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Eigenfunction description of laser beams and orbital angular momentum of light

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The propagation of light beams through astigmatic lens systems is accompanied by a transfer of orbital angular momentum. We develop a method to describe propagating light beams by operators of which the field is an eigenfunction. This method is applied to determine when an astigmatic lens system transforms gaussian beams into other gaussian beams and where in the system angular momentum is transferred. We show that the Gouy phase is equal to the dynamical phase of a quantummechanical harmonic oscillator with time-dependent energy.

1. Introduction

With the transverse intensity gradient of a beam of light corresponds a density of transverse momentum. This in turn may lead to a nonvanishing orbital angular momentum along the beam axis. Recently, it was demonstrated that astigmatic lens systems may change this angular momentum [1]. Since lenses normally do not change the polarization of the light, the spin is conserved. In an experiment [2], Hermite–gaussian beams which possess no orbital angular momentum were transformed into Laguerre–gaussian beams, which have a well-defined orbital angular momentum per photon. Since the total angular momentum of matter and light is conserved, this transformation is accompanied by a small torque on the lens system. The transformation of laser beams using astigmatic lenses has recently been investigated experimentally by Tamm and Weiss [3].

In this paper we present a novel theoretical description of light beams, and we apply it to this type of experiments. The method exploits some well-known similarities between quantum mechanics and paraxial wave optics, and makes use of a state-vector representation developed by Stoler [4]. Instead of giving analytical expressions for the electromagnetic field, we specify the field by (hermitian) operators of which the field is an eigenfunction, and by the corresponding eigenvalues. Two operators suffice to determine the spatial dependence of a monochromatic beam and one operator determines its polarization. The propagation of laser light through optical systems can then be described by the evolution of the 'eigenoperators' of the light field. In this way, we derive conditions under which gaussian beams are transformed into other gaussian beams. A case of special interest occurs when Hermite–gaussians are converted into Laguerre–gaussian modes. Analogously, the change of angular momentum of a field by lenses and lens systems is described by the evolution of the corresponding operator $\hat{L}_z$, in the same manner as in the Heisenberg picture in quantum mechanics. We also demonstrate that the Gouy phase is related to the dynamical phase of a time-dependent harmonic oscillator.

2. Linear and angular momentum of classical electromagnetic fields

The local densities of linear momentum and angular momentum of the source-free classical electromagnetic (em) field in vacuum are given by [5,6]

$$\boldsymbol{p}=\epsilon_0 \boldsymbol{E}\times\boldsymbol{B}\,,$$

$$\boldsymbol{j}=\epsilon_0 \boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B})\,, \tag{1}$$

in terms of the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$. These densities correspond to the Noether currents associated with the invariance of the free Maxwell equations under spatial translations and rotations, respectively [6,7]. The total linear and angular momentum of the em field, defined by

$$\boldsymbol{P}=\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{p}\,,\quad \boldsymbol{J}=\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{j}, \tag{2}$$

are therefore conserved quantities in vacuum.

We now consider monochromatic fields with frequency ω, and we use the complex notation

$$\boldsymbol{E}=(\boldsymbol{E}\,\mathrm{e}^{-\mathrm{i}\omega t}+\boldsymbol{E}^*\,\mathrm{e}^{\mathrm{i}\omega t})/2\,,$$
$$\boldsymbol{B}=(\boldsymbol{B}\,\mathrm{e}^{-\mathrm{i}\omega t}+\boldsymbol{B}^*\,\mathrm{e}^{\mathrm{i}\omega t})/2\,, \tag{3}$$

where the asterisk denotes complex conjugation. Then we can eliminate the magnetic field from eqs. (2) by using the Maxwell equation

$$\mathrm{i}\omega\boldsymbol{B}=\nabla\times\boldsymbol{E}\,. \tag{4}$$

For fields that vanish sufficiently fast for $|\boldsymbol{r}|\to\infty$, partial integration in eqs. (2) leads to the following expression for the total linear momentum

$$\boldsymbol{P}=\frac{\epsilon_0}{2\mathrm{i}\omega}\int \mathrm{d}\boldsymbol{r}\sum_{j=x,y,z}E_j^*\nabla E_j\,, \tag{5}$$

where we used that $\nabla\cdot\boldsymbol{E}=0$. The total momentum is manifestly independent of time. The expression (5) takes a form similar to the quantum-mechanical expression for the expectation value of the linear momentum of a particle. This has been noted before in a reciprocal (Fourier) representation [6–8]. In a similar way the total angular momentum can be written as

$$\boldsymbol{J}=\frac{\epsilon_0}{2\mathrm{i}\omega}\int \mathrm{d}\boldsymbol{r}\sum_{j=x,y,z}E_j^*(\boldsymbol{r}\times\nabla)E_j+\frac{\epsilon_0}{2i\omega}\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{E}^*\times\boldsymbol{E}$$
$$\equiv\boldsymbol{L}+\boldsymbol{S}\,. \tag{6}$$

The interpretation of the first and second term on the right hand side of eq. (6) as orbital angular momentum and spin is, although seemingly obvious, not without fundamental difficulties [6–8].

3. The paraxial approximation

Paraxial wave optics [9–11] describes the propagation of light beams whose transverse dimensions are much smaller than the typical longitudinal distance over which the field changes in magnitude. We start by recalling some results from ref. [9]. The transverse dimensions of the beam have the order of magnitude of the beam waist w_0, which is assumed to be much smaller than the diffraction length $l=kw_0^2$, with $k=\omega/c$ the wave number. One writes for the electric-field component of a beam propagating in the z direction

$$\boldsymbol{E}=\mathrm{e}^{\mathrm{i}kz}\boldsymbol{F}\,. \tag{7}$$

The small parameter w_0/l is used as an expansion parameter, which means that derivatives of $\boldsymbol{F}$ with respect to z can be neglected compared to the transverse derivatives. The field $\boldsymbol{F}$ satisfies to lowest order in w_0/l the paraxial wave equation

$$2\mathrm{i}k\frac{\partial}{\partial z}\boldsymbol{F}=-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\boldsymbol{F}\,, \tag{8}$$

which determines the propagation of the field in the z direction for a given field distribution in a plane $z=z_0$. Furthermore, the z component of $\boldsymbol{F}$ is smaller than the transverse components by a factor w_0/l.

