Marcus versus Stratonovich for Systems with Jump Noise

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Abstract

The famous Itô–Stratonovich dilemma arises when one examines a dynamical system with a multiplicative white noise. In physics literature, this dilemma is often resolved in favour of the Stratonovich prescription because of its two characteristic properties valid for systems driven by Brownian motion: (i) it allows physicists to treat stochastic integrals in the same way as conventional integrals, and (ii) it appears naturally as a result of a small correlation time limit procedure. On the other hand, the Marcus prescription [*IEEE Trans. Inform. Theory* **24**, 164 (1978); *Stochastics* **4**, 223 (1981)] should be used to retain (i) and (ii) for systems driven by a Poisson process, Lévy flights or more general jump processes. In present communication we present an in-depth comparison of the Itô, Stratonovich, and Marcus equations for systems with multiplicative jump noise. By the examples of a real-valued linear system and a complex oscillator with noisy frequency (the Kubo–Anderson oscillator) we compare solutions obtained with the three prescriptions.

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1 Introduction

The Itô–Stratonovich dilemma is a remarkable issue in the theory of stochastic integrals and stochastic differential equations (SDE) with a white Gaussian noise. It has been extensively discussed in physics literature; the references include basic monographs on statistical physics [Gar04, vK07, Ris89, HL84].

The famous Itô formula gives the rule how to change variables in the stochastic Itô integral [Itô44]. In particular the usual integration by parts is not applicable, and the chain rule (also called Newton–Leibniz rule) does not hold in the Itô calculus. The Itô interpretation is preferable, e.g. if the SDE is obtained as a continuous time limit of a discrete time problem, as it takes place in mathematical finance [CT04, SL81] or population biology [Tur77].

Stratonovich [Str66] introduced another form of stochastic integral which can be treated according to the conventional rules of integration. Another important property of a Stratonovich equation concerns its interpretation as a Wong–Zakai small correlation time limit of solutions of differential equations with Gaussian coloured noise [WZ]. Stratonovich prescription is preferable, e.g. in physical kinetics [Ris89, WBL+79, BLS+79, van81].

In general, since the white noise is a mathematical idealisation of a real dynamics, the choice of prescription is not predetermined and may depend on the dynamical properties of the particular system. Thus, Kupferman at el. [KPS04] showed that an adiabatic elimination procedure in a system with inertia and coloured multiplicative noise leads to either an Itô or a Stratonovich equation, depending on whether the noise correlation time tends to zero faster or slower than the particle relaxation time. We refer the reader to a recent review [MM12] for historical background and discussions of some contemporary contributions, and mention the work [KM14] as the newest evidence of the continuing interest to this classical problem.

Until recently, the Itô–Stratonovich dilemma was discussed in the context of Brownian motion. Meanwhile, stochastic systems with multiplicative jump noises also attract increasing attention. They include

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systems driven by a Poisson process, Lévy flights, or general Lévy processes [Hän80b, Gri98, BB00, Sch03, CT04, GS04, Paw06, EZIK09, GPSS10, SRV$^+$10, SPRM11, Sro10]. However, it is not well-known among physicists that both remarkable properties of the Stratonovich integral are violated if the driving process has jumps. In [Mar78, Mar81] S. Marcus fixed this problem by introducing an SDE of a new type, whose solution pertains the features incident to the Stratonovich calculus in the continuous case. Although a Marcus equation (also called canonical equation) has been well treated in mathematical literature [KPP95, Kun04, App09], there is only a very few papers in physics literature, which discuss this issue. Motivated by the investigation of stochastic energetics for jump processes, Kanazawa et al. [KSH12] essentially followed the Wong–Zakai smoothing approach to define an SDE driven by a multiplicative white jump noise. They eventually re-derived a Marcus canonical equation and then applied it to study heat conduction by non-Gaussian noises from two athermal environments [KSH13]. Li et al. in [LMW13, LMW14b] gave an introduction to Marcus calculus via two equivalent constructions used in mathematical and engineering literature [Mar78, Mar81, DF93a, DF93b, SDL13] and developed a path-wise simulation algorithm allowing to compute thermodynamic quantities. Further, in [LMW14a] Li et al. extended the approach by Kupferman et al. [KPS04] to the case of a Poisson coloured noise. Similarly to [KPS04], in certain parameter regimes they obtained either Itô or Marcus canonical equations.

In this paper we present an in-depth comparison of the Itô, Stratonovich, and Marcus equations for systems driven by jump noise. In order to preserve the Markovian nature of solutions, we consider coloured noise being an Ornstein–Uhlenbeck process driven by a Brownian motion or a general Lévy process.

In the pure Brownian case, we recover the Stratonovich equation as a small relaxation time limit of differential equations driven by the Ornstein–Uhlenbeck process. Although the passage to the white noise limit should not depend on the smoothing procedure, in the jump case we give an instructive derivation of the Marcus equation as a limit of the Ornstein–Uhlenbeck coloured jump noise approximations.

We analyse the SDEs in the Itô, Stratonovich and Marcus form for two generic examples, namely for a real-valued linear system with multiplicative white noise and a complex oscillator with noisy frequency (the Kubo–Anderson oscillator). In case of the Kubo–Anderson oscillator we discover a remarkable similarity of solutions to the Stratonovich and Marcus equations. Nonetheless, from the physical point of view, the Marcus equation seems to be a more consistent and natural tool for description of a physical system with bursty dynamics or subject to jump noise.

