Two-Dimensional Failure Modeling with Minimal Repair

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Abstract: In this paper, we discuss two-dimensional failure modeling for a system where degradation is due to age and usage. We extend the concept of minimal repair for the one-dimensional case to the two-dimensional case and characterize the failures over a two-dimensional region under minimal repair. An application of this important result to a manufacturer's servicing costs for a two-dimensional warranty policy is given and we compare the minimal repair strategy with the strategy of replacement of failure. © 2003 Wiley Periodicals, Inc. Naval Research Logistics 51: 345–362, 2004.

1. INTRODUCTION

All systems are unreliable in the sense that they degrade and fail. These systems are often comprised of several components and fail due to the failure of one or more components. A repairable system can be restored to an operational state by either repair or replacement, but if it is nonrepairable, then the only option is replacement on failure. Under minimal repair, a failed multicomponent system is restored to an operational state without affecting the level of deterioration. In other words, the restored system is identical to what it was just before failure. This is appropriate as the rectification of the failed components (through either repair or replacement) has negligible effect on the remaining components.

The bulk of the reliability literature deals with "one-dimensional" failure modeling where the degradation depends solely on the age of the system, and so failures are random points along a one-dimensional time continuum. However, for many systems, the degradation is a function of age and usage. In the case of automobiles, usage corresponds to the distance traveled, and, in

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the case of aircraft landing gear, usage corresponds to the number of landings. In such cases, failures are random points in a two-dimensional plane with age and usage as the two axes. Two-dimensional failure modeling has received some attention in the literature, and two approaches have been proposed.

In the first approach, the two-dimensional problem is effectively reduced to a one-dimensional problem by treating usage as a random function of age (see Lawless, Hu, and Cao [14] and Blischke and Murthy [2]). This approach can be used for both repairable and nonrepairable systems. The concept of minimal repair in this case is essentially the same as for the one-dimensional case proposed by Barlow and Hunter [1]. This approach was used in Iskandar [9] and Blischke and Murthy [2] in the context of warranty cost analysis.

In the second approach, the modeling of system failure involves a bivariate distribution. For a nonrepairable system, rectification involves replacing the failed item by a new one. If the failure is detected immediately and the time to replace is negligible (so that it can be ignored), then failures over the two-dimensional plane can be modeled by a two-dimensional renewal process. This type of process was first studied by Hunter [6], and then Blischke and Murthy [2], Murthy, Iskandar, and Wilson [18], and Hunter [7] have used this model in the context of warranty cost analysis for two-dimensional warranties.

Yang and Nachlas [24] describe the two approaches to two-dimensional failure modeling and use the bivariate failure distribution method to discuss two-dimensional renewal process models for a nonrepairable system. They also show how to obtain the system's point availability when replacement time is nonnegligible and when preventive replacement occurs at a specified age or usage.

The concept of minimal repair in the case of a bivariate failure distribution is the main focus of this paper. We extend the concept of minimal repair for the one-dimensional case to the two-dimensional case and carry out a study of the failures under minimal repair. We then apply these results to the analysis of two-dimensional warranty policies. The outline of the paper is as follows. In Section 2, we give a brief review of one-dimensional failure modeling as a background for the development of the two-dimensional case in Section 3. We discuss two cases—(i) the failed system is always minimally repaired and (ii) the failed system is always replaced by a new item. In Section 3, we introduce the concept of minimal repair for two-dimensional failure modeling, and we then derive some results when failures are always rectified through minimal repair. Following this, we discuss the case where failures are always rectified through replacement, and this allows us to compare the two types of rectification action. We compare one-dimensional and two-dimensional failure modeling in Section 4 and discuss Weibull failure models in Section 5. In Section 6 we consider the application of the two-dimensional minimal repair model to warranty cost analysis. We conclude with a brief discussion of topics for future research in Section 7.

2. ONE-DIMENSIONAL FAILURE MODELING

A system can be either repaired or replaced at each failure and the durations of all such corrective maintenance actions are assumed to be small compared to the times between failures and so can be ignored. Let the nonnegative random variable T_n denote the time of the nth system failure, $n \geq 1$, and $Y_n = T_n - T_{n-1}$ the time between the nth and $(n-1)$th failure (where $T_0 = 0$).

Suppose that T_1 has distribution function

$$F(u) = \Pr\{T_1 \leq u\}, \qquad u \geq 0, \tag{1}$$

and survivor function

$$\bar{F}(u) = \Pr\{T_1 > u\}. \tag{2}$$

When $F(u)$ is differentiable, the failure density function is given by

$$f(u) = dF(u)/du, \tag{3}$$

and the hazard function is defined as

$$r(u) = f(u)/\bar{F}(u). \tag{4}$$

The probability that the system will fail for the first time in the interval $[u, u + \delta u)$, given that it has not failed prior to u, is $r(u)\delta u + o(\delta u)$.

Successive system failures can be modeled using a point process formulation. Let $N_1(s, t)$ denote the number of system failures in the interval $[s, t)$, $0 \le s < t$, with $N_1(0, t)$ abbreviated to $N_1(t)$. The failure intensity function or rate of occurrence of failures (ROCOF) at t is given by

$$\lambda(t) = \lim_{\delta t \to 0} \frac{\mathrm{P}\{N_1(t, t + \delta t) = 1\}}{\delta t}, \tag{5}$$

so the probability that a failure will occur in the interval $[t, t + \delta t)$ is $\lambda(t)\delta t + o(\delta t)$. Assuming simultaneous failures cannot occur, it follows from (5) that $\lambda(t) = dE\{N_1(t)\}/dt$ or

$$\Lambda(t) = E\{N_1(t)\} = \int_0^t \lambda(u)\, du. \tag{6}$$

$\Lambda(t)$ is termed the cumulative intensity function for the failure process.