Within the paraxial approximation we find it convenient to consider quantities of the em field that are defined as averages over planes with $z=$constant. Thus, the linear and angular momentum per unit of length in a plane $z=z_0$ are given by

$$\mathscr{P}(z_0)=\iint \mathrm{d}x\,\mathrm{d}y\,\boldsymbol{p}(x,y,z_0)\,,$$

$$\mathscr{J}(z_0)=\iint \mathrm{d}x\,\mathrm{d}y\,\boldsymbol{j}(x,y,z_0)\,. \tag{9}$$

Since the integrations in these definitions do not extend over all space, the steps leading to the appealing forms (5) and (6) are not necessarily valid here. For instance, the quantities (9) are not independent of time. Therefore, we will always average such quantities over a period $2\pi/\omega$. Then, in the paraxial approximation the (time-averaged) transverse components of $\mathscr{P}$, and the z component of $\mathscr{J}$ can be rewritten to lowest order in w_0/l as

$$\mathscr{P}_x = \sum_{j=x,y} \frac{\epsilon_0}{2\mathrm{i}\omega} \iint \mathrm{d}x\,\mathrm{d}y\, F_j^* \frac{\partial}{\partial x} F_j ,$$

$$\mathscr{P}_y = \sum_{j=x,y} \frac{\epsilon_0}{2\mathrm{i}\omega} \iint \mathrm{d}x\,\mathrm{d}y\, F_j^* \frac{\partial}{\partial y} F_j ,$$

$$\mathscr{J}_z = \sum_{j=x,y} \frac{\epsilon_0}{2\mathrm{i}\omega} \iint \mathrm{d}x\,\mathrm{d}y\, F_j^* \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) F_j + \frac{\epsilon_0}{2\mathrm{i}\omega} \iint \mathrm{d}x\,\mathrm{d}y\, (F_x^* F_y - F_y^* F_x) \equiv \mathscr{L}_z + \mathscr{S}_z . \tag{10}$$

Here we explicitly neglected derivatives of $\boldsymbol{F}$ with respect to z compared with transverse derivatives, and used that the z component of $\boldsymbol{F}$ is small [9]. The z component of the linear momentum per unit length is to lowest order in w_0/l given by

$$\mathscr{P}_z = \sum_{j=x,y} \frac{\epsilon_0}{2\omega} \iint \mathrm{d}x\,\mathrm{d}y\, F_j^* k F_j . \tag{11}$$

From the well-known expression for the local energy density of the em field [5,6],

$$u = \frac{\epsilon_0}{2} (\boldsymbol{E}^2 + c^2 \boldsymbol{B}^2) , \tag{12}$$

we obtain the time-averaged field energy per unit of length $\mathscr{E}$,

$$\mathscr{E} = \mathscr{N}\hbar\omega = \sum_{j=x,y} \frac{\epsilon_0}{2} \iint \mathrm{d}x\,\mathrm{d}y\, F_j^* F_j , \tag{13}$$

where $\mathscr{N}$ is the number of photons per unit of length. Division of quantities like eqs. (10) and (11) by $\mathscr{N}$ yields the value of that quantity per photon. For instance, the momentum in the z direction is, obviously, equal to $\hbar k$ per photon.

4. Operator formalism and notation

4.1. Schrödinger picture

The propagation of light beams can also be described in an operator formalism which is very similar to the operator description of the hamiltonian evolution of quantum-mechanical states. We use here the formalism developed by Stoler [4]. Moreover, the analogy between the expressions (10)–(13) for time-averaged quantities of the classical em field, and the quantum-mechanical expectation values of the same quantities for particles, allows us to extend that formalism (see also ref. [11]). We use the following notations and conventions:

- the field $\boldsymbol{F}(x, y, z)$ is represented by the ket vector $|\boldsymbol{F}(z)\rangle$;
- operators acting upon the field $\boldsymbol{F}$ are represented by corresponding operators, denoted by a caret, acting upon the ket $|\boldsymbol{F}\rangle$;
- the evolution of a field propagating from $z = z_0$ to $z = z_1$ through a given lossless optical system is determined by a unitary operator $\hat{h}$, according to

$$|\boldsymbol{F}(z_1)\rangle = \hat{h} |\boldsymbol{F}(z_0)\rangle ; \tag{14}$$

the coordinate z plays the same role as time in ordinary quantum mechanics;
- the scalar product $\langle \boldsymbol{F}(z) \| \boldsymbol{G}(z)\rangle$ is defined as

$$\langle \boldsymbol{F}(z) \| \boldsymbol{G}(z)\rangle = \frac{\epsilon_0}{2\omega} \iint \mathrm{d}x\,\mathrm{d}y\, \boldsymbol{F}^*(x,y,z) \cdot \boldsymbol{G}(x,y,z) = \sum_{j=x,y} \frac{\epsilon_0}{2\omega} \iint \mathrm{d}x\,\mathrm{d}y\, F_j^*(x,y,z) G_j(x,y,z) , \tag{15}$$

to lowest order in w_0/l.

When $\hat{Q}$ is an hermitian operator, the quantity $\mathscr{Q}(z)$ defined by

$$\mathscr{Q}(z) = \langle \boldsymbol{F}(z) | \hat{Q} | \boldsymbol{F}(z)\rangle \tag{16}$$

is real. For example, when we introduce the operators for transverse momentum

$$\hat{p}_x = \frac{1}{\mathrm{i}} \frac{\partial}{\partial x} , \qquad \hat{p}_y = \frac{1}{\mathrm{i}} \frac{\partial}{\partial y} , \tag{17}$$

then it follows from eqs. (10) that

$$\mathscr{P}_x = \langle \boldsymbol{F} | \hat{p}_x | \boldsymbol{F}\rangle , \qquad \mathscr{P}_y = \langle \boldsymbol{F} | \hat{p}_y | \boldsymbol{F}\rangle . \tag{18}$$

The momentum operators obey the usual commutation relations with the position operators $\hat{x}$ and $\hat{y}$,

$$[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = \mathrm{i} . \tag{19}$$

As another example, we introduce the operator $\hat{L}_z$ for the z component of the orbital angular momentum acting upon the external degrees of freedom of the field, by

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x . \qquad (20)$$

The spin operator $\hat{S}_z$ on the other hand acts upon the vectorial indices of the field,

$$\hat{S}_z |F_j\rangle = \sum_{k=x,y} \mathrm{i}\epsilon_{jkz} |F_k\rangle , \qquad (21)$$

with ϵ_{jkl} the completely anti-symmetrical Levi–Civita tensor. A matrix representation of the spin operator is therefore

$$\hat{S}_z = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} . \qquad (22)$$

The eigenvalues of $\hat{S}_z$ are, of course, equal to ± 1. With these operators substituted for $\hat{Q}$ we can write eqs. (10) for $\mathscr{L}_z$ and $\mathscr{S}_z$ in the form (16).

