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Abstract. We examine the behavior of a one-dimensional superconducting wire exposed to an applied electric current. We use the time-dependent Ginzburg-Landau model to describe the system and retain temperature and applied current as parameters. Through a combination of spectral analysis, asymptotics and canonical numerical computation, we divide this two-dimensional parameter space into a number of regions. In some of them only the normal state or a stationary state or an oscillatory state are stable, while in some of them two states are stable. One of the most interesting features of the analysis is the evident collision of real eigenvalues of the associated PT-symmetric linearization, leading as it does to the emergence of complex elements of the spectrum. In particular this provides an explanation to the emergence of a stable oscillatory state. We show that part of the bifurcation diagram and many of the emerging patterns are directly controlled by this spectrum, while other patterns arise due to nonlinear interaction of the leading eigenfunctions.

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We consider a one-dimensional superconducting wire of finite extent. An electric current is fed into the wire at one of its ends creating a voltage difference across the wire. This is a canonical problem that has received considerable attention since it involves the case of a resistive state in which a normal current and a superconducting current coexist. One of the intriguing phenomena associated with this state is the formation of phase slip centers (PSC). These are points in space-time where the order parameter in the time-dependent Ginzburg Landau equation (TDGL) vanishes. In fact, as was pointed out by Ivlev and Kopnin [1], phase

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slip centers can be thought of as vortices in space-time. The appearance of phase slip centers is related to oscillations found numerically through the emergence of time-periodic solutions. The phase slip centers and the associated oscillations can be indirectly observed experimentally via the appearance of steps in I-V curves ([2], [3], [4]). Another type of behavior found in the resistive state involves stationary solutions of the TDGL [2]. In this case the gauge invariant quantities reach a steady state.

One goal of this paper is to understand the origin of the different patterns observed in the resistive state. Another goal is to compute the bifurcation diagram in the parameter space. In particular we will explain why and when oscillatory solutions emerge. We will also consider the loss of stability of these oscillatory solutions as the temperature is lowered below a critical value that depends on the applied current I . The key idea is that the oscillations appear as a consequence of a Hopf bifurcation driven by a PT-symmetric spectral problem. A crucial role in the analysis is played by the dependence of this spectrum on the applied current. An additional goal is to elucidate the appearance of hysteresis in I-V curves in the present setup.

Our starting point is the time-dependent Ginzburg Landau model that we write in a non-dimensional form:

$$\psi_t + i\varphi\psi = \psi_{xx} + \Gamma\psi - |\psi|^2\psi. \quad (1)$$

Here ψ is the complex-valued order parameter, φ is the electric potential and Γ is proportional to $T_c - T$. Conservation of the current I implies the relation

$$\frac{i}{2}(\psi\psi_x^* - \psi_x\psi^*) - \sigma\varphi_x = I, \quad (2)$$

where σ models the Ohmic resistivity. (In equations (1), (2) and all subsequent equations, we use a variable in subscript to denote a partial derivative.) The wire is assumed to extend along $-L \leq x \leq L$, and it is assumed that $\psi(\pm L, t) = 0$. The main conclusions below are also valid for other boundary conditions such as $\psi_x(\pm L, t) = 0$. In order to concentrate on the main features of the phase transition mechanism, we take $\sigma = L = 1$. This enables us to concentrate on the key parameters I and Γ . Some TDGL models include a factor $\zeta_1/\sqrt{1 + \zeta_2|\psi|^2}$ in front of the left hand side of equation (1). We deal here with the small ζ_2 limit, but our essential results are valid for finite positive ζ_2 .