2 Itô and Stratonovich Calculus for Brownian Motion

The definitions of the Itô and Stratonovich integrals w.r.t. the Brownian motion W are well known. For a non-anticipating stochastic process Y we define the Itô integral as a limit

$$\int_0^t Y_s\,\mathrm{d}W_s := \lim_{n\to\infty}\sum_{k=1}^n Y_{t_{k-1}}(W_{t_k}-W_{t_{k-1}}) \tag{1}$$

and the Stratonovich integral as

$$\int_0^t Y_s\circ\mathrm{d}W_s := \lim_{n\to\infty}\sum_{k=1}^n \frac{Y_{t_k}+Y_{t_{k-1}}}{2}(W_{t_k}-W_{t_{k-1}}), \tag{2}$$

where $0=t_0<\cdots<t_n=t$ is a partition with the vanishing mesh $\max_{0\leqslant k\leqslant n}|t_k-t_{k-1}|\to 0$ as $n\to\infty$. We refer the reader to [Gar04, Chapter 4.2] for a discussion about the mathematical properties and physical interpretations of these objects. We recall here two simple examples of stochastic integrals.

Example 2.1. A straightforward calculation based on the definitions (1) and (2) yields:

$$\int_0^t W_s\,\mathrm{d}W_s = \frac{W_t^2}{2}-\frac{t}{2}, \qquad \int_0^t W_s^2\,\mathrm{d}W_s = \frac{W_t^3}{3}-\int_0^t W_s\,\mathrm{d}s, \tag{3}$$

$$\int_0^t W_s\circ\mathrm{d}W_s = \frac{W_t^2}{2}, \qquad \int_0^t W_s^2\circ\mathrm{d}W_s = \frac{W_t^3}{3}. \tag{4}$$

As we see, the Stratonovich calculus pertains the Newton–Leibniz integration rule.

Consider now the Itô and Stratonovich SDEs with multiplicative noise, see [Gar04, Chapter 4.3]:

$$X_t = x + \int_0^t a(X_s)\,\mathrm{d}s + \int_0^t b(X_s)\,\mathrm{d}W_s \tag{5}$$

and

$$X_t^\circ = x + \int_0^t a(X_s^\circ)\,\mathrm{d}s + \int_0^t b(X_s^\circ)\circ\mathrm{d}W_s. \tag{6}$$

It is well known that the Stratonovich equation can be rewritten in the Itô form as

$$X_t^\circ = x + \int_0^t \Big(a(X_s^\circ) + \frac{1}{2}b'(X_s^\circ)b(X_s^\circ)\Big)\,\mathrm{d}s + \int_0^t b(X_s^\circ)\,\mathrm{d}W_s. \tag{7}$$

For a twice differentiable function F, the chain rules for the solutions of these equations read

$$F(X_t) = F(x) + \int_0^t F'(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t F''(X_s)b^2(X_s)\,\mathrm{d}s, \tag{8}$$

$$F(X_t^\circ) = F(x) + \int_0^t F'(X_s^\circ)\circ\mathrm{d}X_s. \tag{9}$$

With the help of Eqs. (8) and (9) we solve two simple linear stochastic differential equations.

Example 2.2. The equations for the real-valued linear system with multiplicative noise in the Itô and Stratonovich form, see [Gar04, §4.4.2], read

$$X_t = 1 + \int_0^t X_s\,\mathrm{d}W_s \quad\text{and}\quad X_t^\circ = 1 + \int_0^t X_s^\circ\circ\mathrm{d}W_s \tag{10}$$

and have unique solutions

$$X_t = \mathrm{e}^{W_t - \frac{t}{2}} \quad\text{and}\quad X_t^\circ = \mathrm{e}^{W_t}, \tag{11}$$

respectively.

Example 2.3. The Kubo–Anderson oscillator with noisy frequency, see [Gar04, §4.4.3], is described by a complex-valued SDE driven by a Brownian motion with linear drift. Let $w_t = \omega_0 t + \sigma W_t$, $\sigma^2 > 0$ being a noise variance, and $\omega_0 \in \mathbb{R}$ a constant frequency. Consider an SDE in the sense of Itô and Stratonovich:

$$Z_t = Z_0 + \mathrm{i}\int_0^t Z_s\,\mathrm{d}w_s \quad\text{and}\quad Z_t^\circ = Z_0 + \mathrm{i}\int_0^t Z_s^\circ\circ\mathrm{d}w_s \tag{12}$$

It is easy to check with the help of Eqs. (8) and (9), that the solutions to these equations are

$$Z_t = Z_0\mathrm{e}^{\frac{\sigma^2}{2}t}\mathrm{e}^{\mathrm{i}(\omega_0 t + \sigma W_t)} \quad\text{and}\quad Z_t^\circ = Z_0\mathrm{e}^{\mathrm{i}(\omega_0 t + \sigma W_t)}. \tag{13}$$

It is seen from Eq. (13) that the Itô solution has an exponentially increasing amplitude and is not physically relevant.

Along with the Newton–Leibniz rule (9), another important feature of a Stratonovich SDE is that it can be considered as a limit of differential equations driven by smooth approximations of the Brownian motion. This interpretation goes back to Wong and Zakai [WZ]. Instead of polygonal smoothing usually used in the literature [Arn74], we employ coloured noise approximations $\dot W^\tau$ in the form of the Ornstein–Uhlenbeck process with small correlation time τ. They are obtained from the Langevin equation

$$\ddot W_t^\tau = -\frac{1}{\tau}\dot W^\tau + \frac{1}{\tau}\dot W, \tag{14}$$

which is solved explicitly as

$$W_t^\tau = \int_0^t (1 - \mathrm{e}^{-\frac{t-s}{\tau}})\,\mathrm{d}W_s \quad\text{and}\quad \dot W_t^\tau = \frac{1}{\tau}\int_0^t \mathrm{e}^{-\frac{t-s}{\tau}}\,\mathrm{d}W_s. \tag{15}$$