Let H_t denote the history of the failure process up to, but not including, time t (see Lawless and Thiagarajah [15]). The conditional failure intensity function is then given by

$$\lambda_c(t) = \lim_{\delta t \to 0} \frac{\mathrm{P}\{N_1(t, t + \delta t) = 1 \mid H_t\}}{\delta t}, \tag{7}$$

which implies that $\lambda(t) = E\{\lambda_c(t)\}$. Thus, $\lambda(t)$ is the mean of $\lambda_c(t)$ averaged over all sample paths of the failure process (see Hokstad [5]).

The forms for the functions $\lambda_c(t)$ and $\lambda(t)$ depend on the type of corrective maintenance action carried out at each failure. We consider the two cases "always minimally repair" and "always replace."

2.1. Minimal Repair

The concept of minimal repair was first proposed by Barlow and Hunter [1]. When the system is minimally repaired on failure, it is restored to its condition immediately before the failure occurred. Thus, the conditional failure intensity function is unaffected by each failure, so is independent of H_t, and therefore

$$\lambda_c(t) = r(t), \tag{8}$$

where $r(u)$ is the hazard function for T_1 given in (4).

This simple nonrandom form for $\lambda_c(t)$ implies that $\lambda(t) = r(t)$. Hence, the counting process $\{N_1(t),\ t \geq 0\}$ is a nonhomogeneous Poisson process (NHPP) with intensity function $\lambda(t) = r(t)$ and cumulative intensity $\Lambda(t) = \int_0^t r(u)\ du$, so

$$p_n(t) = \mathrm{P}\{N_1(t) = n\} = \frac{\exp[-\Lambda(t)][\Lambda(t)]^n}{n!}, \qquad n \geq 0. \tag{9}$$

See Lawless and Thiagarajah [15] for the conditional failure intensity approach or Murthy [17] and Nakagawa and Kowada [19] for alternative approaches to obtain this result.

2.2. Replacement

On failure, the system is replaced by a new and identical system. Hence, the conditional intensity function returns to its value at time 0 at each failure, and so, for $n \geq 0$ and $T_n \leq t < T_{n+1}$, it follows that

$$\lambda_c(t) = r(t - T_n). \tag{10}$$

The counting process $\{N_1(t),\ t \geq 0\}$ is a renewal process (RP) and the interfailure times Y_n, $n \geq 1$, are i.i.d. random variables with distribution function $F(u)$ (see Lawless and Thiagarajah [15]). It then follows that

$$p_n(t) = \mathrm{P}\{N_1(t) = n\} = F^{(n)}(t) - F^{(n+1)}(t), \qquad n \geq 0, \tag{11}$$

where $F^{(n)}(t)$ is the n-fold convolution of $F(t)$ with itself, and

$$M(t) = E\{N_1(t)\} = \sum_{n=1}^{\infty} F^{(n)}(t). \tag{12}$$

$M(t)$ can also be expressed as the solution of the renewal integral equation

$$M(t) = F(t) + \int_0^t M(t-u)f(u)\ du. \tag{13}$$

(See Cox [4], Ross [21], or Tijms [22] for details.)

Analytical expressions for $M(t)$ can only be found for a few special cases of $F(t)$, and these are obtained by using Laplace transforms in either (12) or (13). In all other cases, a numerical procedure is needed to evaluate $M(t)$. Xie's [23] method, using direct numerical Riemann-Stieltjes integration of (13) has been found to be very fast and accurate.

3. TWO-DIMENSIONAL FAILURE MODELING

We now assume that the degradation of a system depends on its age and usage. Let T_n and X_n, $n \geq 1$, denote the time of the nth system failure and the corresponding usage at that time. $Y_n = T_n - T_{n-1}$ gives the time between the nth and $(n-1)$th failure, and $Z_n = X_n - X_{n-1}$ is the system usage during this period (where $T_0 = 0$, $X_0 = 0$).

In the two-dimensional approach to modeling failures, we begin by assuming that (T_1, X_1) is a nonnegative bivariate random variable with distribution function

$$F(u, v) = \mathrm{P}\{T_1 \leq u, X_1 \leq v\}, \qquad u \geq 0, \quad v \geq 0. \tag{14}$$

The survivor function is given by

$$\bar{F}(u, v) = \Pr\{T_1 > u, X_1 > v\} = \int_u^\infty \int_v^\infty f(s, t)\, dt ds. \tag{15}$$

and, if $F(u, v)$ is differentiable, then the bivariate failure density function is given by

$$f(u, v) = \frac{\partial^2 F(u, v)}{\partial v \partial u}. \tag{16}$$

The hazard function is defined as

$$r(u, v) = f(u, v)/\bar{F}(u, v), \tag{17}$$

so the probability that the first system failure will occur in $[u, u + \delta u) \times [v, v + \delta v)$ given that $T_1 > u$ and $X_1 > v$ is $r(u, v)\delta u \delta v + o(\delta u \delta v)$.

COMMENT: The bulk of the literature on bivariate distributions deals with the case where X_1 and T_1 are viewed as random lifetimes of two different items. The bivariate distribution then models the statistical dependence between the two lifetimes. In contrast, here we are dealing with a single item with X_1 and T_1 denoting the age and usage at first failure. This point is discussed in more detail by Yang and Nachlas [24].