The evolution of a light beam propagating in vacuum is found from the formal solution of eq. (8) [4]. It can be written in the form (14), with $\hat{h}$ the free propagation operator describing evolution,

$$\hat{h} = \exp\left(- \frac{\mathrm{i}(z_1 - z_0)}{2k} \hat{p}^2 \right) , \qquad (23)$$

where we defined $\hat{p}^2 = \hat{p}_x^2 + \hat{p}_y^2$.

4.2. Heisenberg picture

When a field evolves according to eq. (14), the evolution of a quantity $\mathscr{Q}(z)$, defined in eq. (16), is given by

$$\mathscr{Q}(z_1) = \langle \boldsymbol{F}(z_0) | \hat{h}^\dagger \hat{Q} \hat{h} | \boldsymbol{F}(z_0) \rangle , \qquad (24)$$

with $\hat{h}^\dagger$ the Hermitian conjugate of $\hat{h}$. As in the Heisenberg picture in quantum mechanics we can interpret the operator $\hat{h}^\dagger \hat{Q} \hat{h}$ as giving the evolution of the quantity corresponding to the operator $\hat{Q}$, where now the field is described by the state vector $|\boldsymbol{F}(z_0)\rangle$ at a single value z_0 of z. This picture is very useful when examining certain properties of physical quantities of the form (16) that are independent of the fields. For instance, if the operator $\hat{Q}$ commutes with the operator $\hat{p}^2$, it also commutes with the free propagation operator (23). This implies that

$$\hat{h}^\dagger \hat{Q} \hat{h} = \hat{Q} , \qquad (25)$$

which in turn implies that the quantity $\mathscr{Q}(z)$ is conserved under free propagation, i.e.

$$\frac{\mathrm{d}\mathscr{Q}}{\mathrm{d}z} = 0 . \qquad (26)$$

Thus the z components of both orbital angular momentum and spin are conserved during free propagation, since the operators $\hat{L}_z$ and $\hat{S}_z$ both commute with $\hat{p}^2$. This means that their value per unit length is uniform along z. Also the energy per unit length (13) is a conserved quantity, as it is represented by the identity operator. Note that 'conserved' here does not refer to conservation in time, but rather to conservation with the position z.

5. Ideal lenses

A lens is called ideal when it is sufficiently thin, so that propagation within the lens is negligible. An ideal lens modifies an incoming field distribution $\boldsymbol{F}(x, y, z)$ by adding a local phase $\psi(x, y)$ to the field, where ψ is a real function of (x, y) [12]. If the lens has a constant refractive index n, the phase change ψ is due to the local change of the optical path, which is proportional to the local thickness $b(x, y)$ of the lens,

$$\psi(x, y) = kb(x, y)(n-1) . \qquad (27)$$

In the operator formalism of the preceding section, the ideal lens is represented by the operator

$$\hat{T} = \exp[\mathrm{i}\psi(\hat{x}, \hat{y})] . \qquad (28)$$

In particular, a cylindrical lens with its axis oriented at an angle γ with the positive x axis is described by the phase

$$\psi(x, y) = -\frac{k}{2f}(x\cos\gamma + y\sin\gamma)^2 , \qquad (29)$$

with f the focal length of the lens along its axis. The focal length in the orthogonal direction is infinite. Within the Heisenberg picture it is easy to prove that the z component of the spin, $\mathscr{S}_z$, cannot be changed by a lens. Since the operator $\hat{S}_z$ does not act upon the coordinates of the field,

$$\hat{T}^\dagger \hat{S}_z \hat{T} = \hat{S}_z , \qquad (30)$$

for arbitrary $\hat{T}$ defined in eq. (28). The orbital part of the angular momentum does change in general, since

$$\hat{T}^{\dagger}\hat{L}_z\hat{T}=\hat{L}_z+\hat{x}\frac{\partial\psi}{\partial\hat{y}}-\hat{y}\frac{\partial\psi}{\partial\hat{x}}$$
$$\equiv \hat{L}_z+\delta\hat{L}_z\,. \qquad (31)$$

For a general field $|\boldsymbol{F}_{\rm in}\rangle$ incident on the lens, the difference in the value of $\mathscr{L}_z$ in the output field and its value in the input field is

$$\delta\mathscr{L}_z=\langle \boldsymbol{F}_{\rm in}|\,\delta\hat{L}_z\,|\boldsymbol{F}_{\rm in}\rangle\,. \qquad (32)$$

Therefore the angular momentum absorbed by the lens per unit time is $-c\delta\mathscr{L}_z$. This is the total torque on the lens. In terms of the output field $|\boldsymbol{F}_{\rm out}\rangle$ the relation (32) becomes

$$\delta\mathscr{L}_z=-\langle \boldsymbol{F}_{\rm out}|\,\delta\hat{L}_z\,|\boldsymbol{F}_{\rm out}\rangle\,, \qquad (33)$$

which follows from the relation

$$\hat{T}\hat{L}_z\hat{T}^{\dagger}=\hat{L}_z-\delta\hat{L}_z\,. \qquad (34)$$

For a spherical lens the operator $\delta\hat{L}_z$ vanishes. Hence, such a lens can never absorb angular momentum in the z direction from an em field. For a cylindrical lens satisfying eq. (29) one finds

$$\delta\hat{L}_z(\gamma)=\frac{k}{2f}\left[2\hat{x}\hat{y}\cos 2\gamma-(\hat{x}^2-\hat{y}^2)\sin 2\gamma\right]\,. \qquad (35)$$

From eq. (35) it follows that

$$\delta\hat{L}_z(\gamma\pm\pi/2)=-\delta\hat{L}_z(\gamma)\,. \qquad (36)$$

Thus, when a cylindrical lens is rotated over $\pi/2$ about the z axis, the amount of orbital angular momentum absorbed by the lens changes sign. This implies, that for any incoming field there is always an orientation γ_0 of the lens for which $\mathscr{L}_z$ of that field is not changed by the lens. Then it follows from eq. (35) again that the maximum change in the value of $\mathscr{L}_z$, $\delta\mathscr{L}_{\rm max}$, is attained for an orientation angle $\gamma=\gamma_0\pm\pi/4$. In fact, the change in orbital angular momentum of an arbitrary field due to a cylindrical lens can always be written as

$$\delta\mathscr{L}_z=\pm\delta\mathscr{L}_{\rm max}\sin(2\gamma-2\gamma_0)\,. \qquad (37)$$

6. Applications

6.1. Circular fields

As is well known, the operator $\hat{L}_z$ takes the simple form

$$\hat{L}_z=-\mathrm{i}\frac{\partial}{\partial\phi} \qquad (38)$$

in cylindrical coordinates

$$x=r\cos\phi;\quad y=r\sin\phi;\quad z=z\,. \qquad (39)$$

We define a circular field as an eigenfunction of $\hat{L}_z$. These fields have an azimuthal dependence given by $\exp(\mathrm{i}l\phi)$. Since ϕ is a periodic variable, l must be an integer. In a circular field with eigenvalue l the orbital angular momentum per photon amounts to $l\hbar$ [1], and the intensity is independent of ϕ. Moreover, since $\hat{L}_z$ commutes with the free propagation operator, circular fields remain circular during free propagation. We shall denote these fields by kets $|l\rangle$. Examples of circular fields are the Laguerre–gaussian modes [10].

