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Adjoint Functors and Representation Dimensions

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Abstract We study the global dimensions of the coherent functors over two categories that are linked by a pair of adjoint functors. This idea is then exploited to compare the representation dimensions of two algebras. In particular, we show that if an Artin algebra is switched from the other, then they have the same representation dimension.

Keywords adjoint functor, representation dimension, global dimension, tilting module group algebra
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1 Introduction

The importance of adjoint pairs of functors in the category theory, homological algebra and the representation theory is well known. As we know, an adjoint pair of functors between two categories provides a bridge to transfer information on morphism sets of one category to that of the other, and thus produces an elegant linkage between two categories. This philosophy is widely used in establishing a homological and structural relationship between two categories.

As one of the homological invariants of algebras, Auslander introduced in [1] the notion of representation dimension for Artin algebras. It measures homologically how far away an Artin algebra is from representation-finite algebras, a class of algebras which are better understood than the other class of algebras in the representation theory. Unfortunately, before 1998 there were not many investigations on this subject. Since 1998, many advances on the representation dimension have been made. The recent developments on this subject established many interesting connections with different problems in representation theory, as well as with other areas. In particular, there is a connection of representation dimension with the finitistic dimension conjecture (see [2], [3, Theorem 4.2]).

In this paper, we try to use the idea of adjoint functors to compare the representation dimension or the global dimension of an Artin algebra with that of the other. First, we establish results on the relationship of global dimensions in terms of category language. Among them is the following result:

Theorem 1.1 *Suppose* $\mathscr{C}$ *and* $\mathscr{D}$ *are additive* k*-categories, and* $F : \mathscr{C} \longrightarrow \mathscr{D}$ *and* $G : \mathscr{D} \longrightarrow \mathscr{C}$ *are an adjoint pair of additive functors with* F *a left adjoint and* G *a right adjoint. We denote by* $\widehat{\mathscr{C}}$ *the category of coherent functors over* $\mathscr{C}$*. Then*:

(1) *Suppose that* $\mathrm{gl.dim}(\widehat{\mathscr{D}}) \geq 2$*. If* G *is full and quasi-dense, then* $\mathrm{gl.dim}(\widehat{\mathscr{C}}) \leq \mathrm{gl.dim}(\widehat{\mathscr{D}})$.

(2) *Suppose that* $\mathrm{gl.dim}(\widehat{\mathscr{C}}) \geq 2$*. If* F *and* G *are dense, then* $\mathrm{gl.dim}(\widehat{\mathscr{D}}) \leq \mathrm{gl.dim}(\widehat{\mathscr{C}})$.

(3) *Suppose that* $\mathrm{gl.dim}(\widehat{\mathscr{C}}) \geq 2$, $\mathrm{gl.dim}(\widehat{\mathscr{D}}) \geq 2$*, and* F *has a left adjoint functor. If both* F *and* G *are full and quasi-dense, then* $\mathrm{gl.dim}(\widehat{\mathscr{C}}) = \mathrm{gl.dim}(\widehat{\mathscr{D}})$.

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As applications we apply our results to group algebras with trivial intersection subgroups, and stable equivalences of Morita type between self-injective algebras.

Next, we state our results on the representation dimension.

Theorem 1.2 *Let A be an Artin algebra. Then*:

(1) *Suppose that there is an ideal I in A such that $I\,\mathrm{rad}(A) = 0 = \mathrm{rad}(A)I$. If A/I is representation-finite, then* $\mathrm{rep.dim}(A) \le 3$.

(2) *Suppose $T = \tau^{-1}S \oplus P$ is an APR-tilting module with S a simple A-module. If* $\mathrm{inj.dim}(S) = 1$, *then* $\mathrm{rep.dim}(A) = \mathrm{rep.dim}(B)$, *where* $B = \mathrm{End}({}_AT)$.

(3) *Suppose $T = \tau(I) \oplus Q$ is an APR-cotilting module with I a simple A-module. If* $\mathrm{proj.dim}(I) = 1$, *then* $\mathrm{rep.dim}(B) = \mathrm{rep.dim}(A)$, *where* $B = \mathrm{End}({}_AT)$.

Note that the algebras A and B in Theorem 1.2 may not be stably equivalent in general.

The paper is detailed in the following way. In Section 2 we recall some definitions and basic results concerning the adjoint functors and coherent functors. In Section 3 we investigate the global dimensions of coherent functors defined on the categories linked by an adjoint pair of functors. In Section 4 we apply our results to both group algebras and stable equivalences of Morita type. In Section 5 we study the representation dimensions of Artin algebras and their factor algebras by using a canonical adjoint pair of functors. In Section 6 we investigate the representation dimension via APR-tilting functors which are a special kind of adjoint pairs of functors.

2 Preliminary

In this section we recall some basic definitions and results on adjoint functors, which are needed in our discussion.

Given an Artin algebra A, we denote by $\mathrm{rad}(A)$ the Jacobson radical of A, and by A-mod the category of all finitely generated left A-modules. Given an A-module M in A-mod, we denote by $\mathrm{proj.dim}(M)$ the projective dimension of M. The global dimension of A, denoted by $\mathrm{gl.dim}(A)$, is the supremum of all $\mathrm{proj.dim}(M)$ with $M \in A$-mod.

For any Artin algebra A, Auslander in [1] defined the representation dimension of A, denoted by $\mathrm{rep.dim}(A)$, as follows:

$$\begin{aligned}\mathrm{rep.dim}(A) = \inf\{&\mathrm{gl.dim}(\Lambda) \mid \Lambda \text{ is an Artin algebra with } \mathrm{dom.dim}(\Lambda) \ge 2 \text{ and } \mathrm{End}({}_\Lambda T) \text{ is}\\ &\text{Morita equivalent to } A, \text{where } T \text{ is the maximal injective summand of } \Lambda\}.\end{aligned}$$

As was pointed out in [1], this is equivalent to

$$\mathrm{rep.dim}(A) = \inf\{\mathrm{gl.dim}\ \mathrm{End}({}_AM) \mid M \text{ is a generator-cogenerator for } A\text{-mod}\},$$

where an A-module M is called a generator for A-mod if each module in A-mod is a homomorphic image of a direct sum of copies of M, and a cogenerator for A-mod if each module in A-mod is a submodule of a direct sum of copies of M. Clearly, $\mathrm{rep.dim}(A) = \mathrm{rep.dim}(A^{op})$, where A^{op} stands for the opposite algebra of A.

Representation dimension was used in [1] to describe the representation-finite algebras, namely, an Artin algebra is representation-finite if and only if its representation dimension is at most 2. Recently, there are some advances on this subject [3–8]. As have been seen from the definition, the study of representation dimension involves the global dimension, and the latter is linked to coherent functors which have been used to solve problems in algebraic space curves recently by Hartshorne [9]. First, let us recall the definition of coherent functors.

Definition 2.1 *Given an additive k-category $\mathscr{A}$, we say that a functor $F : \mathscr{A}^{op} \longrightarrow Ab$ is coherent if there is an exact sequence $(-, A_1) \longrightarrow (-, A_0) \longrightarrow F \longrightarrow 0$ with $A_i \in \mathscr{A}$ for $i = 0, 1$. Here Ab is the category of all Abelian groups, and $(-, A_i)$ stands for the functor $\mathrm{Hom}_{\mathscr{A}}(-, A_i)$. The full subcategory of the functor category $(\mathscr{A}^{op}, Ab)$ consisting of all coherent functors, is denoted by $\widehat{\mathscr{A}}$.*

By Yoneda's lemma, the projective objects in $\widehat{\mathscr{A}}$ are of the form $(-, X)$ with X an object in $\mathscr{A}$, and each coherent functor F can be determined by a morphism $f : A_1 \longrightarrow A_0$, that is, there is an exact sequence $(-, A_1) \stackrel{(-,f)}{\longrightarrow} (,A_0) \longrightarrow F \longrightarrow 0$ in $\mathscr{A}$. As in the case of a module category, we may define the global dimension of the category $\widehat{\mathscr{A}}$ to be the supremum of the projective dimensions of all functors in $\widehat{\mathscr{A}}$. For more information on coherent functors we refer to [10] and [9].

The precise connection between the global dimension of an Artin algebra and that of the coherent functor category is recorded in the following lemma due to Auslander in [11].

Lemma 2.2 *Let M be in A-mod. Then the category $\widehat{\mathrm{add}(M)}$ and $\mathrm{End}({}_AM)$-mod are equivalent. In particular,* $\mathrm{gl.dim}(\mathrm{End}({}_AM)) = \mathrm{gl.dim}(\widehat{\mathrm{add}(M)})$.

Thus the study of the global dimensions of the categories of coherent functors over two categories, which are linked by a pair of adjoint functors, would have a potential application to the comparison of the representation dimensions of Artin algebras. The following lemma is well-known in [11] for the calculation of global dimensions for endomorphism algebras of modules that are generator-cogenerators.