Successive system failures can be modeled using a two-dimensional point process formulation. Expanding the one-dimensional formulation used in Section 2, we let $N_2(s, t; w, x)$ denote the number of system failures in the rectangle $[s, t) \times [w, x)$, $0 \leq s < t$, $0 \leq w < x$, and we abbreviate $N_2(0, t; 0, x)$ to $N_2(t, x)$. The failure intensity or rate of occurrence of failures (ROCOF) at the point (t, x) is given by the function

$$\lambda(t, x) = \lim_{\substack{\delta t \to 0 \\ \delta x \to 0}} \frac{\mathrm{P}\{N_2(t, t + \delta t; x, x + \delta x) = 1\}}{\delta t \delta x}, \tag{18}$$

so the probability that a failure will occur in $[t, t + \delta t) \times [x, x + \delta x)$ is $\lambda(t, x)\ \delta t\delta x + o(\delta t\delta x)$.

Assuming simultaneous failures cannot occur, it follows from (18) that $\lambda(t, x) = \partial^2 E\{N_2(t, x)\}/\partial t\partial x$ or

$$\Lambda(t, x) = E\{N_2(t, x)\} = \int_0^t \int_0^x \lambda(u, v)\ dv du. \tag{19}$$

Let $H_{t,x}$ denote the history of the failure process up to, but not including, the point (t, x). The conditional failure intensity function is then given by

$$\lambda_c(t, x) = \lim_{\substack{\delta t \to 0 \\ \delta x \to 0}} \frac{\mathrm{P}\{N_2(t, t + \delta t; x, x + \partial x) = 1 \mid H_{t,x}\}}{\delta t \partial x}. \tag{20}$$

The forms for the functions $\lambda_c(t, x)$, $\lambda(t, x)$, and $N_2(t, x)$ are now discussed for the two cases "always minimally repair" and "always replace."

3.1. Minimal Repair

The conditional failure intensity function is unaffected by each failure and so

$$\lambda_c(t, x) = \lambda(t, x) = r(t, x), \tag{21}$$

where $r(u, v)$ is the hazard function for (T_1, X_1) given in (17). We now show that the counting process $\{N_2(t, x), t \geq 0, x \geq 0\}$ is a two-dimensional NHPP with intensity function given by (21), so that

$$E\{N_2(t, x)\} = \Lambda(t, x) = \int_0^t \int_0^x r(u, v)\ dv du. \tag{22}$$

PROPOSITION: If $p_n(t, x) = \mathrm{P}\{N_2(t, x) = n\}$, then, for $n > 0$, $p_n(t, x)$ satisfies the equation

$$\frac{\partial^2 p_n(t, x)}{\partial t \partial x} = [p_{n-1}(t, x) - p_n(t, x)]\lambda(t, x), \tag{23}$$

where $\lambda(t, x) = r(t, x)$.

For $n = 0$, the corresponding equation is

$$\frac{\partial^2 p_0(t, x)}{\partial t \partial x} = -p_0(t, x)\lambda(t, x). \tag{24}$$

The conditions $p_n(t, 0) = p_n(0, x) = 0$ also hold for $n \geq 0$ and $t, x > 0$. The solution to Eqs. (23) and (24) is

$$p_n(t, x) = \frac{\exp[-\Lambda(t, x)]\Lambda(t, x)^n}{n!}, \qquad n \geq 0. \tag{25}$$

PRELIMINARY RESULTS: In order to prove the above proposition, we need the following two results that follow from the basic calculus for two-dimensional probability functions.

(i) Let $H_1(t, x)$ denote any probability function defined over $R_+^2 = \{(t, x): t \geq 0, x \geq 0\}$. Then

$$H_1(t + \delta t, x + \delta x) = H_1(t, x) + \left[\frac{\partial^2 H_1(t, x)}{\partial t \partial x}\right] \delta t \delta x + o(\delta t \delta x), \tag{26}$$

where

$$\frac{\partial^2 H_1(t, x)}{\partial t \partial x} = \lim_{\substack{\delta t \to 0 \\ \delta x \to 0}} \left[\frac{H_1(t + \delta t, x + \delta x) - H_1(t, x)}{\delta t \delta x}\right]. \tag{27}$$

(ii) If $H_2(t, x) = g[H_1(t, x)]$, then

$$\frac{\partial^2 H_2(t, x)}{\partial t \partial x} = g'[H_1(t, x)] \frac{\partial^2 H_1(t, x)}{\partial t \partial x} \tag{28}$$

where $g'(y) = dg(y)/dy$.

PROOF OF PROPOSITION: For $n > 0$, note that n failures occur in $[0, t + \delta t) \times [0, x + \delta x)$ as a result of (i) $(n - 1)$ failures occurring in $[0, t) \times [0, x)$ and one failure occurring in $[t, t + \delta t) \times [x, x + \delta x)$ or (ii) n failures occurring in $[0, t) \times [0, x)$ and no failure occurring in $[t, t + \delta t) \times [x, x + \delta x)$. This yields the following:

$$p_n(t + \delta t, x + \delta x) = p_{n-1}(t, x)\lambda(t, x)\ \delta t \delta x + p_n(t, x)[1 - \lambda(t, x)\delta t \delta x] + o(\delta t \delta x), \tag{29}$$

or

$$p_n(t + \delta t, x + \delta x) - p_n(t, x) = [p_{n-1}(t, x) - p_n(t, x)]\lambda(t, x)\ \delta t \delta x + o(\delta t \delta x), \tag{30}$$

For $n = 0$, a similar argument gives

$$p_0(t + \delta t, x + \delta x) = p_0(t, x)[1 - \lambda(t, x)\delta t \delta x] + o(\delta t \delta x) \tag{31}$$

or

$$p_0(t + \delta t, x + \delta x) - p_0(t, x) = -p_0(t, x)\lambda(t, x)\delta t \delta x + o(\delta t \delta x). \tag{32}$$

Using (26) in (30) and (32), dividing through by $\delta t \delta x$ and taking the limit, we obtain (23) and (24), respectively.