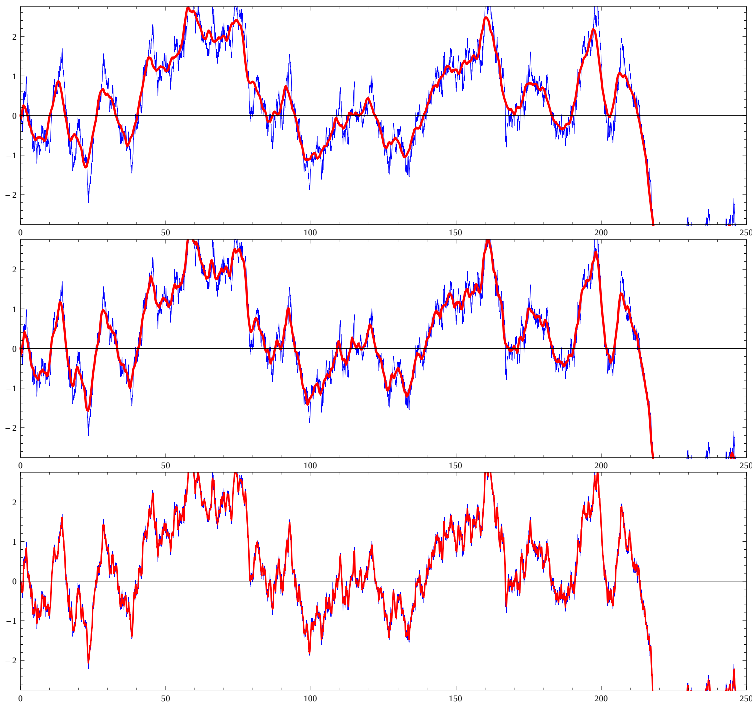

Figure 1: (Colour online). Coloured approximations W^τ (red) of the Brownian motion W (blue) with decreasing relaxation times $\tau = 20, 1$, and 0.1 (from top to bottom).

It can be easily shown that W^τ tends to W as $\tau \to 0$ (see Fig. 1), so that $\dot{W}^\tau$ can be seen as τ-correlated approximations of the delta-correlated white noise $\dot{W}$. Let us now substitute W in Eq. (5) by its approximation W^τ and consider a τ-dependent differential equation

$$X_t^\tau = x + \int_0^t a(X_s^\tau)\,\mathrm{d}s + \int_0^t b(X_s^\tau)\dot{W}_s^\tau\,\mathrm{d}s. \tag{16}$$

Assume that $b(x) > 0$ and define a function $F(x) = \int_0^x \frac{\mathrm{d}y}{b(y)}$ which is strictly monotone and smooth. Then the Newton–Leibniz rule of the conventional calculus gives

$$F(X_t^\tau) = F(x) + \int_0^t F'(X_s^\tau)\,\mathrm{d}X_s^\tau = F(x) + \int_0^t \frac{a(X_s^\tau)}{b(X_s^\tau)}\,\mathrm{d}s + W_t^\tau. \tag{17}$$

Since W^τ converges to W as $\tau \to 0$, X^τ converges to a limit X° which satisfies the equation

$$F(X_t^\circ) = F(x) + \int_0^t \frac{a(X_s^\circ)}{b(X_s^\circ)}\,\mathrm{d}s + W_t. \tag{18}$$

Let G denote the inverse of F, that is $G(F(x)) = x$. Taking into account that $G'(x) = b(x)$ and $G''(x) = b'(x)$, we apply the formula (8) with the function G to the solution $F(X_t^\circ)$ to obtain the equality

$$\begin{aligned} X_t^\circ = G(F(X_t^\circ)) &= x + \int_0^t a(X_s)\,\mathrm{d}s + \int_0^t b(X_s^\circ)\,\mathrm{d}W_s + \frac{1}{2}\int_0^t b'(X_s^\circ)b(X_s^\circ)\,\mathrm{d}s \\ &= x + \int_0^t a(X_s^\circ)\,\mathrm{d}s + \int_0^t b(X_s^\circ)\circ\mathrm{d}W_s. \end{aligned} \tag{19}$$

Hence, the process X° solves the Stratonovich SDE (6).

3 Itô and Stratonovich calculuses for processes with jumps

With the help of formulae (1) and (2) one can also define Itô and Stratonovich integrals for a broader class of processes with jumps, in particular for Lévy processes and for semimartingales, see [Kun04, Chapter 1].

For simplicity we restrict ourselves to integrals and SDEs driven by a Lévy process with finite number of jumps, which is a sum of a Brownian motion with drift and an independent compound Poisson process. Let $P = (P_t)_{t\geqslant 0}$ be a Poisson process with intensity $\lambda > 0$ and arrival times $T_0 = 0$, $(T_m)_{m\geqslant 1}$, such that the waiting times $T_m - T_{m-1}$ are i.i.d. exponentially distributed with the mean λ^{-1}. Let

$$N_t = \sum_{m=1}^{P_t} J_m = \sum_{m=1}^{\infty} J_m \mathbb{I}_{[T_m,\infty)}(t) \tag{20}$$

be a compound Poisson process with the i.i.d. jumps $(J_m)_{m\geqslant 1}$ being independent of P. Here $\mathbb{I}_{[T,\infty)}(t)$ is the indicator function being 1 on $[T,\infty)$ and 0 otherwise. For $\omega_0 \in \mathbb{R}$, $\sigma \geqslant 0$ and a Brownian motion W denote $L_t = \omega_0 t + \sigma W_t + N_t$.

For a trajectory of a random process Y we denote by ΔY_t the jump size of Y at the time instant t, i.e. $\Delta Y_t = Y_t - Y_{t-}$, where $Y_{t-} = \lim_{h\downarrow 0} Y_{t-h}$.

To compare the continuous and jump calculuses, we give a couple of basic integration examples.

