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Formulating Dynamic Multi-rigid-body Contact Problems with Friction as Solvable Linear Complementarity Problems

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Abstract. A linear complementarity formulation for dynamic multi-rigid-body contact problems with Coulomb friction is presented. The formulation, based on explicit Euler integration and polygonal approximation of the friction cone, is guaranteed to have a solution for any number of contacts and contact configuration. A model with the same property is formulated for impact problems with friction and nonzero elasticity coefficient. An explicit Euler scheme based on these formulations is presented and is proved to have uniformly bounded velocities as the stepsize tends to zero for the Newton-Euler formulation in body coordinates.

Key words: Impacts with friction, multi-body dynamics, complementarity problems, contact constraints.

1. Introduction

Dynamic multi-rigid-body contact problems arise in several research areas, especially robotics and virtual reality. They are concerned essentially with modeling the dynamic behavior of several rigid bodies in contact.

The usual approach in treating such problems has been to set up at each discrete moment in time a Linear Complementarity Problem (LCP), whose solution, if it exists, gives the accelerations and contact forces at that moment. The accelerations are subsequently used to advance in time via a numerical integration scheme. For frictionless contact configurations, a solution to the corresponding LCP is known to always exist [1]. However friction cannot be neglected in most applications of interest. Several friction models that determine accelerations have been proposed [1, 2, 3], based either on Coulomb's friction law or on approximations of it. Unfortunately for all these models the existence of a solution of the corresponding LCP can be guaranteed only for some special cases. Several examples are known for which there are

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no accelerations consistent with the contact constraints [3], even in the case where just one contact is involved.

A new approach proposed in [11] is to combine the LCP defining the accelerations with the integration step for the velocities in order to obtain a discrete problem having velocities and contact impulses as unknowns, rather than accelerations. The main advantage is that the new problem has a solution for a much larger class of applications (it is guaranteed to have a solution for one contact with friction, regardless of the configuration).

In this paper, we will modify the model used in [11], such that the new formulation, although identical in spirit with the previous one, will be guaranteed to have a solution, *regardless of the configuration and number of contacts.* To our knowledge this is the first consistent discretization of the dynamic multi-body contact problem with friction that is guaranteed to have a solution for the general case. Although the solution is not be guaranteed to be unique, Lemke's algorithm [8] will always find a solution to our formulation. In addition to the constraints used in [11], our model also incorporates frictionless joint-type constraints.

The discrete model in the Newton-Euler formulation in body coordinates has several properties that are also found in real dynamics systems. This is likely to increase the believability of the simulations, an important goal especially for virtual reality. We prove that the increase in kinetic energy cannot exceed the one obtained in the same configuration, with no contacts enforced. The increase in kinetic energy is at most quadratic with respect to time if the external forces are uniformly bounded and at most exponential if the external forces grow at most linearly with respect to the position and velocity. We hope that these properties will be very helpful in a future paper, where we intend to analyze the behavior of our numerical solution as the stepsize goes to zero.

An important problem in simulating multi-rigid-body-contact with friction is appropriate handling of impact. Several models exist for problems with totally plastic collisions (zero elasticity coefficient) [1, 11], partial elastic or elastic collisions in the absence of friction [10] (restitution coefficient between 0 and 1) as well as for partial elastic or elastic collision with Coulomb friction in two dimensions [6]. In the present paper we develop solvable models in two or three dimensions that take friction into account for collision situations with nonzero restitution coefficient.

We will also show that, for our model, an important class of impacts have a simpler resolution than the general case. For these cases we prove that the kinetic energy after impact cannot exceed the kinetic energy before impact. In [6] the authors claim that the above property is shared by all configurations of the respective two dimensional model. Unfortunately their proof is based on an incorrect proposition, as we will show at the end of Section 3.

The paper is organized as follows. In Section 2 we describe the constraints and the LCP model, and we prove its solvability. In Section 3 we present the model collision resolution and some of its alternatives. In Section 4 we present an explicit Euler step integration algorithm. In Section 5 we prove that the velocities computed by the algorithm are uniformly bounded as the step-size tends to zero. In section 6 we discuss the conclusions. The most technical proofs are deferred to the Appendix.

2. The contact model

2.1. The constraints and the complementarity condition

In our model, a contact constraint is expressed as $\Phi(q) \geq 0$, where $\Phi(q)$ is a continuously differentiable function and q is a vector parameterizing the position (state) of the multi-body system. If the contact has been established for a while, then $\frac{d}{dt}\Phi(q(t)) \geq 0$. In case this quantity is (strictly) is positive, we have take-off, or breaking of the contact. In other words the contact becomes inactive. Assuming that the position is a differentiable function, one gets $\Phi_q(q(t))\dot{q} \geq 0$, or

$$\Phi_q(q(t))v(t) \geq 0 \tag{2.1}$$

with a strict inequality meaning that the contact breaks.

Let us assume now that we perform a numerical simulation at discrete moments in time and that $q^{(l)}$,$v^{(l)}$ and k are the computed positions, velocities and external forces at time step l. Let h be the corresponding integration step-length. Then an explicit Euler step defines the computed positions at time step $l+1$ as $q^{(l+1)} = q^{(l)}+hv^{(l)}$. Inequality (2.1) gives

$$(n(q^{(l+1)}))^T v^{(l+1)} \geq 0, \tag{2.2}$$

where $n = n(q) = \Phi_q(q)^T$ denotes the Jacobian of the constraint or the contact normal. This definition of the contact normal can be extended even for a reasonably small interpenetration [7]. We note that in [11] inequality (2.2) is replaced by a similar inequality which involves only position values

$$(n(q^{(l+1)}))^T q^{(l+1)} \geq \alpha_0, \tag{2.3}$$

where α_0 is chosen to make this inequality a good approximation of the contact constraint. It turns out that the ability of working with (2.2) rather than with (2.3) is going to be essential for the developments in the present paper.

It can be easily seen that the left-hand side of (2.2) is the normal velocity at the contact point. Let $c_n n$ be the normal force exercised by the constraint. Since the contact can only assume compressive force we have $c_n \geq 0$. In case of take-off we have $(n(q^{(l+1)}))^T v^{(l+1)} > 0$ and there is no contact force ($c_n = 0$). This generates a complementarity condition: both the normal force and the normal velocity are nonnegative, and if one of them is positive then the other one must vanish. It is convenient to denote this complementarity condition by writing

$$(n(q^{(l+1)})^T v^{l+1} \geq 0 \qquad \text{compl. to} \qquad c_n \geq 0.$$

A multi-body dynamic system has also joint constraints, which are determined by equality constraints of the type $\Theta(q) = 0$, where $\Theta(q)$ is a continuously differentiable function of q. The corresponding velocity constraint becomes

$$(\nu(q^{(l+1)}))^T v^{(l+1)} = 0; \tag{2.4}$$

where $\nu = \nu(q) = \Theta_q(q)^T$ is the Jacobian of the constraint. The force exercised by such a constraint is $c_\nu \nu$, [5], where c_ν can be any real number (positive or negative).