We now use the chain rule given in (28) to show that (25) satisfies (30) and (32). For $n > 0$, we have

$$\frac{\partial^2 p_n(t, x)}{\partial t \partial x} = -\frac{\partial^2 \Lambda(t, x)}{\partial t \partial x} \frac{\exp[-\Lambda(t, x)]\Lambda(t, x)^n}{n!} + \frac{\exp[-\Lambda(t, x)]\Lambda(t, x)^{n-1}}{(n-1)!} \frac{\partial^2 \Lambda(t, x)}{\partial t \partial x}$$

$$= \left[\frac{\exp[-\Lambda(t, x)]\Lambda(t, x)^{n-1}}{(n-1)!} - \frac{\exp[-\Lambda(t, x)]\Lambda(t, x)^n}{n!}\right] \frac{\partial^2 \Lambda(t, x)}{\partial t \partial x} = [p_{n-1}(t, x) - p_n(t, x)]\lambda(t, x),$$

and so (23) is satisfied. The case $n = 0$ is verified in a similar way. □

3.2. Replacement

At each system failure, the conditional failure intensity function returns to its value at time 0. Hence, if $n \geq 0$ and $T_n \leq t < T_{n+1}$, $X_n \leq x < X_{n+1}$, then

$$\lambda_c(t, x) = r(t - T_n, x - X_n). \tag{33}$$

The counting process $\{N_2(t, x), t \geq 0, x \geq 0\}$ is a two-dimensional RP and the variables (Y_n, Z_n), $n \geq 1$, are i.i.d. with distribution function $F(u, v)$. It then follows that

$$p_n(t, x) = \mathrm{P}\{N_2(t, x) = n\} = F^{(n)}(t, x) - F^{(n+1)}(t, x), \qquad n \geq 0, \tag{34}$$

where $F^{(n)}(t, x)$ is the n-fold bivariate convolution of $F(t, x)$ with itself. The expected number of failures over $[0, t) \times [0, x)$ is given by

$$M(t, x) = E\{N_2(t, x)\} = \sum_{n=1}^{\infty} F^{(n)}(t, x), \tag{35}$$

and this renewal function can also be expressed as the solution of the two-dimensional integral equation

$$M(t, x) = F(t, x) + \int_0^t \int_0^x M(t - u, x - v) f(u, v) \, dv du. \tag{36}$$

Equations (34)–(36) are from Hunter [6], who also describes the bivariate Laplace transform approach for evaluating $M(t, x)$ using (35). It is extremely difficult to obtain analytic expressions for the renewal function using this approach, so computational techniques for $M(t, x)$ are therefore required. Iskandar [8] uses (36) to develop a numerical integration evaluation procedure based on Xie's [23] method.

4. COMPARISON BETWEEN ONE-DIMENSIONAL AND TWO-DIMENSIONAL FAILURE MODELING

We now consider some of the expressions derived in Sections 2 and 3 in order to compare and contrast the two types of failure modeling. For first system failure in the one-dimensional case, (1), (2), and (4) give

$$F(u) + \bar{F}(u) = 1, \tag{37}$$

and

$$\bar{F}(u) = \exp\left[-\int_0^u r(s)\, ds\right]. \tag{38}$$

Note that, for a specified $r(u)$, $\bar{F}(u)$ and $F(u)$ can be uniquely determined using (38) and (37), respectively.

In two-dimensional failure modeling, however, we have

$$F(u, v) + \bar{F}(u, v) < 1, \tag{39}$$

since

$$F(u, v) + \bar{F}(u, v) + \mathrm{P}\{T_1 \le u, X_1 > v\} + \mathrm{P}\{T_1 > u, X_1 \le v\} = 1. \tag{40}$$

Also,

$$\bar{F}(u, v) \ne \exp\left\{-\int_0^u \int_0^v r(s, t)\, dtds\right\}, \tag{41}$$

and we illustrate this through an example.

EXAMPLE 1: Let

$$\bar{F}(u, v) = \bar{G}(u)\bar{H}(v).$$

This implies that T_1 and X_1 are independent random variables with survivor functions $\bar{G}(u)$ and $\bar{H}(v)$. Let the corresponding density and hazard functions be $g(u)$ and $h(v)$, and $r_{T_1}(u)$ and $r_{X_1}(v)$, respectively. The left-hand side of (41) is given by

$$\bar{G}(u)\bar{H}(v) = \exp\left\{-\int_0^u r_{T_1}(s)\, ds\right\}\exp\left\{-\int_0^v r_{X_1}(t)\, dt\right\} = \exp\left\{-\left[\int_0^u r_{T_1}(s)\, ds + \int_0^v r_{X_1}(t)\, dt\right]\right\}.$$

However, since

$$r(s,\ t) = \frac{g(s)h(t)}{\bar{G}(s)\bar{H}(t)} = r_{T_1}(s)r_{X_1}(t),$$

the right-hand side of (41) is given by

$$\exp\left\{-\left[\int_0^u r_{T_1}(s)\,ds \int_0^v r_{X_1}(t)\,dt\right]\right\},$$

implying that the two are not the same. □

Note that, given $r(u,\ v)$, we are then unable to determine $\bar{F}(u,\ v)$ or $F(u,\ v)$ uniquely.

Suppose that we now look at the counting processes for successive system failures. Under minimal repair, (8) and (9) in the one-dimensional failure model and (21), (22), and (25) in the two-dimensional case each define an NHPP, while, for replacement, (10)–(13) and (33)–(36) define a one-dimensional and two-dimensional RP, respectively.