Example 3.1. Let P be a Poisson process. Then the integration in the Itô sense (1) gives

$$\begin{aligned} \int_0^t P_s \,\mathrm{d}P_s &= \sum_{s\leqslant t} P_{s-}\Delta P_s = \frac{P_t^2}{2} - \frac{P_t}{2} \\ \int_0^t P_s^2 \,\mathrm{d}P_s &= \sum_{s\leqslant t} P_{s-}^2\Delta P_s = \frac{P_t^3}{3} - \frac{P_t^2}{2} + \frac{P_t}{6}. \end{aligned} \tag{21}$$

whereas the integration in the Stratonovich sense (2) yields

$$\int_0^t P_s \circ \mathrm{d}P_s = \frac{P_t^2}{2} \quad \text{and} \quad \int_0^t P_s^2 \circ \mathrm{d}P_s = \frac{P_t^3}{3} + \frac{P_t}{2}. \tag{22}$$

As we see, even in the simple case of a squared Poisson process as an integrand, the Stratonovich calculus does not obey the Newton–Leibniz integration rule[1].

Similarly to the previous section, we consider Itô and Stratonovich SDEs, see Eqs. (5) and (6) with a jump process instead of a Brownian motion.

Example 3.2. We solve the equations for the real-valued linear system with the multiplicative Poisson noise in the Itô and Stratonovich form

$$X_t = 1 + \int_0^t X_s \,\mathrm{d}(zP_s) \quad \text{and} \quad X_t^\circ = 1 + \int_0^t X_s^\circ \circ \mathrm{d}(zP_s), \tag{23}$$

where $z \in \mathbb{R}$ is a jump size. The solution of the Itô equation is the so-called stochastic exponent and the solution exists and is unique for $z > -1$:

$$X_t = \prod_{s\leqslant t}(1 + z\Delta P_s) = (1+z)^{P_t}. \tag{24}$$

To solve the Stratonovich equation, we note that at the arrival time T_m the solution satisfies the equality

$$X_{T_m}^\circ = X_{T_m-}^\circ + \frac{X_{T_m-}^\circ + X_{T_m}^\circ}{2} z. \tag{25}$$

This yields for $z \neq 2$

$$X_{T_m}^\circ = \frac{2+z}{2-z} X_{T_m-}^\circ \tag{26}$$

[1] Note that in [Gri98], the so–called Fisk–Stratonovich definition of the Stratonovich integral for jump processes is used. It is different from (2) and leads to a trivial equivalence between the Itô and Stratonovich calculuses in the pure jump Poissonian case.

Consequently, the solution of the Stratonovich SDE is found in the form

$$X_t^\circ = \Big(\frac{2+z}{2-z}\Big)^{P_t}. \tag{27}$$

Example 3.3. Consider the Kubo–Anderson oscillator perturbed by a centred Lévy process $\sigma W_t + z(P_t - \lambda t)$, $\langle \sigma W_t + z(P_t-\lambda t)\rangle = 0$. Denote $l_t = \omega_0 t + \sigma W_t + z(P_t - \lambda t)$ and solve two complex-valued SDEs in the Itô and Stratonovich form:

$$Z_t = Z_0 + \mathrm{i}\int_0^t Z_s\,\mathrm{d}l_s \quad \text{and} \quad Z_t^\circ = Z_0 + \mathrm{i}\int_0^t Z_s^\circ \circ \mathrm{d}l_s. \tag{28}$$

Let us first solve the Itô equation. Between the arrival times of Poisson process P, the solution of the Itô equation coincides with the continuous Itô solution (13). At the arrival time T_m the position of the solution is found from the relation

$$Z_{T_m} = Z_{T_m-} + \mathrm{i}Z_{T_m-}z \quad \text{and thus} \quad Z_{T_m} = (1+\mathrm{i}z)Z_{T_m-}. \tag{29}$$

Combining the continuous and the jump parts of the solution and taking into account that

$$1+\mathrm{i}z = (1+z^2)^{1/2}\mathrm{e}^{\mathrm{i}\varphi(z)}, \quad \text{where} \quad \varphi(z) = \arctan z \in \Big(-\frac{\pi}{2},\frac{\pi}{2}\Big) \text{ for } z\in\mathbb{R}, \tag{30}$$

we finally obtain a physically inappropriate Itô solution with exponentially increasing amplitude

$$Z_t = Z_0(1+z^2)^{\frac{P_t}{2}}\,\mathrm{e}^{\frac{\sigma^2}{2}t}\mathrm{e}^{\mathrm{i}((\omega_0-\lambda z)t+\sigma W_t+\varphi(z)P_t)}. \tag{31}$$

In the Stratonovich case, as in the Example 3.2, at the arrival times of P the jumps of Z° satisfy

$$Z_{T_m}^\circ = \frac{2+\mathrm{i}z}{2-\mathrm{i}z}Z_{T_m-}^\circ \tag{32}$$

whereas between the jumps the solution follows the continuous Stratonovich dynamics considered in Example 2.3. Noting that

$$\frac{2+\mathrm{i}z}{2-\mathrm{i}z} = \mathrm{e}^{\mathrm{i}\psi(z)}, \quad \text{where} \quad \psi(z) = \arcsin\frac{z}{1+\frac{z^2}{4}} \in \Big(-\frac{\pi}{2},\frac{\pi}{2}\Big) \text{ for } z\in\mathbb{R}, \tag{33}$$

we obtain a physically meaningful solution representing stochastic oscillations

$$Z_t^\circ = Z_0\mathrm{e}^{\mathrm{i}((\omega_0-\lambda z)t+\sigma W_t+\psi(z)P_t)}. \tag{34}$$

with constant amplitude.