Lemma 2.3 *Let A be an Artin algebra and let M be a generator-cogenerator for A-mod. Suppose m is a non-negative integer. Then* $\mathrm{gl.dim}(\mathrm{End}({}_AM)) \leq m$ *if and only if for each A-module Y there is an exact sequence*

$$0 \longrightarrow M_{m-2} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow Y \longrightarrow 0,$$

with $M_j \in \mathrm{add}({}_AM)$ for $j = 0, \ldots, m-2$, such that

$$0 \longrightarrow \mathrm{Hom}_A(X, M_{m-2}) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(X, M_1) \longrightarrow \mathrm{Hom}_A(X, M_0) \longrightarrow \mathrm{Hom}_A(X, Y) \longrightarrow 0$$

is exact for all $X \in \mathrm{add}({}_AM)$.

The dual statement of the above lemma is left to the reader. Now, let us recall the definition of adjoint functors from a standard text book (for example see [12]).

Definition 2.4 *Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ and $G : \mathscr{D} \longrightarrow \mathscr{C}$ be two functors. If, for each pair of objects x in $\mathscr{C}$ and y in $\mathscr{D}$, there is a natural bijection*

$$\varphi_{x,y} : \mathrm{Hom}_{\mathscr{D}}(Fx, y) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(x, Gy),$$

then we say that F is left adjoint to G, and G is right adjoint to F. In this case, we denote the pair by (F, G), or (F, G, φ).

Given a category $\mathscr{C}$ we may form the opposite category $\mathscr{C}^{\mathrm{op}}$ of $\mathscr{C}$. Note that if (F, G, φ) is an adjoint pair between $\mathscr{C}$ and $\mathscr{D}$, then (G, F, φ^{-1}) is an adjoint pair between $\mathscr{C}^{\mathrm{op}}$ and $\mathscr{D}^{\mathrm{op}}$.

Given an adjoint pair $(F, G, \varphi) : \mathscr{C} \longrightarrow \mathscr{D}$, there are two natural transformations: the unit η from $Id_{\mathscr{C}}$ to GF, and the co-unit ε from FG to $Id_{\mathscr{D}}$. Since (F, G, φ) and $(F, G, \eta, \varepsilon)$ are determined mutually, we denote the adjoint pair (F, G, φ) sometimes by $(F, G, \eta, \varepsilon)$.

The following lemma collects some results on adjoint functors (see [13, Chapter IV]):

Lemma 2.5 *Let $(F, G, \eta, \varepsilon)$ be an adjoint pair of functors between categories $\mathscr{C}$ and $\mathscr{D}$.*

(1) *$(G, F, \varepsilon, \eta)$ is an adjoint pair between $\mathscr{C}^{\mathrm{op}}$ and $\mathscr{D}^{\mathrm{op}}$.*

(2) *G is faithful if and only if every counit ε_C of the adjunction is epi. G is full if and only if each ε_C is split epi.*

(3) *The composition $F \stackrel{F\eta}{\longrightarrow} FGF \stackrel{\varepsilon F}{\longrightarrow} F$ and the composition $F \stackrel{F\eta}{\longrightarrow} FGF \stackrel{\varepsilon F}{\longrightarrow} F$ are identity.*

(4) *For morphisms $f : Fc \longrightarrow d, g : c \longrightarrow Gd, h : c' \longrightarrow c$ and $k : d \longrightarrow d'$, with c, c' objects in $\mathscr{C}$ and d, d' objects in $\mathscr{D}$, we have*

$$\varphi(fk) = (\varphi f)Gk,\ \varphi((Fh)f) = h(\varphi f),\ \varphi f = \eta_c Gf,\ \varphi^{-1}(g) = (Fg)\varepsilon_d.$$

(5) *If either F or G is full, then $G\varepsilon : GFG \longrightarrow G$ is invertible with inverse $\eta G : G \longrightarrow GFG$.*

Finally, we recall the definition of a stable module category. Given an algebra A, the stable category A-$\underline{\text{mod}}$ of A is defined as follows. The objects of A-$\underline{\text{mod}}$ are the same as those of A-mod, and the morphisms between two objects X and Y are given by $\underline{\text{Hom}}_A(X,Y) = \text{Hom}_A(X,Y)/\mathscr{P}(X,Y)$, where $\mathscr{P}(X,Y)$ is the subspace of $\text{Hom}_A(X,Y)$ consisting of those homomorphisms from X to Y which factor through a projective A-module.

For a detailed study on stable module categories one may refer to [14]. For more information and basic knowledge on homological algebra and representation theory of Artin algebras we refer to the excellent books [15] and [16], respectively.

3 Adjoint Functors and Global Dimensions

As we have seen, the definition of representation dimension of an Artin algebra is closely related to global dimension. So it is of interest to develop a theory about the relationship of global dimensions of coherent functor categories linked by an adjoint pair of functors. This section is devoted to this purpose.

Let us start our consideration with an inclusion functor.

Theorem 3.1 *Let $\mathscr{C}$ be an additive k-category and $\mathscr{C}'$ a full subcategory of $\mathscr{C}$. If the inclusion functor admits a right adjoint functor $F : \mathscr{C} \longrightarrow \mathscr{C}'$ then* $\text{gl.dim}(\widehat{\mathscr{C}'}) \leq \text{gl.dim}(\widehat{\mathscr{C}})$.

Proof If $\text{gl.dim}(\widehat{\mathscr{C}})$ is infinite, then there is nothing to prove. So we assume that $\text{gl.dim}(\widehat{\mathscr{C}}) = m < \infty$.

Let G be a coherent functor in $\widehat{\mathscr{C}'}$, that is, there is a morphism $f : Y_1' \longrightarrow Y_0'$ in $\mathscr{C}'$ such that

$$ {}_{\mathscr{C}'}(-,Y_1') \longrightarrow {}_{\mathscr{C}'}(-,Y_0') \longrightarrow G \longrightarrow 0 $$

is exact on $\mathscr{C}'$. Since Y_1' and Y_0' are also objects in $\mathscr{C}$, we may extend this functor G to a functor $\bar{G}$ in $\widehat{\mathscr{C}}$ such that the restriction of $\bar{G}$ to $\mathscr{C}'$ is the given functor G. Since the global dimension of $\widehat{\mathscr{C}}$ is m, we have the following exact sequence of functors on $\mathscr{C}$:

$$ 0 \longrightarrow {}_{\mathscr{C}}(-,Y_m) \longrightarrow \cdots \longrightarrow {}_{\mathscr{C}}(-,Y_2) \longrightarrow {}_{\mathscr{C}}(-,Y_1') \longrightarrow {}_{\mathscr{C}}(-,Y_0') \longrightarrow \bar{G} \longrightarrow 0, $$

with Y_j in $\mathscr{C}$ for $2 \leq j \leq m$. Note that F is a right adjoint functor of the inclusion functor. This implies that for all $Y' \in \mathscr{C}'$ and $X \in \mathscr{C}$ the adjoint isomorphism ${}_{\mathscr{C}'}(Y',FX) \simeq {}_{\mathscr{C}}(Y',X)$ is functorial. Since $\mathscr{C}'$ is a full subcategory of $\mathscr{C}$, we have the following exact commutative diagram for all $Y' \in \mathscr{C}'$:

$$ \begin{array}{ccccccccccc} 0 \longrightarrow & {}_{\mathscr{C}}(Y',Y_m) & \rightarrow \cdots \rightarrow & {}_{\mathscr{C}}(Y',Y_2) & \longrightarrow {}_{\mathscr{C}}(Y',Y_1') \longrightarrow {}_{\mathscr{C}}(Y',Y_0') \longrightarrow G \rightarrow 0 \\ & \wr| & & \wr| & \\ 0 \longrightarrow & {}_{\mathscr{C}'}(Y',FY_m) & \rightarrow \cdots \rightarrow & {}_{\mathscr{C}'}(Y',FY_2). & \end{array} $$

Note that FY_j is in $\mathscr{C}'$. Thus we have an exact sequence

$$ 0 \rightarrow {}_{\mathscr{C}'}(Y',FY_m) \rightarrow \cdots \rightarrow {}_{\mathscr{C}'}(Y',FY_2) \rightarrow {}_{\mathscr{C}'}(Y',Y_1') \rightarrow {}_{\mathscr{C}'}(Y',Y_0') \rightarrow G \rightarrow 0 $$

for each $Y' \in \mathscr{C}'$. This shows that proj.dim $G \leq m$, and therefore we have that $\text{gl.dim}(\widehat{\mathscr{C}'}) \leq m = \text{gl.dim}(\widehat{\mathscr{C}})$.

The following lemma, due to [17], describes when an inclusion functor has a right (or left) adjoint functor. Before we state the lemma, we recall a few concepts from [18, 19]. A full subcategory $\mathscr{C}$ of a category $\mathscr{D}$ is called contravariantly finite in $\mathscr{D}$ if each object D in $\mathscr{D}$ has a right $\mathscr{C}$-approximation, that is, there is a morphism $f : C \longrightarrow D$ in $\mathscr{D}$ with $C \in \mathscr{C}$ such that $\text{Hom}(C',f) : (C',C) \longrightarrow (C',D)$ is surjective for all $C' \in \mathscr{C}$. Dually, one may define the notion of a left $\mathscr{C}$-approximation and the notion of a covariantly finite subcategory.

