$$

Combining (3.24), (3.25) and (3.2) and using the explicit expression for $k(\varepsilon)$, we obtain estimate (3.23) and Theorem 3.1 is proved.

Remark 3.1. It is essential for Section 4 below to note that $\mathcal{M}_\varepsilon$ satisfies $\mathcal{M}_\varepsilon \subset S_\varepsilon B_\varepsilon$.

4. Uniform exponential attractors for damped hyperbolic equations. In this section, we apply the abstract result obtained in Section 3 in order to construct a uniform family of exponential attractors for the family of equations (2.1). We first give uniform bounds on the solutions of (2.1) in the spaces $\mathcal{E}^\kappa(\varepsilon)$, $\kappa = 0, 1, 2$, and estimates on the difference of solutions of (2.1) (the proofs of the most classical estimates will be given in Section 6).

4.1. Uniform bounds on the solutions.

Theorem 4.1. *Let assumptions (2.2) hold and let $0 < \varepsilon \leq 1$. Then, for every $\xi_u(0) := (u_0, u_0') \in \mathcal{E}(\varepsilon)$, problem (2.1) has a unique solution u such that $\xi_u(t) \in \mathcal{E}(\varepsilon)$, for every $t \geq 0$, and the following estimate is valid:*

$$\|\xi_u(T)\|_{\mathcal{E}(\varepsilon)}^2 + \int_T^{T+1} (\varepsilon\|\partial_t^2 u(t)\|_{H^{-1}}^2 + \|\partial_t u(t)\|_{L^2}^2)\, dt \leq$$

$$\leq Q\left(\|\xi_u(0)\|_{\mathcal{E}(\varepsilon)}^2\right) e^{-\alpha T} + Q(\|g\|_{L^2}), \quad (4.1)$$

where the monotonic function Q and the constant $\alpha > 0$ are independent of $\varepsilon > 0$. Moreover, system (2.1) possesses a global Lyapunov function:

$$\mathcal{U}(\xi_u) := \frac{1}{2}\|\nabla_x u\|_{L^2}^2 + (F(u), 1) - (g, u) + \frac{1}{2}\varepsilon\|\partial_t u\|_{L^2}^2, \quad (4.2)$$

which satisfies the relation

$$\mathcal{U}(\xi_u(T)) - \mathcal{U}(\xi_u(0)) = -\gamma \int_0^T \|\partial_t u(t)\|_{L^2}^2\, dt, \quad (4.3)$$

for every solution $\xi_u(t)$. Here, $F(u) := \int_0^u f(s)\, ds$ and $(\cdot, \cdot)$ denotes the standard inner product in $L^2(\Omega)$.

We now give, based on (4.1), uniform bounds on the solutions in the spaces $\mathcal{E}^1(\varepsilon)$ for more regular initial data.

Theorem 4.2. *Let the assumptions of Theorem 4.1 hold and let, in addition, $\xi_u(0)$ belong to $\mathcal{E}^1(\varepsilon)$. Then, the solution u of problem (2.1) satisfies $\xi_u(t) \in \mathcal{E}^1(\varepsilon)$, for every $t \geq 0$, and the following analogue of estimate (4.1) is valid:*

$$\|\xi_u(T)\|_{\mathcal{E}^1(\varepsilon)}^2 + \int_T^{T+1} (\varepsilon\|\partial_t^2 u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{H^1}^2)\, dt \leq$$

$$\leq Q\left(\|\xi_u(0)\|_{\mathcal{E}^1(\varepsilon)}^2\right) e^{-\alpha T} + Q(\|g\|_{L^2}), \quad (4.4)$$

where the monotonic function Q and the constant $\alpha > 0$ are independent of $\varepsilon > 0$.

We finally give uniform bounds on the solutions in $\mathcal{E}^2(\varepsilon)$.

Theorem 4.3. *Let the assumptions of Theorem 4.1 hold. Let also $\xi_u(0)$ belong to $\mathcal{E}^2(\varepsilon)$ and the compatibility condition*

$$\Delta_x u_0\big|_{\partial\Omega} = g\big|_{\partial\Omega} \quad (4.5)$$

be satisfied. Then, the solution u of equation (2.1) satisfies $\xi_u(t) \in \mathcal{E}^2(\varepsilon)$, for every $t \geq 0$, and the following estimate is valid:

$$\|\xi_u(T)\|_{\mathcal{E}^2(\varepsilon)}^2 + \int_T^{T+1} (\varepsilon\|\partial_t^2 u(t)\|_{H^1}^2 + \|\partial_t u(t)\|_{H^2}^2)\, dt \leq$$

$$\leq Q\left(\|\xi_u(0)\|_{\mathcal{E}^2(\varepsilon)}\right) e^{-\alpha T} + Q(\|g\|_{H^1}), \quad (4.6)$$

where the monotonic function Q and the constant $\alpha > 0$ are independent of $\varepsilon > 0$.

Let us consider, to conclude this section, the limit parabolic problem

$$\gamma\partial_t u = \Delta_x u - f(u) + g, \quad u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0, \quad (4.7)$$

that is associated with (2.1) for $\varepsilon = 0$. It is well known (see e.g. [BV]) that, under the above assumptions, this equation has a unique solution $u(t)$, for every $u_0 \in L^2(\Omega)$. Moreover, it is not difficult to verify that all the estimates of Theorems 4.1-4.3 remain valid for $\varepsilon = 0$ as well (obviously, in that case, the quantity $\partial_t u(0)$ should be expressed through $u(0)$ by using equation (4.7)). To be more precise, the following assertion is valid.

Corollary 4.1. *Let assumptions* (2.2) *hold and let, in addition,* (4.5) *be satisfied. Then, the solution $u(t)$ of problem* (4.7) *satisfies the following estimate:*

$$\|u(T)\|_{H^3}^2 + \|\partial_t u(T)\|_{H^1}^2 + \int_T^{T+1} \|\partial_t u(t)\|_{H^2}^2 \, dt \le$$

$$\le Q(\|u_0\|_{H^3})e^{-\alpha T} + Q(\|g\|_{H^1}), \quad (4.8)$$

where $\alpha > 0$ and Q is an appropriate monotonic function. Moreover, if $u(0) \in H^i(\Omega) \cap H_0^1(\Omega)$, for $i = 1, 2$, then

$$\|u(T)\|_{H^i}^2 + \|\partial_t u(T)\|_{H^{i-2}}^2 + \int_T^{T+1} \|\partial_t u(t)\|_{H^{i-1}}^2 \, dt \le Q(\|u_0\|_{H^i})e^{-\alpha T} + Q(\|g\|_{L^2}).$$

Actually, we will need in the sequel stronger estimates for the solutions of (4.7) and we have the

Corollary 4.2. *Let the assumptions of Theorem 4.3 hold. Then*

$$\|\partial_t^2 u(T)\|_{H^{-1}}^2 + \int_T^{T+1} \left(\|\partial_t^2 u(t)\|_{L^2}^2 + \|(-\Delta_x)^{-1}\partial_t^3 u(t)\|_{L^2}^2 \right) dt \le$$

$$\le Q(\|u_0\|_{H^3})e^{-\alpha T} + Q(\|g\|_{H^1}), \quad (4.9)$$

where $\alpha > 0$ and Q is a monotonic function.

The proof of this corollary is based on classical arguments and is left to the reader.

4.2. Estimates on the difference of solutions.

Theorem 4.4. *Let the assumptions of Theorem 4.1 hold and let $u_1(t)$ and $u_2(t)$ be two solutions of equation* (2.1) *such that $\xi_{u_1}(0)$ and $\xi_{u_2}(0)$ belong to $\mathcal{E}(\varepsilon)$. Then*

$$\|\xi_{u_1}(t) - \xi_{u_2}(t)\|_{\mathcal{E}(\varepsilon)} \le Ce^{Kt}\|\xi_{u_1}(0) - \xi_{u_2}(0)\|_{\mathcal{E}(\varepsilon)}, \quad (4.10)$$

where the constants C and K depend on $\|\xi_{u_i}(0)\|_{\mathcal{E}(\varepsilon)}$, $i = 1, 2$, but are independent of ε.

We then give an (asymptotically) smoothing property for the difference of solutions of (2.1), assuming that their initial values belong to $\mathcal{E}^1(\epsilon)$. To this end, we decompose the function $v(t) := u_1(t) - u_2(t)$ into the sum of an exponentially decaying and a smoothing parts: $v(t) = v_1(t) + v_2(t)$, where the functions v_1 and v_2 satisfy the equations

$$\varepsilon\partial_t^2 v_1 + \gamma\partial_t v_1 - \Delta_x v_1 = 0, \quad \xi_{v_1}(0) = \xi_v(0), \quad (4.11)$$

and

$$\varepsilon\partial_t^2 v_2 + \gamma\partial_t v_2 - \Delta_x v_2 = -l(t)v, \quad \xi_{v_2}(0) = 0, \quad (4.12)$$

respectively, with $l(t) = \int_0^1 f'(su_1(t) + (1-s)u_2(t))ds$.

Theorem 4.5. *Let $u_1(t)$ and $u_2(t)$ be two solutions of (2.1) such that $\xi_{u_1}(0)$ and $\xi_{u_2}(0)$ belong to $\mathcal{E}^1(\epsilon)$ and let the functions $v_1(t)$ and $v_2(t)$ be defined as above. Then, the following estimates hold:*

$$\|\xi_{v_1}(t)\|_{\mathcal{E}(\varepsilon)}^2 \leq Ce^{-\alpha t}\|\xi_v(0)\|_{\mathcal{E}(\varepsilon)}^2, \tag{4.13}$$

$$\|\xi_{v_2}(t)\|_{\mathcal{E}^1(\varepsilon)}^2 \leq C_1 e^{Kt}\|\xi_v(0)\|_{\mathcal{E}(\varepsilon)}^2, \tag{4.14}$$

where all the constants are independent of ε (the constants C and $\alpha > 0$ depend only on equation (2.1) and the constants C_1 and K depend also on $\|\xi_{u_i}(0)\|_{\mathcal{E}^1(\varepsilon)}$, $i = 1, 2$).

We now consider the analogues of Theorems 4.4 and 4.5 for the limit parabolic problem (4.7) ($\varepsilon \to 0$); see e.g. [BV]. We recall that, in contrast to the hyperbolic problem (2.1), the parabolic problem possesses a smoothing property on a finite interval.

Theorem 4.6. *Let the assumptions of Theorem 4.1 hold and let $u_1(t)$ and $u_2(t)$ be two solutions of the limit problem (4.7) such that $u_i(0) \in H_0^1(\Omega)$. Then*

$$\|u_1(t) - u_2(t)\|_{H^1} \leq Ce^{Kt}\|u_1(0) - u_2(0)\|_{H^1}. \tag{4.15}$$

Moreover, for every $t > 0$

$$\|u_1(t) - u_2(t)\|_{H^1}^2 \leq C\frac{1+t}{t}e^{Kt}\|u_1(0) - u_2(0)\|_{L^2}^2, \tag{4.16}$$

where the constants C and K depend on the H^1-norm of the initial data for u_i.