5. WEIBULL FAILURE MODELS

5.1. One-Dimensional Model

We assume that T_1, the system's age at first failure, has a Weibull distribution with

$$F(u) = 1 - \exp\{-(u/\theta)^{\beta}\}, \tag{42}$$

where $\theta > 0$ and $\beta > 0$. The corresponding hazard function is

$$r(u) = (\beta/\theta)(u/\theta)^{\beta-1}, \tag{43}$$

which is decreasing in u when $\beta < 1$, constant when $\beta = 1$, and increasing in u when $\beta > 1$.

Under minimal repair, the ROCOF is given by $\lambda(t) = r(t)$ and the resulting point process is referred to as the Weibull process. From (6), the expected number of system failures in the interval $[0,\ t)$ under minimal repair is given by

$$\Lambda(t) = (t/\theta)^{\beta}. \tag{44}$$

The expected number of system failures in the interval $[0,\ t)$ under replacement is the renewal function $M(t)$ given by either (12) or (13). When $\beta = 1$, we have $M(t) = t/\theta$, but no analytical expression is available for general β. However, in this case, Xie's [23] method can be used to give a fast and accurate numerical evaluation using (13).

5.2. Two-Dimensional Model

Several approaches have been proposed towards the construction of bivariate Weibull models by considering the failure behavior of systems. We confine our attention to a model proposed by Lu and Bhattacharyya [16], where the survivor function is

$$\bar{F}(u,\ v) = \exp\{-[(u/\theta_1)^{\beta_1/\delta} + (v/\theta_2)^{\beta_2/\delta}]^{\delta}\}, \tag{45}$$

with θ_1, θ_2, β_1, $\beta_2 > 0$, and $0 < \delta \le 1$. Note that T_1 and X_1 are not independent. The corresponding failure density function is given by

$$f(u, v) = \alpha_1^{-1}\alpha_2^{-1}\beta_1\beta_2(u/\alpha_1)^{\beta_1/\delta-1}(v/\alpha_2)^{\beta_2/\delta-1}[(u/\alpha_1)^{\beta_1/\delta} + (v/\alpha_2)^{\beta_2/\delta}]^{\delta-2} \times \{[(u/\alpha_1)^{\beta_1/\delta} + (v/\alpha_2)^{\beta_2/\delta}]^{\delta} + 1/\delta - 1\} \times \exp\{-[(u/\alpha_1)^{\beta_1/\delta} + (v/\alpha_2)^{\beta_2/\delta}]^{\delta}\}, \quad (46)$$

and the hazard function is

$$r(u, v) = \alpha_1^{-1}\alpha_2^{-1}\beta_1\beta_2(u/\alpha_1)^{\beta_1/\delta-1}(v/\alpha_2)^{\beta_2/\delta-1}[(u/\alpha_1)^{\beta_1/\delta} + (v/\alpha_2)^{\beta_2/\delta}]^{\delta-2} \times \{[(u/\alpha_1)^{\beta_1/\delta} + (v/\alpha_2)^{\beta_2/\delta}]^{\delta} + 1/\delta - 1\}. \quad (47)$$

The conditional expectation of X_1 given $T_1 = u$ is given by

$$E[X_1 \mid T_1 = u] = \delta\theta_2(u/\theta_1)^{\beta_1(1/\delta-1)}\exp\{(u/\theta_1)^{\beta_1}\} \times \left\{\int_0^\infty w^c(w+z)^{2\delta-2}\exp\{-(w+z)^{\delta}\}\,dw + (1/\delta - 1)\int_0^\infty w^c(w+z)^{\delta-2}\exp\{-(w+z)^{\delta}\}\,dw, \right. \quad (48)$$

where $c = \delta/\beta_2$ and $z = (u/\theta_1)^{\beta_1/\delta}$.

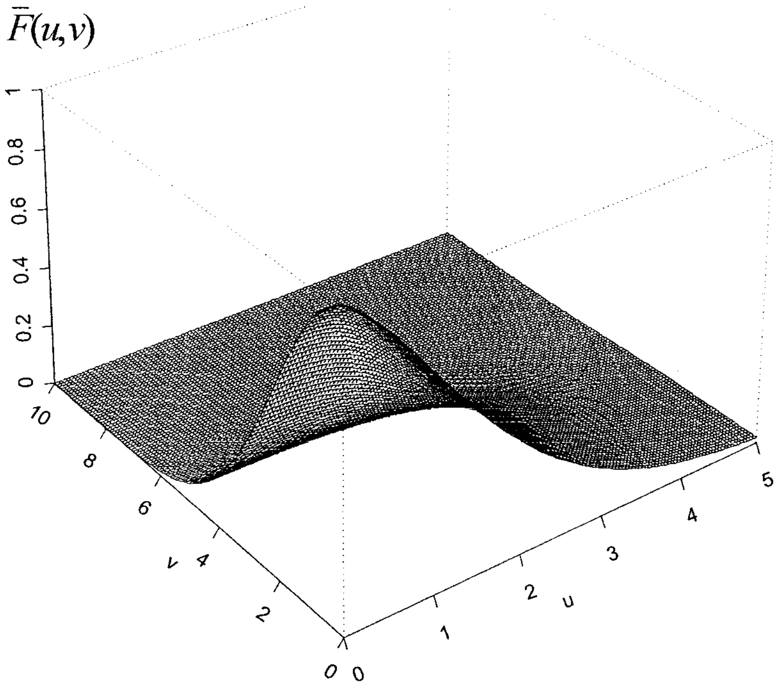

Figure 1. Plot of survivor function $\bar{F}(u, v)$.