In considering friction we use the same approach as in [11]. Namely we consider Coulomb friction and we assume that the contact force has two components: a normal component c_n and a tangential(friction) component. The Coulomb friction law requires the contact force to lie in a circular cone (or elliptic cone for anisotropic friction). This is a difficult condition to deal with since it leads to a nonlinear complementarity problem which is much more difficult to analyze and/or to solve numerically. Therefore the cone is approximated by a polygonal cone [11, 2]

$$\widehat{FC}(q) = \{c_n n + D\beta \,|\, c_n \geq 0, \beta \geq 0, e^T\beta \leq \mu c_n\} \tag{2.5}$$

where $e = [1, 1, ..., 1]^T \in I\!R^u$ and u is the number of edges of the polygonal approximation and μ is the friction coefficient. The u columns of D represent vectors of the space tangent to the contact (in the two dimensional simulations D has just two columns). Also it is assumed that if a vector d is among the columns of D, so is $-d$. The new complementarity conditions can be written as

$$\lambda e + D^T v^{(l+1)} \geq 0 \quad \text{compl. to} \quad \beta \geq 0 \tag{2.6}$$

$$\mu c_n - e^T\beta \geq 0 \quad \text{compl. to} \quad \lambda \geq 0 \tag{2.7}$$

The complementarity of the vectors of (2.6) is understood componentwise. If $\mu c_n - e^T\beta > 0$ then $\lambda = 0$ and therefore $D^T v^{(l+1)} \geq 0$. But if there is a column d of D such that $d^T v^{(l+1)} > 0$, then for the column corresponding to $-d$ we have $-d^T v^{(l+1)} < 0$ contradicting the non negativity of $D^T v^{(l+1)}$. Therefore $D^T v^{(l+1)} = 0$. Since the columns of D^T span the tangent plane at the contact, this means precisely that the tangential velocity is zero. Therefore if the contact force is inside the cone $\widehat{FC}(q)$ the bodies involved in contact do not exhibit relative motion.

By negating the above implication it follows that, if the bodies in contact are in relative motion, then at least one of the elements of $D^T v^{(l+1)}$ is positive, forcing $\mu c_n - e^T\beta = 0$. Also, from the complementarity condition (2.6) we have $\beta^T D^T v^{(l+1)} \leq 0$, so that the friction force exhibits negative work, as expected. It can be shown that as the approximation to the Coulomb friction cone is more accurate (i.e., when $u \to \infty$), the angle between the friction force and the relative velocity tends to π, if the bodies in contact have nonzero tangential contact velocity. Therefore in the limit, the above friction model behaves like the Coulomb friction model.

2.2. The Newton law and the LCP

We will assume that our multi-body system has p contacts and m joint constraints. A superscript (j) will denote an object associated with the contact (j). The superscript (l) on v and q denotes the velocities and positions at time-step (l). The Newton Law for forces is

$$M\frac{dv}{dt} = F_c + F_{ext}$$

where F_c is the force due to the constraints and F_{ext} represents Coriolis and external forces. M is the mass matrix. In an explicit Euler integration setting, this becomes:

$$M(v^{(l+1)} - v^{(l)}) = hF_c + hF_{ext}$$

The constraint forces are composed of the forces presented in the previous subsection. Therefore adding the Newton Law to the complementarity conditions from the previous sections, we get the following (generalized) LCP:

$$M(v^{l+1} - v^{(l)}) - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{(i)} -$$

$$-\sum_{j=1}^{p} \quad (n^{(j)}c_n^{(j)} \quad +D^{(j)}\beta^{(j)}) = hk \qquad (2.8)$$

$$\nu^{(i)^T}v^{l+1} = 0, \qquad i = 1..m \qquad (2.9)$$

$$n^{(j)^T}v^{l+1} \geq 0, \text{ compl. to } c_n^{(j)} \geq 0, \; j = 1..p \qquad (2.10)$$

$$\lambda^{(j)}e^{(j)} + D^{(j)T}v^{l+1} \geq 0, \text{ compl. to } \beta^{(j)} \geq 0, \; j = 1..p \qquad (2.11)$$

$$\mu^{(j)}c_n^{(j)} - e^{(j)^T}\beta^{(j)} \geq 0, \text{ compl. to } \lambda^{(j)} \geq 0, \; j = 1..p \qquad (2.12)$$

Here k represents the external and Coriolis forces. Note that the variables c_n, c_ν, β represent impulses rather than forces (they are forces multiplicated by h, the step-length). The (generalized) LCP can be written in the following matrix form:

$$\begin{bmatrix} M & -\tilde{\nu} & -\tilde{n} & -\tilde{D} & 0 \\ \tilde{\nu}^T & 0 & 0 & 0 & 0 \\ \tilde{n}^T & 0 & 0 & 0 & 0 \\ \tilde{D}^T & 0 & 0 & 0 & \tilde{E} \\ 0 & 0 & \tilde{\mu} & -\tilde{E}^T & 0 \end{bmatrix} \begin{bmatrix} v^{(l+1)} \\ \tilde{c}_\nu \\ \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} + b = \begin{bmatrix} 0 \\ 0 \\ \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} \qquad (2.13)$$

$$\begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix}^T \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} = 0, \qquad \begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} \geq 0.$$

Here $\tilde{\nu} = [\nu^{(1)}, ..., \nu^{(p)}]$, $\tilde{c}_\nu = [c_\nu^{(1)}, ..., c_\nu^{(p)}]^T$, $\tilde{n} = [n^{(1)}, ..., n^{(p)}]$, $\tilde{c}_n = [c_n^{(1)}, ..., c_n^{(p)}]^T$, $\tilde{\beta} = [\beta^{(1)T}, ..., \beta^{(p)T}]$, $\tilde{D} = [D^{(1)}, ..., D^{(p)}]$, $\lambda = [\lambda^{(1)}, ..., \lambda^{(p)}]$, $\tilde{\mu} = diag(\mu^{(1)}, ..., \mu^{(p)})^T$ and $\tilde{E} = diag(e^{(1)}, ..., e^{(p)})$ and

$$b = \left[-Mv^{(l)} - hk, 0, 0, 0, 0\right]^T. \qquad (2.14)$$

To prove that such a problem has always a solution, we first prove the following solvability theorem.