We finally estimate the difference of solutions between the unperturbed ($\varepsilon = 0$) and perturbed ($\varepsilon \neq 0$) problems. To this end, we look for a formal asymptotic expansion of u^ε near $t = 0$ with respect to ε. According to the standard scheme (see e.g. [VLy] for details), we seek for an asymptotic expansion of the form

$$u^\varepsilon(t) := u^0(t) + \varepsilon\tilde{u}^1(\frac{t}{\varepsilon}) + \varepsilon\mathcal{R}(t), \tag{4.17}$$

where $u^0(t)$ is the solution of the limit parabolic problem (4.7), the boundary layer term $\tilde{u}^1(\tau)$, $\tau := t/\varepsilon$, satisfies

$$\partial_\tau^2 \tilde{u}^1 + \gamma\partial_\tau\tilde{u}^1 = 0, \quad \tilde{u}^1(0) = \partial_t u^0(0) - u_0' \quad \text{and} \quad \lim_{\tau \to +\infty}\tilde{u}^1(\tau) = 0, \tag{4.18}$$

i.e. (solving (4.18))

$$\tilde{u}^1(\tau) = -\frac{1}{\gamma}e^{-\gamma\tau}\phi_{u^\varepsilon}(0), \quad \text{where} \quad \phi_{u^\varepsilon}(0) := u_0' - \frac{1}{\gamma}(\Delta_x u_0 - f(u_0) + g) \tag{4.19}$$

(here, we have computed the value of $\partial_t u^0(0)$ from equation (4.7)), and the remainder $\mathcal{R}(t) := \mathcal{R}(t, \varepsilon)$ will be estimated in the next theorem below.

Theorem 4.7. *Let the assumptions of Theorem 4.3 hold. Then, the solution $u^\varepsilon(t)$ of problem (2.1) possesses a decomposition (4.17), where $u^0(t)$ is the solution of problem (4.7), the boundary layer $\tilde{u}^1$ is defined via (4.19) and the remainder $\mathcal{R}$ can be estimated as follows:*

$$\|\xi_\mathcal{R}(t)\|_{\mathcal{E}(\varepsilon)} \le Ce^{Kt}, \quad \xi_\mathcal{R}(t) := [\mathcal{R}(t), \partial_t \mathcal{R}(t)], \qquad (4.20)$$

where the constants C and K depend on $\|\xi_{u^\varepsilon}(0)\|_{\mathcal{E}^2(\varepsilon)}$, but are independent of ε.

Proof. Owing to the construction of the asymptotic expansion, the remainder $\mathcal{R}$ is solution of the following wave equation:

$$\begin{cases} \varepsilon \partial_t^2 \mathcal{R} + \gamma \partial_t \mathcal{R} - \Delta_x \mathcal{R} + l(t)\mathcal{R} = -l(t)\tilde{u}^1(\frac{t}{\epsilon}) + \Delta_x \tilde{u}^1(\frac{t}{\epsilon}) - \partial_t^2 u^0, \\ \mathcal{R}\big|_{t=0} = \frac{1}{\gamma}\phi_{u^\varepsilon}(0), \ \partial_t \mathcal{R}\big|_{t=0} = 0, \ \mathcal{R}\big|_{\partial\Omega} = 0, \end{cases} \qquad (4.21)$$

where $l(t) \equiv l_{u^\varepsilon, u^0}(t) := \int_0^1 f'(u^0(t) + \theta(u^\varepsilon(t) - u^0(t)))d\theta$. We note that, due to Theorem 4.3, $\xi_\mathcal{R}(0) \in \mathcal{E}(\varepsilon)$ and $\|\xi_\mathcal{R}(0)\|_{\mathcal{E}(\varepsilon)} = \frac{1}{\gamma}\|\phi_{u^\varepsilon}(0)\|_{H^1} \le Q(\|\xi_{u^\varepsilon}(0)\|_{\mathcal{E}(\varepsilon)})$. As usual, multiplying equation (4.21) by $\partial_t \mathcal{R}$, integrating over Ω and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\varepsilon\|\partial_t\mathcal{R}(t)\|_{L^2}^2 + \|\nabla_x\mathcal{R}(t)\|_{L^2}^2 + 2\left(\nabla_x\tilde{u}^1(\frac{t}{\epsilon}), \nabla_x\mathcal{R}(t)\right)\right) +$$
$$+ \gamma\|\partial_t\mathcal{R}(t)\|_{L^2}^2 = -\left(l(t)(\mathcal{R}(t) + \tilde{u}^1(\frac{t}{\epsilon})), \partial_t\mathcal{R}(t)\right) + \left(\partial_t\nabla_x\tilde{u}^1(\frac{t}{\epsilon}), \nabla_x\mathcal{R}(t)\right) -$$
$$- \left(\partial_t^2 u^0(t), \partial_t\mathcal{R}(t)\right). \quad (4.22)$$

We note that, due to Theorem 4.2 and Corollary 4.1, we have

$$\|l(t)\|_{L^\infty} \le C(1 + \|u^\varepsilon(t)\|_{H^2} + \|u^0(t)\|_{H^2})^2 \le C_1\left(1 + \|\xi_{u^\varepsilon}(0)\|_{\mathcal{E}^1(\varepsilon)}^2\right), \qquad (4.23)$$

where C_1 is independent of ε. Consequently, we have no difficulty to estimate the first term in the right-hand side of (4.22), namely $\left|\left(l(t)(\mathcal{R}(t) + \tilde{u}^1(\frac{t}{\epsilon})), \partial_t\mathcal{R}(t)\right)\right| \le K\|\nabla_x\mathcal{R}(t)\|_{L^2}^2 + \frac{\gamma}{4}\|\partial_t\mathcal{R}(t)\|_{L^2}^2 + K\|\tilde{u}^1(\frac{t}{\epsilon})\|_{H^1}^2$, where K depends on $\|\xi_{u^\varepsilon}(0)\|_{\mathcal{E}^1(\varepsilon)}$, but is independent of ε. Also, the last term in (4.22) arises no difficulty since, according to Corollary 4.2, we have an estimate of the integral of $\partial_t^2 u^0$. We estimate the second term in the right-hand side of (4.22) in the following way:

$$\left|\left(\partial_t\nabla_x\tilde{u}^1(\frac{t}{\epsilon}), \nabla_x\mathcal{R}(t)\right)\right| \le$$
$$\le \|\partial_t\tilde{u}^1(\frac{t}{\epsilon})\|_{H^1}\|\nabla_x\mathcal{R}(t)\|_{L^2} \le \|\partial_t\tilde{u}^1(\frac{t}{\epsilon})\|_{H^1}\left(1 + \|\nabla_x\mathcal{R}(t)\|_{L^2}^2\right).$$

Inserting these estimates into (4.22), we find

$$\frac{1}{2}\frac{d}{dt}\left(\varepsilon\|\partial_t\mathcal{R}(t)\|_{L^2}^2 + \|\mathcal{R}(t)\|_{H^1}^2 - 2\left(\nabla_x\tilde{u}^1(\frac{t}{\epsilon}), \nabla_x\mathcal{R}(t)\right)\right) +$$
$$+ \frac{\gamma}{2}\|\partial_t\mathcal{R}(t)\|_{L^2}^2 \le K\left(1 + \|\partial_t\tilde{u}^1(\frac{t}{\epsilon})\|_{H^1}\right)\|\mathcal{R}(t)\|_{H^1}^2 +$$
$$+ C\left(\|\partial_t\tilde{u}^1(\frac{t}{\epsilon})\|_{H^1} + \|\tilde{u}^1(\frac{t}{\epsilon})\|_{H^1}^2 + \|\partial_t^2 u^0(t)\|_{L^2}^2\right). \quad (4.24)$$

We now note that, in view of the explicit expression (4.19) of the boundary layer term, we have an estimate of the form

$$\|\tilde{u}^1(\tau)\|_{H^1} + \int_0^\infty \|\partial_t \tilde{u}^1(\frac{t}{\varepsilon})\|_{H^1} \, dt \leq Q(\|\xi_{u^\varepsilon}(0)\|_{\mathcal{E}^2(\varepsilon)}), \qquad (4.25)$$

where Q is independent of ε. Thus, applying Gronwall's inequality to (4.24) and using (4.9) and (4.25), we obtain

$$\varepsilon\|\partial_t \mathcal{R}(t)\|_{L^2}^2 + \|\mathcal{R}(t)\|_{H^1}^2 + \int_t^{t+1} \|\partial_t \mathcal{R}(s)\|_{L^2}^2 \, ds \leq C_2 e^{K_1 t}, \qquad (4.26)$$

where the constants C_2 and K_1 depend on $\|\xi_{u^\varepsilon}(0)\|_{\mathcal{E}^2(\varepsilon)}$, but are independent of ε. Therefore, it only remains to estimate $\|\partial_t \mathcal{R}(t)\|_{H^{-1}}$. In order to do so, we multiply equation (4.21) by $(-\Delta_x)^{-1}\partial_t^2 \mathcal{R}$ and integrate by parts. We then have

$$\frac{d}{dt}\Gamma(t) = -\left(\tilde{u}^1(\frac{t}{\epsilon}), \partial_t^2 \mathcal{R}(t)\right) - \left(l(t)\tilde{u}^1(\frac{t}{\epsilon}), (-\Delta_x)^{-1}\partial_t^2 \mathcal{R}(t)\right) + $$
$$+ \left(\partial_t[l(t)\mathcal{R}(t)], (-\Delta_x)^{-1}\partial_t \mathcal{R}(t)\right) + $$
$$+ \|\partial_t \mathcal{R}(t)\|_{L^2}^2 - \varepsilon\|\partial_t^2 R(t)\|_{H^{-1}}^2 + \left((-\Delta_x)^{-1}\partial_t^3 u^0(t), \partial_t \mathcal{R}(t)\right) \equiv H(t), \quad (4.27)$$

where

$$\Gamma(t) := \frac{\gamma}{2}\|\partial_t \mathcal{R}(t)\|_{H^{-1}}^2 + (\mathcal{R}(t), \partial_t \mathcal{R}(t)) + \left(l(t)\mathcal{R}(t), (-\Delta_x)^{-1}\partial_t \mathcal{R}(t)\right) + $$
$$+ \left(\partial_t^2 u^0(t), (-\Delta_x)^{-1}\partial_t \mathcal{R}(t)\right).$$

It is not difficult to verify, using Hölder's inequality, estimate (4.23) and estimate (4.9) for $\|\partial_t^2 u^0(t)\|_{H^{-1}}$, that

$$\frac{\gamma}{4}\|\partial_t \mathcal{R}(t)\|_{H^{-1}}^2 - C\|\mathcal{R}(t)\|_{H^1}^2 - C_3 \leq \Gamma(t) \leq \gamma\|\partial_t \mathcal{R}(t)\|_{H^{-1}}^2 + C\|\mathcal{R}(t)\|_{H^1}^2 + C_3, \quad (4.28)$$

where the constants C and C_3 depend on $\|\xi_{u^\varepsilon}(0)\|_{\mathcal{E}^2(\varepsilon)}$, but are independent of ε. So, it only remains to estimate the function $H(t)$ in the right-hand side of (4.27). To do so, we note that, due to the embeddings $H^1 \subset L^6$ and $H^2 \subset C$, we can easily deduce from estimates (4.6) and (4.8) the inequality $\|\partial_t l(t)\|_{L^6} \leq C_4$, where C_4 depends only on $\|\xi_{u^\varepsilon}(0)\|_{\mathcal{E}^2(\varepsilon)}$ (and is independent of ε). Consequently, due to Hölder's inequality, the function H satisfies

$$|H(t)| \leq$$
$$\leq C\left(\|\mathcal{R}(t)\|_{H^1}^2 + \|\partial_t \mathcal{R}(t)\|_{L^2}^2 + \|(-\Delta_x)^{-1}\partial_t^3 u^0(t)\|_{L^2}^2 + \varepsilon^{-1}\|\tilde{u}^1(\frac{t}{\varepsilon})\|_{H^1}^2\right), \quad (4.29)$$

where C is independent of ε. We now observe that the explicit expression (4.19) of the boundary layer $\tilde{u}^1$ implies that

$$\varepsilon^{-1}\int_0^\infty \|\tilde{u}^1(\frac{t}{\varepsilon})\|_{H^1}^2 \, dt \leq C\|\phi_{u^\varepsilon}(0)\|_{H^1}^2 \leq C', \qquad (4.30)$$

where C' depends on $\|\xi_{u^\varepsilon}(0)\|_{\mathcal{E}^2(\varepsilon)}$, but is independent of ε. Integrating (4.27) over $[0, T]$ and using estimates (4.9), (4.26) and (4.28)-(4.30), we derive the necessary estimate for $\|\partial_t R(t)\|_{H^{-1}}$. This finishes the proof of Theorem 4.7.

