SEMI-INVARIANTS OF QUIVERS

AIDAN SCHOFIELD

ABSTRACT

Let Q be a quiver without oriented cycles and let α be a dimension vector such that $Gl(\alpha)$ has an open orbit on the representation space $R(\alpha)$. We find representations $\{S_i\}$ and corresponding polynomials $\{P_{S_i,\alpha}\}$ on $R(\alpha)$, which generate the semi-invariants and are algebraically independent.

0. *Notation and generalities*

Let Q be a quiver with *vertex set* Q_0 and *arrow set* Q_1. Typically, we shall write $v \in Q_0$, $a \in Q_1$, and a has a *tail*, ta, and *head*, ha;

$$\underset{ta}{\cdot} \xrightarrow{a} \underset{ha}{\cdot} .$$

We fix an algebraically closed field k. A *representation* R of Q is a family of finite-dimensional vector spaces $\{R(v) : v \in Q_0\}$, together with linear maps $R(a): R(ta) \to R(ha)$. The *dimension vector* of R is the function, $\underline{\dim} R$, defined by $\underline{\dim} R(v) = \dim R(v)$; it lies in Γ, the space of integer-valued functions on Q_0. If R and S are representations then a *morphism* $\phi: R \to S$ is a collection of linear maps $\phi(v): R(v) \to S(v)$ such that, for all $a \in Q_1$, $\phi(ta)\, S(a) = R(a)\, \phi(ha)$. The collection of morphisms is written $\operatorname{Hom}(R, S)$. With these definitions, the category of representations of Q, namely $\underline{\operatorname{Rep}}(Q)$, may be seen to be an abelian category.

We assume throughout this paper that there are no *oriented cycles*; that is, there is no sequence of arrows $a_1, a_2, \ldots, a_n$ such that $ha_i = ta_{i+1}$ and $ha_n = ta_1$. A *path* $p = a_1 a_2 \ldots a_n$ is a sequence of arrows such that $ha_i = ta_{i+1}$ for $i = 1$ to $n-1$. We define $tp = ta_1$ and $hp = ha_n$. The *trivial path* from v to v has head and tail v. We define $[v, w]$ to be the vector space on the basis of paths from v to w. This is finite dimensional given our assumption that there are no oriented cycles. We define a representation P_v by $P_v(w) = [v, w]$, and $P_v(a): [v, ta] \to [v, ha]$ is defined by composition with a. One checks that $\operatorname{Hom}(P_v, R) \simeq R(v)$, so that $\operatorname{Hom}(P_v, -)$ is an exact functor which implies that P_v is a projective representation. The set $\{P_v\}$ is a complete set of indecomposable projective representations up to isomorphism. Similarly, we define the indecomposable injective representations of Q by $I_v(w) = D[w, v]$; here DV for a vector space V is defined to be $\operatorname{Hom}_k(V, k)$. One checks that $\operatorname{Hom}(R, I_v) \simeq DR(v)$. Note that

$$\operatorname{Hom}(P_v, P_w) \simeq [v, w] \simeq DI_w(v) \simeq \operatorname{Hom}(I_v, I_w).$$

So there is a natural equivalence between the category of projective and the category of injective representations of Q. If we define $\Lambda = \oplus P_v$, Λ has a natural algebra structure, the *path algebra* of Q, and representations of Q are modules for Λ. One can show that the above equivalence between the category of projectives and injectives is induced by the functor $D \operatorname{Hom}(\ ,\Lambda)$.

Received 22 June 1988.

1980 *Mathematic Subjects Classification* (1985 *Revision*) 16A64.

The author was supported by NSF grant DMS 8806371.

J. London Math. Soc. (2) 43 (1991) 385–395

Given any representation R, there is a short exact sequence

$$0 \longrightarrow \oplus_a R(ta) \otimes_k P_{ha} \longrightarrow \oplus_v R(v) \otimes_k P_v \longrightarrow R \longrightarrow 0$$

which we call the *canonical projective resolution* of R. For the maps in this see [4]. Dually we may define the *canonical injective resolution* of R. This implies that Ext^i vanishes on $\mathrm{Rep}(Q)$ for $i > 1$.

If R has no projective summands, then $\mathrm{Hom}(R, \Lambda) = 0$; we apply $D\,\mathrm{Hom}(\ ,\Lambda)$ to the canonical projective resolution: this gives a new short exact sequence:

$$0 \longrightarrow \tau R \longrightarrow \oplus_a R(ta) \otimes_k I_{ha} \longrightarrow \oplus_v R(v) \otimes_k I_v \longrightarrow 0;$$

τR is a representation of Q, the *Auslander–Reiten translate* of R and τ is the *Auslander–Reiten functor*, which in this case is equivalent to $D\,\mathrm{Ext}(\ ,\Lambda)$. We shall use τ both for this functor and for the linear map it induces on Γ via dimension vectors. The next two formulae due to Auslander and Reiten are important: we shall refer to them as the duality formulae:

$$\mathrm{Ext}(R, -) \simeq D\,\mathrm{Hom}(-, \tau R) \quad \text{and} \quad \mathrm{Hom}(R, -) \simeq D\,\mathrm{Ext}(\ , \tau R).$$

The corresponding formulae for projective representations have already been noted:

$$\mathrm{Hom}(P_v, -) \simeq D\,\mathrm{Hom}(-, I_v) \quad \text{and} \quad \mathrm{Ext}(P_v, -) = 0 = D\,\mathrm{Ext}(\ , I_v).$$

Let α, β be dimension vectors; we define the *Euler inner product by*

$$\langle \alpha, \beta \rangle = \sum_v \alpha(v)\beta(v) - \sum_a \alpha(ta)\beta(ha).$$

Ringel shows that $\langle \underline{\dim}\, R, \underline{\dim}\, S \rangle = \dim \mathrm{Hom}(R, S) - \dim \mathrm{Ext}(R, S)$. We define the matrix E by the formula

$$\langle \alpha, \beta \rangle = \alpha E \beta^T,$$

where β^T is the transpose of β. Note that $\langle \alpha, \beta \rangle = -\langle \beta, \tau\alpha \rangle$ by the duality formulae.