Lemma 3.2 *Suppose $\mathscr{C}$ and $\mathscr{D}$ are two full subcategories of an Abelian category, which are closed under direct summands and direct sums. If $\mathscr{C}$ is a full subcategory of $\mathscr{D}$, then the* *inclusion functor* $i : \mathscr{C} \longrightarrow \mathscr{D}$ *has a right adjoint if and only if* $\mathscr{C}$ *is contravariantly finite in* $\mathscr{D}$ *and for each object* D*, there is a right* $\mathscr{C}$*-approximation* $f : C \longrightarrow D$ *of* D *such that* $\mathrm{Hom}(C', Ker(f)) = 0$ *for all* C' *in* $\mathscr{C}$*. Dually, the inclusion functor has a left adjoint if and only if* $\mathscr{C}$ *is covariantly finite in* $\mathscr{D}$ *and for each object* D*, there is a left* $\mathscr{C}$*-approximation* $f : D \longrightarrow C$ *of* D *such that* $\mathrm{Hom}(Cok(f), C') = 0$ *for all* C' *in* $\mathscr{C}$*.*

Now, let us also give the dual statement of Theorem 3.1.

Theorem 3.3 *Let* $\mathscr{C}$ *be an additive* k*-category and* $\mathscr{C}'$ *a full subcategory of* $\mathscr{C}$*. If the inclusion functor admits a left adjoint functor* $F : \mathscr{C} \longrightarrow \mathscr{C}'$*, then* $\mathrm{gl.dim}(\widehat{\mathscr{C}'^{\mathrm{op}}}) \le \mathrm{gl.dim}(\widehat{\mathscr{C}^{\mathrm{op}}})$.

We say that a functor F from an additive category $\mathscr{C}$ to another additive category $\mathscr{D}$ is said to be *dense* if for each object $D \in \mathscr{D}$, there is an object $C \in \mathscr{C}$ such that $D \simeq FC$; and *quasi-dense* if for each object $D \in \mathscr{D}$, there is an object $C \in \mathscr{C}$ such that D is a direct summand of FC.

For a general adjoint pair of functors, we can prove the following result:

Theorem 3.4 *Suppose* $\mathscr{C}$ *and* $\mathscr{D}$ *are additive* k*-categories and* $F : \mathscr{C} \longrightarrow \mathscr{D}$ *and* $G : \mathscr{D} \longrightarrow \mathscr{C}$ *an adjoint pair of additive functors with* F *a left adjoint and* G *a right adjoint.*

(1) *Suppose that* $\mathrm{gl.dim}(\widehat{\mathscr{D}}) \ge 2$*. If* G *is full and quasi-dense, then* $\mathrm{gl.dim}(\widehat{\mathscr{C}}) \le \mathrm{gl.dim}(\widehat{\mathscr{D}})$.

(2) *Suppose that* $\mathrm{gl.dim}(\widehat{\mathscr{C}}) \ge 2$*. If* F *and* G *are dense, then* $\mathrm{gl.dim}(\widehat{\mathscr{D}}) \le \mathrm{gl.dim}(\widehat{\mathscr{C}})$.

(3) *Suppose that* $\mathrm{gl.dim}(\widehat{\mathscr{C}}) \ge 2$, $\mathrm{gl.dim}(\widehat{\mathscr{D}}) \ge 2$*, and* F *has a left adjoint functor. If both* F *and* G *are full and quasi-dense, then* $\mathrm{gl.dim}(\widehat{\mathscr{C}}) = \mathrm{gl.dim}(\widehat{\mathscr{D}})$.

Proof (1) We assume that $\mathrm{gl.dim}(\widehat{\mathscr{D}}) = m < \infty$. Let H be a functor in $\widehat{\mathscr{C}}$. Then there is a morphism $f : C_1 \longrightarrow C_0$ such that ${}_{\mathscr{C}}(-, C_1) \xrightarrow{\bar{f}} {}_{\mathscr{C}}(-, C_0) \longrightarrow H \longrightarrow 0$ is exact on $\mathscr{C}$. Since G is quasi-dense, there are D_1 and $D_0 \in \mathscr{D}$ such that $G(D_i) \simeq C_i \oplus C_i'$ for $i = 1, 2$. We form the morphism $\left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right) : C_1 \oplus C_1' \longrightarrow C_0 \oplus C_0'$. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc}
{}_{\mathscr{C}}(-, C_1 \oplus C_1') & \xrightarrow{\left(-,\left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right)\right)} & {}_{\mathscr{C}}(-, C_0 \oplus C_0') & \longrightarrow & H \oplus {}_{\mathscr{C}}(-, C_0') & \longrightarrow & 0 \\
\downarrow \wr & & \downarrow \wr & & & & \\
{}_{\mathscr{C}}(-, G(D_1)) & \longrightarrow & {}_{\mathscr{C}}(-, G(D_0)) & & & & \\
\downarrow \wr & & \downarrow \wr & & & & \\
{}_{\mathscr{D}}(F-, D_1) & \xrightarrow{g'} & {}_{\mathscr{D}}(F-, D_0). & & & &
\end{array}$$

Since G is full, there is a morphism $f' : D_1 \longrightarrow D_0$ such that $Gf' = \left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right) : C_1 \oplus C_1' \longrightarrow C_0 \oplus C_0'$. This means that g' is induced from f'.

Now we consider the coherent functor ${}_{\mathscr{D}}(-, D_1) \xrightarrow{(-,f')} {}_{\mathscr{D}}(-, D_0) \longrightarrow \bar{H} \to 0$. Since $\mathrm{gl.dim}\, \widehat{\mathscr{D}} = m$, there is an exact sequence

$$0 \to {}_{\mathscr{D}}(-, D_m) \to \cdots \to {}_{\mathscr{D}}(-, D_2) \to {}_{\mathscr{D}}(-, D_1) \to {}_{\mathscr{D}}(-, D_0) \to \bar{H} \to 0$$

defined on $\mathscr{D}$, and we get the following exact commutative diagram:

$$\begin{array}{ccccccccccccc}
0 \longrightarrow & {}_{\mathscr{D}}(FC, D_m) & \to \cdots \to & {}_{\mathscr{D}}(FC, D_2) & \longrightarrow & {}_{\mathscr{D}}(FC, D_1) & \longrightarrow & {}_{\mathscr{D}}(FC, D_0) & \longrightarrow & \bar{H}(FC) & \longrightarrow 0 \\
& \mid\wr & & \mid\wr & & \mid\wr & & \mid\wr & & & \\
0 \longrightarrow & {}_{\mathscr{C}}(C, GD_m) & \to \cdots \to & {}_{\mathscr{C}}(C, GD_2) & \longrightarrow & {}_{\mathscr{C}}(C, GD_1) & \longrightarrow & {}_{\mathscr{C}}(C, GD_0) & \longrightarrow & H(C) \oplus {}_{\mathscr{C}}(C, C_0') & \longrightarrow 0,
\end{array}$$

for all $C \in \mathscr{C}$. This shows that the projective dimension of $H \oplus (-, C_0')$ is at most m, and thus the projective dimension of H is at most m. Hence $\mathrm{gl.dim}(\widehat{\mathscr{C}}) \le \mathrm{gl.dim}(\widehat{\mathscr{D}})$.

(2) Suppose H is a coherent functor in $\widehat{\mathscr{D}}$. Then there exists a morphism $f : D_1 \longrightarrow D_0$ in $\mathscr{D}$ such that the sequence ${}_{\mathscr{D}}(-, D_1) \xrightarrow{\bar{f}} {}_{\mathscr{D}}(-, D_0) \longrightarrow H \longrightarrow 0$ is exact on $\mathscr{D}$. This yields

another functor $\bar{H}$ on $\mathscr{C}$ and the following commutative diagram:

$$\begin{array}{ccccccc} {}_{\mathscr{C}}(-,G(D_1)) & \xrightarrow{\overline{Gf}} & {}_{\mathscr{C}}(-,G(D_0)) & \longrightarrow & \bar{H} & \longrightarrow & 0 \\ \downarrow \simeq & & \downarrow \simeq & & & & \\ {}_{\mathscr{D}}(F-,D_1) & \xrightarrow{\bar{f}} & {}_{\mathscr{D}}(F-,D_0). & & & & \end{array}$$

If gl.dim $\widehat{\mathscr{C}} = n \geq 2$, then there is an exact sequence of functors on $\mathscr{C}$:

$$0 \to {}_{\mathscr{C}}(-,C_n) \to \cdots \to {}_{\mathscr{C}}(-,C_2) \to {}_{\mathscr{C}}(-,G(D_1)) \to {}_{\mathscr{C}}(-,G(D_0)) \to \bar{H} \to 0,$$

where $C_i \in \mathscr{C}$. Since G is dense, we may replace each C_i by $G(D_i)$ for some $D_i \in \mathscr{D}$. Thus, by the adjoint isomorphism, we have the following exact sequence:

$$0 \to {}_{\mathscr{D}}(F-,D_n) \to \cdots \to {}_{\mathscr{D}}(F-,D_2) \to {}_{\mathscr{D}}(F-,D_1) \xrightarrow{\bar{f}} {}_{\mathscr{D}}(F-,D_0) \to \bar{H} \to 0.$$

Since F is dense, the image of F is $\mathscr{D}$, and we may consider the functor $(F-,D_i)$ as a functor on $\mathscr{D}$ by putting $D = FC$ for some $C \in \mathscr{C}$, and the morphism is defined in a natural way. The density of F implies also that the natural transformation $(F-,D_i) \longrightarrow (F-,D_{i-1})$ is induced by a homomorphism from $D_i \longrightarrow D_{i-1}$, which is the image of the id_{D_i} under the transformation. Thus the last exact sequence can be considered as sequence of functors on $\mathscr{D}$, and this shows that the projective dimension of H is bounded above by n. Thus gl.dim$(\widehat{\mathscr{D}}) \leq$ gl.dim$(\widehat{\mathscr{C}})$.