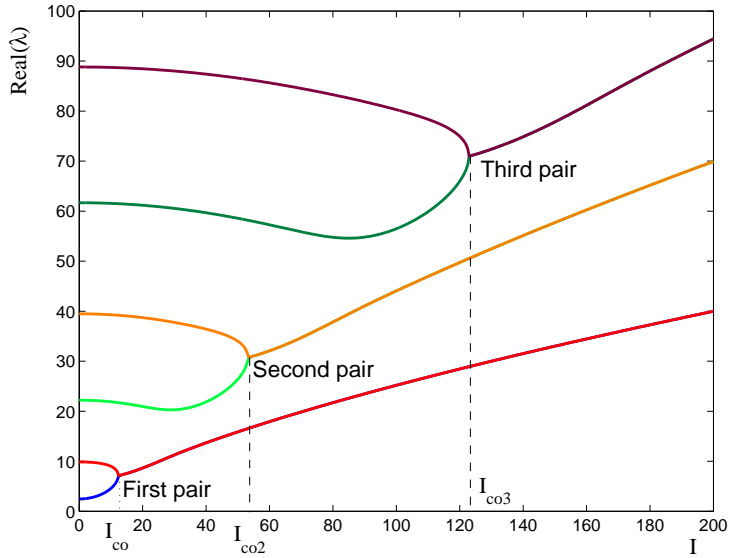


Figure 1: The real parts of the first 6 eigenvalues of the PT-symmetric spectral problem (3).

To understand the phase transition from the normal state to the superconducting state we linearize the TDGL (1) about the normal state $\psi = 0$, $\varphi = -Ix$. Writing $\psi(x, t) = u(x)e^{(\Gamma-\lambda)t}$, we obtain for $u(x)$ the spectral problem

$$M[u] = u_{xx} + ixIu = -\lambda u, \quad u(\pm 1) = 0. \quad (3)$$

The spectral problem (3) is called PT-symmetric, since it is invariant under the joint transformation of $x \rightarrow -x$ (parity) and complex conjugacy (time reversal). The normal state thus loses its stability when $\Gamma > \text{Real}(\lambda(I))$. However, since the spectral problem (3) is not self-adjoint, it is not clear at all that the spectrum is real. On the other hand the PT symmetry provides some useful information on the spectrum. Spectral PT-symmetric problems have attracted some interest in recent years following the numerical observation of Bender and Boettcher [6] that the spectrum of certain PT-symmetric problems *is* real. While ref. [6] considered a problem on the entire line, we deal here with a problem in a finite interval. When $I = 0$ the spectrum is of course real. The PT-symmetry implies that if $(\lambda, u(x))$ is a spectral pair with complex λ , then also $(\lambda^*, u^*(-x))$ is a spectral pair. Since the spectrum depends smoothly on I as long as the eigenvalues remain separated [7], a real eigenvalue cannot split spontaneously into a complex

pair. This implies that at least for small I all eigenvalues are real. However, when the current I is large, the lowest eigenvalues (in absolute value) are shown to satisfy $\lambda = O(iI)$, namely, to leading order they are purely imaginary. This implies that eigenvalues indeed collide as I increases. Specifically we find that the first such collision occurs when the first and second eigenvalues approach each other and collide at a critical value $I_{co} \approx 12.31$.

At the collision point, the geometric multiplicity of the eigenvalue is 1. To find the behavior of the spectrum near I_{co} we set the current I to be $I = I_{co} + \epsilon a$. Here ϵ is a small positive number, and we introduce a to determine through its sign the direction in which we move from I_{co} . We then consider an expansion of the form

$$\lambda = \mu_0 + \epsilon^{1/2}\mu_1 + \epsilon\mu_2 + \dots, \quad u = u_0 + \epsilon^{1/2}u_1 + \epsilon u_2 + \dots \quad (4)$$

The nonanalytic nature of the expansion for λ is a consequence of the Jordan form of the spectral problem at the critical value $I = I_{co}$. The leading order term in (4) is found to be $\mu_0 \approx 0.71$, with an associated eigenfunction u_0 that we normalize by $u_0(0) = 1$. The first order correction μ_1 is conveniently expressed through the auxiliary function $K(x)$ that solves

$$K_{xx} + ixI_{co}K + \mu_0K = u_0, \quad K(\pm 1) = 0. \quad (5)$$

Writing $u_0 = U_r + iU_i$, and defining $a_1 = 2 \int_{-1}^1 xU_rU_i dx$ and $b = \int_{-1}^1 Ku_0 dx$, one obtains $\mu_1^2 = -aa_1/b$. A numerical integration gives $a_1 \approx 0.29$ and $b \approx 0.12$. Since $a_1/b > 0$, we see that when $a < 0$, i.e. when I is a little smaller than I_{co} , there are two real solutions; these are the first two real eigenvalues just before the collision. However, for I beyond I_{co} , that is for $a > 0$, the single eigenvalue μ_0 splits into a pair of complex eigenvalues. It can be further shown that μ_2 is a single real number, i.e. it is the same for both splitting eigenvalues [8]. The analysis above shows that the real part of the leading eigenvalue is not an analytic function of the current at $I = I_{co}$. In fact, its derivative blows up as I_{co} is approached from below. On the other hand, the real part of the first eigenvalue (pair) is a smooth function of I just above I_{co} . This analysis holds for any later collision of real eigenvalues as well. It is in agreement with the numerical calculation presented in Figure 1.