Corollary 4.3. *Let the assumptions of Theorem 4.3 hold and let $u^\varepsilon(t)$ and $u^0(t)$ be solutions of (2.1) and (4.7) respectively such that $u^\varepsilon(0) = u^0(0) = u_0$. Then*

$$\|\xi_{u^\varepsilon - u^0}(t)\|_{\mathcal{E}(\varepsilon)} \leq C_1 \varepsilon e^{Kt} + C_2 e^{-\frac{\gamma}{\varepsilon}t} \left(\|\phi_{u^\varepsilon}(0)\|_{H^{-1}} + \varepsilon^{1/2} \|\phi_{u^\varepsilon}(0)\|_{L^2} \right), \quad (4.31)$$

where $\phi_{u^\varepsilon}(t) := \partial_t u^\varepsilon(t) - \frac{1}{\gamma}\left(\Delta_x u^\varepsilon(t) - f(u^\varepsilon(t)) + g\right)$ and where the constants C_i, $i = 1, 2$, and K depend on $\|\xi_{u^\varepsilon}(0)\|_{\mathcal{E}^2(\varepsilon)}$, but are independent of ε.

Indeed, (4.31) is an immediate corollary of Theorem 4.7.

Corollary 4.4. *Let the assumptions of Theorem 4.7 hold and let $u^\varepsilon(t)$ and $u(t)$ be solutions of problems (2.1) and (4.7) respectively, with $u(0) \in H^3(\Omega) \cap H^1_0(\Omega)$ satisfying the compatibility condition (4.5), but such that $u(0) \neq u^\varepsilon(0) = u_0$. Then*

$$\|\xi_{u^\varepsilon - u}(t)\|_{\mathcal{E}(0)} \leq C_1 \varepsilon e^{Kt} + C_2 \|u^\varepsilon(0) - u(0)\|_{H^1} e^{Kt} +$$
$$+ C_3 e^{-\frac{\gamma}{\varepsilon}t} \left(\|\phi_{u^\varepsilon}(0)\|_{H^{-1}} + \varepsilon^{1/2} \|\phi_{u^\varepsilon}(0)\|_{L^2} \right), \quad (4.32)$$

where the constants C_i, $i = 1, 2, 3$, and K depend on $\|\xi_{u^\varepsilon}(0)\|_{\mathcal{E}^2(\varepsilon)}$ and $\|u^0(0)\|_{H^3}$, but are independent of ε.

Proof. Let $u^0(t)$ be the solution of the parabolic equation with $u^0(0) = u^\varepsilon(0) = u_0$. Then, on the one hand, we have estimate (4.31) for the difference between $u^\varepsilon(t)$ and $u^0(t)$ and, on the other hand, estimate (4.15) implies that

$$\|\xi_u(t) - \xi_{u^0}(t)\|_{\mathcal{E}(0)} \leq C e^{Kt} \|u(0) - u^0(0)\|_{H^1}. \quad (4.33)$$

Combining (4.31) and (4.33), we obtain (4.32).

In the sequel, we will also need to control the evolution of the quantity

$$-\phi_{u^\varepsilon}(t) \equiv \gamma \varepsilon \partial_t^2 u^\varepsilon(t). \quad (4.34)$$

Corollary 4.5. *Let the assumptions of Theorem 4.7 hold. Then*

$$\|\phi_{u^\varepsilon}(t)\|_{H^{-1}} + \varepsilon^{\frac{1}{2}} \|\phi_{u^\varepsilon}(t)\|_{L^2} \leq C \left(e^{-\frac{\gamma}{\varepsilon}t} \|\phi_{u^\varepsilon}(0)\|_{H^{-1}} + \varepsilon \right), \quad (4.35)$$

where C depends on $\|\xi_{u^\varepsilon}(0)\|_{\mathcal{E}^2(\varepsilon)}$, but is independent of ε.

Proof. It follows from the asymptotic expansion (4.17) of u^ε that

$$\gamma \phi_{u^\varepsilon}(t) = -\varepsilon \gamma \partial_t \tilde{u}_1(\frac{t}{\varepsilon}) + \varepsilon \left(\Delta_x \tilde{u}_1(\frac{t}{\varepsilon}) + \Delta_x \mathcal{R}(t) - \gamma \partial_t \mathcal{R}(t) \right) +$$
$$+ \varepsilon l(t)(\tilde{u}^1(\frac{t}{\varepsilon}) + \mathcal{R}(t)), \quad (4.36)$$

where $l(t) \equiv l_{u^\varepsilon, u^0}(t) := \int_0^1 f'(u^0(t) + \theta(u^\varepsilon(t) - u^0(t)))d\theta$. We note that, without loss of generality, we may assume that $t \leq 1$. It then follows from (4.19), (4.20), Theorem 4.2 and Corollary 4.1 (see also (4.33)) that $\gamma \|\phi_{u^\varepsilon}(t)\|_{H^{-1}} \leq e^{-\frac{\gamma}{\varepsilon}t} \|\phi_{u^\varepsilon}(0)\|_{H^{-1}} + C\varepsilon$, where the constant C depends on $\|\xi_{u^\varepsilon}(0)\|_{\mathcal{E}^2(\varepsilon)}$. We then recall that, due to Theorem 4.3, we have the estimate $\|\phi_{u^\varepsilon}(t)\|_{H^1} \leq C_1$. Finally, interpolating between the spaces

H^{-1} and H^1, we derive the necessary estimate for the second term in the left-hand side of (4.35).

Remark 4.1. We note that, due to (4.34), estimate (4.35) can be rewritten as follows:

$$\|\partial_t^2 u^\varepsilon(t)\|_{H^{-1}} + \varepsilon^{\frac{1}{2}}\|\partial_t^2 u^\varepsilon(t)\|_{L^2} \le \frac{C_1}{\varepsilon}e^{-\frac{\gamma}{\varepsilon}t} + C_2, \tag{4.37}$$

where the constants C_i, $i = 1, 2$, depend on $\|\xi_{u^\varepsilon}(0)\|_{\mathcal{E}^2(\varepsilon)}$, but are independent of ε.

4.3. Uniform exponential attractors for damped hyperbolic equations. According to Theorem 4.1, the hyperbolic problem (2.1) with $\varepsilon \ne 0$ generates a semigroup $S_t(\varepsilon) : \mathcal{E}(\varepsilon) \to \mathcal{E}(\varepsilon)$ via the formula

$$S_t(\varepsilon)\xi_u(0) = \xi_u(t), \text{ where } u(t) \text{ is the solution of (2.1).} \tag{4.38}$$

(Here, the $\mathcal{E}^\kappa(\varepsilon)$ denote the spaces defined by (2.4)-(2.5).) Moroever, due to Theorems 4.2 and 4.3, this semigroup is well defined in the phase spaces $\mathcal{E}^1(\varepsilon)$ and $\mathcal{E}^2(\varepsilon) \cap \{(4.5)\}$ as well. Analogously, according to Corollary 4.1, the limit parabolic problem (4.7) corresponding to $\varepsilon = 0$ generates a semigroup $S_t : H_0^1(\Omega) \to H_0^1(\Omega)$ via the formula

$$S_t u_0 = u(t), \quad \text{where } u(t) \text{ is the solution of (4.7).} \tag{4.39}$$

Moreover, this semigroup is well defined in the phase spaces $H^2(\Omega) \cap H_0^1(\Omega)$ and $H^3(\Omega) \cap H_0^1(\Omega) \cap \{(4.5)\}$ as well. We note however that this semigroup and the semigroups $S_t(\varepsilon)$ introduced above are defined in different phase spaces. In order to overcome this difficulty, we introduce an infinite dimensional submanifold $\mathcal{N}^i$ of $\mathcal{E}^i(0)$ by the following expression:

$$\mathcal{N}^i := \{[u, v] \in \mathcal{E}^i(0), \ v = \frac{1}{\gamma}(\Delta_x u - f(u) + g) \equiv \mathcal{N}(u)\}, \tag{4.40}$$

$i = 0, 1, 2$, and we define a semigroup $S_t(0) : \mathcal{N}^i \to \mathcal{N}^i$ by the following expression:

$$S_t(0)[u, v] = [S_t u, \mathcal{N}(S_t u)]. \tag{4.41}$$

We easily check that $\mathcal{N} \in C^1(H^i(\Omega), H^{i-2}(\Omega))$, $i = 1, 2, 3$ (see e.g. [BV]). Moreover, the following estimates are satisfied, for $i = 1, 2, 3$:

$$\|\mathcal{N}(u)\|_{H^{i-2}} + \|D_u \mathcal{N}(u)\|_{\mathcal{L}(H^i, H^{i-2})} \le Q(\|u\|_{H^i}), \ \forall u \in H^i(\Omega), \tag{4.42}$$

for an appropriate monotonic function Q. Thus, $\mathcal{N}^i$ is indeed a C^1-submanifold of $\mathcal{E}^i(0)$ and the semigroups S_t and $S_t(0)$ are conjugated by the diffeomorphism $\Phi : u \mapsto [u, \mathcal{N}(u)]$. Therefore, the assertions of Theorem 4.6 remain valid for the semigroup $S_t(0)$:

Corollary 4.6. *Let the assumptions of Theorem 4.6 hold and let ξ_1 and ξ_2 belong to $\mathcal{N}^0$. Then*

$$\begin{cases} \|S_t(0)\xi_1 - S_t(0)\xi_2\|_{\mathcal{E}(0)} \le Ce^{Kt}\|\xi_1 - \xi_2\|_{\mathcal{E}(0)}, \\ \|S_t(0)\xi_1 - S_t(0)\xi_2\|_{\mathcal{E}^1(0)}^2 \le C\frac{1+t}{t}e^{Kt}\|\xi_1 - \xi_2\|_{\mathcal{E}(0)}^2, \ t > 0, \end{cases} \tag{4.43}$$

where the constants C and K depend on $\|\xi_i\|_{\mathcal{E}(0)}$, $i = 1, 2$.

The main result of this section is the following theorem.