Theorem 2.1. Consider a (mixed) LCP of the form

$$\begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} = \begin{pmatrix} M & -F & -H \\ F^T & 0 & 0 \\ H^T & 0 & N \end{pmatrix} \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix} + \begin{pmatrix} -k \\ 0 \\ b \end{pmatrix} \qquad (2.15)$$

$$s \geq 0; \quad \lambda \geq 0; \quad \lambda^T s = 0, \qquad (2.16)$$

where M, N, F, H are given matrices and b, k are given vectors of the appropriate dimension.

If M is a symmetric positive definite matrix, N a copositive matrix [8, Def. 3.8.1] and b a nonnegative vector having only nonnegative elements (in particular all components of b can be 0), then the above

LCP has a solution. Lemke's algorithm, with precautions taken against cycling, will always find a solution λ of the LCP obtained by eliminating x and y, and then a solution (x, y, λ) of the original LCP can be recovered by solving for x and y in the first two rows of (2.15).

Proof See Appendix A.

Theorem 2.2. For any positive definite mass matrix M and nonnegative friction coefficients, the LCP (2.13) has a solution which can be computed by Lemke's algorithm.

Proof Let N be the matrix made of the last three sets of rows and columns of the matrix form 2.13. Since the matrix is the sum between a positive semidefinite matrix (having $\tilde{E}$ and $-\tilde{E}^T$ as its blocks) and a matrix with nonnegative coefficients ($\tilde{\mu}$) it follows that it is copositive. By defining $H = [\tilde{n}\ \tilde{D}\ 0]$, the conclusion follows after using Theorem 2.1 . □

We denote the solution set of LCP (2.13) by $\mathcal{L}(v^{(l)}, hk)$, so that we can write (2.13) as

$$(v^{(l+1)}, \tilde{c}_\nu, \tilde{c}_n, \tilde{\beta}, \tilde{\lambda}) \in \mathcal{L}(v^{(l)}, hk) \tag{2.17}$$

The solution set of the LCP (2.13) also depends on $q^{(l)}$ and h since $\tilde{\nu}, \tilde{n}, M, \tilde{D}$ are evaluated at $q^{(l+1)} = q^{(l)} + hv^{(l)}$ but we will focus on the dependency of (2.13) on the free term (2.8). The previous theorem proves precisely that the set $\mathcal{L}(v^{(l)}, hk)$ is nonempty and that a point in it can be found by Lemke's algorithm.

2.3. Energy properties of the formulation

In this subsection, we assume that the mass matrix is constant during the simulation. This happens with the Newton-Euler formulation in body coordinates [9]. This assumption allows us to prove a property of the formulation that reproduces a similar property in real dynamics systems.

Theorem 2.3. The kinetic energy at the new step, $\frac{1}{2} v^{(l+1)^T} M v^{(l+1)}$, cannot exceed the kinetic energy $\frac{1}{2}(v^{(l)} + hM^{-1}k)^T M (v^{(l)} + hM^{-1}k)$ of the system in the same configuration, with no constraints enforced.

Proof From (2.8) we have, after multiplying with $v^{(l+1)^T}$

$$v^{(l+1)^T} M (v^{(l+1)} - v^{(l)}) = \textstyle\sum_{i=1}^m v^{(l+1)^T} \nu^{(i)} c_\nu^{(i)} +$$
$$+ \sum_{j=1}^{p} (v^{(l+1)^T} n^{(j)} c_n^{(j)} + v^{(l+1)^T} D^{(j)} \beta^{(j)}) + h v^{(l+1)^T} k. \tag{2.18}$$

Using (2.9), (2.10) and (2.13) we get

$$v^{(l+1)^T} M(v^{(l+1)} - v^{(l)}) = \sum_{j=1}^{p} (v^{(l+1)^T} D^{(j)} \beta^{(j)}) + h v^{(l+1)^T} k \qquad (2.19)$$

From (2.11), (2.12) and (2.13) it follows that $v^{(l+1)^T} D^{(j)} \beta^{(j)} = -\lambda^{(j)} e^T \beta^{(j)} = -\mu^{(j)} \lambda^{(j)} c_n^{(j)}$. Therefore

$$v^{(l+1)^T} M(v^{(l+1)} - v^{(l)}) = -\sum_{j=1}^{p} (\mu^{(j)} \lambda^{(j)} c_n^{(j)}) + h v^{(l+1)^T} k \leq h v^{(l+1)^T} k \qquad (2.20)$$

Let $\gamma_2 = M^{1/2} v^{(l+1)}, \gamma_1 = M^{1/2} v^{(l)}, \chi = M^{-1/2} k$. Then (2.20) can be written as

$$\gamma_2^T (\gamma_2 - \gamma_1) \leq h \gamma_2^T \chi \qquad (2.21)$$

or $\gamma_2^T \gamma_2 \leq \gamma_2^T (\gamma_1 + h\chi) \leq \|\gamma_2\| \|\gamma_1 + h\chi\|$, the last inequality resulting from the Cauchy inequality. Therefore $\|\gamma_2\| \leq \|\gamma_1 + h\chi\|$. Taking the squares, dividing by 2 and replacing with the original variables yields

$$\frac{1}{2} v^{(l+1)^T} M v^{(l+1)} \leq \frac{1}{2} (v^{(l)} + hM^{-1}k)^T M (v^{(l)} + hM^{-1}k) \qquad (2.22)$$

which proves the claim. □

3. The collision model

Most existing models for instantaneous inelastic collision can be seen as particular cases of contact dynamics problems, for $h = 0$, cf. [1, 11]. In this section, we will present a simple partially elastic collision model with friction based on the approach used for frictionless collision in [10]. For two dimensions the approach is similar with the one in [6] but, in addition, a solution to the model is guaranteed to be found. We will not consider cases for which portions of the tangential impulse is reversible (like in the highly elastic materials used in making superballs [6]) although our model can be easily modified to include some form of tangential impulse restitution.

The collision is assumed to have two phases: compression and decompression. The collision will generate a new contact (between the bodies that collide) , which, in the compression phase, is added to those already existent on the contact list. Each contact is characterized by an elasticity coefficient, e_j, $j = 1..p$.