In this case it could be instructive to determine the oscillator's line shape. Assume that $|Z_0| = 1$ and determine the relaxation function

$$\begin{aligned}\Phi^\circ(t) &= \langle \overline{Z_0^\circ}Z_t^\circ\rangle = \langle \mathrm{e}^{\mathrm{i}(\omega_0-\lambda z)t+\sigma W_t+\psi(z)P_t)}\rangle \\ &= \mathrm{e}^{\mathrm{i}(\omega_0-\lambda z)t}\langle \mathrm{e}^{\mathrm{i}\sigma W_t}\rangle\langle \mathrm{e}^{\mathrm{i}\psi(z)P_t}\rangle = \mathrm{e}^{\mathrm{i}(\omega_0-\lambda z)t}\mathrm{e}^{-t\frac{\sigma^2}{2}}\mathrm{e}^{\lambda t(\mathrm{e}^{\mathrm{i}\psi(z)}-1)} \\ &= \mathrm{e}^{-t(\frac{\sigma^2}{2}+\lambda(1-\cos\psi(z)))}\mathrm{e}^{\mathrm{i}t(\omega_0+\lambda\sin\psi(z)-\lambda z)} = \mathrm{e}^{-\gamma^\circ t+\mathrm{i}(\omega_0+\omega^\circ)t},\end{aligned} \tag{35}$$

where

$$\begin{aligned}\gamma^\circ &= \frac{\sigma^2}{2} + \lambda(1-\cos\psi(z)),\\ \omega^\circ &= \lambda(\sin\psi(z) - z).\end{aligned} \tag{36}$$

Then the line shape (see Kubo [Kub63], Eqs. (2.6) and (3.6)) has the Lorenzian form

$$I^\circ(\omega) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty \mathrm{e}^{-\mathrm{i}\omega t}\Phi^\circ(t)\,\mathrm{d}t = \frac{1}{\pi}\frac{\gamma^\circ}{(\gamma^\circ)^2+(\omega-\omega_0-\omega^\circ)^2}. \tag{37}$$

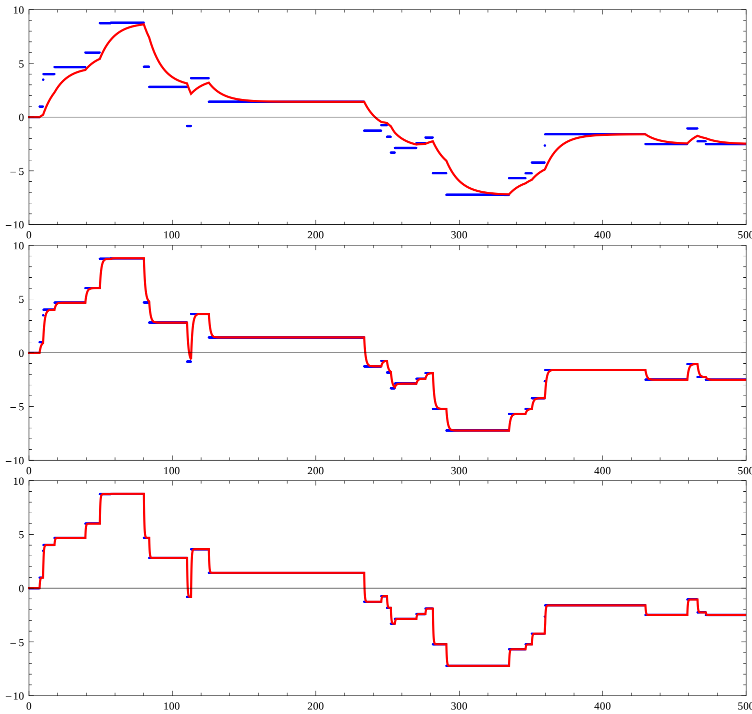

Figure 2: (Colour online) Coloured approximations N^τ (red) of the compound Poisson process N (blue) with decreasing relaxation times $\tau = 100, 1$, and 0.3 (from top to bottom).

4 Coloured jump noise and Marcus SDEs

As we demonstrated in the previous section, the Stratonovich calculus in the jump case does not pertain the Newton–Leibniz change of variables rule. Now we study if it is consistent with the small correlation limit of the coloured noise approximations.

For simplicity consider a compound Poisson process $N = (N_t)_{t\geqslant 0}$ defined in (20). As in Section 3, consider the coloured jump noise $\dot N^\tau$, being a derivative of the solution of then Langevin equation with the relaxation time $\tau > 0$ driven by N:

$$\ddot N_t^\tau = -\frac{1}{\tau}\dot N_t^\tau + \frac{1}{\tau}N_t, \quad N_0^\tau = 0. \tag{38}$$

Clearly

$$N_t^\tau = \int_0^t (1-\mathrm{e}^{-\frac{t-s}{\tau}})\,\mathrm{d}N_s = \sum_{m=1}^\infty J_m(1-\mathrm{e}^{-\frac{t-T_m}{\tau}})\mathbb{I}_{[T_m,\infty)}(t) \tag{39}$$

and

$$\dot N_t^\tau = \frac{1}{\tau}\int_0^t \mathrm{e}^{-\frac{t-s}{\tau}}\,\mathrm{d}N_s = \sum_{m=1}^\infty \frac{J_m}{\tau}\mathrm{e}^{-\frac{t-T_m}{\tau}}\mathbb{I}_{[T_m,\infty)}(t). \tag{40}$$

The approximation N_t^τ converges to N_t as $\tau\to 0$ on the time intervals between the jumps, and monotonically 'glues together' the discontinuities (see Fig. 2). For $\tau > 0$ consider a random differential equation driven by the multiplicative smoothed process N^τ:

$$X_t^\tau = x + \int_0^t a(X_s^\tau)\,\mathrm{d}s + \int_0^t b(X_s^\tau)\,\mathrm{d}N_s^\tau. \tag{41}$$

Let us study the limiting behaviour of X^τ in the limit $\tau\downarrow 0$. Clearly, between the jumps of N and for small τ, the solution X^τ moves along the external field a. Put for simplicity $a = 0$. Then the equation for X^τ

takes the form

$$\dot{X}^\tau_t = b(X^\tau_t)\dot{N}^\tau_t = b(X^\tau_t)\sum_{m=1}^\infty \frac{J_m}{\tau}\mathrm{e}^{-\frac{t-T_m}{\tau}}\mathbb{I}_{[T_m,\infty)}(t), \tag{42}$$