The preceding material may be found in Gabriel [4], Ringel [12, 13] and Dlab and Ringel [3].

Let α be a fixed dimension vector; we define

$$R(\alpha) = \times_a {}^{\alpha(ta)}k^{\alpha(ha)},$$

where ${}^{\alpha(ta)}k^{\alpha(ha)}$ denotes the space of $\alpha(ta) \times \alpha(ha)$ matrices. A point p of $R(\alpha)$ defines a representation R_p of the dimension vector α in the natural way. Define

$$G\ell(\alpha) = \times_v G\ell(\alpha(v)),$$

where $G\ell(n)$ is the space of invertible $n \times n$ matrices. Then $G\ell(\alpha)$ acts in the natural way on $R(\alpha)$, and two representations R_p, R_q are isomorphic if and only if p and q are conjugate under $G\ell(\alpha)$.

If $G\ell(\alpha)$ has an open orbit on $R(\alpha)$, the corresponding representation is denoted by $G(\alpha)$. In this case, α is said to be a *pre-homogeneous dimension vector*. If the representation $G(\alpha)$ is indecomposable, then Kac [8] shows that it has endomorphism ring k. In this case, α is said to be a *real Schur root*. In general

$$G(\alpha) \equiv \bigoplus_{i-1}^{s} G(\alpha_i)^{n_i}$$

for real Schur roots α_i. By the Artin–Voigt lemma, $\mathrm{Ext}(G(\alpha), G(\alpha)) = 0$.

The space $\mathrm{G}\ell(\alpha)$ acts on the ring $P(\alpha)$ of polynomial functions on $R(\alpha)$. A semi-invariant polynomial f satisfies $g(f) = \chi(g) f$ for some character χ on $\mathrm{G}\ell(\alpha)$. The characters of $\mathrm{G}\ell(\alpha)$ are of the form $(\det)^{\gamma}$ for some vector γ in Γ, since the only characters of $\mathrm{G}\ell(n)$ are powers of det, the determinant function.

We intend to determine the semi-invariant polynomials for $\mathrm{G}\ell(\alpha)$ when α is a pre-homogeneous dimension vector. In this case, the theorem of Sato and Kimura [**13**] shows that if $\{c_1, \ldots, c_n\}$ are the components of codimension 1 in the complement of the open orbit and they are defined by polynomials $\{f_1, \ldots, f_n\}$, then these polynomials are semi-invariant, algebraically independent, and every semi-invariant is a product of them up to a scalar. A reference for the material here is Kac [**8**].

1. *Determinantal semi-invariants*

Let α, β be dimension vectors such that $\langle \alpha, \beta \rangle = 0$; we define a polynomial $P_{\alpha,\beta}$ on $R(\alpha) \times R(\beta)$ which is semi-invariant with respect to the action of $\mathrm{G}\ell(\alpha) \times \mathrm{G}\ell(\beta)$.

Let $(p, q) \in R(\alpha) \times R(\beta)$. We attempt to construct a non-zero homomorphism from R_p to R_q. This amounts to solving the following system of homogeneous linear equations:

$$R_p(a) X^{ha} - X^{ta} R_q(a) = 0 \quad \text{for all} \quad a \in Q_1, \tag{1}$$

where X^v is an $\alpha(v) \times \beta(v)$ matrix of indeterminants. If $\langle \alpha, \beta \rangle = 0$, the number of indeterminants, $\sum_v \alpha(v) \beta(v)$, is equal to $\sum_a \alpha(ta) \beta(ha)$, the number of equations. If $M_{\alpha,\beta}(p, q)$ is the matrix of coefficients of (1), we define $P_{\alpha,\beta}(p, q) = \det M_{\alpha,\beta}(p, q)$.

THEOREM 1.1. *$P_{\alpha,\beta}(p,q)$ is a polynomial semi-invariant for the action of $\mathrm{G}\ell(\alpha) \times \mathrm{G}\ell(\beta)$ on $R(\alpha) \times R(\beta)$. It vanishes at (p, q) if and only if* $\mathrm{Hom}(R_p, R_q) \neq 0$.

Proof. Consider the map $\phi: R(\alpha) \times R(\beta) \to \mathrm{Hom}_k(\oplus_v^{\alpha(v)} k^{\beta(v)}, \oplus_a^{\alpha(ta)} k^{\beta(ha)})$ given by $\phi(p, q) = M_{\alpha,\beta}(p, q)$.

If we give $\mathrm{Hom}_k(\oplus_v^{\alpha(v)} k^{\beta(v)}, \oplus_a^{\alpha(ha)} k^{\beta(ta)})$ the natural structure of a $\mathrm{G}\ell(\alpha) \times \mathrm{G}\ell(\beta)$ module, ϕ is a homomorphism of $\mathrm{G}\ell(\alpha) \times \mathrm{G}\ell(\beta)$ modules. It follows at once that $P_{\alpha,\beta} = \det M_{\alpha,\beta}$ is a polynomial semi-invariant.

There is a non-zero homomorphism from R_p to R_q if and only if (1) has a non-zero solution, and this happens if and only if $P_{\alpha,\beta}(p, q) = 0$.

Given a representation M of dimension vector α, there is some point p' in $R(\alpha)$ such that $M \simeq R_{p'}$; we define $P_{M,\beta}(q) = P_{\alpha,\beta}(p', q)$. Strictly speaking, $P_{M,\beta}$ is only defined up to multiplication by a non-zero scalar. It is a polynomial semi-invariant for the action of $\mathrm{G}\ell(\beta)$ on $R(\beta)$. *If* N is a representation of the dimension vector β, we define $P_{\alpha,N}$ similarly.