Theorem 4.8. *Let assumptions* (2.2) *hold. Then, for every* $\varepsilon \in [0,1]$, *the semigroup* $S_t(\varepsilon)$ *generated by problem* (2.1) *possesses an exponential attractor* $\mathcal{M}_\varepsilon$ *which attracts exponentially the bounded subsets of* $\mathcal{E}^2(\varepsilon)$ *in the topology of* $\mathcal{E}(\varepsilon)$, *i.e. there exists a function* Q *and a constant* $\alpha > 0$ *that are independent of* ε *such that, for every* $B \subset \mathcal{E}^2(\varepsilon)$

$$\text{dist}_{\mathcal{E}(\varepsilon)}\left(S_t(\varepsilon)B, \mathcal{M}_\varepsilon\right) \leq Q(\|B\|_{\mathcal{E}^2(\varepsilon)})e^{-\alpha t}. \tag{4.44}$$

(Of course, for $\varepsilon = 0$, *we take* $B \subset \mathcal{N}^2 \subset \mathcal{E}^2(0)$.*) Moreover, the fractal dimension of* $\mathcal{M}_\varepsilon$ *in* $\mathcal{E}(\varepsilon)$ *is uniformly bounded with respect to* ε *and the symmetric distance between* $\mathcal{M}_\varepsilon$ *and* $\mathcal{M}_0$ *in* $\mathcal{E}(\varepsilon)$ *satisfies*

$$\text{dist}_{sym,\mathcal{E}(\varepsilon)}\left(\mathcal{M}_\varepsilon, \mathcal{M}_0\right) \leq C\varepsilon^\tau, \tag{4.45}$$

where the constants $C > 0$ *and* $0 < \tau < 1$ *are independent of* ε.

Proof. We apply Theorem 3.1 to our problem. To this end, we first note that, owing to Theorem 4.3, the balls

$$B_\varepsilon := \{\xi \in \mathcal{E}^2(\varepsilon), \ \|\xi\|^2_{\mathcal{E}^2(\varepsilon)} \leq 2Q(\|g\|_{H^1})\}, \tag{4.46}$$

where Q is the same as in (4.6), are uniformly (with respect to ε) absorbing sets for the semigroups $S_t(\varepsilon)$ in $\mathcal{E}^2(\varepsilon)$. Consequently, it is sufficient to verify the exponential attraction to $\mathcal{M}_\varepsilon$ *only* for these bounded subsets of $\mathcal{E}^2(\varepsilon)$. We complete the family $B_\varepsilon \subset \mathcal{E}^2(\varepsilon)$ with the set

$$B_0 := \{[u,v] \in \mathcal{N}^2, \ \|u\|^2_{H^3} \leq 2Q(\|g\|_{H^1})\}. \tag{4.47}$$

We also note that, without loss of generality, we may assume that the functions Q in (4.6) and (4.8) are the same and that, consequently, B_0 is an absorbing set for $S_t(0)$ in $\mathcal{N}^2$ as well. Moreover, it follows from estimates (4.6) and (4.8) that there exists T_0 independent of $\varepsilon \in [0,1]$ such that

$$S_t(\varepsilon)B_\varepsilon \subset B_\varepsilon, \ \ \text{if } t \geq T_0. \tag{4.48}$$

Let us now fix $T \geq T_0$ (independent of ε) such that, for every $\varepsilon \in (0,1]$ and for every $\xi_{u_1}(0), \ \xi_{u_2}(0) \in B_\varepsilon$, the function $v_1(t)$ defined by equation (4.11) satisfies the estimate

$$\|\xi_{v_1}(T)\|_{\mathcal{E}(\varepsilon)} \leq \kappa\|\xi_{u_1}(0) - \xi_{u_2}(0)\|_{\mathcal{E}(\varepsilon)}, \tag{4.49}$$

where $\kappa < \frac{1}{2}$ (it is possible to do so thanks to (4.13)).

We then set $S_\varepsilon := S_T(\varepsilon)$ and construct the exponential attractors $\mathcal{M}_\varepsilon^d$ for the discrete semigroups generated by these maps. To this end, we need to verify the assumptions of Theorem 3.1.

Assumption (3.1) is obviously satisfied for the family of energy spaces $\mathcal{E}(\varepsilon)$ and $\mathcal{E}^1(\varepsilon)$ defined by (2.4). Moreover, the L^2-norm (and even the H^1-norm) of $\mathcal{N}(u)$ is uniformly bounded on B_0 and, consequently

$$\|b_0\|^2_{\mathcal{E}(\varepsilon)} \leq \|b_0\|^2_{\mathcal{E}(0)} + C_1\varepsilon, \ \ \forall b_0 \in B_0. \tag{4.50}$$

Thus, (3.2) is also valid (with ε replaced by $\varepsilon^{\frac{1}{2}}$).

Let us now verify (3.3). To this end, we define the operators $\mathcal{C}_\varepsilon : B_\varepsilon \to \mathcal{E}(\varepsilon)$, $\varepsilon > 0$, as follows: $\mathcal{C}_\varepsilon \xi_u(0) := \xi_{v_1}(T)$, where $v_1(t)$ is the solution of the following linear problem:

$$\varepsilon \partial_t^2 v_1 + \gamma \partial_t v_1 - \Delta_x v_1 = 0, \quad \xi_{v_1}(0) = \xi_u(0). \tag{4.51}$$

Thus, $\mathcal{K}_\varepsilon := S_\varepsilon - \mathcal{C}_\varepsilon$. For $\varepsilon = 0$, we set $\mathcal{C}_0 = 0$. Then, estimates (3.3) hold thanks to Theorem 4.5 and Corollary 4.6 and thanks to our choice of T.

Let us finally verify (3.4). To this end, we set

$$\Pi_\varepsilon := \Pi, \quad \Pi[u, v] := [u, \mathcal{N}(u)]. \tag{4.52}$$

Obviously, $\Pi_\varepsilon B_\varepsilon = B_0$ and estimate (3.4) is an immediate corollary of (4.31) (the boundary layer term disappears because we start from time $t = T > 0$).

Thus, all the assumptions of Theorem 3.1 are satisfied and, consequently, there exists a family of discrete exponential attractors $\mathcal{M}_\varepsilon^d$ which satisfy conditions (3.5)–(3.7). As usual (see e.g. [EFNT]), we finally construct the continuous attractors as follows:

$$\mathcal{M}_\varepsilon := \cup_{t \in [0,T]} S_t(\varepsilon) \mathcal{M}_\varepsilon^d. \tag{4.53}$$

We note that, by construction, $\mathcal{M}_\varepsilon \subset S_T(\varepsilon) B_\varepsilon$ (see Remark 3.1). Consequently (thanks to (4.35)), the boundary layer term in (4.32) will be of order ε. Therefore, (4.32) implies that

$$\text{dist}_{sym,\mathcal{E}(0)} (\mathcal{M}_\varepsilon, \mathcal{M}_0) \leq C_1 e^{KT} \left(\varepsilon + \text{dist}_{sym,\mathcal{E}(\varepsilon)} \left(\mathcal{M}_\varepsilon^d, \mathcal{M}_0^d \right) \right) + C_2 \varepsilon, \tag{4.54}$$

and, thanks to (3.7), (4.50) and (4.54), we find estimate (4.45) (where, compared with (3.7), the exponent τ is replaced by $\min\{\frac{1}{2}, \tau\}$). The exponential attraction (4.44) is an immediate corollary of (3.5) and of the uniform Lipschitz continuity (4.10) and (4.43). Thus, there only remains to verify the boundedness of the fractal dimensions. As usual, this property is an immediate corollary of the uniform (with respect to ε) time Lipschitz continuity of the solutions of (2.1), which is established in Lemma 4.1 below. Moreover, thanks to Remark 3.1 and (4.53), this Lipschitz continuity is indeed necessary for the trajectories $\xi_u(t)$ with $\xi_u(0) \in B_\varepsilon$ for $t \geq T$ *only*.

Lemma 4.1. *Let the assumptions of Theorem 4.3 hold and let $u(t)$ be the solution of (2.1). Then, for every $0 < T \leq t$ and $0 < s \leq 1$, $\frac{\|\xi_u(t+s) - \xi_u(t)\|_{\mathcal{E}(\varepsilon)}}{s} \leq Q_T(\|\xi_u(0)\|_{\mathcal{E}^2(\varepsilon)})$, where the function Q_T depends on T, but is independent of ε.*

The proof of this lemma follows immediately from (4.6) and (4.37).

5. Transitivity of exponential attraction and exponential attraction in $\mathcal{E}(\varepsilon)$. In this section, we formulate and prove an abstract result on the transitivity of exponential attraction and then apply this theorem to our problem to deduce that the attractors $\mathcal{M}_\varepsilon$ attract uniformly (with respect to ε) not only the bounded subsets of $\mathcal{E}^2(\varepsilon)$, but also the bounded subsets of $\mathcal{E}(\varepsilon)$.

Theorem 5.1. *Let $(\mathcal{M}, d)$ be a metric space and let S_t be a semigroup acting on this space such that*

$$d(S_t m_1, S_t m_2) \leq C e^{Kt} d(m_1, m_2), \tag{5.1}$$

for appropriate constants C and K. We further assume that there exist three subsets M_1, M_2, $M_3 \subset \mathcal{M}$ such that

$$\text{dist}_{\mathcal{M}}\left(S_t M_1, M_2\right) \le C_1 e^{-\alpha_1 t}, \quad \text{dist}_{\mathcal{M}}\left(S_t M_2, M_3\right) \le C_2 e^{-\alpha_2 t}. \tag{5.2}$$

Then

$$\text{dist}_{\mathcal{M}}\left(S_t M_1, M_3\right) \le C' e^{-\alpha' t}, \tag{5.3}$$

where $C' = CC_1 + C_2$ and $\alpha' = \frac{\alpha_1 \alpha_2}{K + \alpha_1 + \alpha_2}$.

Proof. Let m_1 belong to M_1 and let us set $t = t_1 + t_2$, where $t_i \ge 0$, $i = 1, 2$, will be fixed below. Owing to the first estimate of (5.2), there exists $m_2 \in M_2$ such that

$$d(S_{t_1} m_1, m_2) \le C_1 e^{-\alpha_1 t_1}. \tag{5.4}$$

Then, estimate (5.1) implies that

$$d(S_t m_1, S_{t_2} m_2) \le CC_1 e^{Kt_2 - \alpha_1 t_1}. \tag{5.5}$$

On the other hand, using the second estimate of (5.2), we deduce that there exists $m_3 \in M_3$ such that

$$d(S_{t_2} m_2, m_3) \le C_2 e^{-\alpha_2 t_2}. \tag{5.6}$$

Combining (5.4)–(5.6) and noting that $m_1 \in M_1$ and $t_1 \in [0, t]$ is arbitrary, we obtain

$$\text{dist}_{\mathcal{M}}\left(S_t M_1, M_3\right) \le \inf_{t_1 + t_2 = t}\left(CC_1 e^{Kt_2 - \alpha_1 t_1} + C_2 e^{-\alpha_2 t_2}\right). \tag{5.7}$$

Fixing the values t_i in an optimal way (i.e. such that $Kt_1 - \alpha_1 t_2 = \alpha_2 t_2$), we obtain (5.3). $\qquad\blacksquare$

We are now in a position to prove the

Proposition 5.1. *Let the assumptions of Theorem 4.8 hold. Then, there exist a function Q and constants $R > 0$ and $\alpha > 0$ that are independent of $\varepsilon \in [0, 1]$ such that, for every bounded subset $B \subset \mathcal{E}(\varepsilon)$*

$$\text{dist}_{\mathcal{E}(\varepsilon)}\left(S_t(\varepsilon)B, B_{\mathcal{E}^2(\varepsilon)}(0, R)\right) \le Q(\|B\|_{\mathcal{E}(\varepsilon)})e^{-\alpha t}. \tag{5.8}$$

Corollary 5.1. *Let the assumptions of Theorem 4.8 hold. Then, there exist a function Q and a positive constant $\alpha > 0$ such that, for every $\varepsilon \in [0, 1]$ and for every bounded subset $B \subset \mathcal{E}(\varepsilon)$ (for $\varepsilon = 0$, obviously, $B \subset \mathcal{N}^0 \subset \mathcal{E}(0)$)*

$$\text{dist}_{\mathcal{E}(\varepsilon)}\left(S_t(\varepsilon)B, \mathcal{M}_\varepsilon\right) \le Q(\|B\|_{\mathcal{E}(\varepsilon)})e^{-\alpha t}, \tag{5.9}$$

where the attractors $\mathcal{M}_\varepsilon$ have been defined in Theorem 4.8.