We can now calculate the effect of an arbitrary ideal lens on the orbital angular momentum of a circular field. The change $\delta\mathscr{L}_z$ in orbital angular momentum of an arbitrary input field $|\boldsymbol{F}\rangle$, eq. (32), can be rewritten in cylindrical coordinates:

$$\delta\mathscr{L}_z=\langle\boldsymbol{F}|\,\partial\psi/\partial\phi\,|\boldsymbol{F}\rangle$$
$$=\frac{\epsilon_0}{2\omega}\int_0^\infty \mathrm{d}r\int_0^{2\pi} r\,\mathrm{d}\phi\,\boldsymbol{F}^*\!\cdot\!\boldsymbol{F}\frac{\partial\psi}{\partial\phi}$$
$$=-\frac{\epsilon_0}{2\omega}\int_0^\infty \mathrm{d}r\int_0^{2\pi} r\,\mathrm{d}\phi\,\psi\frac{\partial}{\partial\phi}(\boldsymbol{F}^*\!\cdot\!\boldsymbol{F})\,, \qquad (40)$$

where we used partial integration to obtain the last equality. From this equality it is clear that no ideal lens can change the orbital angular momentum of a circular field. Hence

$$\delta\mathscr{L}_z=\langle l|\,\delta\hat{L}_z\,|l\rangle=0\,. \qquad (41)$$

With eq. (33), this also shows that the lens cannot have changed the orbital angular momentum of a field when the *out*coming beam is circular.

6.2. A cylindrical lens system

In this subsection we consider a particular optical system, which has been used in the experiments mentioned in the Introduction [2]. The system consists of two identical cylindrical lenses, oriented along the x axis, with focal length f, and separated from each other by a distance $2d$ (fig. 1). In ref. [3] a slightly different configuration was used, in that the two lenses had different focal lengths. Our results can easily be generalized to include this case. However, an arbitrary conversion of a gaussian beam can already be realized with lenses with equal focal lengths (see sect. 7). The propagation operator $\hat{h}$ for this system is

$$\hat{h}=\exp\left(\frac{-\mathrm{i}k\hat{x}^2}{2f}\right)\exp\left(\frac{-\mathrm{i}d\hat{p}^2}{k}\right)\exp\left(\frac{-\mathrm{i}k\hat{x}^2}{2f}\right). \tag{42}$$

The action of the system can be specified by giving the evolution of the coordinate operators $\hat{x}$ and $\hat{y}$, and of the momentum operators $\hat{p}_x$ and $\hat{p}_y$ in the Heisenberg picture,

$$\hat{h}^\dagger\hat{x}\hat{h}=\hat{x}\left(1-\frac{2d}{f}\right)+\frac{2d}{k}\hat{p}_x\,,$$

$$\hat{h}^\dagger\hat{y}\hat{h}=\hat{y}+\frac{2d}{k}\hat{p}_y\,,$$

$$\hat{h}^\dagger\hat{p}_x\hat{h}=\hat{p}_x\left(1-\frac{2d}{f}\right)-\frac{2k}{f}\left(1-\frac{d}{f}\right)\hat{x}\,,$$

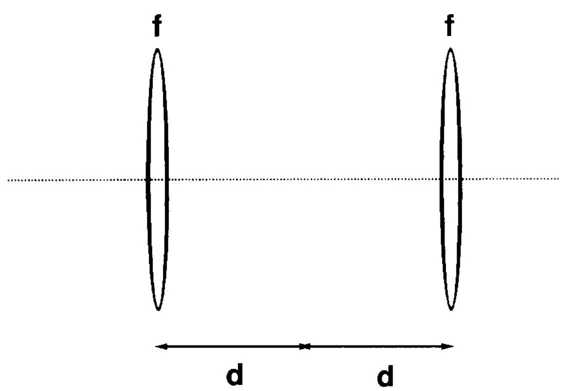

Fig. 1. Configuration of two parallel cylindrical lenses with focal length f at mutual distance $2d$.

$$\hat{h}^\dagger\hat{p}_y\hat{h}=\hat{p}_y\,, \tag{43}$$

obtained by means of the Baker–Campbell–Hausdorff formula [13]

$$\exp(B)\,A\exp(-B)$$

$$=A+[B,A]+\frac{1}{2!}[B,[B,A]]+\cdots\,. \tag{44}$$

From eqs. (43) one finds that the orbital angular momentum changes according to

$$\hat{h}^\dagger\hat{L}_z\hat{h}=\hat{L}_z\left(1-\frac{2d^2}{f^2}\right)+\frac{2k}{f}\left(1-\frac{d}{f}\right)\hat{x}\hat{y}+\frac{4d^2}{kf}\hat{p}_x\hat{p}_y$$

$$+(\hat{x}\hat{p}_y+\hat{y}\hat{p}_x)\left(\frac{2d}{f}-\frac{2d^2}{f^2}\right). \tag{45}$$

In particular, for incoming circular fields $|l\rangle$ only the first term on the right hand side of eq. (45) contributes, and one finds for the change in $\mathscr{L}_z$

$$\delta\mathscr{L}_z=-\frac{2d^2}{f^2}\langle l|\hat{L}_z|l\rangle\,, \tag{46}$$

so that the lens system absorbs an amount $2l\hbar d^2/f^2$ of angular momentum per photon. However, the output field cannot be circular. In fact, when both in- and outcoming fields would be circular, then neither of the two lenses could have changed $\mathscr{L}_z$, as shown in section 6.1; furthermore, during the free propagation between the lenses $\mathscr{L}_z$ is conserved (section 4.2), so that the orbital angular momentum of in- and outcoming fields would have to be the same. This is in contradiction with eq. (46), since the multiplication factor is always negative.

From similar arguments is clear that the change in angular momentum of an incoming circular field takes place exclusively at the second lens.

Moreover, eq. (46) shows that we must have $d=f/\sqrt{2}$ in order that the lens system changes a circular field $|l\rangle$ with $l\neq0$ into a field with zero angular momentum. Conversely, the same condition is required to produce a circular field $|l\rangle$ with $l\neq0$ from an input field that carries no angular momentum (cf. ref. [2]).