We give another interpretation of these polynomials next.

If $0 \to P \xrightarrow{\phi} Q \to M \to 0$ is a projective resolution of M, and q is a point of $R(\beta)$, we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}(M, R_q) \longrightarrow \mathrm{Hom}(Q, R_q) \xrightarrow{(\phi, R_q)} \mathrm{Hom}(P, R_q) \longrightarrow \mathrm{Ext}(M, R_q) \longrightarrow 0.$$

Therefore, $\mathrm{Hom}(M, R_q) \neq 0$ if and only if (ϕ, R_q) is not invertible. Similarly, if $0 \to N \to I \xrightarrow{\mu} J \to 0$ is an injective coresolution of N, $\mathrm{Hom}(R_p, N) \neq 0$ if and only if (R_p, μ) is not invertible. These remarks together with the next lemma give our new interpretation of $P_{M,\beta}$ and $P_{\alpha,N}$.

LEMMA 1.2. *Let*

$$0 \longrightarrow \oplus_a k^{\alpha(ta)} \oplus_k P_{ha} \xrightarrow{\phi_p} \oplus_v k^{\alpha(v)} \otimes_k P_v \longrightarrow R_p \longrightarrow 0$$

be the canonical projective resolution of R_p. *Then* $(\phi_p, R_q) = M_{\alpha,\beta}(p, q)$.

A similar result holds for the canonical injective resolution of R_q.

Proof. We have the canonical isomorphisms: $\operatorname{Hom}(P_v, R_q) \simeq R_q(v) \simeq k^{\beta(v)}$. Therefore,

$$\operatorname{Hom}(\oplus_v k^{\alpha(v)} \otimes P_v, R_q) \simeq \oplus_v {}^{\alpha(v)}k^{\beta(v)},$$

and

$$(\operatorname{Hom}(\oplus_a k^{\alpha(ta)} \otimes P_{ha}, R_q) \simeq \oplus_a {}^{\alpha(ta)}k^{\beta(ha)}.)$$

After making these identifications, the result is a simple matter to check. The dual result for an injective coresolution of R_q is the same. This means that any projective resolution may be used in this way to calculate $P_{M,\beta} = \det M_{M,\beta}$. For if

$$0 \longrightarrow P \longrightarrow Q \longrightarrow M \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow P' \longrightarrow Q' \longrightarrow M \longrightarrow 0$$

are two projective resolutions of M, it is an easy matter to show that the map $\phi \oplus I_{Q'}$ is associated to $\phi' \oplus I_Q$; in consequence $\det(\phi, R_q) = \det(\phi \oplus I_{Q'}, R_q)$ which differs by a unit from $\det(\phi' \oplus I_Q, R_q) = \det(\phi', R_q)$. Similar remarks hold for injective coresolutions.

The next lemma is a simple consequence of this result. Note that $\langle \alpha, \beta \rangle = 0$ if and only if $\langle \beta, \tau\alpha \rangle = 0$.

LEMMA 1.3. $P_{M,\beta} = P_{\beta,\tau M}$ *up to a non-zero scalar for a representation* M, *having no projective summands.*

Proof. If $0 \to P \to Q \to M \to 0$ is the canonical projective resolution of M, then

$$0 \longrightarrow \tau M \longrightarrow D\operatorname{Hom}(P, \Lambda) \longrightarrow D\operatorname{Hom}(Q, \Lambda) \longrightarrow 0$$

is an injective coresolution of τM. The result follows.

For the projective representation P_v, $\operatorname{Hom}(P_v, N) = 0$ if and only if $N(v) = 0$, which happens if and only if $\operatorname{Hom}(N, I_v) = 0$. These are equivalent to either of the conditions, $\langle \pi_v, \beta \rangle = 0$ or $\langle \beta, \iota_v \rangle = 0$, since $\operatorname{Ext}(P_v, -) = 0 = \operatorname{Ext}(-, I_v)$. Here π_v is the dimension vector of P_v and ι_v is the dimension vector of I_v.

Finally, we calculate the weight of the semi-invariant $P_{\alpha,\beta}$ and hence of the semi-invariants $P_{M,\beta}$ and $P_{\alpha,N}$. Recall the definition of the matrix E from section 0.

LEMMA 1.4. *The weight of the semi-invariant* $P_{\alpha,\beta}$ *on* $R(\alpha) \times R(\beta)$ *is* $\det^{-\beta E^T} \cdot \det^{\alpha E}$. *Therefore, the semi-invariant* $P_{M,\beta}$ *has weight* $\det^{\alpha E}$ *and* $P_{\alpha,N}$ *has weight* $\det^{-\beta E^T}$.

Proof. The matrix $M_{\alpha,\beta}$ is an element of $\operatorname{Hom}(\oplus_v^{\alpha(v)} k^{\beta(v)}, \oplus_a^{\alpha(ta)} k^{\beta(ha)})$. Choose a basis for each ${}^{\alpha(v)}k^{\beta(v)}$ and ${}^{\alpha(ta)}k^{\beta(ha)}$. This represents $M_{\alpha,\beta}$ as an explicit matrix. Let

(c,d) be a central element in $\mathrm{G}\ell(\alpha)\times\mathrm{G}\ell(\beta)$: so $c=(\ldots,c_v,\ldots)$ and $d=(\ldots,d_v,\ldots)$. Then operating by (c,d) on $M_{\alpha,\beta}$ multiplies any given monomial in the determinant of $M_{\alpha,\beta}$ by

$$\prod_a (d_{ha}\,c_{ta}^{-1})^{\beta(ha)\alpha(ta)}\left(\prod_v (d_v\,c_v^{-1})^{\beta(v)\alpha(v)}\right)^{-1}.$$

Therefore, $P_{\alpha,\beta}=\det M_{\alpha,\beta}$ transforms by this character on the centre of $\mathrm{G}\ell(\alpha)\times\mathrm{G}\ell(\beta)$. The only character of $\mathrm{G}\ell(\alpha)\times\mathrm{G}\ell(\beta)$ that induces this central character is $\det^{-\beta E^T}\det^{\alpha E}$.