Proof of Proposition 5.1. The case $\varepsilon = 0$ is straightforward, due to the smoothing property for parabolic equations. We will thus consider below the case $\varepsilon \ne 0$ only.

We will prove estimate (5.8) in several steps, using the spaces $\mathcal{E}^\kappa(\varepsilon)$, with fractional order $0 < \kappa < 1$. The most complicated step is to prove the analogue of (5.8) for the space $\mathcal{E}^\kappa(\varepsilon)$, with $0 < \kappa < \frac{1}{2}$, instead of $\mathcal{E}^2(\varepsilon)$. So, we will concentrate on this point only.

Step 1. We decompose (following [BV]) the nonlinearity f into a sum $f = f_1 + f_2$ such that

$$f_1'(v) \geq 0, \quad f_1(0) = f_1'(0) = 0, \quad |f_2(v)| + |f_2'(v)| + |f_2''(v)| \leq C, \qquad (5.10)$$

where C is independent of v (such a decomposition exists, thanks to assumptions (2.2)). We then decompose a solution $u(t)$ of (2.1) into the sum $u(t) = v(t) + w(t)$, where $v(t)$ is the solution of

$$\varepsilon \partial_t^2 v + \gamma \partial_t v - \Delta_x v + f_1(v) = 0, \quad \xi_v\big|_{t=0} = \xi_u\big|_{t=0}, \quad v\big|_{\partial\Omega} = 0. \qquad (5.11)$$

Consequently, w satisfies the equation

$$\varepsilon \partial_t^2 w + \gamma \partial_t w - \Delta_x w + [f_1(v+w) - f_1(v)] = g - f_2(u), \quad \xi_w\big|_{t=0} = 0, \quad w\big|_{\partial\Omega} = 0. \quad (5.12)$$

Arguing as in the proof of Theorem 4.1, we can prove that the function $v(t)$ satisfies the estimate

$$\int_T^{T+1} \|\partial_t v(t)\|_{L^2}^2 \, dt + \|\xi_v(T)\|_{\mathcal{E}(\varepsilon)}^2 \leq Q\left(\|\xi_u(0)\|_{\mathcal{E}(\varepsilon)}\right) e^{-\alpha T}, \qquad (5.13)$$

where Q and $\alpha > 0$ are independent of ε.

We now estimate the solution w of (5.12). To this end, we multiply this equation by $(-\Delta_x)^\kappa \theta(t)$, $\theta(t) := \partial_t w(t) + \alpha w(t)$, where α is a fixed small strictly positive number and $0 < \kappa < \frac{1}{2}$, integrate over Ω and integrate by parts. We obtain, after standard transformations

$$\frac{1}{2}\frac{d}{dt}[\varepsilon\|\theta(t)\|_{\kappa,2}^2 + \|w(t)\|_{1+\kappa,2}^2 + (\Phi_v(t)w(t), (-\Delta_x)^\kappa w(t))] +$$
$$+ (\gamma - \alpha\varepsilon)\|\theta(t)\|_{\kappa,2}^2 + \alpha\|w(t)\|_{1+\kappa,2}^2 - \alpha(\gamma - \alpha\varepsilon)(w(t), (-\Delta_x)^\kappa\theta(t)) =$$
$$= (g - f_2(u(t)), \partial_t(-\Delta_x)^\kappa\theta(t)) - \alpha\left(\Phi_v(t)w(t), (-\Delta_x)^\kappa w(t)\right) +$$
$$+ (\partial_t\Phi_v(t)w(t), (-\Delta_x)^\kappa w(t)) + (\Phi_v(t)\partial_t w(t), (-\Delta_x)^\kappa w(t)), \quad (5.14)$$

where $\Phi_v := \int_0^1 f_1'(v+sw)\,ds$. We only estimate the last two terms in the right-hand side of (5.14) (the other terms are easier to estimate). Thanks to Hölder's inequality and the Sobolev embedding theorems, we have, for $\kappa < \frac{1}{2}$

$$|(\partial_t\Phi_v(t)w(t), (-\Delta_x)^\kappa w(t))| \leq C\|\partial_t\Phi_v(t)\|_{L^{3/2}}\|w(t)\|_{\kappa,2}\|w(t)\|_{1+\kappa,2} \leq$$
$$\leq \mu\|w(t)\|_{\kappa,2}^2 + C_\mu\|\partial_t\Phi_v(t)\|_{L^{3/2}}^2\|w(t)\|_{1+\kappa,2}^2, \quad (5.15)$$

for an arbitrary $\mu > 0$. We note that, owing to the dissipative integrals for u and v (see (4.3) and (5.13)), we have

$$\int_0^\infty \|\partial_t\Phi_v(t)\|_{L^{3/2}}^2 \, dt \leq$$
$$\leq Q(\|\xi_u(0)\|_{\mathcal{E}(\varepsilon)}) \int_0^\infty (\|\partial_t u(t)\|_{L^2}^2 + \|\partial_t v(t)\|_{L^2}^2) \, dt \leq Q'(\|\xi_u(0)\|_{\mathcal{E}(\varepsilon)}) < \infty. \quad (5.16)$$

We now estimate the last term in the right-hand side of (5.14), splitting it into two terms again:

$$(\Phi_v(t)\partial_t w(t), (-\Delta_x)^\kappa w(t)) = ((\Phi_v(t) - f_1'(v(t)))\partial_t w(t), (-\Delta_x)^\kappa w(t)) +$$
$$+ (f_1'(v(t))\partial_t w(t), (-\Delta_x)^\kappa w(t)) . \quad (5.17)$$

The second term in the right-hand side of (5.17) can be estimated using Hölder's inequality and appropriate embedding theorems:

$$|(f_1'(v(t))\partial_t w(t), (-\Delta_x)^\kappa w(t))| \le C\|f_1'(v(t))\|_{L^3}\|\partial_t w(t)\|_{\kappa,2}\|w(t)\|_{1+\kappa,2} \le$$
$$\le \mu\|\partial_t w(t)\|_{\kappa,2}^2 + C_\mu\|f_1'(v(t))\|_{L^3}^2\|w(t)\|_{1+\kappa,2}^2. \quad (5.18)$$

(Here, we have also implicitly used the assumption $\kappa < \frac{1}{2}$.) We note that, thanks to (5.10), we have $|f_1'(v)| \le C|v|(1+|v|)$ and, consequently, due to (5.13), we have the analogue of a "dissipative" integral:

$$\int_0^\infty \|f_1'(v(t))\|_{L^3}^2 \, dt \le Q(\|\xi_u(0)\|_{\mathcal{E}(\varepsilon)}) < \infty. \quad (5.19)$$

So, there only remains to estimate the first term in the right-hand side of (5.17). To this end, we note that, according to (2.2), we have

$$|\Phi_v(t) - f_1'(v(t))| \le C|w(t)|(1 + |u(t)| + |v(t)|), \quad (5.20)$$

and, consequently, thanks to Hölder's inequality and the Sobolev embedding theorems, it follows that

$$|((\Phi_v(t) - f_1'(v(t)))\partial_t w(t), (-\Delta_x)^\kappa w(t))| \le$$
$$\le C(1 + \|u(t)\|_{H^1} + \|v(t)\|_{H^1})\|\partial_t w(t)\|_{L^2}\|w(t)\|_{L^{1+\kappa}}^2 \le$$
$$\le \mu\|w(t)\|_{L^{1+\kappa}}^2 + C_\mu\|\partial_t w(t)\|_{L^2}^2\|w(t)\|_{L^{1+\kappa}}^2. \quad (5.21)$$

Inserting estimates (5.15)–(5.21) into (5.14) and arguing as in the proof of Theorem 4.2, we obtain the estimate

$$\varepsilon\|\partial_t w(t)\|_{\kappa,2}^2 + \|w(t)\|_{1+\kappa,2}^2 + \int_t^{t+1} \|\partial_t w(s)\|_{\kappa,2}^2 \, ds \le$$
$$\le Q(\|\xi_u(0)\|_{\mathcal{E}(\varepsilon)})e^{-\alpha t} + Q(\|g\|_{H^1}), \quad (5.22)$$

where Q and $\alpha > 0$ are independent of ε.

Multiplying equation (5.12) by $(-\Delta_x)^{\kappa-1}\partial_t^2 w$, arguing as in the proof of Theorem 4.1 (Step 2) and using (5.22), we finally find

$$\|\xi_w(t)\|_{\mathcal{E}^\kappa(\varepsilon)} \le Q(\|\xi_u(0)\|_{\mathcal{E}(\varepsilon)})e^{-\alpha t} + Q(\|g\|_{H^1}) \quad (5.23)$$

(the derivation of (5.23) is standard and simpler than that of (5.22) because we can use the additional regularity of $w(t)$ obtained above (in (5.22)); so, we leave the details to the reader). Estimates (5.13) and (5.23) imply that

$$\mathrm{dist}_{\mathcal{E}(\varepsilon)}\left(S_t(\varepsilon)B, B_{\mathcal{E}^\kappa(\varepsilon)}(0, R)\right) \le Q(\|B\|_{\mathcal{E}(\varepsilon)})e^{-\alpha t}, \quad (5.24)$$

for every bounded subset $B \subset \mathcal{E}(\varepsilon)$ and for $R = 2Q(\|g\|_{H^1})$.

Step 2. Starting now from $\xi_u(0) \in \mathcal{E}^\kappa(\varepsilon)$, with κ close to $\frac{1}{2}$, and arguing as in the first step (but the proof is simpler since the solutions are more regular), we have

$$\operatorname{dist}_{\mathcal{E}(\varepsilon)} \left(S_t(\varepsilon)B, B_{\mathcal{E}^1(\varepsilon)}(0, R') \right) \le Q(\|B\|_{\mathcal{E}^\kappa(\varepsilon)})e^{-\alpha t}, \qquad (5.25)$$

for every bounded subset $B \subset \mathcal{E}^\kappa(\varepsilon)$.