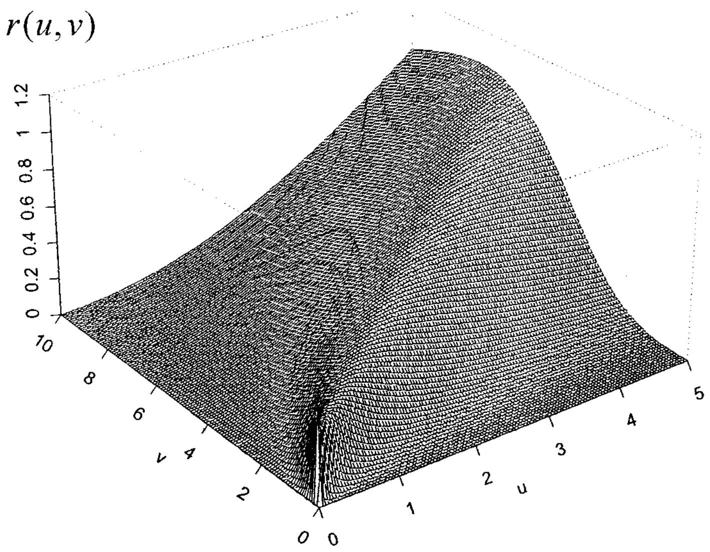

Figure 2. Plot of hazard function $r(u, v)$.

EXAMPLE 2: Let the model parameters be as follows:

$$\theta_1 = 2, \qquad \theta_2 = 3, \qquad \beta_1 = 1.5, \qquad \beta_2 = 2.0, \qquad \delta = 0.5.$$

The units for age and usage are years and 10,000 km, respectively. The expected age and usage at first system failure are given by

$$E(T_1) = \theta_1\Gamma(1/\beta_1 + 1) = 1.81 \quad \text{(years)} \quad \text{and} \quad E(X_1) = \theta_2\Gamma(1/\beta_2 + 1) = 2.66 \quad (10^3 \text{ km}).$$

Figure 1 is a plot of the survivor function $\bar{F}(u, v)$, and Figure 2 is a plot of the hazard function $r(u, v)$. Note that $r(u, v)$ increases as u (age) and v (usage) increases since β_1 and β_2 are greater than 1.

MINIMAL REPAIR: The expected number of system failures in the rectangle $[0, t) \times [0, x)$ under minimal repair is

$$E\{N_2(t, x)\} = \Lambda(t, x) = \int_0^t \int_0^x r(u, v)\, dv du, \tag{49}$$

where $r(u, v)$ is given by (47).

REPLACEMENT: The expected number of system failures in the rectangle $[0, t) \times [0, x)$ under replacement is given by the renewal function $M(t, x)$ [(35) or (36)]. For general β_1 and β_2, we need to evaluate $M(t, x)$ numerically.

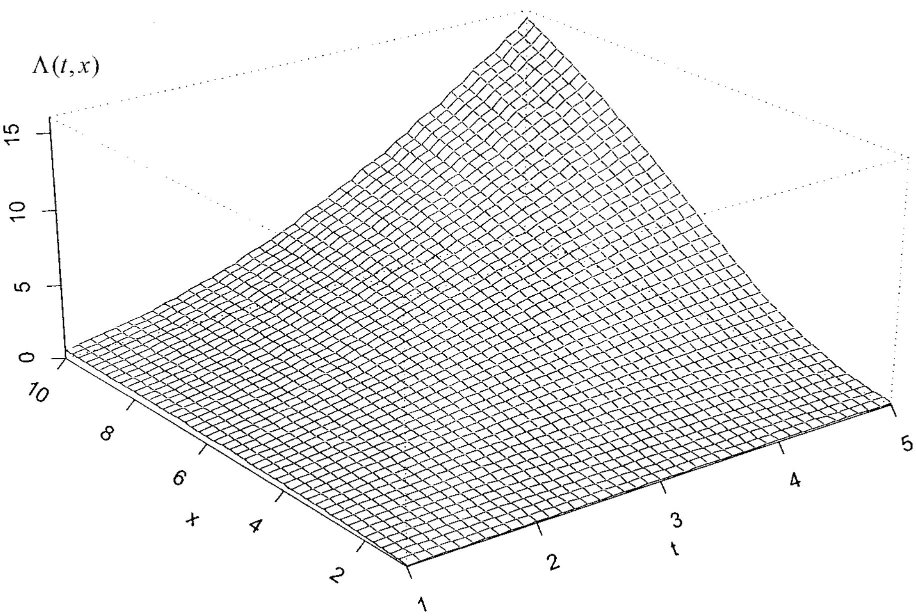

Figure 3. Plot of cumulative intensity function $\Lambda(t, x)$.

EXAMPLE 2 [cont.]: Figure 3 is a plot of $\Lambda(t, x)$, and Figure 4 is a plot of $M(t, x)$, obtained using the two-dimensional renewal equation solver from Iskandar [8]. Note that $\Lambda(t, x) > M(t, x)$ for all $t > 0$, $x > 0$.

SPECIAL CASE [$\delta = 1$]: In this case T_1 and X_1 are independent and

$$\bar{F}(u, v) = \exp\{-[(u/\theta_1)^{\beta_1} + (v/\theta_2)^{\beta_2}]\}. \tag{50}$$

The hazard function is given by

$$r(u, v) = (\beta_1\beta_2/\theta_1\theta_2)(u/\theta_1)^{\beta_1-1}(v/\theta_2)^{\beta_2-1}, \tag{51}$$

which is increasing in u and v for $\beta_1 > 1$ and $\beta_2 > 1$.