2. *Perpendicular categories*

Let R be a representation of Q. We define $R^\perp$, the *right perpendicular category* of R, to be the full subcategory of representations S such that

$$\operatorname{Hom}(R,S)=0=\operatorname{Ext}(R,S).$$

Similarly, we define ${}^\perp R$ the *left perpendicular category* of R to be the full subcategory of representations T such that $\operatorname{Hom}(T,R)=0=\operatorname{Ext}(T,R)$. Given a set of representations $\{R_i: i\in I\}$ we define $\{R_i\}^\perp$ and ${}^\perp\{R_i\}$ in the obvious way. These categories are closed under direct sums, direct summands, extensions, images, kernels and cokernels. Therefore they are abelian subcategories. The last three properties require a slight check.

These notions are connected to the determinantal semi-invariants in the following way. We see that $P_{R,\beta}$ is defined and non-zero at a point q in $R(\beta)$ if and only if R_q lies in $R^\perp$. Also, $P_{T,\alpha}$ is defined in $R(\alpha)$ and non-zero on the orbit corresponding to R if and only if T lies in ${}^\perp R$.

We recall the duality formulae from Section 0. These imply at once the following two results on perpendicular categories.

THEOREM 2.1. *Assume that R has no projective direct summands. Then $R^\perp={}^\perp\tau R$.*

THEOREM 2.2. *Assume that the support of R is the whole of Q. Then $\tau: {}^\perp R\to R^\perp$ is an equivalence of categories.*

Let α be a pre-homogeneous dimension vector; then the open orbit of $\mathrm{G}\ell(\alpha)$ on $R(\alpha)$ defines a representation $G(\alpha)$. By Kac [**8**],

$$G(\alpha)\simeq\bigoplus_{i=1}^{s} G(\alpha_i)^{n_i},$$

where α_i is a real Schur root. Further,

$$\operatorname{Ext}(G(\alpha),\,G(\alpha))=0$$

and so $\operatorname{Ext}(G(\alpha_i),\,G(\alpha_j))=0$ for all i and j. By a result of Happel and Ringel [**6**, Lemma 4.2] we know that any map from $G(\alpha_i)$ to $G(\alpha_j)$ is either injective or surjective. It follows that the dimension vectors $\{\alpha_i\}$ are partially ordered by $\alpha_i\geqslant\alpha_j$ if and only if there is a non-zero homomorphism from $G(\alpha_i)$ to $G(\alpha_j)$. We may choose the subscripts in such a way so that $i<j$ implies that $G(\alpha_j)\in G(\alpha_i)^\perp$. It follows that in order to study $G(\alpha)^\perp$ we need only inductively study the categories $G(\alpha_i)^\perp$ for a real Schur root α_i. This turns out to be technically simpler.

The next two results were also proved by Geigle and Lenzing [5].

THEOREM 2.3. *Let Q be a quiver with n vertices; let α be a real Schur root. Then $G(\alpha)^{\perp}$ is naturally equivalent to representations of a quiver $Q(\alpha^{\perp})$ having no oriented cycles with $n-1$ vertices. The inverse functor from* Rep $(Q(\alpha^{\perp}))$ *to $G(\alpha)^{\perp}$ is a full exact embedding into* Rep (Q).

Proof. If $G(\alpha)$ is a projective representation, this result is clear since $G(\alpha) \simeq P_v$ for some vertex v, and $G(\alpha)^{\perp}$ is the full subcategory of representations whose support does not contain v.

In the case where $G(\alpha)$ is not projective, we shall prove that $G(\alpha)^{\perp}$ has enough projective objects. Consider Λ as the free representation of rank 1; let

$$\dim(\mathrm{Ext}(G(\alpha), \Lambda)) = s,$$

and let

$$0 \longrightarrow \Lambda \longrightarrow \Lambda^{\sim} \longrightarrow (G(\alpha))^s \longrightarrow 0 \tag{2}$$

be the exact sequence such that no copy of $G(\alpha)$ splits off. Since $G(\alpha)$ is not projective. Hom $(G(\alpha), \Lambda) = 0$. Apply Hom $(G(\alpha), \cdot)$ to the exact sequence to obtain

$$0 \longrightarrow \mathrm{Hom}(G(\alpha), \Lambda^{\sim}) \longrightarrow \mathrm{Hom}(G(\alpha), G(\alpha)^s) \longrightarrow \mathrm{Ext}(G(\alpha), \Lambda) \longrightarrow \mathrm{Ext}(G(\alpha), \Lambda^{\sim}) \longrightarrow 0.$$

Since the middle map is invertible by construction, $\Lambda^{\sim}$ must lie in $G(\alpha)^{\perp}$. Also, it must be a projective object, because Ext $(\Lambda, \cdot)$ and Ext $(G(\alpha), \cdot)$ both vanish on $G(\alpha)^{\perp}$, and so must Ext $(\Lambda^{\sim}, \cdot)$. Since $\Lambda^{\sim} \oplus G(\alpha)$ is a tilting module for Λ, it has precisely n non-isomorphic direct summands (see Bongartz [2]); therefore, $\Lambda^{\sim}$ must have $n-1$ non-isomorphic direct summands. To show that $\Lambda^{\sim}$ is a generator in $G(\alpha)^{\perp}$, we apply Hom $(\cdot, S)$ to (2) for S in $G(\alpha)^{\perp}$; we find the exact sequence

$$0 = \mathrm{Hom}(G(\alpha), S) \longrightarrow \mathrm{Hom}(\Lambda^{\sim}, S) \longrightarrow \mathrm{Hom}(\Lambda, S) \longrightarrow \mathrm{Ext}(G(\alpha), S) = 0.$$

It follows that every homomorphism from Λ to S lifts to a homomorphism from $\Lambda^{\sim}$ to S; since Λ^t maps onto S for some t, so does $(\Lambda^{\sim})^t$. Since $\Lambda^{\sim}$ is a projective generator in $G(\alpha)^{\perp}$, $G(\alpha)^{\perp}$ is naturally equivalent to modules for End $(\Lambda^{\sim})$.