3.1. The Compression Phase

Suppose that the collision is a proper collision, in the sense that it has a nonzero normal velocity. In other words, there is at least a contact, say defined by index $j = 1$, such that ${n^{(1)}}^T v^- < 0$, where v^- denotes the pre-collision velocities.

In the compression phase, the dynamic system will respond with constraint impulses (generated by joints, contacts, or friction). We denote the impulses by the same symbols as in (2.13) but with superscript c. Let v^c be the velocity at the end of the compression phase. At the end of compression, each contact from the list is either maintained $c_n^c \geq 0, n^T v^c = 0$, or is breaking, $c_n^c = 0, n^T v^c \geq 0$. Therefore we recover the same complementarity conditions as in (2.10). Conservation of impulse requires

$$M(v^c - v^-) - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{c(i)} - \sum_{j=1}^{p} (n^{(j)} c_n^{c(j)} + D^{(j)} \beta^{c(j)}) = 0 \qquad (3.1)$$

The conditions on joints and friction being identical with (2.9), (2.11) and (2.12) with the appropriate complementarity conditions,it follows that

$$(v^c, \tilde{c}_\nu^c, \tilde{c}_n^c, \tilde{\beta}^c, \tilde{\lambda}^c) \in \mathcal{L}(v^-, 0). \qquad (3.2)$$

By Theorem 2.2, Lemke's algorithm will find a solution in $\mathcal{L}(v^-, 0)$.

Also, it is worth mentioning that $v^c \neq v^-$, since ${n^{(1)}}^T v^- < 0$ and ${n^{(1)}}^T v^c \geq 0$.

3.2. The decompression phase

Let v^+ be the velocity after the decompression phase. Each contact constraint will generate a decompression impulse, dependent on the coefficient of restitution e_j. For instance, the contact constraint j will generate an impulse $c_n^{d(j)} = e_j c_n^{c(j)} + c_n^{x(j)}$. The additional impulse $c_n^{x(j)} \geq 0$ is necessary to prevent interpenetration. Either the contact breaks, and then $n^{(j)} v^+ \geq 0$, $c_n^{x(j)} = 0$, or it is maintained and then $n^{(j)} v^+ = 0$, $c_n^{x(j)} \geq 0$. Therefore the same type of complementarity as (2.10),

$$({n^{(j)}}^T) v^+ \geq 0 \qquad \text{compl. to} \qquad c_n^{x(j)} \geq 0.$$

is generated. Hence solving for the post-collision velocity will be similar to (2.13). The restitution impulse is

$$F^r = \sum_{j=1}^{p} e_j n^{(j)} c_n^{c(j)}. \qquad (3.3)$$

We assume that the tangential impulse is irreversible and does not contribute to the restitution force. The conservation of impulse requires

$$M(v^+ - v^c) - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{x(i)} - \sum_{j=1}^{p} (n^{(j)} c_n^{x(j)} + D^{(j)} \beta^{x(j)}) = F^r. \quad (3.4)$$

The same friction model gives rise to equations analogue to (2.11), (2.12). By Theorem 2.2 the post-collision velocity, v^+ , can be found by Lemke's algorithm since

$$(v^+, \tilde{c}_\nu^x, \tilde{c}_n^x, \tilde{\beta}^x, \tilde{\lambda^x}) \in \mathcal{L}(v^c, F^r). \quad (3.5)$$

There are two special cases worth noting. One is when $e_j = 0$, $j = 1..p$ (totally inelastic collision). Then $F^r = 0$ and $v^+ = v^c$ is a solution in $\mathcal{L}(v^c, 0)$ with all constraint impulses being 0, and $\tilde{\lambda}^x = \tilde{\lambda}^c$. In this case we can say that there is no decompression phase.

The other case concerns frictionless contacts. Our remark is an extension of a similar one [10]. Namely, we assume

- (a) all new contacts generated by collision have the same elasticity coefficient ϵ. Usually there is only one such contact, but some special cases might have several simultaneous collisions, like a block falling flat on a tabletop.

- (b) The elasticity coefficients characterizing the other contacts are less than ϵ, $e_j \leq \epsilon$, $1 \leq j \leq p$.

- (c) The pre-collision velocities satisfy the contact constraints exactly, $(n^{(j)}(q^-))^T v^- = 0$, for the contacts j that were maintained immediately before the collision. (this would be the case for a suitably adapted implicit integration scheme for example).

Under these assumptions a solution of the problem (3.5) is $v^+ = (1 + \epsilon)v^c - v^-$, with $c_n^{x(i)} = (\epsilon - e_i)c_n^{c(i)}$. This means that the only contacts that break at decompression are the ones originating from collision.

A disadvantage of the decompression phase is that for each collision, two LCP's need to be solved. Based on the previous observation, a simplified model for for partially inelastic collision would be to simply use $v^+ = (1 + \epsilon)v^c - v^-$, where ϵ is the elasticity coefficient of the contacts generated by collision.

Lemma 3.1. If, for the decompression phase, $v^+ = (1 + \epsilon)v^c - v^-$, then ${v^+}^T M v^+ \leq v^- M v^-$.

Proof Let $\gamma_2 = M^{\frac{1}{2}} v^c$, $\gamma_1 = M^{\frac{1}{2}} v^-$, $\gamma_3 = M^{\frac{1}{2}} v^+$. Then since $(v^c, \tilde{c}_\nu^c, \tilde{c}_n^c, \tilde{\beta}^c, \tilde{\lambda}^c) \in \mathcal{L}(v^-, 0)$, we have $\gamma_2^T(\gamma_2 - \gamma_1) \leq 0$ by (2.21) (since $h = 0$). Obviously, $\gamma_3 = \gamma_2 + \epsilon(\gamma_2 - \gamma_1)$. Therefore

$$\|\gamma_3\|^2 = \gamma_2^T \gamma_2 + \epsilon^2 (\gamma_2 - \gamma_1)^T (\gamma_2 - \gamma_1) + 2\epsilon \gamma_2^T (\gamma_2 - \gamma_1) \leq \gamma_2^T \gamma_2 + \epsilon^2 (\gamma_2 - \gamma_1)^T \leq \tag{3.6}$$

$$\gamma_2^T \gamma_2 + (\gamma_2 - \gamma_1)^T (\gamma_2 - \gamma_1) \leq \gamma_2^T \gamma_2 + (\gamma_2 - \gamma_1)^T (\gamma_2 - \gamma_1) - 2\gamma_2^T (\gamma_2 - \gamma_1) = \|\gamma_1\|^2. \tag{3.7}$$

Hence $\|\gamma_3\| \leq \|\gamma_1\|$ which proves the claim. □

The property in the conclusion of the Lemma means that the kinetic energy cannot increase by resolving collisions using $v^+ = (1+\epsilon)v^c - v^-$. This is a desirable property for any collision resolution.