We computed the first few eigenvalues numerically as they increase past special collision points. Increasing I beyond I_{co} , the first two eigenvalues move as a complex pair according to the PT-symmetry. The real parts of the first six eigenvalues as a function of I are plotted in Figure 1. We see there that respective pairs of eigenvalues collide at successive critical values of I .

The normal state becomes unstable at that value of Γ for which $\Gamma - \text{Real}(\lambda) = 0$. For $I < I_{co}$ the first eigenvalue $\lambda(I)$ is real. When the temperature is sufficiently low, i.e. when $\Gamma = \lambda(I)$, the normal state loses stability. Proceeding to high order terms in the bifurcation expansion it is found that the bifurcation branch that emerges at $\Gamma = \Gamma_1(I) = \lambda(I)$ is stable for $I < I_k \approx 10.92$. In this regime, i.e. when $I < I_k$ and $\Gamma > \Gamma(I)$, the bifurcating solution converges to a stationary solution. By ‘stationary’ here we mean that writing $\psi = f e^{i\chi}$, the gauge invariant quantities $f(x, t)$, $q(x, t) = \chi_x(x, t)$ and $\theta(x, t) = \chi_t(x, t) - \varphi(x, t)$ converge to stationary functions $f_0(x)$, $q_0(x)$, $\theta_0(x)$. Once I crosses the critical collision value I_{co} and the eigenvalue splits into a conjugate complex pair, the phase transition temperature is determined by the condition $\Gamma = \text{Real}(\lambda(I)) = \Gamma_1(I)$. Thus, for $I > I_{co}$ a Hopf bifurcation occurs and the solution to the full TDGL is periodic.

Consider now a current $I > I_{co}$. When Γ is below $\Gamma_1(I)$, the positive real part of the spectrum dominates, and the normal state is stable. Increasing Γ with I fixed we see that when $\Gamma = \Gamma_1(I)$ a Hopf bifurcation into a periodic solution takes place as explained above. In addition to determining the bifurcation curve $\Gamma = \Gamma(I)$, the spectral problem (3) can also be used to compute the bifurcating branch, which is always stable, in the periodic regime. To see this, fix a current I greater than the critical value I_{co} . Let the ground state of equation (3) consist of the eigenvalue pair $\lambda_r \pm i\lambda_i$, with associated eigenfunctions $w_1(x)$ and $w_2(x)$ related by $w_1(x) = w_2^*(-x)$. We normalize both eigenfunctions by the condition $w_i(0) = 1$. Set the temperature to be slightly below the critical value determined by $\Gamma = \Gamma_1(I)$, by selecting $\Gamma = \lambda_r + \epsilon^2$. Neglecting a short time interval during which transients decay, the asymptotic solution of the full TDGL (1)-(2) is found to be of the form

$$u(x, t) = \epsilon A \left(\exp \left(i(\omega\epsilon^2 + \lambda_i)t \right) w_1(x) + \exp \left(-i(\omega\epsilon^2 + \lambda_i)t \right) w_2(x) + O(\epsilon^3) \right). \quad (6)$$

The amplitude A and frequency ω are constants that are determined by the ground state w_1 and w_2 . They are computed numerically for each current I . For instance, when $I = 20$ we found $A = 0.921$, $\omega = 1.8$. One can draw a number of conclusions from the expression (6). First, the period of the oscillations is not exactly the imaginary component of the eigenvalue, but rather it is has a correction due to the nonlinear interaction of w_1 and w_2 . Secondly, the solution at $x = 0$ is $u(0, t) \approx 2\epsilon A \cos((\lambda_i + \omega\epsilon^2)t)$. Therefore we obtain a phase slip center that is periodic in time at $x = 0$. Ivlev and Kopnin [1] made the nice observation that a PSC can be thought of as a vortex in space-time. In this sense, the solution structure given in equation (6) indicates that the PSCs constitute a periodic placement of degree-one space-time vortices with period $P = \pi/(\lambda_i + \omega\epsilon^2)$. The curve $\Gamma_1(I)$ is depicted by the solid line in Figure 2.