6.3. A paradox

Geometrical optics suggests a simple way to convert a circular field $|l\rangle$ into the opposite state $|-l\rangle$.

This is achieved when the system inverts the x direction, without changing the y direction. For a well-collimated beam this simple reflection is produced by the configuration of fig. 2, with $d=f$. This configuration has been termed a π convertor [2]. The picture of x inversion is confirmed by eqs. (43), provided that the last terms of the first two equations are negligible. This requires that

$$\delta \equiv d/kw^2 \ll 1 , \tag{47}$$

where w is the transverse dimension of the incoming beam. Then the lens system transforms a field $|\boldsymbol{F}\rangle$ according to

$$\boldsymbol{F}(x,y) \to \boldsymbol{F}(-x,y) , \tag{48}$$

apart from a phase factor, and a small $\mathcal{O}(\delta)$ correction. When applied to a circular field $|l\rangle$, this transformation (48) yields another circular field $|-l\rangle$. This simple geometrical picture is confirmed by eq. (46), which shows that for $d=f$ the momentum transfer is exactly $-2\hbar l$ per photon. However, this transformation is in contradiction with the conclusion of section 6.2, namely that for a circular input field the output field cannot be circular. In fact, the conversion is only exact in the geometrical optics limit $k\to\infty$ (so that $\delta\to 0$). Hence, when the input field is exactly the circular state $|l\rangle$, then the output field must be written as $|-l\rangle + \delta|s\rangle$, where $|s\rangle$ stands for some superposition of circular fields. Then the input field has an angular momentum per photon equal to $l\hbar$, and for the output field this value is exactly $-l\hbar$ (when $d=f$). Furthermore, the transfer of angular momentum must occur at the second lens, as we showed in section 6.2.

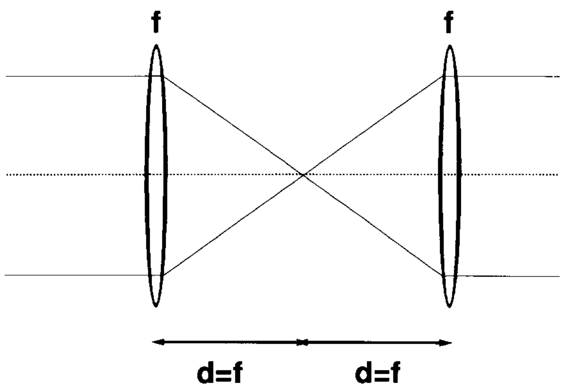

Fig. 2. In the geometrical optics limit the lens system of fig. 1. inverts the image in one direction when $d=f$.

This raises another problem: on the one hand the second lens transfers $2l\hbar$ per photon when δ is small but finite, while on the other hand there can be no transfer when the output field is circular (i.e. for $\delta=0$). This discontinuity can be explained from eq. (29). The phase that the lens adds to the field is of the order of $1/\delta$. Therefore the effect of the second lens on the orbital angular momentum can be arbitrarily large for $\delta\to 0$. Thus, for an *exact* circular output field there is no transfer of $\mathcal{L}_z$, but small $\mathcal{O}(\delta)$ corrections to a pure circular field give in general rise to a finite transfer of $\mathcal{L}_z$ of order unity in units of $\hbar$ per photon. More precisely, since the output field is given by $|-l\rangle + \delta|s\rangle$, one finds from eqs. (33) and (35) the finite change in $\mathcal{L}_z$ by the second cylindrical lens in the limit $\delta\to 0$,

$$\delta\mathcal{L}_z = -2\,\mathrm{Re}\langle s|\hat{x}\hat{y}/w^2|-l\rangle . \tag{49}$$

A similar argument is valid when the *input* field is not exactly circular.

We conclude that the total transfer of angular momentum is close to $-2\hbar l$ per photon when the input field is nearly circular, and when d is close to f. However, the location where the angular momentum is transferred is very sensitive to the input state: a small admixture of other states can give a drastic variation in the distribution of the transfer over the two lenses.

For noncircular input states, eq. (45) shows that the total amount of angular momentum transfer depends in a very sensitive way on d (or f). This is due to the second term in eq. (45), which is proportional to $1/\delta$. When the average of xy vanishes, which is the case for circular fields, this term does not contribute, and the sensitivity disappears.

7. Eigenoperators of fields

We earlier defined circular fields by requiring them to be eigenfunctions of the operator $\hat{L}_z$. In a similar way we can specify a field $|\boldsymbol{F}(z)\rangle$, for given wave number k, by giving three commuting hermitian operators of which the field is an eigenmode, and the corresponding eigenvalues. One of the operators, $\hat{S}(z)$, refers to the vector character (polarization or spin) of the field. The other two operators $\hat{K}_i(z)$, for

$i=1, 2$, act upon the translational degrees of freedom of the field, and determine its (x,y) dependence. Thus, the field $|\boldsymbol{F}(z)\rangle$ is uniquely determined, up to a normalization factor and an arbitrary phase, by

$$\hat{S}(z)|\boldsymbol{F}(z)\rangle = s|\boldsymbol{F}(z)\rangle ,$$
$$\hat{K}_i(z)|\boldsymbol{F}(z)\rangle = \mu_i|\boldsymbol{F}(z)\rangle . \quad (50)$$

The operators $\hat{S}$ and $\hat{K}_i$ may be called eigenoperators of the field. The eigenvalue of the polarization operator $\hat{S}$ equals $s=\pm 1$, and the eigenvalues μ_i are real, and independent of z (see below). When a field propagates through a given optical system, it evolves according to eq. (14), and it will in general no longer satisfy eqs. (50). Instead, the field $\hat{h}|\boldsymbol{F}\rangle$ is an eigenfunction of the operators

$$\hat{S}' = \hat{h}\hat{S}\hat{h}^\dagger ,$$
$$\hat{K}'_i = \hat{h}\hat{K}_i\hat{h}^\dagger , \quad (51)$$

with the same eigenvalues as in eqs. (50). The evolution of an em field is thus also described (up to a phase factor) by the evolution (eq. (51)) of its eigenoperators. This description is sometimes more convenient than dealing with the explicit spatial dependence of the fields. Notice that the operators $\hat{K}_i(z)$ as a function of z do not obey the Heisenberg equation, but rather the Liouville–Von Neumann equation.

Since the optical systems we consider in this paper do not contain polarization-changing elements, polarization is conserved, and will henceforth be left out of consideration. From now on we only discuss scalar fields.