(3) Since (F,G) is an adjoint pair with G full and quasi-dense, we have gl.dim$(\widehat{\mathscr{C}}) \leq$ gl.dim$(\widehat{\mathscr{D}})$ by (1). If H is the left adjoint of F, then we have an adjoint pair (H,F) with F full and quasi-dense, and therefore gl.dim$(\widehat{\mathscr{D}}) \leq$ gl.dim$(\widehat{\mathscr{C}})$ by (1).

Remark In the above result, we assume that gl.dim$(\widehat{\mathscr{D}}) \geq 2$. However, if gl.dim$(\widehat{\mathscr{D}}) \leq 1$, then one can see that gl.dim$(\widehat{\mathscr{C}}) \leq 2$. This remark is valid also for the later statements in the paper.

Of particular interest is the following special case. The proof is similar to that of Theorem 3.4; we leave it to the reader.

Theorem 3.5 *Suppose A and B are two algebras. Let $\mathscr{C} = \mathrm{add}({}_AW)$ for an A-module W and $\mathscr{D} = \mathrm{add}({}_BM)$ for a B-module ${}_BM$. Suppose that $F : \mathscr{C} \longrightarrow \mathscr{D}$ and $G : \mathscr{D} \longrightarrow \mathscr{C}$ are an adjoint pair of additive functors with F a left adjoint and G a right adjoint. Assume that ${}_AA \in \mathscr{C}$ and $F({}_AA) ={}_B B$. If G is quasi-dense and if* gl.dim$(\widehat{\mathscr{D}}) \geq 2$, *then* gl.dim$(\widehat{\mathscr{C}}) \leq$ gl.dim$(\widehat{\mathscr{D}})$.

Next, we consider the case where the unit of an adjoint pair splits.

Theorem 3.6 *Suppose $\mathscr{C}$ and $\mathscr{D}$ are additive k-categories and $F : \mathscr{C} \longrightarrow \mathscr{D}$ and $G : \mathscr{D} \longrightarrow \mathscr{C}$ an adjoint pair of additive functors with F a left adjoint and G a right adjoint. If there is an endofunctor $E : \mathscr{C} \longrightarrow \mathscr{C}$ such that GF is naturally equivalent to $Id_{\mathscr{C}} \oplus E$, then* gl.dim$(\widehat{\mathscr{C}}) \leq$ gl.dim$(\widehat{\mathscr{D}})$.

Dually, suppose $\mathscr{C}$ and $\mathscr{D}$ are additive k-categories and $F : \mathscr{C} \longrightarrow \mathscr{D}$ and $G : \mathscr{D} \longrightarrow \mathscr{C}$ an adjoint pair of additive functors with F a left adjoint and G a right adjoint. If there is an endofunctor $E : \mathscr{D} \longrightarrow \mathscr{D}$ such that FG is naturally equivalent to $Id_{\mathscr{D}} \oplus E$, then gl.dim$(\widehat{\mathscr{D}}^{\mathrm{op}}) \leq$ gl.dim$(\widehat{\mathscr{C}}^{\mathrm{op}})$.

Proof Suppose that gl.dim$(\widehat{\mathscr{D}}) = n < \infty$. Let H be a coherent functor in $\widehat{\mathscr{C}}$. Then there is a homomorphism $f : C_1 \longrightarrow C_0$ in $\mathscr{C}$ such that the following sequence of functors on $\mathscr{C}$ is exact:

$${}_{\mathscr{C}}(-,C_1) \stackrel{(-,f)}{\longrightarrow} {}_{\mathscr{C}}(-,C_0) \longrightarrow H \longrightarrow 0.$$

From $F(f) : F(C_1) \longrightarrow F(C_0)$ we have a functor H' in $\widehat{\mathscr{D}}$:

$${}_{\mathscr{D}}(-,F(C_1)) \stackrel{(-,F(f))}{\longrightarrow} {}_{\mathscr{D}}(-,F(C_0)) \longrightarrow H' \longrightarrow 0.$$

Since gl.dim$(\widehat{\mathscr{D}}) = n$, we know that the above projective presentation of H' can be extended to a projective resolution of H':

$$0 \longrightarrow {}_{\mathscr{D}}(-, D_n) \longrightarrow \cdots \longrightarrow {}_{\mathscr{D}}(-, D_2) \longrightarrow {}_{\mathscr{D}}(-, F(C_1)) \stackrel{(-,Ff)}{\longrightarrow} {}_{\mathscr{D}}(-, F(C_0)) \longrightarrow H' \longrightarrow 0,$$

where all D_i are in $\mathscr{D}$. Now we apply the functors to the module FC for an arbitrary C in $\mathscr{C}$ and get the following exact sequence:

$${}_{\mathscr{D}}(FC, D_n) \longrightarrow \cdots \longrightarrow {}_{\mathscr{D}}(FC, D_2) \longrightarrow {}_{\mathscr{D}}(FC, F(C_1)) \stackrel{(FC,Ff)}{\longrightarrow} {}_{\mathscr{D}}(FC, F(C_0) \longrightarrow H'(FC) \longrightarrow 0.$$

Since (F, G) is an adjoint pair, we have the following exact sequence:

$${}_{\mathscr{C}}(C, GD_n) \longrightarrow \cdots \longrightarrow {}_{\mathscr{C}}(C, GD_2) \longrightarrow {}_{\mathscr{C}}(C, GF(C_1)) \stackrel{(C,GFf)}{\longrightarrow} {}_{\mathscr{C}}(C, GF(C_0)) \longrightarrow H'(FC) \longrightarrow 0.$$

By our assumption, we have that $GF(f) \simeq f \oplus E(f)$, this implies that $H'(FC)$ is isomorphic to $H(C) \oplus E(FC)$. Since C is an arbitrary object in $\mathscr{C}$, the functor $H \oplus E(F-)$ has projective dimension at most n. Thus the direct summand H has also projective dimension at most n. This shows that gl.dim$(\widehat{\mathscr{C}}) \leq$ gl.dim$(\widehat{\mathscr{D}})$.

The dual statement follows from the following fact: If $(F, G, \eta, \varepsilon)$ is an adjoint pair between $\mathscr{C}$ and $\mathscr{D}$, then $(G, F, \varepsilon, \eta)$ is an adjoint pair between the categories $\mathscr{D}^{\rm op}$ and $\mathscr{C}^{\rm op}$.

Lemma 3.7 *Suppose $\mathscr{C}$ and $\mathscr{D}$ are additive k-categories and $F : \mathscr{C} \longrightarrow \mathscr{D}$ and $G : \mathscr{D} \longrightarrow \mathscr{C}$ an adjoint pair of additive functors with F a left adjoint and G a right adjoint. Let M be an object in $\mathscr{D}$. If G is full, then* $\mathrm{End}({}_{\mathscr{C}}GM) \simeq \mathrm{End}({}_{\mathscr{D}}FGM)$. *In particular, the two rings* $\mathrm{End}({}_{\mathscr{C}}GM)$ *and* $\mathrm{End}({}_{\mathscr{D}}FGM)$ *have the same global dimension.*

Proof Since (F, G) is an adjoint pair with G full, we know from Lemma 2.5 that $GFGM \simeq GM$. Now the lemma follows from the adjunction.

4 Examples

In this section we display several examples to show that the previous results can be used to compare the global dimensions or representation dimensions of Artin algebras.

4.1 Adjoint Functors from Representation Theory of Finite Groups

Now we prove the following result on the representation dimensions of group algebras. Given a group Q, we denote by kQ the group algebra of Q over a field k. (Here I use the letter "Q" to denote a group because Q is the first letter of the pronunciation of the word "group" in Chinese.) Recall that a Sylow p-subgroup P of Q is called a trivial intersection subgroup of Q if, for any $g \in Q$, the intersection $P \cap gPg^{-1}$ is either P or 1.

Proposition 4.1 *Let k be an algebraically closed field of characteristic p. Suppose Q is a finite group such that the group algebra kQ of Q is representation-infinite, and suppose P is a trivial intersection Sylow p-subgroup of Q. Let $H = N_Q(P)$. Then* rep.dim$(kH) \leq$ rep.dim(kQ).

Proof Suppose that $U_1, \ldots, U_n$ are non-projective indecomposable kQ-modules such that gl.dim $(\mathrm{End}_{kQ}(\bigoplus_j U_j \oplus kQ)) =$ rep.dim(kQ). Suppose that V_j is the corresponding kH-module under Green correspondence. Let F and G denote the induction and restriction functor, respectively. Then $F(V_j) = U_j \oplus R_j$ and $G(U_j) = V_j \oplus Q_j$, where Q_j and R_j are projective kQ and kH-modules, respectively. Note that the projectivity of the modules Q_j and R_j follows from the trivial intersection property of P (see [20, Theorem 1, p. 71]).