Finally, Ext $(S, T) \simeq \mathrm{Ext}_{G(\alpha)^{\perp}}(S, T)$ for S and T in $G(\alpha)^{\perp}$, because any extension of S on T must also lie in $G(\alpha)^{\perp}$; therefore, $\mathrm{Ext}_{G(\alpha)^{\perp}}(S, \cdot)$ is a right exact functor, for all S in $G(\alpha)^{\perp}$, which implies that End $(\Lambda^{\sim})$ must be hereditary. All hereditary algebras over an algebraically closed field are Morita equivalent to path algebras of quivers; therefore, $G(\alpha)^{\perp}$ is naturally equivalent to the representations of the quiver corresponding to End $(\Lambda^{\sim})$. This quiver has $n-1$ vertices because $\Lambda^{\sim}$ has $n-1$ non-isomorphic direct summands. It has no oriented cycles because its path algebra is Morita equivalent to End $(\Lambda^{\sim})$ and must be finite dimensional.

Let α be some dimension vector for the quiver Q. We define

$$\alpha^{\perp} = \{\beta \in \Gamma : \langle \alpha, \beta \rangle = 0\}.$$

Then we have the following result.

THEOREM 2.4. *Let α be a real Schur root, and let $Q(\alpha^{\perp})$ be the quiver such that $G(\alpha)^{\perp}$ is naturally equivalent to* Rep $(Q(\alpha^{\perp}))$. *Let $\{\alpha_j : j = 1, \ldots, n-1\}$ be the dimension vectors of the simple objects of $G(\alpha)^{\perp}$. Then, the linear map $\Phi : \Gamma(Q(\alpha^{\perp})) \to \alpha^{\perp}$ given by $\phi(\ldots, r_j, \ldots) = \sum r_j \alpha_j$ is an isometry with respect to the Euler forms.*

Proof. It is enough to check that the inner products between the dimension vectors $\{\alpha_j\}$ are correct. Let S_j be the simple object in $G(\alpha)^{\perp}$ of dimension vector α_j: then the inner products between the dimension vectors $\{\alpha_j\}$ are determined by the dimensions of the Hom and Ext groups between the objects $\{S_j\}$. But these are the same in each category.

The next result is the natural extension of Theorem 2.3.

THEOREM 2.5. *Let Q be a quiver having n vertices. Let α be a pre-homogeneous dimension vector. Let*

$$G(\alpha) \simeq \bigoplus_{i=1}^{s} G(\alpha_i)^{n_i},$$

where each α_i is a real Schur root, and $\alpha_i \neq a_j$ for $i \neq j$. Then $G(\alpha)^{\perp}$ is naturally equivalent to representations of a quiver having no oriented cycles and $n-s$ vertices. The same result holds for ${}^{\perp}G(\alpha)$.

Proof. Now $G(\alpha)^{\perp} = \{G(\alpha_i):\ i = 1, \ldots, s\}^{\perp}$. We assume that the suffices are ordered so that $i < j$ implies that $G(\alpha_j) \in G(\alpha_i)^{\perp}$. This is possible by [**6**, Corollary 4.2]. But $G(\alpha_1)^{\perp}$ is equivalent to the category of representations of a quiver without oriented cycles on $n-1$ vertices, $Q(\alpha_1^{\perp})$. Moreover, the Schur roots $\{\alpha_i : i > 1\}$ correspond to real Schur roots $\{\beta_i : i > 1\}$ on $Q(\alpha_1^{\perp})$. It follows that $\{G(\alpha_i) : i = 1, \ldots, s\}^{\perp}$ is equivalent to $\{G(\beta_i) : i = 2, \ldots, s\}^{\perp}$ and the result follows by induction.

The dual result may be proved by turning all the arrows round.

3. *A slice theorem*

First we need a notation. If $H \subset G$ is an inclusion of semisimple algebraic groups and W is an affine variety on which H acts, then the action of H on $G \times W$ given by $(g, w)h = (gh, h^{-1}w)$ has an orbit space $G \times {}^{H} W$ since $G \times W$ is an affine variety. The space $G \times {}^{H} W$ has a natural map to G/H induced by $(g, w) \to gH$; the fibres of this map are copies of W.

Let α be a real Schur root. We use the notation of Theorem 2.4, so the simple objects of $G(\alpha)^{\perp}$ have dimension vector $\{\alpha_j : j = 1, \ldots, r-1\}$. Let the corresponding vertices of $Q(\alpha^{\perp})$ be $\{w_j\}$. Let γ be a dimension vector for $Q(\alpha^{\perp})$ and let $\Phi(\gamma) = \beta$ be the corresponding dimension vector for Q. Let C_{α} be the closed codimension 1 subset of $R(\beta)$ where $P_{G(\alpha), \beta} = 0$. We intend to show that there is an embedding $\mathrm{G}\ell(\gamma) \hookrightarrow \mathrm{G}\ell(\beta)$ and that $R(\gamma)$ may be found as a $\mathrm{G}\ell(\gamma)$-stable subvariety of $R(\beta)$ so that the natural map from $\mathrm{G}\ell(\beta) \times {}^{\mathrm{G}\ell(\gamma)} R(\gamma)$ to $R(\beta) - C_{\alpha}$ is biregular. This may be thought of as a strong version of Luna's étale slice theorem [**11**] for this special case, because the slice actually exists in the Zariski topology: $R(\beta) - C_{\alpha}$ is an open neighbourhood of its unique closed orbit which corresponds to the unique semisimple representation of dimension vector γ and this has isotropy group $\mathrm{G}\ell(\gamma)$.