7.1. Hermite- and Laguerre-gaussian modes

We apply the eigenoperator description to Hermite–gaussian (HG) and Laguerre–gaussian (LG) modes, which are commonly used in laser physics [10]. We first give the eigenoperators of these fields in their focal plane, which is the plane where the transverse dimension of the beam has a minimum. The focal plane is chosen to be the plane $z=0$. In an arbitrary plane $z=z_0$ the fields can then be constructed formally by letting the fields propagate freely from $z=0$ to $z=z_0$. In the next section we will show that the modes defined in this way are indeed the HG and LG modes.

We define a HG field in its focal plane by the eigenvalue equations

$$\frac{1}{2}\left(\frac{k}{\beta}\hat{x}^2 + \frac{\beta}{k}\hat{p}_x^2\right)|H^\beta_{nm}(0)\rangle = (n+\tfrac{1}{2})|H^\beta_{nm}(0)\rangle ,$$
$$\frac{1}{2}\left(\frac{k}{\beta}\hat{y}^2 + \frac{\beta}{k}\hat{p}_y^2\right)|H^\beta_{nm}(0)\rangle = (m+\tfrac{1}{2})|H^\beta_{nm}(0)\rangle . \quad (52)$$

Thus a HG field $|H^\beta_{nm}(0)\rangle$ is an eigenfunction of a 'Hamiltonian' of two identical harmonic oscillators in the x and y direction, with 'energy' $n+\frac{1}{2}$ and $m+\frac{1}{2}$, respectively. The parameter β can be identified with the Rayleigh range or half the confocal parameter of the beam [10]. Analogously, LG fields with the same Rayleigh range β can be defined by

$$\hat{L}_z|L^\beta_{lN}(0)\rangle = l|L^\beta_{lN}(0)\rangle ,$$
$$\frac{1}{2}\left(\frac{k}{\beta}\hat{r}^2 + \frac{\beta}{k}\hat{p}^2\right)|L^\beta_{lN}(0)\rangle = (N+1)|L^\beta_{lN}(0)\rangle , \quad (53)$$

where $\hat{r}^2 = \hat{x}^2 + \hat{y}^2$. Thus a LG field $|L^\beta_{lN}(0)\rangle$ is an eigenfunction of $\hat{L}_z$, with eigenvalue l, and of a 2-D degenerate harmonic oscillator, with 'energy' $N+1$. The latter operator is just the sum of the two harmonic oscillators in eqs. (52). From quantum mechanics it is known that one obtains solutions only for $|l|\le N$. The two eigenvalue equations (53) determine the ϕ and r dependence of LG fields. By applying the free propagation operator (23) to the fields at $z=0$ we obtain the HG and LG modes for arbitrary z,

$$|H^\beta_{nm}(z)\rangle = \exp\left(\frac{-\mathrm{i}z}{2k}\hat{p}^2\right)|H^\beta_{nm}(0)\rangle ,$$
$$|L^\beta_{lN}(z)\rangle = \exp\left(\frac{-\mathrm{i}z}{2k}\hat{p}^2\right)|L^\beta_{lN}(0)\rangle . \quad (54)$$

For $z\neq 0$ these fields are no longer eigenfunctions of the same, or of any other, harmonic oscillator. Namely, we find that the eigenoperators of the HG mode evolve during free propagation as

$$\hat{K}_1(z) = \frac{1}{2}\left[\frac{k}{\beta}\hat{x}^2 + \left(\frac{\beta}{k} + \frac{z^2}{k\beta}\right)\hat{p}_x^2 - \frac{z}{\beta}\hat{t}_x\right],$$
$$\hat{K}_2(z) = \frac{1}{2}\left[\frac{k}{\beta}\hat{y}^2 + \left(\frac{\beta}{k} + \frac{z^2}{k\beta}\right)\hat{p}_y^2 - \frac{z}{\beta}\hat{t}_y\right], \quad (55)$$

and the eigenvalues remain $n+\frac{1}{2}$ and $m+\frac{1}{2}$. Relation (55) follows from eqs. (51) with eq. (23) substituted for $\hat{h}$ while using the BCH formula (44). We defined here the operators $\hat{t}_x$ and $\hat{t}_y$ and their sum $\hat{t}$ by

$$\hat{t}_x=\hat{x}\hat{p}_x+\hat{p}_x\hat{x}\,,$$

$$\hat{t}_y=\hat{y}\hat{p}_y+\hat{p}_y\hat{y}\,,$$

$$\hat{t}=\hat{t}_x+\hat{t}_y\,. \tag{56}$$

The LG mode $|L^{\beta}_{lN}(z)\rangle$ is an eigenmode of $\hat{L}_z$ with eigenvalue l for all z, since $\hat{L}_z$ commutes with $\hat{p}^2$. Its second eigenoperator is given by $\hat{K}(z)=\hat{K}_z(z)+\hat{K}_2(z)$, and the eigenvalue remains $N+1$.

7.2. Gaussian modes

We have shown that HG and LG modes have one eigenoperator in common, of the general form

$$\hat{K}(\boldsymbol{a})=a_1\hat{r}^2+a_2\hat{p}^2+a_3\hat{t}\,, \tag{57}$$

which is invariant under rotations about the z axis. In their focal plane both fields are eigenfunctions of a 2D degenerate harmonic oscillator (DHO), since there a_3 vanishes. Fields with an eigenoperator of the form (57) with arbitrary values a_1, a_2 and a_3 will be called gaussian fields. This is in accordance with textbook nomenclature [10], where gaussian fields are written as an eigenfunction of a DHO multiplied by a position-dependent phase factor (see next section). Such fields remain gaussian during their free propagation. Furthermore, it is easily proved that any gaussian field can be written as a linear superposition of HG modes $|H^{\beta}_{nm}(z)\rangle$ with $n+m=$constant, and with a properly selected value of β and of the position of the focal plane. The converse statement is obviously also true. Thus, any HG mode $|H^{\beta}_{nm}(z)\rangle$ that is rotated about the z axis can be written as such a superposition. The same is true for the LG mode $|L^{\beta}_{lN}(z)\rangle$, where $n+m$ must equal N.

7.3. Mode convertors

The eigenoperator (57) of a gaussian field will be transformed by a cylindrical lens into an operator that is no longer invariant under rotations about the z axis. Then the output field is not gaussian. However, in special cases, a lens system may convert a gaussian field into another gaussian field. Then we call the lens system a mode convertor [1,2]. The condition for a mode convertor is, that the operator

$$\hat{K}'=\hat{h}\hat{K}(\boldsymbol{a})\hat{h}^{\dagger} \tag{58}$$

be of the form (57) again. We substitute for $\hat{h}$ the evolution operator (42) of the lens system from section 6.2. For a given incoming field, this yields the conditions under which the lens system is a mode convertor for this field. We obtain

$$d=k\frac{a_3}{a_1}\,,$$

$$f=k\frac{a_3a_2}{a_2a_1-2a_3^2}\,, \tag{59}$$

in terms of the coefficients a_i of the eigenoperator (57) of the incoming field. For incoming HG modes with eigenoperators (55) we can substitute $a_1=k/\beta$ and $a_3=-z/\beta$ in the conditions (59), which then become

$$z=-d\,,$$

$$f=d\frac{\beta^2+d^2}{\beta^2-d^2}\,. \tag{60}$$

The first condition means that for a mode convertor the focal plane of the incoming HG mode must be halfway between the two lenses [2].