Remark In [6] the authors claim that the property from the above Lemma holds for all two dimensional configurations. This statement is based on the claim [6, pg 480] "Since G is symmetric and positive semidefinite and ϵ is a diagonal matrix consisting of elements $0 \leq \epsilon_N \leq 1$, it follows that

$$\frac{1}{2}\Lambda_C^T \epsilon G \epsilon \Lambda_C - \frac{1}{2}\Lambda_C^T G \Lambda_C \leq 0.''$$

The last inequality is violated for $\Lambda_C = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $G = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$ and $\epsilon = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$ although all these matrices satisfy the assumptions. This makes invalid the previous statement and the proof of the kinetic energy inequality in [6].

4. Strategy for integrating the equations of motion

Various numerical schemes can be devised based on the previous sections. In what follows we propose a simple Euler integration method to find a numerical solution for the time interval $[0, T]$.

$v = v^0$, $q = q^0$; $time = 0$;
while $(time < T)$
  $q_{new} = q + hv$;
  Find $(v_{new}, \tilde{c}_\nu, \tilde{c}_n, \tilde{\beta}, \tilde{\lambda}) \in \mathcal{L}(v, hk)$ $(\tilde{\nu}, \tilde{D}, \tilde{n}, M$ are evaluated
    at q_{new}, k is evaluated at $(q, v))$
  if (no collision detected between $time$ and $time + h$)
    $time = time + h$, $q = q_{new}$, $v = v_{new}$;

else

Estimate the collision time $time_{new}$, collision position and velocity q_{new} and v^-;
Find $(v^c, \tilde{c}^c_\nu, \tilde{c}^c_n, \tilde{\beta}^c, \tilde{\lambda^c}) \in \mathcal{L}(v^-, 0)$;
Find $(v^+, \tilde{c}^x_\nu, \tilde{c}^x_n, \tilde{\beta}^x, \tilde{\lambda^x}) \in \mathcal{L}(v^c, F^r)$, with F^r defined in (3.3);
(Alternately, choose $v^+ = (1+e)v^c - v^-$)
$time = time_{new}$, $v = v^+$, $q = q_{new}$.

end if

end while

Several details need to be specified. We assume that the initial velocity v^0 is consistent with the constraints, i.e. that $(n^{(j)}(q_0))^T v^0 \geq 0$ and $(\nu^{(i)}(q_0))^T v^0 = 0$ for $i = 1..m,\ j = 1..p$. If that were not true, a collision would take place. The algorithm can be easily modified to handle an initial collision. At each step an LCP of the form (2.13) is solved. To set it up, $\tilde{\nu}, \tilde{D}, \tilde{n}, k, M$ need to be specified. Following the discussion in subsection 2.1, all these qualities are evaluated at q_{new}. Variants of this scheme would be to evaluate the data or parts of it at q or $q + \frac{h}{2}v$ [11]. The integration time-step does not have to be constant during the integration an can be changed at each step. One can easily devise a variant of this algorithm which uses a more sophisticated integration method (Runge-Kutta for example).

Also, the collision detection rules need to be specified. A (theoretically) simple collision rule would be to check for $\Phi^{(i)}(q) \leq 0$ (interpenetration). One problem is that the LCP decides take-off based on the condition $\Phi^{(i)}_q(q)v > 0$ but this does not necessarily mean that there was no interpenetration, for $\Phi^{(i)}(q)$ can be negative, due to numerical error. If a new collision between the bodies occurs before $\Phi^{(i)}(q) > 0$ it cannot be detected based on the sign of $\Phi^{(i)}(q)$. A solution would be to use a Nonlinear rather than a Linear Complementarity Problem [11] which enforces the constraint $\Phi^{(i)}(q) \geq 0$ with the cost of additional computation. Another solution, in the case when interpenetration persists even though the LCP (2.13) dictates take-off, would be to check for $\Phi^{(i)}_q(q)v \leq 0$ (normal velocity becomes negative) as a collision condition. This ensures that either there is no interpenetration, or there is an instantaneous decrease in interpenetration. This aspect shows the necessity of a good integration of the geometric and dynamic algorithms, for the outcome can be contradictory due to the discrete nature of the numerical schemes.

Assume that the contact constraint (i) is exactly satisfied at the current time moment, $\Phi^{(i)}(q^{(l+1)}) = 0$ and the LCP results in

$n^{(i)^T}(q^{(l+1)})v^{(l+1)} > 0$. Then $\frac{d}{dh}\Phi^{(i)}(q^{(l)} + hv^{(l+1)}) > 0$. Therefore take-off is certain at least on a small time interval following the current moment. This is an important property which shows the robustness of the formulation in the vicinity of the constraint manifold.

There is a class of configurations for which these algorithms cannot be guaranteed to sweep the interval $[0, T]$ in finite time. One example would be a ball bouncing on the floor with a restitution coefficient strictly between 0 and 1. The ball will sustain an infinite number of collisions in finite time, and the algorithm cannot get past the moment the ball has 0 velocity. One solution to this problem is to require that the restitution coefficient is zero for normal collision velocities less than a certain value.

5. Uniform boundedness of the numerical solution with respect to integration time step

In this section we analyze the properties of our formulation for the case of the Newton-Euler [9] equations in body coordinates. The main advantage in using this formulation is that the mass matrix is constant throughout the simulation. The drawback is that the Coriolis force, which is quadratic in velocity, has to be taken into account, which complicates the regularity analysis, because the assumption that the force is bounded, used in [11] is not obvious in this case. For one body, the Newton-Euler equations in body coordinates are [9, §2.4]

$$m\dot{v}^b + \omega^b \times mv^b = f^b \tag{5.1}$$

$$\mathcal{I}\dot{\omega}^b + \omega^b \times \mathcal{I}\omega^b = \tau^b \tag{5.2}$$

where m is the mass of the body, $\mathcal{I}$ is the inertia matrix (which is diagonal for a suitable choice of coordinates). The vectors v^b and ω^b represent the translational and rotational velocities in body coordinates. The term $f_c = -[(\omega^b \times mv^b)^T,\ (\omega^b \times \mathcal{I}\omega^b)^T]^T$ is the Coriolis force. An important property of the Coriolis force is $[v^{b^T}, \omega^{b^T}]f_c = 0$. This reflects the fact that the Coriolis force produces no work (it is perpendicular to the velocity), and will be a significant asset in proving the uniform boundedness of velocities. It should also be mentioned that for two dimensional simulations, the Newton-Euler formulation in global coordinates [9, §2.4], generates a constant mass matrix and does not exhibit Coriolis force.