Step 3. Starting from $\xi_u(0) \in \mathcal{E}^1(\varepsilon)$, we find similarly

$$\operatorname{dist}_{\mathcal{E}(\varepsilon)} \left(S_t(\varepsilon)B, B_{\mathcal{E}^2(\varepsilon)}(0, R'') \right) \le Q(\|B\|_{\mathcal{E}^1(\varepsilon)})e^{-\alpha t}, \qquad (5.26)$$

for every bounded subset $B \subset \mathcal{E}^1(\varepsilon)$. Finally, using (5.24)–(5.26) and the transitivity of exponential attraction (Theorem 5.1), we obtain (5.8). This finishes the proof of Proposition 5.1.

6. Proofs of the main estimates.

Proof of Theorem 4.1. The proof of existence and uniqueness of solutions can be found, for instance, in [BV]. The existence of a Lyapunov function $\mathcal{U}$ and relation (4.3) are also obtained in [BV]. So, it remains to verify the uniform estimate (4.1). We will give below a formal derivation of this estimate only; it can be justified by considering Galerkin approximations and consists of two steps.

Step 1. Multiplying equation (2.1) by $\partial_t u(t) + \beta u(t)$, $\beta > 0$ small enough, integrating over Ω and arguing in a standard way, we find

$$\int_T^{T+1} \|\partial_t u(t)\|_{L^2}^2 \, dt + \varepsilon \|\partial_t u(T)\|_{L^2}^2 + \|u(T)\|_{H^1}^2 \le$$
$$\le Q \left(\varepsilon \|\partial_t u(0)\|_{L^2}^2 + \|u(0)\|_{H^1}^2 \right) e^{-\alpha T} + Q(\|g\|_{L^2}), \quad (6.1)$$

where Q and $\alpha > 0$ are independent of ε (see e.g. [BV] for details).

Step 2. Thus, it only remains to estimate $\|\partial_t u(t)\|_{H^{-1}}$. In order to do so, we multiply equation (2.1) by $(-\Delta_x)^{-1}\partial_t^2 u(t)$ and obtain, integrating by parts

$$\frac{d}{dt}\Big[\frac{\gamma}{2}\|\partial_t u(t)\|_{H^{-1}}^2 + (u(t), \partial_t u(t)) + (f(u(t)) - g, (-\Delta_x)^{-1}\partial_t u(t))\Big] +$$
$$+ \varepsilon \|\partial_t^2 u(t)\|_{H^{-1}}^2 + \Big[\frac{\gamma}{2}\|\partial_t u(t)\|_{H^{-1}}^2 + (u(t), \partial_t u(t)) +$$
$$+ (f(u(t)) - g, (-\Delta_x)^{-1}\partial_t u(t))\Big] \le h(t), \quad (6.2)$$

where

$$h(t) := \Big[\frac{\gamma}{2}\|\partial_t u(t)\|_{H^{-1}}^2 + (u(t), \partial_t u(t)) + (f(u(t)) - g, (-\Delta_x)^{-1}\partial_t u(t))\Big] +$$
$$+ \|\partial_t u(t)\|_{L^2}^2 + (f'(u(t))\partial_t u(t), (-\Delta_x)^{-1}\partial_t u(t)). \quad (6.3)$$

It follows from the growth restriction $(2.2)_4$ on f, from (4.2), from the Sobolev embedding $H^1 \subset L^6$ and from estimate (6.1) that

$$\int_T^{T+1} |h(t)| \, dt \le Q(\varepsilon \|\partial_t u(0)\|_{L^2}^2 + \|u(0)\|_{H^1}^2)e^{-\alpha T} + Q(\|g\|_{L^2}). \qquad (6.4)$$

Indeed, the only difficulty to derive (6.4) is to estimate the last term in (6.3). Applying Hölder's inequality, together with the embedding $H^1 \subset L^6$, to this term, we have

$$|(f'(u(t))\partial_t u(t), (-\Delta_x)^{-1}\partial_t u(t))| \leq$$
$$\leq \|f'(u(t))\|_{L^3}\|\partial_t u(t)\|_{L^2}\|(-\Delta_x)^{-1}\partial_t u(t)\|_{L^6} \leq$$
$$\leq C(1 + \|u(t)\|_{H^1}^2)\|\partial_t u(t)\|_{L^2}^2. \quad (6.5)$$

Integrating (6.5) over $[T, T+1]$ and estimating the right-hand side of the inequality that we obtain by using (6.1), we find the right-hand side of (6.4). The other terms in (6.3) can be estimated analogously.

Applying now Gronwall's inequality to (6.2) and using estimate (6.4), we obtain, after simple transformations

$$Y(T) \leq CY(0)e^{-T} + Q_1\left(\varepsilon\|\partial_t u(0)\|_{L^2}^2 + \|u(0)\|_{H^1}^2\right)e^{-\alpha T} + Q_1(\|g\|_{L^2}), \quad (6.6)$$

where $Y(t) := \frac{\gamma}{2}\|\partial_t u(t)\|_{H^{-1}}^2 + (u(t), \partial_t u(t)) + (f(u(t)) - g, (-\Delta_x)^{-1}\partial_t u(t))$ and where $C > 0$ and the monotonic function Q_1 are independent of $\varepsilon \in (0, 1)$. It follows from the growth restriction $(2.2)_4$ and from the Sobolev embedding theorems that, analogously to (6.5)

$$C_1\|\partial_t u(t)\|_{H^{-1}}^2 - Q_2(\|u(t)\|_{H^1}) \leq Y(t) \leq C_2\|\partial_t u(t)\|_{H^{-1}}^2 + Q_2(\|u(t)\|_{H^1}), \quad (6.7)$$

for positive constants C_i and a monotonic function Q_2 that are independent of ε. Combining estimates (6.1), (6.6) and (6.7), we can find the necessary estimates for $\|\partial_t u(t)\|_{H^{-1}}$. The estimates for $\varepsilon\|\partial_t^2 u(t)\|_{H^{-1}}^2$ can be easily derived by then integrating (6.2) over $[T, T+1]$. This finishes the proof of Theorem 4.1.

Proof of Theorem 4.2. We again only give a formal derivation of (4.4). We introduce the solution $G = G(x)$ of the following problem:

$$\Delta_x G = g, \quad G\big|_{\partial\Omega} = 0. \quad (6.8)$$

We set $w(t) := u(t) - G$. This function obviously satisfies the equation

$$\varepsilon\partial_t^2 w + \gamma\partial_t w - \Delta_x w + f(w+G) = 0, \quad w\big|_{t=0} = u_0 - G, \quad \partial_t w\big|_{t=0} = u_0', \quad w\big|_{\partial\Omega} = 0. \quad (6.9)$$

We multiply equation (6.9) by $-\Delta_x(\partial_t w(t) + \beta w(t))$ and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}[\varepsilon\|\partial_t\nabla_x w(t)\|_{L^2}^2 + \|\Delta_x w(t)\|_{L^2}^2 - 2\beta\varepsilon(\partial_t w(t), \Delta_x w(t))] +$$
$$+ (\gamma - \beta\varepsilon)\|\partial_t\nabla_x w(t)\|_{L^2}^2 + \beta\|\Delta_x w(t)\|_{L^2}^2 - \gamma\beta(\partial_t\nabla_x w(t), \nabla_x w(t)) =$$
$$= \beta(f(u(t)), \Delta_x w(t)) + (f(w(t) + G), \Delta_x\partial_t w(t)). \quad (6.10)$$

We transform the last term of (6.10) (the other terms are easier to treat) as follows:

$$(f(w+G), \Delta_x\partial_t w) = \partial_t(f(w+G), \Delta_x w) - (f'(w+G)\partial_t w, \Delta_x w), \quad (6.11)$$

and we estimate the last term of this equality via Hölder's inequality:

$$|(f'(w+G)\partial_t w, \Delta_x w)| \leq \|f'(w+G)\|_{L^6}\|\partial_t u\|_{L^3}\|\Delta_x w\|_{L^2}. \quad (6.12)$$

We recall that, due to the growth restriction $(2.2)_4$, we have an estimate of the form $\|f'(w+G)\|_{L^6} \leq C\left(1 + \|g\|_{L^2}^2 + \|w\|_{L^{12}}^2\right)$. Inserting these estimates into (6.12), we find, using classical interpolation results (in particular, we have $\|w\|_{L^{12}}^2 \leq C\|w\|_{H^1}^{3/2}\|w\|_{H^2}^{1/2}$ and $\|\partial_t w\|_{L^3} \leq C\|\partial_t w\|_{L^2}^{1/2}\|\partial_t w\|_{H^1}^{1/2}$)

$$|(f'(w+G)\partial_t w, \Delta_x w)| \leq Q(\|\xi_u(t)\|_{\mathcal{E}(\varepsilon)})\|\Delta_x w\|_{L^2}^{3/2}\|\partial_t u\|_{L^2}^{1/2}\|\nabla_x \partial_t w\|_{L^2}^{1/2}.$$

Applying Young's inequality, we finally obtain, for every $\mu > 0$

$$|(f'(w(t)+G)\partial_t w(t), \Delta_x w(t))| \leq \mu\left(\|\partial_t \nabla_x w(t)\|_{L^2}^2 + \|\Delta_x w(t)\|_{L^2}^2\right) +$$
$$+ Q_\mu(\|\xi_u(t)\|_{\mathcal{E}(\varepsilon)})\|\partial_t u(t)\|_{L^2}^2\|\Delta_x w(t)\|_{L^2}^2.$$

Fixing $\mu > 0$ small enough (but independently of ε) and inserting this estimate into the right-hand side of (6.11), we derive the inequality

$$\frac{1}{2}\frac{d}{dt}\left[\varepsilon\|\partial_t \nabla_x w(t)\|_{L^2}^2 + \|\Delta_x w(t)\|_{L^2}^2 - 2\beta\varepsilon(\partial_t w(t), \Delta_x w(t)) - \right.$$
$$\left. - 2(f(u(t)), \Delta_x w(t))\right] + \beta_1\left(\|\partial_t \nabla_x w(t)\|_{L^2}^2 + \|\Delta_x w(t)\|_{L^2}^2\right) \leq$$
$$\leq Q(\|\xi_u(t)\|_{\mathcal{E}(\varepsilon)})\|\partial_t u(t)\|_{L^2}^2\|\Delta_x w(t)\|_{L^2}^2 + C\|f(u(t))\|_{L^2}^2, \quad (6.13)$$

for a sufficiently small $\beta_1 > 0$ that is independent of ε. We set

$$Y(t) = \varepsilon\|\partial_t \nabla_x w(t)\|_{L^2}^2 + \|\Delta_x w(t)\|_{L^2}^2 - 2\beta\varepsilon(\partial_t w(t), \Delta_x w(t)) - 2(f(u(t)), \Delta_x w(t)).$$

Then, due to Hölder's inequality, due to the growth restriction $(2.2)_4$ and due to classical embedding theorems, we have (for a sufficiently small $\beta > 0$)

$$C_1\left(\varepsilon\|\partial_t w(t)\|_{H^1}^2 + \|w(t)\|_{H^2}^2\right) - Q_1(\|u(t)\|_{H^1}) \leq Y(t) \leq$$
$$\leq C_2\left(\varepsilon\|\partial_t w(t)\|_{H^1}^2 + \|w(t)\|_{H^2}^2\right) + Q_1(\|u(t)\|_{H^1}), \quad (6.14)$$

for appropriate positive constants C_i and monotonic function Q_1 that are independent of ε.