This is a random non-autonomous differential equation with piece-wise smooth right-hand side. It is natural to solve it sequentially on the inter-jump intervals (T_m, T_{m+1}). On this time interval the equation has the form

$$\dot{X}^\tau_t = b(X^\tau_t)\Big[\sum_{k=1}^{m-1}\frac{J_k}{\tau}\mathrm{e}^{-\frac{t-T_k}{\tau}} + \frac{J_m}{\tau}\mathrm{e}^{-\frac{t-T_m}{\tau}}\Big]. \tag{43}$$

and the terms $\sum_{k=1}^{m-1} J_k \mathrm{e}^{-\frac{t-T_k}{\tau}}$ can be neglected for τ small enough such that $\tau \ll \frac{1}{\lambda} = \langle T_m - T_{m-1}\rangle$. Then the equation reduces to

$$X^\tau_t = X^\tau_{T_m-} + \int_{T_m}^{T_m+t} b(X^\tau_s)\frac{J_m}{\tau}\mathrm{e}^{-\frac{s-T_m}{\tau}}\,\mathrm{d}s. \tag{44}$$

For convenience, we perform the time shift at T_m, denote $U^\tau_t = X^\tau_{T_m+t}$ and consider the equation

$$U^\tau_t = X^\tau_{T_m-} + \int_0^t b(U^\tau_s)\frac{J_m}{\tau}\mathrm{e}^{-\frac{s}{\tau}}\,\mathrm{d}s, \quad t\in[0, T_{m+1}-T_m), \tag{45}$$

in the limit $\tau \to 0$. To capture the fast change of the solution caused by the jump of N of the size J_m we perform a time stretching transformation

$$s = -\tau\ln(1-u), \quad u\in[0,1), \quad u = 1-\mathrm{e}^{-s/\tau}, \quad s\geqslant 0. \tag{46}$$

which transforms (45) into

$$U^\tau_t = X^\tau_{T_m-} + \int_0^{1-\mathrm{e}^{-t/\tau}} b(U^\tau_{-\tau\ln(1-u)})J_m\,\mathrm{d}u \tag{47}$$

Denote

$$Y^\tau_u = U^\tau_{-\tau\ln(1-u)} \quad \text{or equivalently} \quad U^\tau_t = Y^\tau_{1-\mathrm{e}^{-t/\tau}}. \tag{48}$$

Then (47) can be rewritten in terms of the process Y^τ as

$$Y^\tau_{1-\mathrm{e}^{-t/\tau}} = X^\tau_{T_m-} + \int_0^{1-\mathrm{e}^{-t/\tau}} b(Y^\tau_u)J_m\,\mathrm{d}u. \tag{49}$$

It is natural to assume that $X^\tau \to X^\diamond$ in the limit $\tau\to 0$. Passing to the limit in equation (49) for any $t>0$ we recover the identity

$$Y^0_1 = X^\diamond_{T_m-} + \int_0^1 b(Y^0_u)J_m\,\mathrm{d}u. \tag{50}$$

The value Y^0_1 determines position the of the limiting solution $X^\diamond$ after the jump of the size J_m. Eq. (50) is the integral form of the ordinary non-linear differential equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}y(u;x,z) &= b(y(u,x,z))z,\\ y(0;x,z) &= x,\end{aligned} \tag{51}$$

with time $u\in[0,1]$, a parameter $z = J_m$ and the initial value x being equal to the value of the solution $X^\diamond_{T_m-}$ just before the jump. Eq. (51) plays a particular role in the theory of Marcus equations. Indeed, for any x and any z let us denote its solution evaluated at $u=1$ by $\phi^z(x) := y(1;x,z)$. Then $X^\diamond_{T_m} = \phi^{J_m}(X^\diamond_{T_m}-)$ and the instantaneous jump occurs along the curve $y(u;X_{T_m-},J_m)$, $u\in[0,1]$, see Figure 3.

Overall, coming back to the process X^τ and taking into account the drift a we find that in the limit $\tau\to 0$ the continuous dynamics of X^τ obeys the following equation with jumps, which is known to be the Marcus (canonical) equation:

$$X^\diamond_t = x + \int_0^t a(X^\diamond_s)\,\mathrm{d}s + \sum_{m\colon T_m\leqslant t}\Big(\phi^{J_m}(X^\diamond_{T_m-}) - X^\diamond_{T_m-}\Big). \tag{52}$$

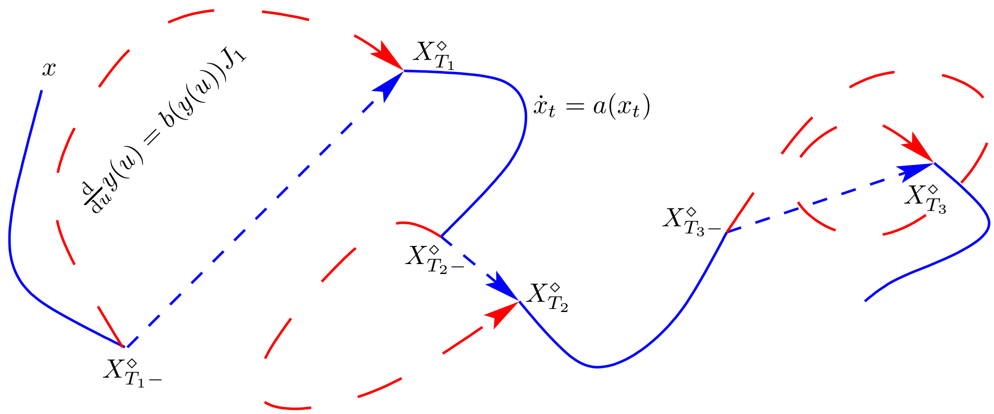

Figure 3: (Colour online) Schematic representation of a solution of Marcus SDE $\mathrm{d}X^\diamond = a(X^\diamond)\,\mathrm{d}t + b(X^\diamond)\diamond \mathrm{d}N$ driven by the jump noise $N_t = \sum_{m=1}^{P_t} J_m$. The solution $X^\diamond$ is a jump process with jumps $X^\diamond_{T_m} - X^\diamond_{T_m-}$ (dashed blue lines). These jumps are obtained as a small relaxation time limit of fast curvilinear motions along the solutions of an auxiliary non-linear ODE $\frac{\mathrm{d}}{\mathrm{d}u}y(u) = b(y(u))J_m$ with the initial value $y(0) = X^\diamond_{T_m-}$ (dashed red lines). Between the jumps of N, the solution $X^\diamond$ moves along the force field a (solid blue lines).