So far we have concentrated on the smooth bifurcation of the normal state into a periodic state or into a stationary state. It turns out, though, that there are regions in the parameter plane where two metastable states coexist. The transition between them is nonsmooth, and therefore it is associated with hysteresis. We already pointed out above that for $I_{co} > I > I_k$ the normal state bifurcates into an unstable branch. This hints that the phase transition there is nonsmooth. Indeed, we identified another curve in the phase plane, that we call $\Gamma_2(I)$, above which the stationary state is stable. The curve $\Gamma_2(I)$ is depicted as a dashed line in Figure 2.

To understand the loss of stability of the periodic state, we recall that the Hopf bifurcation that led to it was triggered by the normal current contribution to the potential term $i\varphi\psi$ in equation (1). Near the transition curve $\Gamma = \Gamma_1(I)$ the magnitude $|\psi|$ of the order parameter is still small, and essentially the entire current is normal. As the temperature is lowered (i.e. Γ increases), $|\psi|$ grows and so does the supercurrent, implying via (2) that the normal current decreases. This effectively returns the system to the small I regime where the bifurcation to a steady state is favored. We thus obtain a third bifurcation curve $\Gamma_3(I)$ where the periodic state loses its stability.

At this point we make reference to Figure 2 and consider the different regimes in the (I, Γ) plane. The solid curve provides the critical temperature $\Gamma = \Gamma_1(I)$ along which the normal state loses its stability. A stable stationary state exists above the dashed line that represents a second

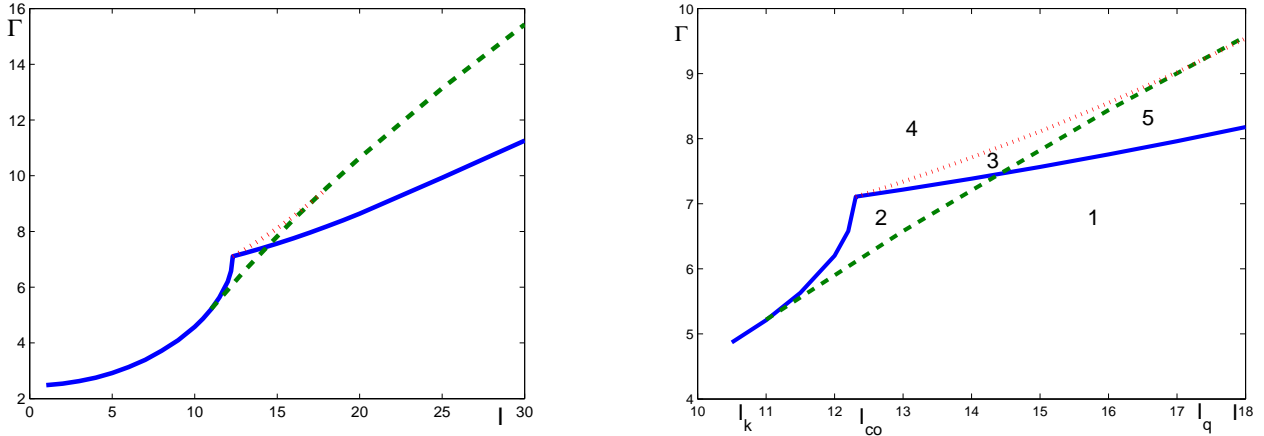


Figure 2: The phase diagram of the different stable states in the temperature-current plane. The parameter Γ is proportional to $T_c - T$. The curves $\Gamma_1(I)$, $\Gamma_2(I)$, $\Gamma_3(I)$ are drawn with solid line, dashed line and dotted line, respectively. The meaning of the different curves and regimes is explained in detail in the text.