7.4. Production of circular fields

We now wish to investigate when it is possible to convert a HG mode $|H^{\beta}_{nm}(z)\rangle$ into a field that has the orbital angular momentum operator $\hat{L}_z$ as an eigenoperator, that is, into a circular field.

The two eigenoperators of the HG mode were given in eqs. (55). Eigenoperators of this form will keep the same form during their propagation through the cylindrical lens system of section 6.2. The reason is (i) that the evolution operator (42) of the lens system consists of exponentials of the operators occurring in (55) and (ii) that these operators form a closed (Lie) algebra. In fact, they generate the direct product SU(1,1)×SU(1,1) [13]. The operators (55) will therefore evolve, according to eqs. (51), within that algebra during the entire propagation. This algebra does not contain $\hat{L}_z$. Therefore, the lens system satisfying eq. (42) alone is not sufficient to produce an eigenfunction of $\hat{L}_z$ from a HG mode $|H^{\beta}_{nm}(z)\rangle$. Both eigenoperators of this field are and remain of the form (55).

However, application of a rotation about the z axis to a HG mode before it enters the lens system will transform eigenoperators in such a way that they will belong to a larger algebra which does contain $\hat{L}_z$. The reason is, that rotations around the z axis are generated by $\hat{L}_z$: the rotation operator $\hat{R}_\gamma$ describing a rotation around the z axis over an angle γ is given by

$$\hat{R}_\gamma = \exp(\mathrm{i}\gamma\hat{L}_z) . \tag{61}$$

If the HG mode $|H^{\beta}_{nm}(z)\rangle$, rotated over an angle γ and having passed the lens system, is to be an eigenfunction of $\hat{L}_z$, we must require $\hat{L}_z$ to be a linear combination of the two eigenoperators of the output field

$$\hat{K}'_i = \hat{h}\hat{R}_\gamma\hat{K}_i\hat{R}_\gamma^{-1}\hat{h}^\dagger , \tag{62}$$

for $i=1, 2$, with $\hat{K}_1$ and $\hat{K}_2$ the eigenoperators of the incoming field, defined in eqs. (55). Conversely, the operator

$$\hat{L}'_z = \hat{R}_\gamma^{-1}\hat{h}^\dagger\hat{L}_z\hat{h}\hat{R}_\gamma \tag{63}$$

must be a linear combination of the operators $\hat{K}_1$ and $\hat{K}_2$. Using eq. (45) one finds

$$\begin{aligned}\hat{R}_\gamma^{-1}\hat{h}^\dagger\hat{L}_z\hat{h}\hat{R}_\gamma &= \hat{L}_z\left(1-\frac{2d^2}{f^2}\right) \\ &+\left(\frac{2d}{f}-\frac{2d^2}{f^2}\right)[(\hat{x}\hat{p}_y+\hat{y}\hat{p}_x)\cos 2\gamma \\ &+\sin 2\gamma(\hat{y}\hat{p}_y-\hat{x}\hat{p}_x)] \\ &+\frac{4d^2}{kf}[\hat{p}_x\hat{p}_y\cos 2\gamma+\tfrac{1}{2}\sin 2\gamma(p_y^2-\hat{p}_x^2)] \\ &+\frac{2k}{f}\left(1-\frac{d}{f}\right)[\hat{x}\hat{y}\cos 2\gamma+\tfrac{1}{2}\sin 2\gamma(\hat{y}^2-\hat{x}^2)] .\end{aligned} \tag{64}$$

Inspection of eq. (64) shows that this expression is indeed a linear combination of $\hat{K}_1$ and $\hat{K}_2$:

$$\hat{R}_\gamma^{-1}\hat{h}^\dagger\hat{L}_z\hat{h}\hat{R}_\gamma = \pm(\hat{K}_2-\hat{K}_1) , \tag{65}$$

if and only if

$$\gamma=\pm\pi/4 , \quad f=d\sqrt{2} , \quad z=-d , \quad \beta=d(1+\sqrt{2}) . \tag{66}$$

These restrictions are the same as those given for the '$\pi/2$ convertor' [2]. Here we have proved that the rotation angle γ must equal $\pm\pi/4$, and that the focal plane of the HG mode must be half way between the two lenses. For the eigenvalue l of $\hat{L}_z$ one finds, from eq. (65),

$$l=\pm(m-n) , \tag{67}$$

because the eigenvalues corresponding to $\hat{K}_2$ and $\hat{K}_1$ are $m+\frac{1}{2}$ and $n+\frac{1}{2}$, respectively. Since the relations (66) are consistent with eqs. (60), the lens system acts in this case also as a mode convertor. That is, the outcoming mode is not just an eigenfunction of $\hat{L}_z$, it is in fact the LG field $|L^{\beta}_{l,n+m}(d)\rangle$.

8. Properties of gaussian modes

We show here that the eigenoperators provide a simple way to explain that a freely propagating field, which is an eigenfunction of a 2D degenerate harmonic oscillator (DHO) in one plane, can be written as the product of a local phase factor and an eigenfunction of another DHO, in any other plane z=constant. Subsequently, we show that the Gouy phase of gaussian beams [10] can be identified with the dynamical phase of a quantum-mechanical DHO with time-dependent energy.