Recalling that according to the definition of the Itô integral we have

$$\int_0^t b(X^\diamond_s)\,\mathrm{d}N_s = \sum_{s\leqslant t} b(X^\diamond_{s-})\Delta N_s = \sum_{m\colon T_m\leqslant t} b(X^\diamond_{T_m-})J_m, \tag{53}$$

we can rewrite (52) as an Itô equation with a correction term

$$X^\diamond_t = x + \int_0^t a(X^\diamond_s)\,\mathrm{d}s + \int_0^t b(X^\diamond_s)\,\mathrm{d}N_s + \sum_{m\colon T_m\leqslant t}\Big(\phi^{J_m}(X^\diamond_{T_m-}) - X^\diamond_{T_m-} - b(X^\diamond_{T_m-})J_m\Big). \tag{54}$$

The last two terms in the formula (54) are abbreviated as the Marcus 'integral' $\int_0^t b(X^\diamond_s)\diamond \mathrm{d}N_s$ with respect to N. The equation (53) is thus formally written as

$$X^\diamond_t = x + \int_0^t a(X^\diamond_s)\,\mathrm{d}s + \int_0^t b(X^\diamond_s)\diamond \mathrm{d}N_s. \tag{55}$$

Now it is easy to obtain the stochastic equation for the colour noise limit of the dynamics driven by the Brownian motion with drift and a compound Poisson process N. Let $L_t = \omega_0 t + \sigma W_t + N_t$. Then on the intervals between the jumps of N the solution evolves according to the continuous Stratonovich equation and is inter-dispersed with jumps calculated with the help of the mapping $\phi^z(x)$. Eventually we obtain the equation

$$\begin{aligned} X^\diamond_t &= x + \int_0^t a(X^\diamond_s)\,\mathrm{d}s + \int_0^t b(X^\diamond_s)\diamond \mathrm{d}L_s \\ &= x + \int_0^t a(X^\diamond_s)\,\mathrm{d}s + \int_0^t b(X^\diamond_s)\circ \mathrm{d}(\sigma W_s + \omega_0 s) \\ &\quad + \int_0^t b(X^\diamond_s)\,\mathrm{d}N_s + \sum_{m\colon T_m\leqslant t}\Big(\phi^{J_m}(X^\diamond_{T_m-}) - X^\diamond_{T_m-} - b(X^\diamond_{T_m-})J_m\Big) \end{aligned} \tag{56}$$

One can prove (see, e.g. [KPP95, Kun04, App09]) that the compound Poisson process N in (56) can be replaced by a Lévy process Z with infinitely many jumps, e.g. a Lévy flights process. In this case, the Marcus correction term contains a sum over infinitely many jumps of Z and $X^\diamond$ satisfies

$$X_t^\diamond = x + \int_0^t a(X_s^\diamond)\,\mathrm{d}s + \int_0^t b(X_s^\diamond)\circ \mathrm{d}(\sigma W_s + \omega_0 s) \\ + \int_0^t b(X_s^\diamond)\,\mathrm{d}Z_s + \sum_{s\leqslant t}\Big(\phi^{\Delta Z_s}(X_{s-}^\diamond) - X_{s-}^\diamond - b(X_{s-}^\diamond)\Delta Z_s\Big). \tag{57}$$

It is clear that for the additive noise, $b(x) \equiv \mathrm{Const}$, the Marcus, Stratonovich and Itô equations coincide. For multiplicative continuous noise, the Marcus equation coincides with the Stratonovich equation and differs from the Itô one. In the case of multiplicative jump noise, all three equations are different.

Also note that the Marcus 'integral' is not an integral but an abbreviation of an Itô integral and a correction sum from the last line in (56) or (57). This is why we are not able to calculate expressions like $\int_0^t P_s \diamond \mathrm{d}P_s$ in the Marcus sense.

Fortunately, the chain rule can be still applied to solutions of Marcus SDEs. Indeed, for any twice differentiable function F we can write

$$F(X_t^\diamond) = F(x) + \int_0^t F'(X_s^\diamond)a(X_s^\diamond)\,\mathrm{d}s + \int_0^t F'(X_s^\diamond)b(X_s^\diamond)\diamond \mathrm{d}L_s \\ = F(x) + \int_0^t F'(X_s^\diamond)\diamond X_s^\diamond. \tag{58}$$

where the term $\int_0^t F'(X_s^\diamond)b(X_s^\diamond)\diamond \mathrm{d}L_s$ can be understood as a small relaxation limit of the integrals $\int_0^t F'(X_s^\tau)b(X_s^\tau)\,\mathrm{d}L_s^\tau$.

Thus, the Marcus calculus enjoys all the properties one would expect from the Stratonovich integration rule, namely, the conventional change of variables formula and the validity of the coloured noise approximations.