To set this up we need some categorical generalities. Let Q be a quiver; we define $\underline{\mathrm{Add}}(Q)$ to be the additive k-category freely generated by Q. So, $\underline{\mathrm{Rep}}(Q)$ is the category of covariant k-functors to $\mathrm{mod}\, k$ from $\underline{\mathrm{Add}}(Q)$. If R is a representation and μ is a map in $\underline{\mathrm{Add}}(Q)$, we extend notation by writing $R(\mu)$ for the image of μ under the functor that R defines. It is clear that $\underline{\mathrm{Add}}(Q)$ is dual to the category of projective representations of Q. So we may regard representations of Q as contravariant functors from the category of projective representations of Q to $\mathrm{mod}\, k$. We intend to define a map from the category of projective representations of Q to the category of projective representations of $Q(\alpha^\perp)$ which induces the equivalence between $\underline{\text{Rep}}(Q(\alpha^\perp))$ and $G(\alpha)^\perp$.

THEOREM 3.1. *Let α be a real Schur root such that $G(\alpha)$ is not projective. Let P be a projective representation of Q. Let* $\dim \operatorname{Ext}(G(\alpha), P) = s$. *Define $P^\sim$ by the following exact sequence:*

$$0 \longrightarrow P \longrightarrow P^\sim \longrightarrow G(\alpha)^s \longrightarrow 0 \tag{3}$$

and no copy of $G(\alpha)$ splits off. Then

(i) *$P^\sim$ lies in $G(\alpha)^\perp$,*
(ii) $\operatorname{Hom}(P, S) \simeq \operatorname{Hom}(P^\sim, S)$ *for all S in $G(\alpha)^\perp$,*
(iii) *$P^\sim$ is a projective object of $G(\alpha)^\perp$,*
(iv) *$(\,)^\sim$ extends to a functor from projective representations of Q to projective objects in $G(\alpha)^\perp$.*

Proof. (i) Applying $\operatorname{Hom}(G(\alpha), -)$ to (3) gives

$$0 = \operatorname{Hom}(G(\alpha), P) \longrightarrow \operatorname{Hom}(G(\alpha), P^\sim) \longrightarrow \operatorname{Hom}(G(\alpha), G(\alpha)^s)$$
$$\longrightarrow \operatorname{Ext}(G(\alpha), P) \longrightarrow \operatorname{Ext}(G(\alpha), P^\sim) \longrightarrow 0.$$

By construction, the map from $\operatorname{Hom}(G(\alpha), G(\alpha)^s)$ to $\operatorname{Ext}(G(\alpha), P)$ is an isomorphism, so $P^\sim$ lies in $G(\alpha)^\perp$.

(ii) Apply $\operatorname{Hom}(\,, S)$ to (3):

$$0 \longrightarrow \operatorname{Hom}(G(\alpha), S) \longrightarrow \operatorname{Hom}(P^\sim, S) \longrightarrow \operatorname{Hom}(P, S)$$
$$\longrightarrow \operatorname{Ext}(G(\alpha), S) = 0 \longrightarrow \operatorname{Ext}(P^\sim, S) \longrightarrow \operatorname{Ext}(P, S) = 0.$$

This proves both (i) and (ii).

(iv) The functor $\operatorname{Hom}(P, Q)$ embeds in $\operatorname{Hom}(P, Q^\sim)$ since Q embeds in $Q^\sim$; then $\operatorname{Hom}(P, Q^\sim) \simeq \operatorname{Hom}(P^\sim, Q^\sim)$ by (ii). This proves (iv).

By composing $(\,)^\sim$ with an equivalence between $G(\alpha)^\perp$ and $\underline{\text{Rep}}(Q(\alpha^\perp))$, we obtain a functor Φ from the projective representations of Q to the projective representations of $Q(\alpha^\perp)$. Part (ii) of Theorem 3.1 shows that the pullback along Φ defines the equivalence between $\operatorname{Rep}(Q(\alpha^\perp))$ and $G(\alpha)^\perp$.

We fix isomorphisms $\Phi(P_{v_i}) \simeq \oplus_j (P_{w_j})^{s_{ij}}$. If we have a representation of $Q(\alpha^\perp)$ of dimension vector γ, we define by pullback along Φ a representation of Q of dimension vector β that lies in $G(\alpha)^\perp$. We have

$$\beta(v_i) = \sum_j s_{ij}\,\gamma(w_j).$$

For all points q in $R(\gamma)$, we have

$$\operatorname{Hom}(\Phi(P_{v_i}), R_q) \simeq \operatorname{Hom}(\oplus_j (P_{w_j})^{s_{ij}}, R_q) = \oplus_j (k^{\gamma(w_j)})^{s_{ij}},$$

a fixed vector space independent of the choice of the point q. Therefore, pullback along Φ defines a map $\Phi\colon R(\gamma) \to R(\beta) - C_\alpha$. The decompositions $k^{\beta(v_i)} = \oplus_j (k^{\gamma(w_i)})^{s_{ij}}$ giving an embedding $G\ell(\gamma) \hookrightarrow G\ell(\beta)$ under which $\Phi(R(\gamma))$ is a $G\ell(\gamma)$-stable subvariety of $R(\beta) - C_\alpha$.

THEOREM 3.2. $R(\beta) - C_\alpha \simeq G\ell(\beta) \times^{G\ell(\gamma)} R(\gamma)$.

Proof. Let p be a point in $R(\beta)-C_\alpha$; then R_p lies in $G(\alpha)^\perp$ and so it is isomorphic to some representation $R_{p'}$ for p' in $\Phi(R(\gamma))$. So, the map from $\mathrm{G}\ell(\beta)\times^{\mathrm{G}\ell(\gamma)}R(\gamma)$ to $R(\beta)-C_\alpha$ is surjective.