Inequality (6.13) now implies that

$$\frac{d}{dt}Y(t) + (\beta_1 - Q(\|\xi_u(t)\|_{\mathcal{E}(\varepsilon)})\|\partial_t u(t)\|_{L^2}^2)Y(t) \leq$$
$$\leq \left(\beta_1 + Q(\|\xi_u(t)\|_{\mathcal{E}(\varepsilon)})\|\partial_t u(t)\|_{L^2}^2\right)\left[2\beta\varepsilon|(\partial_t w(t), \Delta_x w(t))| + 2|(f(u(t)), \Delta_x w(t))|\right] +$$
$$+ C\|f(u(t))\|_{L^2}^2. \quad (6.15)$$

Applying Hölder's inequality to the right-hand side of (6.15), using (6.14) and the obvious estimate $\|f(u(t))\|_{L^2}^2 \leq Q(\|\xi_u(0)\|_{\mathcal{E}(\varepsilon)})e^{-\alpha t} + Q(\|g\|_{L^2})$, we find

$$\frac{d}{dt}Y(t) + B(t)Y(t) \leq \left(Q_2(\|\xi_u(0)\|_{\mathcal{E}(\varepsilon)})e^{-\alpha t} + Q_2(\|g\|_{L^2})\right)\left(1 + \|\partial_t u(t)\|_{L^2}^2\right), \quad (6.16)$$

where $B(t) := \frac{1}{2}\beta_1 - 2Q(\|\xi_u(t)\|_{\mathcal{E}(\varepsilon)})\|\partial_t u(t)\|_{L^2}^2$.

We note that it easily follows from (4.3) that $\int_0^T \|\partial_t u(t)\|_{L^2}^2\, dt \leq Q_3(\|\xi_u(0)\|_{\mathcal{E}(\varepsilon)})$, for a function Q_3 that is independent of $T \geq 0$ and ε, and, consequently

$$\int_0^T B(t)\, dt \geq \frac{1}{2}\beta_1 T - Q_4(\|\xi_u(0)\|_{\mathcal{E}(\varepsilon)}), \qquad (6.17)$$

where $\beta_1 > 0$ and Q_4 are independent of T and ε. Applying then Gronwall's inequality to (6.16) and using (6.17), we find, after simple computations

$$\varepsilon\|\partial_t w(T)\|_{H^1}^2 + \|w(T)\|_{H^2}^2 \leq Q\left(\varepsilon\|\partial_t u(0)\|_{H^1}^2 + \|u(0)\|_{H^2}^2\right)e^{-\alpha T} + Q(\|g\|_{L^2}), \quad (6.18)$$

where Q and $\alpha > 0$ are independent of ε.

A similar estimate for $\int_T^{T+1} \|\partial_t u(t)\|_{H^1}^2\, dt$ can be easily derived by now integrating (6.13) over $[T, T+1]$ and by using (6.18). So, there only remains to estimate $\|\partial_t u(t)\|_{L^2}$ and $\varepsilon\int_T^{T+1} \|\partial_t^2 u(t)\|_{L^2}^2\, dt$. To this end, we multiply equation (6.9) by $\partial_t^2 w(t)$ and argue as in Step 2 of the proof of Theorem 4.1. We also note that this reasoning is simpler than that performed in the proof of Theorem 4.1 because we already have an estimate of the L^∞-norm of the solution $u(t)$ (due to (6.18) and due to the embedding $H^2 \subset C$) and, consequently, we do not need to worry about the growth of f. That is the reason why we leave the rigorous proof of these estimates to the reader. This finishes the proof of Theorem 4.2.

Proof of Theorem 4.3. As above, we only give a formal derivation of (4.6). We set, as in the proof of Theorem 4.2, $w(t) := u(t) - G$, where G is the solution of (6.8). Then, due to equation (2.1) and due to the compatibility condition (4.5), the function w satisfies equation (6.9), together with the following boundary conditions:

$$w(t)\big|_{\partial\Omega} = \Delta_x w(t)\big|_{\partial\Omega} = 0. \qquad (6.19)$$

Here, we have implicitly used the condition $f(0) = 0$.

Multiplying equation (6.9) by $\Delta_x^2(\partial_t w(t) + \beta w(t))$, integrating by parts using (6.19), and arguing as in the proof of Theorem 4.2, we obtain

$$\frac{1}{2}\frac{d}{dt}[\varepsilon\|\partial_t \Delta_x w(t)\|_{L^2}^2 + \|\nabla_x \Delta_x w(t)\|_{L^2}^2 - 2\beta\varepsilon(\partial_t \nabla_x w(t), \nabla_x \Delta_x w(t))] +$$
$$+ (\gamma - \beta\varepsilon)\|\partial_t \Delta_x w(t)\|_{L^2}^2 + \beta\|\nabla_x \Delta_x w(t)\|_{L^2}^2 - \gamma\beta(\partial_t \nabla_x w(t), \nabla_x \Delta_x w(t)) =$$
$$= -\beta(f(u(t)), \Delta_x^2 w(t)) + (f'(u(t))\Delta_x u(t), \Delta_x \partial_t w(t)) +$$
$$+ (f''(u(t))|\nabla_x u(t)|^2, \partial_t \Delta_x w(t)). \quad (6.20)$$

We set $Y(t) := \varepsilon\|\partial_t \Delta_x w(t)\|_{L^2}^2 + \|\nabla_x \Delta_x w(t)\|_{L^2}^2 - 2\beta\varepsilon(\partial_t \nabla_x w(t), \nabla_x \Delta_x w(t))$. Then, for a sufficiently small $\beta > 0$, we easily have

$$C_1\left(\varepsilon\|\partial_t \Delta_x w(t)\|_{L^2}^2 + \|\nabla_x \Delta_x w(t)\|_{L^2}^2\right) \leq Y(t) \leq$$
$$\leq C_2\left(\varepsilon\|\partial_t \Delta_x w(t)\|_{L^2}^2 + \|\nabla_x \Delta_x w(t)\|_{L^2}^2\right), \quad (6.21)$$

for positive constants C_i that are independent of ε.

We again estimate the most complicated term in the right-hand side of (6.20) only (the other terms can be estimated analogously). Using Hölder's inequality, we find, for every $\mu > 0$

$$|(f'(u(t))\Delta_x u(t), \Delta_x \partial_t w(t))| \leq C\|f'(u(t))\|_{L^\infty}\|u(t)\|_{H^2}\|\partial_t w(t)\|_{H^2} \leq$$
$$\leq C_\nu\|f'(u(t))\|_{L^\infty}^2\|u(t)\|_{H^2}^2 + \nu\|\partial_t w(t)\|_{H^2}^2. \quad (6.22)$$

Noting that $H^2 \subset C$, we have

$$\|f'(u(t))\|_{L^\infty}^2 \leq Q(\|u(t)\|_{H^2}), \tag{6.23}$$

for an appropriate monotonic function Q depending only on f. Estimating the H^2-norm of $u(t)$ using Theorem 4.2, inserting (6.22) and (6.23) into (6.20) and using (6.21), we find, after standard calculations $\frac{d}{dt}Y(t)+\beta_1 Y(t) \leq Q(\|\xi_u(0)\|_{\mathcal{E}^1(\varepsilon)})e^{-\alpha t}+Q(\|g\|_{L^2})$, for a positive constant β_1 and a function Q that are independent of ε. Applying now Gronwall's inequality and using (6.21) again, we finally obtain

$$\int_T^{T+1} \|\partial_t w(t)\|_{H^2}^2\, dt + \varepsilon\|\partial_t w(T)\|_{H^2}^2 + \|w(T)\|_{H^3}^2 \leq$$
$$\leq Q\left(\varepsilon\|\partial_t u(0)\|_{H^2}^2 + \|u(0)\|_{H^3}^2\right)e^{-\alpha T} + Q(\|g\|_{L^2}), \tag{6.24}$$

for appropriate monotonic function Q and positive constant α that are independent of ε. Noting that $\|u(t)\|_{H^3} \leq \|w(t)\|_{H^3} + \|g\|_{H^1}$, (6.24) then implies an analogous (to (6.24)) estimate for $u(t)$.

Multiplying finally equation (6.9) by $\Delta_x \partial_t^2 w$ and arguing as in Step 2 of the proof of Theorem 4.1, we have the estimates for $\|\partial_t u(t)\|_{H^1}$ and $\int_T^{T+1}\varepsilon\|\partial_t^2 u(t)\|_{H^1}^2 dt$. This finishes the proof of Theorem 4.3.

Proof of Theorem 4.4. We set $v(t) = u_1(t) - u_2(t)$. Then, we have

$$\varepsilon\partial_t^2 v + \gamma\partial_t v - \Delta_x v + l(t)v = 0, \quad \xi_v(0) = \xi_{u_1}(0) - \xi_{u_2}(0), \tag{6.25}$$

where $l(t) := \int_0^1 f'(su_1(t) + (1-s)u_2(t))\, ds$.

We note that, according to the growth restrictions on f', the Sobolev embedding theorems and estimate (4.1)

$$\|l(t)\|_{L^3} \leq C(1 + \|u_1(t)\|_{H^1} + \|u_2(t)\|_{H^1})^2 \leq Q(\|\xi_{u_1}(0)\|_{\mathcal{E}(\varepsilon)} + \|\xi_{u_2}(0)\|_{\mathcal{E}(\varepsilon)}). \tag{6.26}$$

Multiplying now equation (6.25) by $\partial_t v$, integrating over Ω and using the obvious estimate

$$|(l(t)v(t), \partial_t v(t))| \leq C\|l(t)\|_{L^3}(\|v(t)\|_{H^1}^2 + \|\partial_t v(t)\|_{L^2}^2), \tag{6.27}$$

we obtain, after standard transformations (see e.g. [BV])

$$\int_T^{T+1} \|\partial_t v(t)\|_{L^2}^2\, dt + \varepsilon\|\partial_t v(T)\|_{L^2}^2 + \|\partial_t v(T)\|_{H^1}^2 \leq$$
$$\leq Ce^{KT}\left(\varepsilon\|\partial_t v(0)\|_{L^2}^2 + \|v(0)\|_{H^1}^2\right). \tag{6.28}$$

In order to obtain an estimate for $\|\partial_t v(t)\|_{H^{-1}}$, we multiply equation (6.25) by $(-\Delta_x)^{-1}\partial_t^2 v$ and integrate over Ω. We then have

$$\frac{d}{dt}\left[\frac{\gamma}{2}\|\partial_t v(t)\|_{H^{-1}}^2 + (v(t), \partial_t v(t)) + (l(t)v(t), (-\Delta_x)^{-1}\partial_t v(t))\right]+$$
$$+ \varepsilon\|\partial_t^2 v(t)\|_{H^{-1}}^2 \leq \|\partial_t v(t)\|_{L^2}^2 + (l'(t)v(t), (-\Delta_x)^{-1}\partial_t v(t))+$$
$$+ (l(t)\partial_t v(t), (-\Delta_x)^{-1}\partial_t v(t)) \equiv h(t). \tag{6.29}$$