Now we define $\mathscr{C} = \mathrm{add}(\bigoplus_j V_j \oplus kH)$ and $\mathscr{D} = \mathrm{add}(\bigoplus_j U_j \oplus kQ)$. Clearly, F is a functor from $\mathscr{C}$ to $\mathscr{D}$, and G is a functor from $\mathscr{D}$ to $\mathscr{C}$. Moreover, the pair (F, G) is an adjoint pair, and G is quasi-dense. Note that $F(kH) = kQ$, and that kH and kQ have the same representation type by [20, Theorem 1, p. 81]. So gl.dim$(\widehat{\mathscr{C}}) \leq$ gl.dim$(\widehat{\mathscr{D}}) =$ rep.dim(kQ) by Theorem 3.5, this means that rep.dim$(kH) \leq$ gl.dim$(\widehat{\mathscr{C}}) \leq$ rep.dim(kQ) by definition.

In general, given a subgroup H of a finite group Q, the induction $F = kQ \otimes_{kH} -$ and the restriction $G = \mathrm{Hom}_{kQ}({}_{kQ}kQ_{kH}, -)$ is an adjoint pair. For this adjoint pair, the unit

$Id_{kH\text{-mod}} \longrightarrow GF$ is split injective, that is, each component is split injective. By [21, VII, Theorem 4.3, pp. 46–47], we may associate a kH-module V' with each kH-module V. Note that $V' = \oplus_{t\in T\setminus H} t\otimes V$, where T is a transversal of H in Q. If $f : V \longrightarrow W$ is a homomorphism between kH-modules V and W, we define a kH-module homomorphism $f' : V' \longrightarrow W'$ by sending $t \otimes v$ to $t \otimes (v)f$. In this way, we have a functor $E' : kH\text{-mod} \longrightarrow kH\text{-mod}$ and a natural equivalence $GF \longrightarrow Id_{kH\text{-mod}} \oplus E'$.

Proposition 4.2 *Let H be a subgroup of a finite group Q, and let V be a kH-module such that* $\text{add}(GFV) \subseteq \text{add}(V)$ *(for example, the inertia group of V is Q). Then* $\text{gl.dim}(\text{End}(_{kH}V)) \leq \text{gl.dim}(\text{End}(_{kQ}FV))$. *If, in addition, $[Q : H]$ is invertible in k, then* $\text{gl.dim}(\text{End}(_{kH}V)) = \text{gl.dim}(\text{End}(_{kQ}FV))$.

Proof Since $\text{add}(GFV) \subseteq \text{add}(V)$, the restriction functor G can be considered as a functor from $\text{add}(FV)$ to $\text{add}(V)$. In case the inertia group of the kH-module V is Q, we know that each $t\otimes V$ is isomorphic to V as kH-modules. So we have $\text{add}(GFV) = \text{add}(V)$. Thus the pair (F, G) is an adjoint pair between the categories $\text{add}(V)$ and $\text{add}(FV)$. Now the first statement of our result follows from Theorem 3.6 by the above observation.

For the second statement, we note that under the assumption, there is a natural equivalence $FG \longrightarrow Id_{kQ\text{-mod}} \oplus E'$ with E' an endofunctor from kQ-mod to itself by [21, VII, Theorem 4.3]. Thus $\text{gl.dim}(\text{End}(_{kQ}FV)^{\text{op}}) \leq \text{gl.dim}(\text{End}(_{kH}V)^{\text{op}})$ by Theorem 3.6. Note that for a Noether ring A, we have $\text{gl.dim}(A) = \text{gl.dim}(A^{\text{op}})$. Thus the second statement follows.

Corollary 4.3 *If Q is a finite group and if H is any subgroup contained in the center of Q, then, for any kH-module V,* $\text{gl.dim}(\text{End}(_{kH}V)) \leq \text{gl.dim}(\text{End}(_{kQ}FV))$.

Proof As H lies in the center of Q, we have that $\text{add}(GFV) = \text{add}(V)$ for any kH-module V. Thus the corollary follows from Proposition 4.2.

The above result suggests the following conjecture:

Conjecture 4.4 *Let k be a field. If Q is a finite group and H is a subgroup of Q, then* $\text{rep.dim}(kH) \leq \text{rep.dim}(kQ)$.

4.2 Adjoint Functors Induced from Stable Equivalences of Morita Type

In this subsection, we shall use the idea in the previous section to compare the global dimensions of two endomorphism algebras of modules which are connected by an adjoint pair of functors arising from a stable equivalence of Morita type.

Let us first recall the definition of a stable equivalence of Morita type [22].

Definition 4.5 *Two algebras A and B are said to be stably equivalent if there is an equivalence $F : A\text{-}\underline{\text{mod}} \rightarrow B\text{-}\underline{\text{mod}}$ of the stable categories.*

Two algebras A and B are stably equivalent of Morita type if there exist an A-B-bimodule $_AM_B$ and a B-A-bimodule $_BN_A$ such that

(1) M and N are projective as left and right modules, respectively; and,

(2) $M\otimes_B N \simeq A\oplus P$ as A-A-bimodules for some projective A-A-bimodule P, and $N\otimes_A M \simeq B \oplus Q$ as B-B-bimodules for some projective B-B-bimodule Q.

Suppose A and B are stably equivalent of Morita type. If A and B are self-injective algebras, then the functors $(F := N\otimes_A -, G := M\otimes_A B-)$ are an adjoint pair. In fact, by [23, Lemma 4.8], we may further assume that $(M \otimes_B -, N \otimes_A -)$ is also an adjoint pair.

Theorem 4.6 *Let A and B be two self-injective k-algebras without separable summands. Suppose that A and B are stably equivalent of Morita type under the adjoint pair (F, G). If X is an A-module, then* $\text{gl.dim}(\text{End}(A \oplus X)) = \text{gl.dim}(\text{End}(B \oplus FX))$.

Proof Let $\mathscr{C} = \text{add}(A \oplus X)$ and $\mathscr{D} = \text{add}(B \oplus FX)$. It follows from the definition of a stable equivalence of Morita type that $GF(X)$ lies in $\mathscr{C}$. Thus we have an adjoint pair of functors (F, G) between the categories $\mathscr{C}$ and $\mathscr{D}$. If we define $E = P \otimes_A -$, then $GF \simeq Id_{\mathscr{C}} \oplus E$. Thus, by Theorem 3.6 and Lemma 2.2, we have that $\text{gl.dim}(\text{End}(A \oplus X)) \leq \text{gl.dim}(\text{End}(B \oplus FX))$.

On the other hand, we may assume that (G, F) is an adjoint pair which defines a stable equivalence of Morita type between A and B. Similarly, we can prove that gl.dim(End$(B \oplus FX)) \leq$ gl.dim(End$(A \oplus X)$). This finishes the proof.

Let us remark that an alternative proof of the above result follows from [24, Corollary 3.3(1)] since stable equivalences of Morita type preserve global dimension.

5 Representation Dimensions between Algebras and Their Factor Algebras

In this section, we shall use the adjunction isomorphism of an adjoint pair of functors to compare the representation dimensions of an algebra and its factor algebras. Our general question is: If I is an ideal in A such that $I^2 = 0$ and rep.dim$(A/I) \leq 2$, is rep.dim$(A) \leq 3$?

While we cannot answer this question in general, we get some results when we impose certain restrictions on I. Our first result in this direction is the following result:

Proposition 5.1 *Let A be an indecomposable algebra and let P be a left ideal in A such that P is an indecomposable projective-injective A-module. Then there is a generator-cogenerator M for A-mod such that*

(1) rep.dim$(A/soc(P)) \leq$ gl.dim(End$(_AM)$).

(2) *If* rep.dim$(A/soc(P)) \geq 2$, *then* rep.dim$(A) \leq$ gl.dim(End$(_AM)$) = rep.dim $(A/soc(P))$.

(3) *If A is non-semi-simple, then the representation dimension of the one-point extension of A by a simple injective A-module coincides with that of A.*

Proof Note that (a) $I := \mathrm{soc}(P)$ is an ideal in A and (b) each indecomposable A-module X is either an A/I-module (that is, $IX = 0$), or isomorphic with P.

(1) Let W be a generator-cogenerator for A/I-mod such that $m :=$ rep.dim(A/I) = gl.dim $(\mathrm{End_{A/I}(W)})$. Then $M := W \oplus P$ is a generator-cogenerator for A-mod. We define $\mathscr{C} = \mathrm{add}(W \oplus P)$, $\mathscr{C}' = \mathrm{add}(W)$ and $F = (A/I) \otimes_A - : \mathscr{C} \to \mathscr{C}'$. Clearly, $FP \simeq P/IP$, which is a projective A/I-module, thus in $\mathscr{C}'$. This shows that F is a functor from $\mathscr{C}$ to $\mathscr{C}'$. Moreover, F is a left adjoint functor of the inclusion functor since $\mathrm{Hom}_{A/I}((A/I) \otimes_A X, Y') \simeq \mathrm{Hom}_A(X, \mathrm{Hom}_{A/I}(A/I, Y')) \simeq \mathrm{Hom}_A(X, Y')$ for $X \in A$-mod and $Y' \in A/I$-mod.