From (22), the expected number of system failures over the rectangle $[0, t) \times [0, x)$, under minimal repair, is given by

$$E\{N_2(t, x)\} = \Lambda(t, x) = (t/\theta_1)^{\beta_1}(x/\theta_2)^{\beta_2}. \tag{52}$$

Unfortunately, the two-dimensional renewal function cannot be expressed analytically. However, when $\beta_1 = \beta_2 = 1$, then from (35) using Laplace transforms, we have

$$M(t, x) = \sum_{n=1}^{\infty} P_n(t/\theta_1)P_n(x/\theta_2), \tag{53}$$

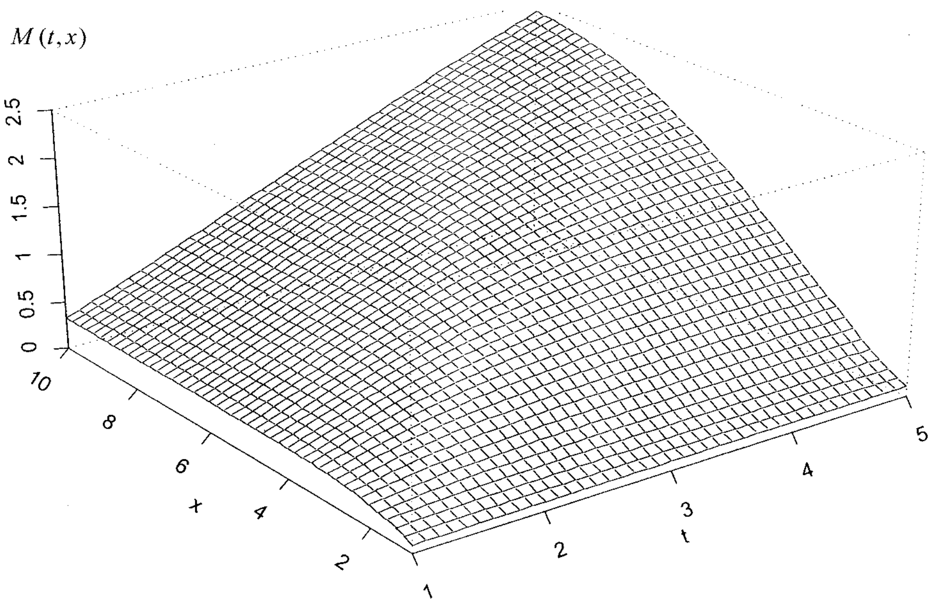

Figure 4. Plot of renewal function $M(t, x)$.

where

$$P_n(y) = \frac{1}{\Gamma(n)} \int_0^y u^{n-1} e^{-u}\, du \tag{54}$$

is the incomplete gamma function (see Hunter [6, 7] for more details).

6. WARRANTY COST ANALYSIS

Many different kinds of two-dimensional warranties have been studied and details can be found in Blischke and Murthy [2]. We confine our attention to the following 2-D Free Replacement Warranty (FRW) policy where the manufacturer provides free repairs or replacements for any failed items up to time K or up to usage L, whichever occurs first, from the time of the initial purchase. The warranty region is a rectangle and many products and sold with this type of warranty (e.g., automobiles, aircraft components, etc.).

We consider the case where the product is repairable and is sold with an FRW policy. The failure distribution is given by a two-dimensional distribution function $F(u, v)$. In this case, the manufacturer has several options and two of these are as follows: (i) Replace all failed items by new ones, and (ii) repair fall failures using minimal repair. The expected costs for these two strategies are different, and they have a significant impact on the manufacturer's total profit.

REPLACEMENT: Let C_r denote the cost to the manufacturer of replacing a failed item by a new one, and let $E[C_T^r(K, L)]$ denote the expected total servicing cost per unit sold. We assume that a failure results in an instantaneous claim and that all claims are valid. We also

Table 1. Expected number of failures under the AR strategy $[M(K, L)]$.

		Usage L (10^3 km)				
		1	3	5	8	10
Age K (years)	1	0.095	0.301	0.328	0.330	0.330
	2	0.105	0.588	0.801	0.840	0.841
	3	0.106	0.704	1.181	1.374	1.388
	4	0.107	0.740	1.387	1.855	1.927
	5	0.107	0.750	1.474	2.218	2.418

assume that the time to replace is negligible so that it can be ignored. Replacements under this policy then occur according to a two-dimensional RP associated with $F(u, v)$. Following Blischke and Murthy [2], the expected total cost to the manufacturer per unit sold is

$$E[C_T^r(K, L)] = C_r[1 + M(K, L)], \tag{55}$$

where $M(t, x)$ is the renewal function given by (36). Note that we have included the cost of the initial sale so that this is the total cost to the manufacturer. The manufacturer needs to ensure that the product sale price is larger than $E[C_T^r(K, L)]$ in order to make a profit.

MINIMAL REPAIR: Let C_m denote the cost of each minimal repair. With the same assumptions about claims as in the always replace case and negligible repair times, failures over the warranty region occur according to a two-dimensional NHPP with intensity function $\lambda(t, x)$ given by (21). If $E[C_T^m(K, L)]$ denotes the expected total cost per unit sold then

$$E[C_T^m(K, L)] = C_r + C_m\Lambda(K, L), \tag{55}$$

with $\Lambda(t, x)$ given by (22).

RELATIVE COMPARISON: Define

$$\Psi(K, L) = \frac{E[C_T^m(K, L)]}{E[C_T^r(K, L)]} = \frac{1 + (C_m/C_r)\Lambda(K, L)}{1 + M(K, L)}. \tag{56}$$

This is a function of the warranty parameters (K and L) and the relative cost ratio (C_m/C_r). If $\Psi(K, L) < 1$, then minimal repair yields a smaller expected cost whereas if $\Psi(K, L) > 1$, then replacement is the better strategy. If $\Psi(K, L) = 1$, then either strategy can be used.