We consider a field $|F_N(0)\rangle$ that is an eigenfunction of the DHO

$$\hat{K}(0)=\frac{1}{2}\left[\frac{\hat{r}^2}{\alpha_0}+\alpha_0\hat{p}^2\right], \tag{68}$$

with eigenvalue $N+1$. When the field propagates through the vacuum, it becomes an eigenfunction of the operator

$$\hat{K}(z)\equiv\exp\left(\frac{-\mathrm{i}z}{2k}\hat{p}^2\right)\frac{1}{2}\left[\frac{\hat{r}^2}{\alpha_0}+\alpha_0\hat{p}^2\right]\exp\left(\frac{\mathrm{i}z}{2k}\hat{p}^2\right), \tag{69}$$

at position z. We now search for parameters R and α such that this eigenoperator $\hat{K}(z)$ is again a DHO up to a position-dependent phase factor. More precisely, we require

$$\hat{K}(z)=\exp\left(\frac{\mathrm{i}k}{2R}\hat{r}^2\right)\frac{1}{2}\left[\frac{\hat{r}^2}{\alpha}+\alpha\hat{p}^2\right]\exp\left(\frac{-\mathrm{i}k}{2R}\hat{r}^2\right). \tag{70}$$

This relation implies that the field $|F(z)\rangle$ at position z is the product of the phase factor $\exp(ik\hat{r}^2/2R)$ and an eigenfunction of a DHO of the form (68) with α_0 replaced by $\alpha(z)$. Its eigenvalue is $N+1$. Using the BCH formula (44) we find the unique solution of eq. (70) to be

$$R(z)=\frac{z^2+\alpha_0^2k^2}{z},$$

$$\alpha(z)=\frac{z^2+\alpha_0^2k^2}{k^2\alpha_0}. \tag{71}$$

For HG and LG modes we may substitute the value $\alpha_0=\beta/k$, according to eqs. (52) and (53). Then we find

$$R(z)=\frac{z^2+\beta^2}{z},$$

$$\alpha(z)=\frac{z^2+\beta^2}{k\beta}. \tag{72}$$

These parameters are, indeed, equal to the local radius of curvature of these modes, and to the square of the local beam radius, respectively [10].

8.1. Gouy phase

As just shown we can rewrite the field $|F_N(z)\rangle$ as

$$|F_N(z)\rangle=\exp\left(\frac{ik\hat{r}^2}{2R(z)}\right)|H_N(\alpha,z)\rangle, \tag{73}$$

where $|H_N(\alpha,z)\rangle$ is an eigenfunction of a DHO with eigenvalue $N+1$. Using the fact that $|F_N(z)\rangle$ satisfies the paraxial wave equation, we find, after substituting eq. (73),

$$\begin{aligned}2ik\frac{\mathrm{d}}{\mathrm{d}z}|H_N(\alpha,z)\rangle&=\left[k^2\frac{1}{R^2}\frac{\partial R}{\partial z}\hat{r}^2\right.\\&\quad\left.+\exp\left(\frac{-ik}{2R}\hat{r}^2\right)\hat{p}^2\exp\left(\frac{+ik}{2R}\hat{r}^2\right)\right]|H_N(\alpha,z)\rangle\\&=\left[\frac{\hat{r}^2}{\alpha^2}+\hat{p}^2+\frac{k}{R}\hat{t}\right]|H_N(\alpha,z)\rangle\\&=\left[\frac{\hat{r}^2}{\alpha^2}+\hat{p}^2+\frac{k}{2\alpha}\frac{\partial\alpha}{\partial z}\hat{t}\right]|H_N(\alpha,z)\rangle,\end{aligned} \tag{74}$$

with $\hat{t}$ defined in eq. (56). The field distribution corresponding to $|H_N(\alpha,z)\rangle$ can be written as a real function of x and y, represented by the ket $|G_N(\alpha)\rangle$, multiplied by an overall z-dependent phase factor,

$$|H_N(\alpha,z)\rangle=\exp(-i\phi_N(z))|G_N(\alpha)\rangle. \tag{75}$$

The last term on the right hand side of eq. (74) being proportional to $\mathrm{d}\alpha/\mathrm{d}z$ results from the z dependence of the real part $|G_N(\alpha)\rangle$ in eq. (75) through $\alpha(z)$. Indeed one can show that

$$\hat{t}|G_N(\alpha)\rangle=4i\alpha\frac{\partial}{\partial\alpha}|G_N(\alpha)\rangle. \tag{76}$$

The remaining two terms on the right hand side of eq. (74) therefore determine the explicit z dependence of $|H_N(\alpha,z)\rangle$, and hence the phase $\phi_N(z)$. This remaining part has the form of a Hamiltonian for an harmonic oscillator with constant mass and position-dependent frequency. The ket $|H_N(\alpha,z)\rangle$ is an eigenfunction of this Hamiltonian with position-dependent eigenvalue ('energy') $2(N+1)/\alpha(z)$. Therefore, the phase $\phi_N(z)$ is given by

$$\phi_N(z)=\int_0^z \mathrm{d}s\,\frac{2(N+1)}{2k\alpha(s)}. \tag{77}$$

For HG and LG modes the substitution of $\alpha_0=\beta/k$ leads to the well-known expression for the Gouy phase [10]:

$$\phi(z)=\tan^{-1}(z/\beta), \tag{78}$$

and $\phi_N=(N+1)\phi$. Thus HG and LG modes evolve, apart form the phase factor in eq. (73) in the same way as the state of a quantum-mechanical harmonic oscillator with time-dependent energy in the adiabatic regime, when it starts in an eigenstate. Namely, in the adiabatic approximation one neglects the time derivative of the hamiltonian. Here, on the other hand, the presence of the third term in eq. (74) proportional to the derivative of $\alpha(z)$ ensures that the ket $|H_N(\alpha,z)\rangle$ remains an *exact* eigenfunction of the Hamiltonian.

9. Conclusions

We discussed the change in orbital angular momentum of light beams during their passage through systems of cylindrical lenses. We presented a formalism which makes use of some formal analogies between quantum mechanics and paraxial wave optics. For instance, a light field can be represented by a state vector, with the coordinate z as a propagation variable, which plays the same role as time in quantum mechanics [4]. We showed that physical quantities such as the linear and angular momentum per unit beam length can be given in the form of expectation values of corresponding operators in the state associated with the field.

In the 'Schrödinger picture', the propagation of fields along the z axis is described by a unitary operator. In the 'Heisenberg picture' the evolution of physical quantities rather than of the fields is given. By using the Heisenberg picture, we proved that, e.g., the z component of orbital angular momentum is uniform along z for arbitrary freely propagating light beams. Furthermore, we calculated the amount of angular momentum absorbed by ideal lenses, and described effects of cylindrical lenses for arbitrary incident fields.

We then discussed the action of cylindrical lens systems on the linear and orbital angular momentum of light beams, and determined exactly where and how much angular momentum is transferred. We also found a peculiar paradox for a particular lens system in the geometrical optics limit. It turns out that this limit displays a singularity for lens systems.

We argued that a light field is determined at each position z by three eigenoperators which have the field as an eigenfunction, and by its eigenvalues. Propagation of a field can alternatively be described by giving the evolution of its eigenoperators. In particular, we applied this formalism to define Hermite–gaussian (HG) and Laguerre–gaussian (LG) modes [10], and the definition for more general gaussian modes arises naturally. Using this formalism we were able to find in a straightforward way when an astigmatic lens system converts one gaussain mode into another. We also derived the necessary and sufficient conditions under which HG modes can be converted into LG modes and conversely. This possibility was derived in a completely different way in ref. [2].

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