If the map is not injective then it is not injective on some $\mathrm{G}\ell(\beta)$ orbit; it would follow from this that the stabilizer of a point in the intersection of this orbit with $R(\gamma)$ is larger than its stabilizer in $\mathrm{G}\ell(\gamma)$. However, each of these stabilizers is the group of units of the endomorphism ring of the corresponding representation.

This proves that the map is bijective and in characteristic 0 a bijective morphism of smooth varieties is biregular. In characteristic p, a further argument is required which we shall sketch below. What we shall show is that if C is a commutative k-algebra then the morphism between the C-points of the two varieties is an isomorphism. It follows at once that the morphism is biregular by taking C to be the co-ordinate ring of one of the varieties.

It is enough to check this when C is a local algebra. We consider representations of Q over C. These are modules for $\Lambda\otimes C$ which are free as C-modules, where Λ is the path algebra of Q; $G(\alpha)\otimes C$ is such a representation. We define $(G(\alpha)\otimes C)^\perp$ to be the full subcategory of such representations over C such that $\mathrm{Hom}(G(\alpha)\otimes C,-)=0=\mathrm{Ext}(G(\alpha)\otimes C,-)$. Now $\Lambda^{\sim}\otimes C$ is still a projective generator for this category and the map $\Lambda\otimes C\to\mathrm{End}(\Lambda^{\sim}\otimes C)=\mathrm{End}(\Lambda^{\sim})\otimes C$ is still an epimorphism in the category of rings since $\Lambda\to\mathrm{End}(\Lambda^{\sim})$ is an epimorphism. So, $(G(\alpha)\otimes C)^\perp$ is naturally equivalent to the category of $\mathrm{End}(\Lambda^{\sim})\otimes C$ modules free as C-modules.

We have an induced map $\Phi_C: R(\gamma)(C)\to R(\beta)(C)$ and since every point of $R(\beta)(C)-C_\alpha(C)$ corresponds to a representation in $(G(\alpha)\otimes C)^\perp$, the above discussion shows that it lies in a $\mathrm{G}\ell(\beta)(C)$ orbit that intersects non-trivially with $R(\gamma)(C)$. So the induced morphism from $(\mathrm{G}\ell(\beta)\times^{\mathrm{G}\ell(\gamma)}R(\gamma))(C)$ is surjective. It is also injective because the stabilizer in $\mathrm{G}\ell(\gamma)(C)$ of a point q in $R(\gamma)(C)$ is the group of units of $\mathrm{End}(R_q)$ and this is also the stabilizer in $\mathrm{G}\ell(\beta)(C)$ since the corresponding representation in $(G(\alpha)\otimes C)^\perp$ has the same endomorphism ring.

Thus we have shown that the morphism induces a bijection on the C-points of the varieties for any commutative local k-algebra and hence for any commutative k-algebra and so the morphism is biregular.

4. *The main theorem*

We fix our notation for the rest of the paper. Let α be a prehomogeneous dimension vector; so

$$G(\alpha)\simeq\bigoplus_{i=1}^{s}G(\alpha_i)^{n_i},$$

where each α_i is a real Schur root. We assume that α is a *sincere* dimension vector: that is, $\alpha(v)\neq 0$ for all $v\in Q_0$. Let $\{\beta_j: j=1,\dots,n-s\}$ be the dimension vector of the $n-s$ simple objects in ${}^\perp G(\alpha)$. So each β_j is a real Schur root by inductive application of Theorem 2.4, and $G(\alpha)$ lies in $G(\beta_j)^\perp$. Let γ_j be the dimension vector in $Q(\beta_j^\perp)$ that corresponds to α. We shall need to know later that γ_j is sincere on $Q(\beta_j^\perp)$. The next two results give a proof of this.

LEMMA 4.1. *Let δ be a sincere prehomogeneous dimension vector. Then, $G(\delta)$ is a faithful representation.*

Proof. Let $\sum \lambda_i p_i$ be a non-zero sum of distinct paths from v to w. We wish to show that it has a non-trivial action on $G(\delta)$.

Let $\lambda_1 \neq 0$. We choose a representation R of dimension vector δ so that $R(a) \neq 0$ for all $a \in Q_1$ occurring in the path p_1 and their composition is also non-zero; that is, $R(p_1)$ is non-zero. Further, $R(a') = 0$ for all other arrows. Then $R(\sum \lambda_i p_i) = R(\lambda_1 p_1) \neq 0$. It follows that $R_q(\sum \lambda_i p_i) \neq 0$ for q on an open subset of $R(\delta)$ and therefore $G(\delta)(\sum \lambda_i p_i) \neq 0$ as required.

LEMMA 4.2. *Let δ be a sincere prehomogeneous dimension vector. Let $G(\varepsilon)$ be a simple object in ${}^{\perp}G(\delta)$. Let γ be the dimension vector for $Q(\varepsilon^{\perp})$ corresponding to δ. Then γ is sincere.*

Proof. We have to show that $\mathrm{Hom}(P, G(\delta)) \neq 0$ for each indecomposable projective object P in $G(\beta)^{\perp}$. Certainly, $\mathrm{Ext}(P, G(\delta)) = 0$, since P is projective in $G(\varepsilon)^{\perp}$; so $\mathrm{Hom}(P, G(\delta)) = 0$ implies that $P \in {}^{\perp}G(\delta)$.

On the other hand, Lemma 4.1 implies that there is an embedding $\phi: \Lambda \to G(\delta)^t$ for some t; for example, take $t = \dim \mathrm{Hom}(\Lambda, G(\delta))$ and ϕ to be the canonical map from Λ to $G(\delta)^t$. We have an induced map $\phi^{\sim}: \Lambda^{\sim} \to G(\delta)^t$ by Theorem 3.1 (ii).Now P is a direct summand of $\Lambda^{\sim}$, and $\mathrm{Hom}(P, G(\delta)) = 0$ implies that P lies in the kernel of $\phi^{\sim}$; so $\mathrm{P} \cap \Lambda = 0$ and P embeds in $G(\varepsilon)^u$ for some integer u. Since we assumed that $G(\varepsilon)$ is simple in $\perp G(\delta)$, this implies that $P \cong G(\varepsilon)^r$ for some integer r, which is absurd.