We consider a dynamical system under the action of three types of external forces: forces whose magnitudes increase at most linearly with the distances (e.g. elastic forces), forces whose magnitude increase at most linearly with the velocity (e.g. viscous friction), and forces which are bounded during the simulation time interval $[0\ T]$. The following theorem proves that the kinetic energy obtained by the algorithm in the previous section is uniformly bounded on the time-interval $[0\ T]$. The step-size is not necessarily constant during the integration, but it is bounded by h. In other words for a given h, the time steps h_l can be different during the integration but they have to be smaller than h (h is just an upper bound on the individual step–sizes, and not necessarily the time–step).

Theorem 5.1. Assume that the algorithm from the previous section solves a finite number of collisions between 0 and T and that the collision resolution method ensures that the kinetic energy after the collision is no greater than the one before collision. Let $v(0), v(T)$ and $q(0), q(T)$ be the velocity and position at the beginning and the end of the simulation period.
a) If the external force is of the form $k(v,q) = f_c(v) + k_1(v,q) + k_2(v) + k_3(q)$, where $\|k_1(v,q)\| \leq d_1$, $\|k_2(v)\| \leq d_2\|v\|$ and $\|k_3(q)\| \leq d_3\|q\|$, $\forall q, v$, then for all sufficiently small h

$$v^T(T)Mv(T) \leq (v^T(0)Mv(0) + \|q(0)\|^2)e^{2cT} + 2cTe^{2cT}. \qquad (5.3)$$

b) If $k_2(v) = k_3(q) = 0$, then for all sufficiently small h

$$\sqrt{v^T(T)Mv(T)} \leq \sqrt{v^T(0)Mv(0)} + cT + 1. \qquad (5.4)$$

In both cases, c is a constant that depends solely on $d_i > 0, i = 1..3$ and on the mass matrix M.

Proof See Appendix B.

The forces considered in the previous theorem cover most cases of external forces in real dynamical systems. Therefore we see that for fairly general situations our algorithm provides solutions that are uniformly bounded as $h \downarrow 0$ for the Newton-Euler formulation. If $k_2(v) = -d_2 v$, $d_2 > 0$ (damping force) and $k_3(q) = 0$ it can be shown that the velocities satisfy a bound like (5.4). Another interesting point is that if the external forces (except for the Coriolis force) are uniformly bounded as $T \to \infty$, then the increase in velocity is at most linear with respect to time, which is also true in real dynamical systems.

6. Conclusions

A discrete model for solving multi-rigid-body contact problems with friction has been presented based on Linear Complementarity Problems (LCP). The friction model is an approximation to Coulomb friction model and can be made as accurate as desired. The formulation guarantees that the contact problem always has a solution which will be found by Lemke's algorithm. It also has the property that the kinetic energy increase after one integration step in time cannot exceed the increase for the case where no constraint is active. Also we have formulated a model for treating impact with friction for restitution coefficients between 0 and 1. The model has a compression and a decompression phase, each needing Lemke's algorithm for finding a solution (which is guaranteed to exist).

We have discussed an explicit Euler integration strategy. If the collision resolution is guaranteed to yield a smaller kinetic energy after the collision, then, under some reasonable conditions on the external forces (at most linear increase with respect to velocity and position) the velocities given by the numerical algorithm are uniformly bounded as $h \downarrow 0$ for the Newton-Euler formulation in body coordinates. Also the velocities increase at most linearly with the time-interval considered if the external force is uniformly bounded. An example of a collision resolution that guarantees a non-increasing kinetic energy is given in Section 3.

It is not necessary that the integration procedure use the Newton-Euler formulation or an Euler integration scheme. Both the integration procedure and coordinates can be changed maintaining the solvability of the corresponding LCPs. The Euler integration approach has the advantage that it guarantees the uniform boundedness of the velocities.

Several other issues arise in the practical implementation of the numerical method. Lemke's algorithm will not only compute a solution to the LCP, but it will also compute a basis of (A.1) [8]. If a basis is known, then a solution can be computed by solving a linear system. In many simulations changes of basis are relatively rare compared to the total number of integration steps. Therefore, the same basis can be used for several steps with the cost of just a linear solve and Lemke's algorithm will be called just when the necessity of a change of basis is detected (a negative slack). This observation has been used before [4, 3] with significant improvements of the computational effort. The advantage of our setting is that it guarantees that a change of basis which yields a solution will be computed by Lemke's algorithm.

7. Acknowledgments

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Appendix

A. Proof of Theorem 2.1

We can assume without loss of generality that F^T is full row rank. If not, we can consider just a maximal set of independent rows as being F^T. The components of y corresponding to the dependent variables can be set to 0, and λ will be in the same linear space determined by (2.15).

After solving for x in (2.15), the LCP becomes

$$0 = F^T M^{-1} F y + F^T M^{-1} H \lambda + F^T M^{-1} k \qquad \text{(A.1)}$$

$$s = H^T M^{-1} F y + H^T M^{-1} H \lambda + N\lambda + H^T M^{-1} k + b \qquad \text{(A.2)}$$

$$s \geq 0; \quad \lambda \geq 0; \quad \lambda^T s = 0, \qquad \text{(A.3)}$$

Since F is full row rank $F^T M^{-1} F$ is nonsingular. Therefore we can solve for y from the first equation to get the LCP:

$$s = (G + N)\lambda + g; \tag{A.4}$$

$$s \geq 0; \quad \lambda \geq 0; \quad \lambda^T s = 0, \tag{A.5}$$

where

$$G = H^T M^{-1} H - H^T M^{-1} F (F^T M^{-1} F)^{-1} F^T M^{-1} H,$$
$$g = -H^T M^{-1} F (F^T M^{-1} F)^{-1} F^T M^{-1} k + H^T M^{-1} k + b.$$

By construction G is the Schur complement of N in the big matrix of (2.15), if $N = 0$ [8, Def. 2.3.4]. Since the matrix in (2.15) is positive semidefinite for $N = 0$, so is G [8, Thm 4.1.5]. Therefore $G + N$ is copositive.