Example 4.1. Consider an SDE in the sense of Marcus for a real-valued linear system driven by the Lévy process $L_t = \omega_0 t + \sigma W_t + N_t$ (compare with Example 3.2)

$$X_t^\diamond = 1 + \int_0^t X_s^\diamond \diamond \mathrm{d}L_s. \tag{59}$$

Since the conventional Newton–Leibniz integration formula applies here, we get

$$X_t^\diamond = \mathrm{e}^{L_t} = \mathrm{e}^{\omega_0 t + \sigma W_t + N_t}. \tag{60}$$

In particular, in the Poisson case $L = zP$, $z \in \mathbb{R}$, we obtain (compare with Eqs. (24) and (27))

$$X_t^\diamond = \mathrm{e}^{zP_t}. \tag{61}$$

Example 4.2. Consider the Kubo–Anderson oscillator with Marcus multiplicative noise $l_t = \omega_0 + \sigma W_t + z(P_t - \lambda t)$ (compare with Example 3.3):

$$Z_t^\diamond = Z_0 + \mathrm{i}\int_0^t Z_s^\diamond \diamond \mathrm{d}l_s. \tag{62}$$

The solution to this equation is the conventional exponent

$$Z_t^\diamond = Z_0 \mathrm{e}^{\mathrm{i}(\omega_0 t + \sigma W_t + z(P_t - \lambda t))}, \tag{63}$$

and this solution is physically meaningful. As in the Stratonovich case, assume that $|Z_0| = 1$ and determine the relaxation function

$$\begin{aligned} \Phi^\diamond(t) &= \langle \overline{Z_0^\diamond} Z_t^\diamond \rangle = \langle \mathrm{e}^{\mathrm{i}((\omega_0 - \lambda z)t + \sigma W_t + zP_t)} \rangle \\ &= \mathrm{e}^{-t(\frac{\sigma^2}{2} + \lambda(1-\cos z))}\mathrm{e}^{\mathrm{i}t(\omega_0 + \lambda\sin z - \lambda z)} = \mathrm{e}^{-\gamma^\diamond t + \mathrm{i}(\omega_0 + \omega^\diamond)t}, \\ \gamma^\diamond &= \frac{\sigma^2}{2} + \lambda(1 - \cos z), \\ \omega^\diamond &= \lambda(\sin z - z). \end{aligned} \tag{64}$$

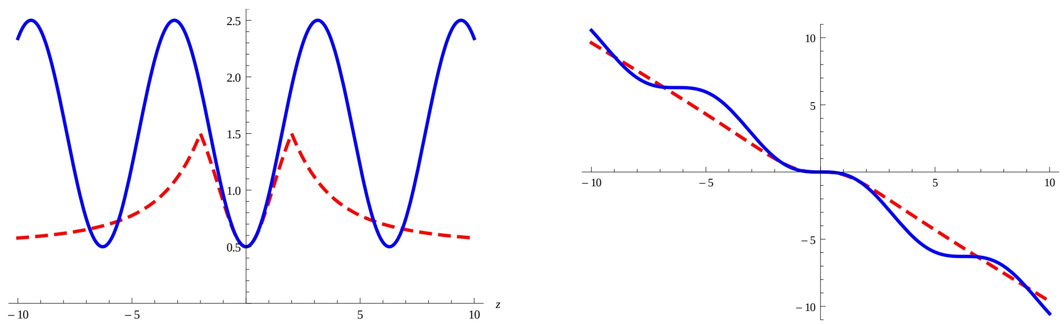

Figure 4: (Colour online) The dependence of the spectral line widths γ° and $\gamma^\diamond$ (left) and phase shifts ω° and $\omega^\diamond$ (right) on the jump size $z \in \mathbb{R}$ for the Stratonovich (red, dashed) and Marcus (blue) Kubo–Anderson oscillators; $\sigma = 1$, $\lambda = 1$.

Then, similar to the Stratonovich case, Eq. (37), the line shape is obtained as

$$I^\diamond(\omega) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty \mathrm{e}^{-\mathrm{i}\omega t}\Phi^\diamond(t)\,\mathrm{d}t = \frac{1}{\pi}\frac{\gamma^\diamond}{(\gamma^\diamond)^2 + (\omega - \omega_0 - \omega^\diamond)^2}. \tag{65}$$

The line widths and the frequency shifts in Stratonovich and Marcus cases are shown in Fig. 4 and compared in the next Section.

5 Discussion

For systems with jump noises or bursty fluctuations the Marcus integration plays the same role as the Stratonovich integration for systems driven by Brownian motion. In this paper, we derived the Stratonovich equation as a small correlation time limit of differential equations driven by Gaussian Ornstein–Uhlenbeck coloured noise. Analogously, we introduced a Marcus canonical equation as a limit of equations driven by Lévy Ornstein–Uhlenbeck coloured noise. In contrast to the Wong–Zakai polygonal approximation scheme, the Ornstein–Uhlenbeck approximations are non-anticipating functions in the sense that they do not account for future events, see [Gar04, §4.2.4]. They can be also treated within the theory of Markov processes and Fokker–Planck equation.

We solved explicitly the Itô, Stratonovich and Marcus equations for two generic linear systems driven by Brownian motion inter-dispersed by Poisson jumps. As expected, the Marcus interpretation is consistent with the conventional integration rules. The Itô interpretation of the Kubo–Anderson oscillator demonstrates a physically inappropriate solution with exponentially increasing amplitude. The Stratonovich and Marcus solutions reveal remarkably similar properties. Both solutions do not leave the unit circle on a complex plane and have a Lorenzian spectral line shape. However, the frequency shifts and the line widths exhibit different behaviour as functions of the jump size z. In particular, in the Marcus case the line width is a periodic function, whereas in the Stratonovich case it attains its maxima at $z = \pm 2$ and decreases monotonically at larger $|z|$.

We note that in the theory of Brownian motion, in dependence of the phenomenon considered, another important prescription is physically relevant, namely the Hänggi–Klimontovich prescription, or the so-called post-point scheme, see [TM77, Hän78, Hän80a, Kli90, Kli94, DH05], Very recent studies go even beyond the Itô, Stratonovich or Hänggi–Klimontovich prescriptions [YA12, SCY$^+$12]. It would be interesting to extend these approaches also to non-Gaussian jump noises. Another interesting research direction would be to give a thermodynamical interpretation in case of discontinuous processes. Here we may refer to the two papers [LL07, Sok10] on this issue in the theory of Brownian motion.

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