Thus gl.dim $\widehat{\mathscr{C}'^{\mathrm{op}}} \leq$ gl.dim $\widehat{\mathscr{C}^{\mathrm{op}}}$ by Theorem 3.3. Since gl.dim$(\mathrm{End_A(W)})$ = gl.dim $\widehat{\mathscr{C}'}$ and gl.dim$(\mathrm{End_A(W)})$ = gl.dim$(\mathrm{End_A(W)^{op}})$, we have that rep.dim$(A/\mathrm{soc}(P)) \leq$gl.dim(End$(_AM)$).

(2) Suppose $m \geq 2$; one has to prove that for each $X \in A$-mod, there is an exact sequence

$$0 \to Y_{m-2} \to \cdots \to Y_1 \to Y_0 \to X \to 0,$$

with $Y_j \in \mathscr{C}$ such that

$$0 \to (-, Y_{m-2}) \to \cdots \to (-, Y_1) \to (-, Y_0) \to (-, X) \to 0$$

is exact on $\mathscr{C}$. However, this follows from the existence of such exact sequences for A/I-modules X. Thus rep.dim$(A) \leq$ gl.dim(End$(_AM)) \leq m$ = rep.dim(A/I).

Let $F := {}_A(A/I) \otimes_{A/I} -$ be the inclusion functor. Then F has a right adjoint functor $\mathrm{Hom}_A(A/I, -)$ which is given by $G = \mathrm{Hom}_A(A/I, X) = \{x \in X \mid Ix = 0\}$. (Note that $\mathrm{Hom}_A(A/I, P) = \mathrm{rad}(P)$ which is an injective A/I-module, thus the functor G is well-defined.) Now it follows from Theorem 3.1 that gl.dim$(\widehat{\mathscr{C}'}) \leq$ gl.dim$(\widehat{\mathscr{C}})$. This implies that $m \leq$ gl.dim(End$(_AM)$). Thus m = gl.dim(End$(_AM)$), and rep.dim$(A) \leq$ gl.dim(End$(_AM)$) = rep.dim(A/I). So (2) follows.

(3) Since the indecomposable projective module $P(\omega)$ over the one-point extension with rad$(P(\omega))$ being the simple injective A-module is projective-injective, we may apply (2) together with [7, Proposition 6.1] to get the statement (3).

Remarks (1) The statement 5.1(2) is not true if rep.dim$(A/\mathrm{soc}(P)) = 0$. The two-by-two upper triangular matrix algebra provides a desired counter-example.

(2) An alternative proof of the statement 5.1(3) is also given in [4].

(3) In case (2) of the above proposition, if there is a generator-cogenerator V for A-mod such that rad(P) and P/soc(P) are contained in add(V), and that rep.dim(A)=gl.dim(End $({}_AV)$), then rep.dim(A) = rep.dim$(A/\text{soc}(P))$. In fact, suppose V is such a generator-cogen-erator for A-mod. Then we know that $(A/I)\otimes_A V$ is a generator-cogenerator for A/I-mod, and that the functors $F := {}_{A/I}(A/I)\otimes_A -$ and $G := \text{Hom}_{A/I}(A/I,-)$ define an adjoint pair between $\mathscr{X} = \text{add}(V)$ and $\mathscr{Y} = \text{add}((A/I)\otimes_A V)$. In fact, G is an embedding functor. Thus gl.dim $\widehat{\mathscr{Y}^{\text{op}}} \leq$ gl.dim $\widehat{\mathscr{X}^{\text{op}}}$ by Theorem 3.3. This means that gl.dim(End$_{A/I}(A/I\otimes_A V)$) $\leq$ gl.dim(End$({}_AV)$) = rep.dim(A). By definition, we have rep.dim$(A/I) \leq$ gl.dim(End$_{A/I}(A/I\otimes_A V)$)$\leq$ rep.dim(A). Thus rep.dim(A)= rep.dim(A/I).

Theorem 5.2 *Let A be an Artin algebra and I an ideal in A such that $I\,rad(A) = 0 = rad(A)I$. If A/I is representation-finite, then* rep.dim$(A) \leq 3$.

Proof Let $B = A/I$. Note that we have an adjoint pair (F,G) of functors given by the restriction functor $F = {}_AB\otimes_B - : B\text{-mod} \longrightarrow A\text{-mod}$ and $G = \text{Hom}_A({}_AB_B,-) : A\text{-mod} \longrightarrow B\text{-mod}$. The canonical surjection from A to A/I defines an injective homomorphism $\beta_X : \text{Hom}_A({}_AB_A,X) \longrightarrow X$ of A-modules. We denote the A-module $\text{Hom}_A({}_AB_A,X)$ by X_+, it is also a B-module. The adjunction $\text{Hom}_A({}_AY,{}_AX) \simeq \text{Hom}_B({}_BY,X_+)$, with $Y\in B$-mod, shows that

$(*)$ Given any B-module Y, A-module X and a homomorphism $f : {}_AY \longrightarrow {}_AX$, there is a B-homomorphism $f' : {}_BY \longrightarrow X_+$ such that $f = f'\beta_X$.

Let $M_1,\dots,M_n$ be a complete list of all non-isomorphic indecomposable B-modules. We define $V = A\oplus D(A)\oplus\bigoplus_{i=1}^n M_i$ and shall show that for any A-module X, there is an exact sequence

$$0\longrightarrow V_1\longrightarrow V_0\longrightarrow X\longrightarrow 0,$$

with $V_i\in\text{add}(V)$ such that

$$0\longrightarrow (V,V_1)\longrightarrow (V,V_0)\longrightarrow (V,X)\longrightarrow 0$$

is exact.

If X is in add(V), then there is nothing to prove. So we assume that X is an indecomposable A-module which does not belongs to add(V). Let C be the cokernel of β_X, and let $\pi' : P\longrightarrow C$ be a projective cover of the A-module C and $\pi : P\longrightarrow X$ a lifting of π' along the canonical projection $\mu : X\longrightarrow C$. Then we may have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega_A(C) & = & \Omega_A(C) & & \\
 & & & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & X_+ & \xrightarrow{(1,0)} & X_+\oplus P & \xrightarrow{(0,1)^T} & P & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle (\beta_X,\pi)^T} & & \downarrow{\scriptstyle \pi'} & & \\
0 & \longrightarrow & X_+ & \xrightarrow{\beta_X} & X & \xrightarrow{\mu} & C & \longrightarrow & 0.\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Now we define $V_1 = \Omega_A(C)$ and $V_0 = X_+\oplus P$ and shall prove that the exact sequence

$$0\longrightarrow V_1\xrightarrow{i} V_0\xrightarrow{(\beta_X,\pi)^T} X\longrightarrow 0$$

has the desired property.

Clearly, $X_+ \oplus P$ lies in add(V). To see that $\Omega_A(C)$ belongs to add(V), we note that $I\Omega_A(C) \subseteq I\,\mathrm{rad}(P) \subseteq I\,\mathrm{rad}(A)P = 0$, that is, $\Omega_A(C)$ is a B-module.

If V' is a projective A-module, then we have an exact sequence

$$(**)\quad 0 \longrightarrow (V', V_1) \longrightarrow (V', V_0) \longrightarrow (V', X) \longrightarrow 0.$$

If V' is a B-module, then any homomorphism from V' to X factors through β_X by $(*)$, and thus through $(\beta_X, \pi)^T$. So we get the exact sequence $(**)$ again.

If V' is an indecomposable injective A-module, then any non-zero homomorphism from V' to X factors through $V'/\mathrm{soc}(V') \simeq D(e\,\mathrm{rad}(A))$, for some primitive idempotent element e in A. Since $\mathrm{rad}(A)I = 0$, we know that $V'/\mathrm{soc}(V')$ is a B-module. Thus, any homomorphism from V' to X factors through β_X by $(*)$, and then through $(\beta_X, \pi)^T$. So we also get the exact sequence $(**)$ in this case. This finishes the proof.

Let us mention that the argument in [8] shows the following result:

Proposition 5.3 *Let A be an algebra and I an ideal in A with $I\,\mathrm{rad}(A) = 0$. If A is self-injective, then* rep.dim$(A) \leq$ rep.dim$(A/I) + 1$. *In particular, if A is a self-injective algebra with an ideal I such that (A/I) is representation-finite, then* rep.dim$(A) \leq 3$.

6 APR-tilting and Representation Dimension

As a typical example of adjoint functors, we consider the tilting functors in this section and compare the representation dimensions of two algebras A and B when B is an endomorphism algebra of a tilting module over A. First we recall the definition of a tilting module.

Definition 6.1 *Let A be an Artin algebra. An A-module T is called a tilting module if the following conditions are fulfilled*:

(1) *The projective dimension of T is at most* 1;

(2) $\mathrm{Ext}^1_A(T, T) = 0$;

(3) *There is an exact sequence* $0 \longrightarrow {}_AA \longrightarrow T' \longrightarrow T'' \longrightarrow 0$ *such that both T' and T'' belong to* add(T).

Dually, one has the notion of a co-tilting module.