It is convenient to denote the above LCP as LCP(G+N,g) and to call λ its solution (once λ is found then $s = (G + N)\lambda + g$). Let now z be a solution of LCP(G+N,0). Then $z^T G z = 0$. Since G is positive semidefinite and symmetric, it follows that $Gz = 0$. Let $w = -(F^T M^{-1} F)^{-1} F^T M^{-1} H z$ Then we have from the definition of G and the last relation

$$0 = F^T M^{-1} F w + F^T M^{-1} H z \tag{A.6}$$

$$0 = H^T M^{-1} F w + H^T M^{-1} H z. \tag{A.7}$$

By multiplying the first relation by w^T and the last relation by z^T and adding them up, we obtain $(Hz + Fw)^T M^{-1} (Hz + Fw) = 0$. Since M is symmetric and positive definite, it follows that $Hz + Fw = 0$ and $z^T g = w^T F^T M^{-1} k + z^T H^T M^{-1} k + z^T b = (Fw + Hz)^T M^{-1} k + z^T b \geq 0$.

Therefore, we have proved that if z is a solution of LCP(G+N,0) then $g^T z \geq 0$. Therefore, by Corollary 4.4.12 of [8] Lemke's algorithm, with precautions taken against degeneracy will find a solution λ to the LCP(G+N,g) and, by solving for x and y in the first two rows of (2.15), a solution (x, y, λ) to the initial LCP. $\Box$.

B. Proof of Theorem 5.1

Lemma B.1. Let $\{z_n\}_{n=0}^N$, $\{w_n\}_{n=0}^N$, be two finite nonnegative sequences and $\{h_n\}_{n=0}^{N-1}$ a positive sequence such that $\sum_{j=0}^{N-1} h_j = T$. Let c_i, $i = 1..5$ be positive constants. Assume that the sequences satisfy the following inequalities:

$$z_{n+1}^2 \leq z_n^2 + c_1 h_n z_n^2 + c_2 h_n w_n + c_3 h_n z_n + c_4 h_n \tag{B.1}$$

$$w_{n+1} \leq w_n + c_5 h_n z_n \tag{B.2}$$

for $0 \leq n < N-1$. Let $c = \max 2c_i$, $i = 1..5$. Then $z_i^2 \leq (z_0^2 + w_0)e^{2cT} + 2cTe^{2cT}$, $w_i \leq (z_0^2 + w_0)e^{2cT} + 2cTe^{2cT}$, $\forall i \leq N$.

Proof Using the inequality $2z_n \leq z_n^2 + 1$ we can write

$$\begin{aligned} z_{n+1}^2 &\leq z_n^2 + c_1 h_n z_n^2 + c_2 h_n w_n + c_3 h_n z_n + c_4 h_n && \text{(B.3)} \\ &\leq z_n^2 + c_1 h_n z_n^2 + c_2 h_n w_n + c_3 h_n (z_n^2/2 + 1/2) + c_4 h_n && \text{(B.4)} \\ &\leq z_n^2 + c h_n z_n^2 + c h_n w_n + c h_n. && \text{(B.5)} \end{aligned}$$

We also have that

$$\begin{aligned} w_{n+1} &\leq w_n + c_5 h_n z_n \leq w_n + c_5 h_n (z_n^2/2 + 1/2) && \text{(B.6)} \\ &\leq w_n + c h_n z_n^2 + c h_n w_n + c h_n. && \text{(B.7)} \end{aligned}$$

Let $r_n = z_n^2 + w_n$. By adding (B.5) with (B.7) we obtain

$$r_{n+1} \leq r_n + 2ch_n r_n + 2ch_n \leq r_n e^{2ch_n} + 2ch_n. \qquad \text{(B.8)}$$

Let $u_n = \sum_{i=0}^n h_i$. We will prove that $r_n \leq (r_0 + 2cu_{n-1})e^{2cu_{n-1}}$ by induction. It is obvious for $n = 1$. Assume it is true for n. Then

$$\begin{aligned} r_{n+1} &\leq r_n e^{2ch_n} + 2ch_n \leq (r_0 + 2cu_{n-1})e^{2cu_{n-1}} e^{2ch_n} + 2ch_n && \text{(B.9)} \\ &\leq (r_0 + 2cu_{n-1})e^{2cu_n} + 2ch_n e^{2cu_n} = (r_0 + 2cu_n)e^{2cu_n} && \text{(B.10)} \end{aligned}$$

Using the fact that $u_i \leq T$ for $0 \leq i \leq N-1$, we can write $z_i^2 \leq r_i \leq (z_0^2 + w_0 + 2cT)e^{2cT}$, $w_i \leq r_i \leq (z_0^2 + w_0 + 2cT)e^{2cT}$, for $0 \leq i \leq N$, which proves the claim. □.

Proof of Theorem 5.1 a) By Theorem 2.3 we have

$$v^{(l+1)^T} M v^{(l+1)} \leq (v^{(l)} + h_l M^{-1} k)^T M (v^{(l)} + h_l M^{-1} k).$$

Let $\gamma^{(l)} = M^{1/2} v^{(l)}$ and $\chi^{(l)} = M^{-1/2} k(q^{(l)}, v^{(l)})$. Then the last relation can be rewritten as

$$\|\gamma^{(l+1)}\|^2 \leq \|\gamma^{(l)} + h_l \chi^{(l)}\|^2 = \|\gamma^{(l)}\|^2 + 2\gamma^{(l)^T}\chi + h_l^2 \|\chi^{(l)}\|^2. \qquad \text{(B.11)}$$