It follows that $\mathrm{Hom}(P, G(\delta)) \neq 0$ for an indecomposable projective object P in $G(\beta)^{\perp}$, and this means that γ, the dimension vector corresponding to δ for $Q(\varepsilon^{\perp})$, is sincere.

We are in a position to prove the main theorem.

THEOREM 4.3. *Let α be a sincere prehomogeneous dimension vector. Let*

$$G(\alpha) \simeq \bigoplus_{i=1}^{s} G(\alpha_i)^{n_i}$$

for real Schur roots $\{\alpha_i\}$. Let $\{\beta_j : 1, \ldots, n-s\}$ be the dimension vectors of the simple objects in ${}^{\perp}G(\alpha)$. Then the semi-invariants $P_{G(\beta_i), \alpha}$ are algebraically independent and generate all the semi-invariants.

Proof. By [8], the complement of the open orbit contains precisely $n-s$ distinct codimension 1 components. By [14], if $\{f_j : j = 1, \ldots, n-s\}$ are the defining polynomials of these $n-s$ components, they are algebraically independent and generate the ring of semi-invariants. It follows that we only have to prove that each $P_{G(\beta_j), \alpha}$ is one of these f_j.

Let $C_j = \{p : P_{G(\beta_j), \alpha}(p) = 0\}$. Then, by Theorem 3.2,

$$R(\alpha) - C_j \equiv \mathrm{G}\ell(\alpha) \times^{\mathrm{G}\ell(\gamma)} R(\gamma),$$

where γ is the dimension vector for $Q(\beta_j^{\perp})$ corresponding to α. By Lemma 4.2, γ is a sincere dimension vector and it is also prehomogeneous for $Q(\beta_j^{\perp})$ since the open orbit in $R(\alpha)$ must intersect $R(\gamma)$ in an open orbit for $\mathrm{G}\ell(\gamma)$.

There are $n-s-1$ codimension 1 components $\{D_l : l = 1, \ldots, n-s-1\}$ of the complement of the open orbit in $R(\gamma)$. By induction these are given by the vanishing of polynomials of the form $P_{G(\delta_l), \gamma}$, $l = 1, \ldots, n-s-1$.

Let R_l be the representation in $G(\beta_j)^{\perp}$ corresponding to $G(\delta_l)$; then $P_{R_l,\alpha}$ vanishes possibly in C_j and definitely on $G\ell(\alpha)\cdot D_l$, but nowhere else. It follows that each $G\ell(\alpha)\cdot D_l$ is of codimension 1 and they are distinct since their intersections with $R(\gamma)$ are distinct. So C_j must be irreducible. One may also see that $G\ell(\alpha)\cdot D_l \simeq G\ell(\alpha)\times^{G\ell(\gamma)} D_l$ has codimension 1 by a direct dimension calculation.

All that remains to show is that $P_{G(\beta_j),\alpha}$ is a reduced polynomial. The weight of the semi-invariant $P_{G(\beta_j),\alpha}$ is $(\det)^{\beta_j E}$ by Lemma 1.4. But $\mathrm{hcf}_v\{\beta_j(v)\} = 1$ since β_j is a real Schur root, and the matrix E is invertible, so $(\det)^{\beta_j E}$ cannot be a power of some other weight, and $P_{G(\beta_j),\alpha}$ cannot be a power either.

References

1. M. Auslander and I. Reiten, 'Representation theory of Artin algebras III', *Comm. Algebra* 3 (1975) 239–294.
2. K. Bongartz, 'Tilted algebras', *Representation of algebras*, Lecture Notes in Mathematics 903 (Springer, Berlin, 1981) 26–38.
3. V. Dlab and C. M. Ringel, 'Indecomposable representations of graphs and algebras', Memoir 173 (American Mathematical Society, Providence, 1976).
4. P. Gabriel, 'Auslander–Reiten sequences and representation-finite algebras', *Proceedings, ICRA*-II, Lecture Notes in Mathematics 831 (Springer, Berlin, 1979) 1–71.
5. W. Geigle and H. Lenzing, 'Perpendicular categories with applications to representations and sheaves', *J. Algebra* to appear.
6. D. Happel and C. M. Ringel, 'Tilted algebras', *Trans. Amer. Math. Soc.* 274 (1982) 399–443.
7. V. G. Kac, 'Infinite root systems, representations of graphs and invariant theory', *Invent. Math.* 56 (1980) 57–92.
8. V. G. Kac, 'Infinite root systems, representations of graphs and invariant theory II', *J. Algebra* 78 (1982) 141–162.
9. V. G. Kac, 'Root systems, representations of quivers and invariant theory', *Proceedings, CIME, Montecatini*, Lecture Notes in Mathematics 996 (Springer, Berlin 1982) 74–108.
10. F. Kirwan, 'Cohomology of quotients in symplectic and algebraic geometry', Mathematics Notes 31 (University Press, Princeton, 1984).
11. D. Luna, 'Slices étales', *Bull. Soc. Math. France Mem.* 33 (1973) 81–105.
12. C. M. Ringel, 'Representations of K species and bimodules', *J. Algebra* 41 (1976) 269–302.
13. C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics 1099 (Springer, Berlin, 1984).
14. M. Sato and T. Kimura, 'A classification of irreducible pre-homogeneous vector spaces and their relative invariants', Mathematics Notes 31 (University Press, Princeton, 1984).

Department of Mathematics
Yale University
New Haven
USA

New address:

School of Mathematics
University Walk
Bristol BS8 1TW