Estimate (6.2), together with Hölder's inequality and the growth restrictions on f, gives

$$\int_T^{T+1} |h(t)|\, dt \le$$
$$\le Q(\|\xi_{u_1}(0)\|_{\mathcal{E}(\varepsilon)} + \|\xi_{u_2}(0)\|_{\mathcal{E}(\varepsilon)})e^{KT}\left(\varepsilon\|\partial_t v(0)\|_{L^2}^2 + \|v(0)\|_{H^1}^2\right). \quad (6.30)$$

To do so, we only estimate the second term in $h(t)$ (the other terms can be estimated analogously). It follows from the growth restrictions on f'' that

$$\|l'(t)\|_{L^{3/2}} \le C(1 + \|u_1(t)\|_{H^1} + \|u_2(t)\|_{H^1})(\|\partial_t u_1(t)\|_{L^2} + \|\partial_t u_2(t)\|_{L^2}). \quad (6.31)$$

Thus, due to (4.1) and (6.28) and due to Hölder's inequality

$$|(l'(t)v(t), (-\Delta_x)^{-1}\partial_t v(t))| \le$$
$$\le C\|l'(t)\|_{L^{3/2}}\|v(t)\|_{H^1}\|\partial_t v(t)\|_{L^2} \le C'\|l'(t)\|_{L^{3/2}}^2\|v(t)\|_{H^1}^2 + \|\partial_t v(t)\|_{L^2}^2 \le$$
$$\le Q(\|\xi_{u_1}(0)\|_{\mathcal{E}(\varepsilon)} + \|\xi_{u_2}(0)\|_{\mathcal{E}(\varepsilon)})(\varepsilon\|\partial_t v(0)\|_{L^2}^2 + \|v(0)\|_{H^1}^2)\times$$
$$\times e^{Kt}(\|\partial_t u_1(t)\|_{L^2}^2 + \|\partial_t u_2(t)\|_{L^2}^2) + \|\partial_t v(t)\|_{L^2}^2. \quad (6.32)$$

Integrating (6.32) over $[T, T+1]$ and using estimates (6.7) and (6.28), we obtain (6.30) (for this part of h). Integrating finally (6.29) and using (6.30), we find, after standard calculations, the necessary estimate for $\|\partial_t v(t)\|_{H^{-1}}$. This finishes the proof of Theorem 4.4.

Proof of Theorem 4.5. We will only prove estimate (4.14) (estimate (4.13) is straightforward, v_1 being the solution of a linear equation). Multiplying equation (4.12) by $-\Delta_x \partial_t v_2(t)$, we have

$$\frac{1}{2}\frac{d}{dt}\left(\varepsilon\|\nabla_x \partial_t v_2(t)\|_{L^2}^2 + \|\Delta_x v_2(t)\|_{L^2}^2\right) + \gamma\|\nabla_x \partial_t v_2(t)\|_{L^2}^2 =$$
$$= (l(t)v(t), \Delta_x \partial_t v_2(t)). \quad (6.33)$$

We recall that $H^2 \subset C$; therefore, it follows from Theorem 4.2 that

$$\|l(t)\|_{L^\infty} + \|\nabla_x l(t)\|_{L^6} \le Q\left(\|\xi_{u_1}(0)\|_{\mathcal{E}^1(\varepsilon)} + \|\xi_{u_2}(0)\|_{\mathcal{E}^1(\varepsilon)}\right). \quad (6.34)$$

Thus, the right-hand side of (6.33) can be estimated as follows:

$$|(l(t)v(t), \Delta_x \partial_t v_2(t))| \le |(\nabla_x l(t)v(t), \partial_t \nabla_x v_2(t))| + |(l(t)\nabla_x v(t), \partial_t \nabla_x v_2(t))| \le$$
$$\le \frac{\gamma}{2}\|\nabla_x \partial_t v_2(t)\|_{L^2}^2 + C'\|v(t)\|_{H^1}^2, \quad (6.35)$$

where the constant C' depends on $\|\xi_{u_i}(0)\|_{\mathcal{E}^1(\varepsilon)}$, but is independent of ε. Inserting this estimate into (6.33) and using (4.10), we have

$$\int_T^{T+1} \|\partial_t v_2(t)\|_{H^1}^2\, dt + \varepsilon\|\partial_t v_2(T)\|_{H^1}^2 + \|v_2(T)\|_{H^2}^2 \le Ce^{KT}\|\xi_v(0)\|_{\mathcal{E}(\varepsilon)}^2. \quad (6.36)$$

So, it only remains to estimate $\|\partial_t v_2(t)\|_{L^2}^2$. To this end, we multiply (as above) equation (4.12) by $\partial_t^2 v_2(t)$ and integrate over Ω. We obtain

$$\frac{d}{dt}\left(\frac{\gamma}{2}\|\partial_t v_2(t)\|_{L^2}^2 - (\Delta_x v_2(t), \partial_t v_2(t)) + (l(t)v(t), \partial_t v_2(t))\right) + \varepsilon\|\partial_t^2 v_2(t)\|_{L^2}^2 =$$
$$= \|\nabla_x \partial_t v_2(t)\|_{L^2}^2 + (\partial_t l(t)v(t), \partial_t v_2(t)) + (l(t)\partial_t v(t), \partial_t v_2(t)) \equiv h(t). \quad (6.37)$$

We note that, using the growth restriction $(2.2)_4$, estimate (4.4) and the embedding $H^2 \subset C$, we have, arguing as in (6.31), the estimate

$$\|\partial_t l(t)\|_{L^{3/2}} \leq Q\left(\|\xi_{u_1}(0)\|_{\mathcal{E}^1(\varepsilon)} + \|\xi_{u_2}(0)\|_{\mathcal{E}^1(\varepsilon)}\right), \quad (6.38)$$

and, consequently, due to (6.34) and (6.38)

$$|h(t)| \leq C_1(\|v(t)\|_{H^1}^2 + \|\partial_t v(t)\|_{L^2}^2 + \|\partial_t v_2(t)\|_{H^1}^2). \quad (6.39)$$

Inserting (6.39) into (6.37), integrating the inequality that we obtain over $[0, T]$ and using (6.28) and (6.36), we find the necessary estimate for $\|\partial_t v_2(t)\|_{L^2}$. This finishes the proof of Theorem 4.5.

REFERENCES

[BV] A. V. Babin and M. I. Vishik, "Attractors of evolutionary equations", Nauka, Moscow, 1989; English translation: North-Holland, Amsterdam, 1992.

[BN] A. Babin and B. Nicolaenko, *Exponential attractors of reaction-diffusion systems in an unbounded domain*, J. Dyn. Diff. Eqns., 7 (1995), 567-590.

[CV] V. V. Chepyzhov and M. I. Vishik, "Attractors for equations of mathematical physics", AMS, Providence, RI, 2002.

[EFNT] A. Eden, C. Foias, B. Nicolaenko and R. Temam, "Exponential attractors for dissipative evolution equations", John-Wiley, New York, 1994.

[EfM1] M. Efendiev and A. Miranville, *Finite dimensional attractors for reaction-diffusion equations in $\mathbb{R}^n$ with a strong nonlinearity*, Disc. Cont. Dyn. Systems, 15 (2) (1999), 399-424.

[EfM2] M. Efendiev and A. Miranville, *On the dimension of the global attractor for dissipative reaction-diffusion systems*, Appl. Math. Letters, To appear.

[EfMZ1] M. Efendiev, A. Miranville and S. Zelik, *Exponential attractors for a nonlinear reaction-diffusion system in $\mathbb{R}^3$*, C. R. Acad. Sci. Paris, 330 (2000), 713-718.

[EfMZ2] M. Efendiev, A. Miranville and S. Zelik, *Infinite dimensional exponential attractors for a non-autonomous reaction-diffusion system*, Math. Nachrichten, To appear.

[EfMZ3] M. Efendiev, A. Miranville and S. Zelik, *Exponential attractors for a singularly perturbed Cahn-Hilliard system*, Math. Nachrichten, To appear.

[FG1] P. Fabrie and C. Galusinski, *Exponential attractors for the slightly compressible 2D Navier-Stokes equations*, Disc. Cont. Dyn. Systems, 2 (3) (1996), 315-348.

[FG2] P. Fabrie and C. Galusinski, *Exponential attractors for a partially dissipative reaction system*, Asymptotic Anal., 12 (1996), 329-354.

[FGM] P. Fabrie, C. Galusinski and A. Miranville, *Uniform inertial sets for damped wave equations*, Disc. Cont. Dyn. Systems, 6 (2) (2000), 393-418.

[FM] P. Fabrie and A. Miranville, *Exponential attractors for nonautonomous first-order evolution equations*, Disc. Cont. Dyn. Systems, 4 (2) (1998), 225-240.

[FN] P. Fabrie and B. Nicolaenko, *Attracteurs exponentiels pour un modèle de convection naturelle en milieu poreux partiellement dissipatif*, Asymptotic Anal., 12 (1996), 295-327.

[FoST] C. Foias, G. Sell and R. Temam, *Inertial manifolds for nonlinear evolution equations*, J. Diff. Eqns., 73 (1988), 309-353.

[H] J. Hale, "Asymptotic behavior of dissipative systems", Math. Surveys and Mon., Vol. 25, Amer. Math. Soc., Providence, RI, 1987.

[KT] A. N. Kolmogorov and V. M. Tikhomirov, *ε-entropy and ε-capacity of sets in functional spaces*, In "Selected works of A. N. Kolmogorov", Vol. 3 (1993), Dordrecht: Kluver.

[Ko] I. N. Kostin, *Rate of attraction to a non-hyperbolic attractor*, Asymptotic Anal., 16 (3-4) (1998), 203-222.

[L] O. Ladyzhenskaya, "Attractors for semigroups of evolution equations", Cambridge University Press, 1991.

[LdN] Le Dung and B. Nicolaenko, *Exponential attractors in Banach spaces*, J. Dyn. Diff. Eqns., 13 (4) (2001), 791-806.

[M1] A. Miranville, *Exponential attractors for a class of evolution equations by a decomposition method*, C. R. Acad. Sci. Paris, 328 (1999), 145-150.

[M2] A. Miranville, *Exponential attractors for a class of evolution equations by a decomposition method. II. The nonautonomous case*, C. R. Acad. Sci. Paris, 328 (1999), 907-912.

[MZ] A. Miranville and S. Zelik, *Robust exponential attractors for singularly perturbed phase-field type equations*, Electronic J. Diff. Eqns., 2002 (63) (2002), 1-28.

[R] G. Raugel, *Singularly perturbed hyperbolic equations revisited*, in "Proceedings of Equadiff 99", (eds. B. Fiedler, K. Gröger and J. Sprekels), World Scientific (2000).

[SH] A. M. Stuart and A. R. Humphries, "Dynamical systems and numerical analysis", Cambridge Monographs on Applied and Computational Mathematics 2, Cambridge University Press, Cambridge, 1996.

[T] R. Temam, "Infinite dimensional dynamical systems in physics and mechanics", 2d ed., Springer-Verlag, New York, 1997.

[VLy] M. I. Vishik and L. A. Lyusternik, *Regular degeneration and boundary layer for linear differential equations with small parameter*, Amer. Math. Soc. Trans., 20 (2) (1962), 239-364.

E-mail address: {fabrie,galusins}@math.u-bordeaux.fr
E-mail address: {miranv,zelik}@mathlabo.univ-poitiers.fr