By the definition of the Coriolis force we have $v^T f_c = 0$. Therefore

$$\begin{aligned} \gamma^{(l)^T}\chi &= v^{(l)^T}(f_c(v^{(l)}) + k_1(v^{(l)}, q^{(l)}) + k_2(v^{(l)}) + k_3(q^{(l)})) && \text{(B.12)} \\ &= v^{(l)^T}(k_1(v^{(l)}, q^{(l)}) + k_2(v^{(l)}) + k_3(q^{(l)})) && \text{(B.13)} \\ &= \gamma^{(l)^T}(\chi_1(v^{(l)}, q^{(l)}) + \chi_2(v^{(l)}) + \chi_3(q^{(l)})), && \text{(B.14)} \end{aligned}$$

where $\chi_i = M^{-1/2}k_i, i = 1,2,3$. By using the inequalities in the hypothesis, we get

$$\gamma^{(l)^T}(\chi_1(v^{(l)}, q^{(l)}) + \chi_2(v^{(l)}) + \chi_3(q^{(l)})) \leq \\ \|\gamma^{(l)}\|(c_1 + c_2\|\gamma^{(l)}\| + c_3\|q^{(l)}\|), \quad \text{(B.15)}$$

$$\|\chi^{(l)}\| \leq (c_1 + c_2\|\gamma^{(l)}\| + c_3\|q^{(l)}\| + c_4\|\gamma^{(l)}\|^2), \quad \text{(B.16)}$$

where the constants $c_i, i = 1..4$ depend only on the constants $d_i, i = 1..4$ and the mass matrix. The term $c_4\|\gamma^{(l)}\|^2$ is a bound on the Coriolis force. Therefore, according to (B.11) we can write

$$\|\gamma^{(l+1)}\|^2 \leq \|\gamma^{(l)}\|^2 + 2\|\gamma^{(l)}\|(c_1 + c_2\|\gamma^{(l)}\| + c_3\|q^{(l)}\|) \\ +h_l^2\|\chi^{(l)}\|^2. \quad \text{(B.17)}$$

Denoting $z_l = \|\gamma^{(l)}\|$ and $w_l = \|q^{(l)}\|$ we get

$$z_{l+1}^2 \leq z_l^2 + c_1h_lz_l + c_2h_lz_l^2 + c_3h_lw_lz_l + h_l^2\|\chi^{(l)}\|^2. \quad \text{(B.18)}$$

Also, $q^{(l+1)} = q^{(l)} + h_lv^{(l)} = q^{(l)} + h_lM^{-1/2}\gamma^{(l)}$ and therefore

$$w_{l+1} \leq w_l + h_lc_5z_l, \quad \text{(B.19)}$$

where c_5 depends only on the mass matrix and $w_l = \|q^{(l)}\|$. Let $c = \max 2c_i,\ i = 1..5$. We choose h such that

$$(c_1 + c_2(\|\gamma_0\|^2 + \|q_0\| + 2cT)^{\frac{1}{2}}e^{cT} \\ +(c_3 + c_4)(\|\gamma_0\|^2 + \|q_0\| + 2cT)e^{2cT})^2 \leq \tfrac{c}{2h}. \quad \text{(B.20)}$$

We will prove by induction that

$$z_{l+1}^2 \leq z_l^2 + c_1h_lz_l + c_2h_lz_l^2 + c_3h_lw_lz_l + \frac{c}{2}h_l. \quad \text{(B.21)}$$

for $0 \leq l \leq N$. From (B.18) it follows that it is sufficient to prove by induction that

$$h_l\|\chi^{(l)}\|^2 \leq \frac{c}{2}. \quad \text{(B.22)}$$

For $l = 0$ this follows from (B.16) and (B.20) with

$$\|\chi^0\| \leq (c_1 + c_2\|\gamma^{(0)}\| + c_3\|q^{(0)}\| + c_4\|\gamma^{(0)}\|^2) \leq \frac{c}{2h_l}$$

Assume that (B.22), and hence (B.21), holds for some $0 \leq n < N-1$. From (B.19) it follows that the sequences $w_l, z_l,\ 0 \leq l \leq n-1$ satisfy the conditions from Lemma B.1. Since $\sum_{l=0}^{n-1} h_l \leq T$, we have

$$z_l^2 \leq (z_0^2 + w_0)e^{2cT} + 2cTe^{2cT}, \qquad 0 \leq l \leq n-1,$$
$$w_l \leq (z_0^2 + w_0)e^{2cT} + 2cTe^{2cT}, \qquad 0 \leq l \leq n-1.$$

Using (B.20) and (B.16) we deduce that $h_n\|\chi^{(n)}\|^2 \leq \frac{c}{2}$ which completes the induction.

Therefore w_l, z_l satisfy (B.21) and (B.19) and the assumptions of Lemma B.1, for $0 \leq l \leq N$, so that the conclusion follows.

b) A similar sequence of inequalities shows that

$$\|\gamma^{(l+1)}\|^2 \leq \|\gamma^{(l)}\|^2 + 2h_l c_1\|\gamma^{(l)}\| + h_l\|\chi^{(l)}\|^2 \tag{B.23}$$

Here c_1 is a constant dependent on d_1 and the mass matrix, M. Following the hypothesis on the external force, we have that

$$\|\chi^{(l)}\| \leq c_1 + c_4\|\gamma^{(l)}\|^2 \tag{B.24}$$

where c_4 depends only on the mass matrix. Let $c = 2\max\{c_1, c_4\}$. We choose h such that

$$h(c_1 + c_4(\sqrt{\|\gamma_0\|^2 + 1} + cT)^2)^2 \leq \frac{c}{2}. \tag{B.25}$$

Let $z_l = \|\gamma^{(l)}\|$. We will prove by induction that

$$\sqrt{z_n^2 + 1} \leq \sqrt{z_0^2 + 1} + c\sum_{l=0}^{n-1} h_l \tag{B.26}$$

for $0 \leq n \leq N$. For $n = 0$ the claim is obvious. Assume it is true for $n - 1 \leq N - 1$. Then by virtue of (B.23) we can write

$$z_n^2 \leq z_{n-1}^2 + 2h_{n-1}c_1 z_{n-1} + h_l^2\|\chi^{(l)}\|^2. \tag{B.27}$$

From (B.24) and the induction hypothesis it follows that:

$$h_{n-1}\|\chi^{(n-1)}\|^2 \leq h(c_1 + c_4 z_{n-1}^2)^2$$
$$\leq h(c_1 + c_4(\sqrt{\|\gamma_0\|^2 + 1} + cT)^2)^2 \leq \frac{c}{2}.$$

Therefore, from (B.27) we deduce that

$$z_n^2 \leq z_{n-1}^2 + 2h_{n-1}c_1 z_{n-1} + h_{n-1}\frac{c}{2}.$$

Adding 1 to both sides, applying Cauchy's inequality and the induction hypothesis, we get

$$z_n^2 + 1 \leq z_{n-1}^2 + 1 + 2h_{n-1}c\sqrt{z_{n-1}^2 + 1} \leq (\sqrt{z_{n-1}^2 + 1} + h_{n-1}c)^2$$
$$\leq (\sqrt{z_0^2 + 1} + c\sum_{l=1}^{n-2} h_l + ch_{n-1})^2$$

which proves that (B.26) is true for n. The Theorem is proved by taking $n = N$ in (B.26) . □

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