curve $\Gamma_2(I)$. For $I < I_k$ the normal state bifurcates into a stable stationary superconducting state. For $I > I_{co}$, on the other hand, the normal state bifurcates into a state that exhibits time-periodic oscillations. When $I > I_{co}$, and the temperature is further lowered (Γ is increased), the periodic state loses its stability at a third critical temperature $\Gamma = \Gamma_3(I)$ represented by the dotted line in the figure. The curves $\Gamma_2(I)$ and $\Gamma_3(I)$ intersect at I_q . For $I > I_q$, the curves $\Gamma_2(I)$ and $\Gamma_3(I)$ coalesce. The frame on the left depicts the bifurcation curves over a large (I, Γ) area, while the frame on the right concentrates on the interesting area near the point $(I_{co}, \Gamma_1(I_{co}))$, where $\Gamma_1(I_{co}) \approx 7.11$. The parameter plane is partitioned into 5 domains. In domains 1,4 and 5 there is a single stable state - the normal state in region 1, a stationary state in region 4 and a periodic state in region 5. In region 2 there are two metastable states - normal and stationary, while in region 3 a stationary and a periodic state are both metastable.

We proceed to draw two further conclusions related to the bifurcation diagram. It is useful to do so in the context of I-V curves. These curves are measured or computed for a fixed Γ , while the current I is raised or lowered adiabatically. When this process cuts through the metastable

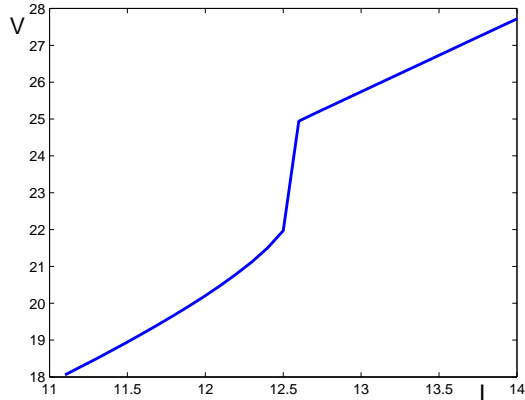


Figure 3: The I-V curve for $\Gamma = 6.3$ and I increasing. Notice that, although the periodic state does not exist for such temperature, and thus there is no PSC here, the I-V curve does exhibit a jump discontinuity at $I \approx 12.57$

regions 2 and 3 in Figure 2 a hysteresis is expected in the I-V curve. While such a hysteresis was predicted a long time ago, we point out that it is not always observed experimentally [4]. As can be seen in Figure 2, the metastable regions are quite small, and therefore it requires careful tuning to pass through them. Another comment relates to the formation of PSCs. These points in space-time where $|\psi|$ vanishes are often associated in the literature with jump discontinuities in the I-V curve. However, this identification works only in one way, and not all such jumps imply the presence of a PSC. For instance, we depict in Figure 3 the I-V curve for $\Gamma = 6.3$ and I slowly increasing. For this Γ one never crosses an area in the parameter plane where the periodic state is stable, and therefore there is no PSC. Nonetheless, the I-V curve exhibits a clear discontinuity at about $I \approx 12.57$. The actual rule for lack of smoothness in I-V curves is that a jump discontinuity indicates a nonsmooth phase transition, while a discontinuity in the derivative indicates a continuous phase transition.

To summarize, using a combination of asymptotic expansions, spectral analysis and canonical numerical computation applied to the time-dependent Ginzburg-Landau model, we have presented a full analysis of the behavior of a one-dimensional superconducting wire exposed to an applied electric current. In particular, retaining temperature and applied current as param-

eters, we have decomposed this two-dimensional parameter space into regions of stability of a normal, stationary and oscillatory state. The collision of real eigenvalues and the consequent emergence of complex spectrum in the associated linearized problem provides the explanation for the Hopf bifurcation leading to the appearance of the oscillatory state and the associated phase slip centers. From the theoretical standpoint, it also reveals a physically significant setting where PT-symmetry does *not* lead to reality of the spectrum, in contrast to its common role [6], [9]. The boundary of the basin of attraction of the normal state has been given precisely in terms of the real part of the leading eigenvalue in this linearized problem. The boundary between the basins of attraction of the oscillatory and stationary states has been calculated near the triple point using asymptotics, and has been computed numerically beyond this. In so doing, we have identified small regions in the parameter space where hysteresis should be anticipated. Finally the asymptotic structure of the periodic solution bifurcating off of the normal state has been developed for I above the first collision value I_{co} and for Γ just above the real part of the first eigenvalue. This expansion reveals the period of the oscillations and location of PSCs along the x -axis

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