Given a tilting module T, we have a torsion pair $(\mathscr{F}(T), \mathscr{T}(T))$ in A-mod:

$$\mathscr{T}(T) := \{X \in A\text{-mod} \mid \mathrm{Ext}^1_A(T, X) = 0\} = \{X \in A\text{-mod} \mid X \text{ is generated by } {}_AT\},$$

$$\mathscr{F}(T) := \{X \in A\text{-mod} \mid \mathrm{Hom}_A(T, X) = 0\} = \{X \in A\text{-mod} \mid X \text{ is cogenerated by } \tau T\},$$

where τ is the Auslander–Reiten translation DTr, and where $\mathscr{T}(T)$ is the torsion class and $\mathscr{F}(T)$ is the torsion-free class. Let B be the endomorphism algebra of T. Then there is a torsion pair $(\mathscr{Y}(T), \mathscr{X}(T))$ in B-mod:

$$\mathscr{Y}(T) := \{Y \in B\text{-mod} \mid \mathrm{Tor}^B_1(T, Y) = 0\} = \{Y \in B\text{-mod} \mid Y \text{ is cogenerated by } D(T_B)\},$$

$$\mathscr{X}(T) := \{Y \in B\text{-mod} \mid T \otimes_B Y = 0\} = \{Y \in B\text{-mod} \mid Y \text{ is generated by } \tau^{-1}D(T_B)\},$$

where $\tau^{-1} = $ TrD is the inverse of the Auslader-Reiten translation.

The following result describes the relationship between these subcategories (see [25]):

Theorem 6.2 [Brenner–Butler theorem] *The functor* $F = \mathrm{Hom}_A(T, -) : \mathscr{T}(T) \longrightarrow \mathscr{Y}(T)$ *is an equivalence with the inverse* $G = T \otimes_B -$, *and the functor* $E = \mathrm{Ext}^1_A(T, -) : \mathscr{F}(T) \longrightarrow \mathscr{X}(T)$ *is an equivalence with the inverse* $Z = \mathrm{Tor}^B_1(T, -)$.

Note that the image of A-mod under the functor F is $\mathscr{X}(T)$, and the image of A-mod under the functor E is $\mathscr{Y}(T)$.

A special case of tilting modules is the BB-tilting modules, due to Brenner and Butler, which is a proper generalization of the APR-tilting module. Let us recall the definitions and basic properties.

Let A be an Artin algebra and S a simple non-injective A-module with the properties: (a) proj.dim $\tau^{-1}S \leq 1$; and, (b) $\mathrm{Ext}^1_A(S,S) = 0$. We denote the projective cover of S by $P(S)$. Then we may assume that ${}_AA = P(S)\oplus P$ such that there is no direct summand of P isomorphic to $P(S)$ and define $T = \tau^{-1}S \oplus P$. It is well known that T is a tilting module. Such a tilting module is called a BB-tilting module. In particular, if S is a projective non-injective simple module, then T is automatically a BB-tilting, this special case was first studied in [26], and the tilting module of this form is called an APR-tilting module in the literature.

Let T be an APR-tilting A-module. Then we have:

(1) $\mathscr{F}(T)$= add(S), and $\mathscr{X}(T)$= add$(E(S))$.

(2) $E(S)$ is a simple B-module. The minimal projective presentation of the B-module $E(S)$ is given by

$$0 \longrightarrow FP_0 \longrightarrow F(\tau^{-1}S) \longrightarrow E(S) \longrightarrow 0,$$

which is induced from the Auslander–Reiten sequence $0 \longrightarrow S \longrightarrow P_0 \longrightarrow \tau^{-1}S \longrightarrow 0$ with P_0 projective. Moreover, If inj.dim$(S) \leq 1$, then $E(S)$ is injective.

(3) If $E(S)$ is injective, then $\mathscr{Y}(T)$= add(B-ind$\backslash E(S)$), where B-ind denotes the subcategory consisting of all the indecomposable B-modules.

(4) If inj.dim$(S) \leq 1$, then $E(S)$ is injective, and the torsion pair $(\mathscr{Y}(T), \mathscr{X}(T))$ splits, that is, every indecomposable B-module lies either in $\mathscr{X}(T)$ or in $\mathscr{Y}(T)$.

(5) If Q is an indecomposable injective A-module which is not isomorphic to the injective hull of S, then FQ is an injective B-module.

When we consider the representation dimensions of Artin algebras, we would like to compare the representation dimension of the algebra A with that of B, where B is the endomorphism algebra of a tilting A-module T. In general, one could not hope that their representation dimensions are equal. This can be seen from the following example: If A is a tame hereditary algebra, one may have a tilting A-module T which contains both non-zero preprojective and non-zero preinjective direct summand. Then $B = \mathrm{End}({}_AT)$ is representation-finite. Hence rep.dim$(A) = 3$ and rep.dim$(B) = 2$. However, we shall prove that certain APR-tilting modules preserve the representation dimension.

Proposition 6.3 *Let A be an Artin algebra, $T = \tau^{-1}S \oplus P$ an APR-tilting module and B = End$({}_AT)$. If* inj.dim$(S) = 1$*, then* rep.dim$(B) \leq$ rep.dim(A).

Proof Note that for an APR-tilting module T, the torsion class $\mathscr{T}(T)$ can be described as $\{X \in A\text{-mod} \mid \mathrm{Hom}_A(X,S) = 0\}$. Suppose $V = U \oplus P \oplus D(A) \oplus V'$, with $V' \in \mathscr{T}(T)$, and $U \in \mathrm{add}(S)$, is an A-module which gives the representation dimension of A, that is, rep.dim$(A) = m =$ gl.dim(End$({}_AV)$). We shall prove that for the module $M = FP(S) \oplus FP \oplus FD(A) \oplus FV' \oplus F(\tau^{-1}S) \oplus D\mathrm{Hom}_A(\tau^{-1}S,T)$ we have gl.dim(End$({}_BM)$) $\leq m$. Note that M is a generator-cogenerator for B-mod by (4) and (5). If $m \leq 2$, then A is representation-finite with a non-trivial radical, this means that the categories $\mathscr{X}(T)$ and $\mathscr{Y}(T)$ are finite, so B is representation-finite, and therefore, rep.dim$(B) \leq$ rep.dim(A). We may assume that $m \geq 3$.

Suppose Y is a B-module which has no direct summands in add(M). Then Y lies in $\mathscr{Y}(T)$ by (4) and (1), and there is an X in $\mathscr{T}(T)$ such that $FX = Y$. Since gl.dim(End$({}_AV)) = m$, there is an exact sequence

$$0 \longrightarrow V_{m-2} \xrightarrow{f_{m-2}} \cdots \longrightarrow V_1 \xrightarrow{f_1} V_0 \xrightarrow{f_0} X \longrightarrow 0$$

with $V_i \in$ add(V) such that $0 \longrightarrow (U, V_{m-2}) \cdots \longrightarrow (U, V_1) \longrightarrow (U, V_0) \longrightarrow (U, X) \longrightarrow 0$ is exact for all $U \in$add(V).

Let K_i be the image of f_i. Then we have an exact sequence

$$0 \longrightarrow K_{i+1} \xrightarrow{\mu_i} V_i \xrightarrow{\pi_i} K_i \longrightarrow 0$$

with $f_i = \pi_i\mu_{i-1}$. From this exact sequence we have the following exact sequence:

$$0 \longrightarrow FK_{i+1} \longrightarrow FV_i \longrightarrow FK_i \longrightarrow EK_{i+1}.$$

Since the image of A-mod under the functor E belongs to $\mathscr{X}(T) = \mathrm{add}(E(S))$, the cokernel C_i of $F\pi_i$ is a semi-simple B-module by (2). We may assume that the minimal projective presentation of C_i is

$$0 \longrightarrow FP_0^{l_i} \xrightarrow{\beta_i} F\tau^{-1}S^{l_i} \xrightarrow{\gamma_i} E(S)^{l_i} = C_i \longrightarrow 0.$$

Let $\alpha_i : F(\tau^{-1}S)^{l_i} \longrightarrow FK_i$ be a lifting of γ_i.

Now we consider the following exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & FK_{i+1} & \longrightarrow & FK_{i+1} \oplus FP_0^{l_i} & \longrightarrow & FP_0^{l_i} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \beta_i & & \\
0 & \longrightarrow & FV_i & \xrightarrow{(1,0)} & FV_i \oplus F(\tau^{-1}S)^{l_i} & \xrightarrow{(0,1)^T} & F(\tau^{-1}S)^{l_i} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \gamma_i & & \\
0 & \longrightarrow & \mathrm{Im}(F\pi_i) & \longrightarrow & FK_i & \longrightarrow & C_i & \longrightarrow & 0,
\end{array}$$

and shall show that the exact sequence

$$(\clubsuit) \quad 0 \longrightarrow FK_{i+1} \oplus FP_0^{l_i} \xrightarrow{\left(\begin{smallmatrix} F\mu_i & 0 \\ 0 & \beta_i \end{smallmatrix}\right)} FV_i \oplus F(\tau^{-1}S)^{l_i} \xrightarrow{\left(\begin{smallmatrix} F\pi_i \\ \alpha_i \end{smallmatrix}\right)} FK_i \longrightarrow 0$$

has the following property: For any L in $\mathrm{add}(M)$, the sequence

$$(*) \qquad 0 \longrightarrow (L, FK_{i+1} \oplus FP_0)^{l_i}) \longrightarrow (L, FV_i \oplus F(\tau^{-1}S)^{l_i}) \longrightarrow (L, FK_i) \longrightarrow 0$$

is exact.