Table 2. Expected number of failures under the AMR strategy $[\Lambda(K, L)]$.

		Usage L (10^3 km)				
		1	3	5	8	10
Age K (years)	1	0.104	0.390	0.491	0.562	0.592
	2	0.121	0.931	1.601	2.163	2.409
	3	0.127	1.332	2.987	4.758	5.578
	4	0.131	1.612	4.364	8.124	10.010
	5	0.134	1.820	5.601	11.970	15.588

Table 3.1. $\Psi(K, L) = C_T^m(K, L)/C_T^r(K, L)$ for $C_m/C_r = 1$.

		Usage L (10^3 km)				
		1	3	5	8	10
Age K (years)	1	1.008	1.069	1.123	1.174	1.197
	2	1.015	1.216	1.444	1.719	1.852
	3	1.019	1.368	1.829	1.426	2.755
	4	1.022	1.501	2.247	3.186	3.762
	5	1.024	1.611	2.669	4.030	4.824

EXAMPLE 3: Let $F(u, v)$ be the two-dimensional Weibull distribution of Example 2. Tables 1 and 2 give the expected number of failures under warranty for the servicing strategies "always replace" (AR) and "always minimally repair" (AMR), over a range of values of K and L. As expected, the values in Table 1 are always smaller than the corresponding values in Table 2.

Tables 3.1–3.3 show values of $\Psi(K, L)$ for three different ratios of C_m/C_r. The values indicated in bold type are those for which $\Psi(K, L) < 1$, so they correspond to the (K, L) combinations where AMR is better than AR. In Table 3.1, $C_m/C_r = 1$, and we see that $\Psi(K, L) > 1$ for all values of K and L, implying that AR is the preferred strategy. This is to be expected since the cost of each repair is the same as the cost of a replacement item. When $C_m/C_r = 0.5$, we see from Table 3.2 that AMR is better than AR for small values of either K or L. If K and L are both large, then AR is the preferred strategy. Table 3.3 shows that, as the ratio of C_m to C_r decreases, AMR is better even for medium values of K and L.

7. CONCLUSIONS AND TOPICS FOR FUTURE RESEARCH

In this paper, we have given a brief review of one-dimensional failure modeling, discussing the cases where a failed system is either always minimally repaired or replaced by a new system. These two types of corrective maintenance action lead to a nonhomogeneous Poisson process (NHPP) and a renewal process (RP), respectively, for the number of system failures over time.

We have then discussed two-dimensional failure modeling when system degradation is due to age and usage. Replacing a system at every failure leads to two-dimensional RP model; but the main focus of the paper has been to show that the one-dimensional minimal repair concept of unchanging failure rate can be extended to two dimensions, and this again produces an NHPP for system failures.

We have applied this important result to the analysis of warranty servicing costs for an automobile where the manufacturer provides free repairs for failed components up to time K or up to usage L, whichever comes first, from the initial purchase.

Table 3.2. $\Psi(K, L) = C_T^m(K, L)/C_T^r(K, L)$ for $C_m/C_r = 0.5$.

		Usage L (10^3 km)				
		1	3	5	8	10
Age K (years)	1	**0.961**	**0.919**	**0.938**	**0.963**	**0.974**
	2	**0.960**	**0.923**	**0.999**	1.131	1.198
	3	**0.961**	**0.978**	1.144	1.424	1.587
	4	**0.063**	1.038	1.333	1.773	2.052
	5	**0.964**	1.092	1.537	2.171	2.558

Table 3.3. $\Psi(K, L) = C_T^m(K, L)/C_T^r(K, L)$ for $C_m/C_r = 0.33$.

		Usage L (10^3 km)				
		1	3	5	8	10
Age K (years)	1	**0.945**	**0.869**	**0.876**	**0.893**	**0.900**
	2	**0.942**	**0.825**	**0.851**	**0.936**	**0.980**
	3	**0.942**	**0.847**	**0.915**	1.089	1.197
	4	**0.943**	**0.884**	1.028	1.299	1.482
	5	**0.943**	**0.918**	1.159	1.551	1.903

Topics for future research include the following:

1. We can consider a system that can be minimally repaired at each failure with probability p and with probability $(1 - p)$ it needs to be replaced by a new one. The one-dimensional failure model for this case has been studied by Brown and Proschan [3].
2. In this paper we assumed that the characteristics in the two-dimensional failure model were both continuous random variables. However, one of these characteristics could be a discrete variable, such as number of uses, and in this case a mixed bivariate failure distribution would be needed in the modeling. Alternatively, we could extend the failure model to deal with more than two characteristics.
3. In this paper we have looked at the "always minimally repair" (AMR) and the "always replace" (AR) strategies for warranty servicing. However, expected warranty servicing costs can be reduced by using a more complex repair–replace strategy involving repair and replacement in different parts of the warranty region. In the one-dimensional case, such strategies have been studied by Nguyen and Murthy [20], Jack and Murthy [12], and Jack and Van der Duyn Schouten [13]. The two-dimensional case has been studied by Iskandar and Murthy [10] and Iskandar, Murthy, and Jack [11] but using only the one-dimensional approach to failure modeling. The two-dimensional failure modeling techniques developed in this paper can now be used to extend this work and this is currently under investigation.
4. Finally, the problem of selecting the proper form for the bivariate failure intensity based on two-dimensional failure data and the estimation of model parameters are topics currently under investigation by the authors.

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