Clearly, this is true for all projective B-modules. If $E(S) = D\mathrm{Hom}_A(\tau^{-1}S, T)$, then, since $E(S)$ is a torsion module and FK_i is a torsion-free module with respect to the torsion pair $(\mathscr{Y}(T), \mathscr{X}(T))$, we have that $\mathrm{Hom}_B(E(S), FK_i) = 0$. So it is sufficient to show that for any L in $\mathrm{add}\big(FV' \oplus FD(A)\big)$ the sequence $(*)$ is exact. Let f be a homomorphism from L to FK_i, with $L = FV'$, or $L = FD(A)$. Then there is a morphism $g : X' \longrightarrow K_i'$, with $X' = V'$ or $X' = D(A)$ such that $FX' = L$ and $Fg = f$ since F is an equivalence. By the choice of the sequence for X, we have a homomorphism $h : X' \longrightarrow V_i$ such that the following diagram is commutative:

$$\begin{array}{ccc}
 & X' & \\
{}^{h}\swarrow & & \searrow^{(g,0)} \\
V_i & \xrightarrow{\pi_i} & K_i = K_i' \oplus S_i.
\end{array}$$

This shows that $f = Fg = F(g, 0) = (Fh)(F\pi_i)$. Thus the sequence $(*)$ is exact for $L = FV'$ or $L = FD(A)$.

Now we define $T_i = F(\tau^{-1}S)^{l_i}$ and $Q_i = FP_0^{l_i}$, and concatenate all short exact sequences $(\clubsuit)$. This provides us with an exact sequence:

$$\begin{aligned}
&0 \longrightarrow FV_{m-2} \oplus Q_{m-3} \longrightarrow FV_{m-3} \oplus T_{m-3} \oplus Q_{m-4} \longrightarrow \cdots \\
&\longrightarrow FV_2 \oplus T_2 \oplus Q_1 \xrightarrow{d_2} FV_1 \oplus T_1 \oplus Q_0 \xrightarrow{d_1} FV_0 \oplus T_0 \xrightarrow{d_0} FX = Y \longrightarrow 0 \qquad (**)
\end{aligned}$$

with $\ker(d_i) = FK_{i+1} \oplus Q_i$. As a consequence of the property $(*)$, we see that that this sequence has the following property: When we apply $\mathrm{Hom}_B(Y', -)$ to $(**)$ for any $Y' \in \mathrm{add}(M)$ we always get an exact sequence.

Thus gl.dim(End($_BM$))$\leq m$, this implies that rep.dim$(B) \leq$ rep.dim(A).

Let us state the dual statement of 6.3. Suppose I is a non-projective simple A-module with inj.dim$(\tau_A I) \leq 1$ and $\mathrm{Ext}^1(I, I) = 0$. If $D(A) = Q(I) \oplus Q$ and there is no direct summand of Q isomorphic to I, where $Q(I)$ stands for the injective envelope of the A-module I, then $T = \tau I \oplus Q$ is a BB-cotilting A-module. Let B be the endomorphism algebra of T. A special case is that I is a non-projective simple injective A-module. In this case we have the notion of a APR-cotilting module. The following is a dual statement of the previous proposition:

Proposition 6.4 *Let A be an Artin algebra and $T = \tau I \oplus Q$ an APR-cotilting module. If* $\text{proj.dim}(_AI) = 1$, *then* $\text{rep.dim}(B) \leq \text{rep.dim}(A)$, *where* $B = \text{End}(_AT)$.

As a corollary of the above discussion, we have

Theorem 6.5 *Let A be an Artin algebra.*

(1) *Suppose* $T = \tau^{-1}S \oplus P$ *is an APR-tilting module and* $B = \text{End}(_AT)$. *If* $\text{inj.dim}(_AS) = 1$, *then* $\text{rep.dim}(A) = \text{rep.dim}(B)$.

(2) *Suppose* $T = \tau I \oplus Q$ *is an APR-cotilting module and* $B = \text{End}(_AT)$. *If* $\text{proj.dim}(_AI) = 1$, *then* $\text{rep.dim}(B) = \text{rep.dim}(A)$.

Proof By Proposition 6.3, we have $\text{rep.dim}(B) \leq \text{rep.dim}(A)$. Under our assumption, the B-module $E(S)$ is a simple injective non-projective module with projective dimension at most one by (3). Let U be the direct sum of one copy of each indecomposable injective B-module not isomorphic to $E(S)$. Then the endomorphism algebra of the associated APR-cotilting B-module $T' = E(S) \oplus U$ is isomorphic to A^{op}, the opposite algebra of A by [26, Proposition 2.5]. This implies that $\text{rep.dim}(A^{\text{op}}) \leq \text{rep.dim}(B)$ by 6.4. Since $\text{rep.dim}(A) = \text{rep.dim}(A^{\text{op}})$, we have proved that $\text{rep.dim}(A) = \text{rep.dim}(B)$.

To state our next result, it is convenient to introduce the following notion:

Definition 6.6 *An APR-tilting A-module $T = \tau^{-1}S \oplus P$ is called a switch APR-tilting module if* $\text{inj.dim}(_AS) = 1$. *Dually, an APR-cotilting A-module $T = \tau I \oplus Q$ is called a switch APR-cotilting module if* $\text{proj.dim}(_AI) = 1$. *An algebra B is said to be switched from an algebra A if there is a finite series of algebras $A = A_1, A_2, \ldots, A_s = B$ with a switch APR-tilting or a switch APR-cotilting module T_i over A_i for each i such that* $A_{i+1} = \text{End}(_{A_i}T_i)$.

With this notion in hand, our main result in this section can be stated as follows:

Theorem 6.7 *If an Artin algebra B is switched from an Artin algebra A, then* $\text{rep.dim}(B) = \text{rep.dim}(A)$.

Next, we apply Theorem 6.7 to the finitistic dimension. Recall that the finitistic dimension of an Artin algebra A is defined to be the supremum of projective dimensions of finitely generated A-modules having finite projective dimension. The finitistic dimension conjecture says that for any Artin algebra A the finitistic dimension of A is finite. Note that it was shown in [27] that for a generalized tilting A-module T the finitistic dimension of A is finite if and only if the finitistic dimension of $\text{End}(_AT)$ is finite. For more information on advances of the conjecture we refer to [3, 28, 29] and the references therein.

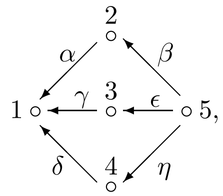
We have the following result:

Corollary 6.8 *Suppose an Artin algebra B is switched from an Artin algebra A and C is a subalgebra of B with* $\text{rad}(C)$ *a left ideal in B. If* $\text{rep.dim}(A) \leq 3$, *then the finitistic dimension of C is finite.*

Proof By Theorem 6.7, $\text{rep.dim}(B) = \text{rep.dim}(A)$. Since $\text{rep.dim}(B) \leq 3$ implies that the finitistic dimension of C is finite by a result in [3], the corollary follows immediately.

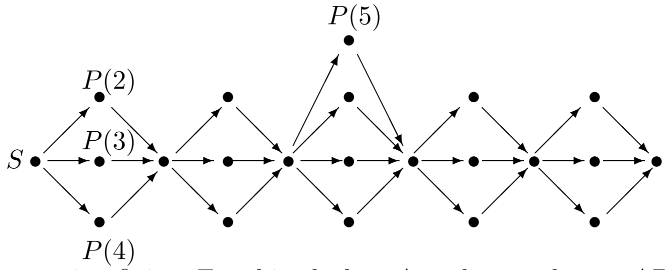
Let us remark that the condition "$\text{inj.dim}(S) = 1$" in Theorem 6.5 can not be dropped. This can be seen from the following example.

Let A be the algebra (over a field k) given by the quiver with relations:



$\beta\alpha = \epsilon\gamma, \quad \beta\alpha = \eta\delta.$

The Auslander-Reiten quiver of this algebra is as follows:



Thus A is representation-finite. For this algebra A we have only one APR-tilting module. The endomorphism algebra B of the APR-tilting module is representation-infinite. So A and B have different representation dimension. One can check that the injective dimension of the simple projective A-module S is 2. This implies that the condition for the simple module S in the previous results is necessary.

Although the two algebras A and B have different representation dimensions, their trivial extensions do have the same representation dimension (see [8]).

We should note that even if an algebra B is obtained from an algebra A by an APR-tilting module satisfying the conditions in Theorem 6.5, the two algebras A and B may not be stably equivalent. This can be seen from two hereditary algebras since it was shown in [30] that two stably equivalent hereditary algebras with no projective-injective modules are isomorphic. As a simple example, we take A to be the path algebra (over a field k) of the quiver $\circ \longrightarrow \circ \longleftarrow \circ$ and B to be the path algebra of the quiver $\circ \longleftarrow \circ \longrightarrow \circ$, then B is obtained from A by APR-tilting, and A and B are not stably equivalent.

Finally, let us remark that Assem, Platzeck and Trepode have recently determined the representation dimension of tilted algebras and the laura algebras in [31]. An iterated application of Theorem 6.7 might give another proof of their result for the case of tilted algebras.

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