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RECTIFIABLE AND UNRECTIFIABLE OSCILLATIONS FOR A GENERALIZATION OF THE RIEMANN–WEBER VERSION OF EULER DIFFERENTIAL EQUATION

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Abstract. The following generalization

$$y'' + \frac{1}{x^\alpha}\left\{\frac{1}{4} + \frac{\lambda}{|\ln x|^\beta}\right\} y = 0 \text{ on } (0,b), \quad b \in (0,1), \quad \alpha \geq 2, \quad \beta > 0, \quad \lambda > 0$$

(for $\alpha = 2$ we suppose that $\beta = 2$ and $\lambda > 1/4$), of the Riemann–Weber version of Euler differential equation is introduced and it is considered together with a suitable boundary layer condition depending on α near $x = 0$. It is shown that this problem is rectifiable (resp., unrectifiable) oscillatory on $(0,b)$ provided $\alpha \in [2,4)$ (resp., $\alpha \geq 4$). It is a kind of geometrical oscillations on $(0,b)$ for which the finite (resp., infinite) length of the graphs of all its solutions is proposed.

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1. Introduction and the Statement of the Main Problem

Let $y = y(x)$ be a real function. The Riemann–Weber version of Euler differential equation is the following second order linear differential equation

$$y'' + \frac{1}{x^2}\left\{\frac{1}{4} + \frac{\lambda}{(\ln x)^2}\right\} y = 0, \quad \lambda > 1/4. \tag{1}$$

It plays an essential role in the study of oscillations for second order nonlinear differential equations of Euler type: $x^2 y'' + g(y) = 0$. For details see [13] and [14]. All solutions of (1) are written explicitly as

$$y(x) = \sqrt{x \ln \frac{1}{x}}\left(c_1 \cos\left(\rho \ln \ln \frac{1}{x}\right) + c_2 \sin\left(\rho \ln \ln \frac{1}{x}\right)\right), \tag{2}$$

where $\rho = \sqrt{\lambda - 1/4}$. Furthermore, it is easy to check that all solutions y of (1) satisfy the following boundary layer condition near $x = 0$,

$$\begin{cases} \text{there are } c > 0 \text{ and } d > 0 \text{ both depending on } y \text{ such that} \\ |y(x)| \leq c x^\delta, \text{ for all } x \in (0,d), \text{ where } \delta = \frac{1}{4}. \end{cases} \tag{3}$$

Let $y = y(x)$ be a real function defined on the interval $\overline{I}$, where $I = (0,b)$ and $b \in (0,1)$. Let $\alpha \geq 2$, $\beta > 0$, and $\lambda > 0$, where for $\alpha = 2$ we suppose

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that $\beta = 2$ and $\lambda > 1/4$. We are concerned with a generalization of the linear problem (1), (3),

$$y'' + \frac{1}{x^\alpha}\left\{\frac{1}{4} + \frac{\lambda}{|\ln x|^\beta}\right\} y = 0 \quad \text{on } I, \tag{4}$$

with the following boundary layer condition near $x = 0$,

$$\left\{ \begin{array}{l} \text{there are } c > 0 \text{ and } d \in I \text{ both depending on } y \text{ such that} \\ |y(x)| \le c x^\delta, \text{ for all } x \in (0, d), \text{ where } \delta > \frac{\alpha - 2}{2}. \end{array} \right. \tag{5}$$

Here δ may depend on y. It is clear that the linear problem (1), (3) is a particular case of (4), (5) especially as $\alpha = \beta = 2$, $\lambda > 1/4$ and $\delta = 1/4$. However, in contrast to (1), (3), the main problem (4), (5) does not admit the explicit form of its solutions.

In order to state the main result of the paper, we need some basic definitions. Let us remark that the functional space of all solutions for the linear equations appearing in the paper is taken to be the space $C(\bar{I}) \cap C^2(I)$.

Definition 1.1. A real function $y = y(x)$ is said to be *oscillatory* (resp., *nonoscillatory*) on I if it has an infinite (resp., a finite) number of zeros on I. A linear problem $y'' + f(x)y = 0$ is said to be *oscillatory* (resp., *nonoscillatory*) on I if all its nontrivial solutions are oscillatory (resp., nonoscillatory) on I.

Let $G(y)$ be the graph of y and let $length(G(y))$ be the length of $G(y)$, both determined by

$$G(y) = \{(t, y(t)) : 0 \le t \le b\} \subseteq \mathbf{R}^2,$$

$$length(G(y)) = \sup \sum_{i=1}^{m} \|(t_i, y(t_i)) - (t_{i-1}, y(t_{i-1}))\|_2,$$

where the supremum is taken over all dissections $0 = t_0 < t_1 < \ldots < t_m = b$ of the interval $\bar{I}$. Here $\| \; \|_2$ denotes the norm in $\mathbf{R}^2$.

Definition 1.2. The graph $G(y)$ is said to be a *rectifiable* curve in $\mathbf{R}^2$ if $length(G(y)) < \infty$. In contrary, $G(y)$ is said to be an *unrectifiable* curve in $\mathbf{R}^2$. See for instance [1, Chapter 5.2].

With the help of two above definitions, we can introduce a kind of geometrical oscillations for a real function y, which includes the rectifiability and unrectifiability of its graph $G(y)$.

Definition 1.3. An oscillatory function $y = y(x)$ on I is said to be *rectifiable* (resp., *unrectifiable*) *oscillatory* on I if its graph $G(y)$ is a rectifiable (resp., unrectifiable) curve in $\mathbf{R}^2$. A linear problem $y'' + f(x)y = 0$ is said to be *rectifiable* (resp., *unrectifiable*) *oscillatory* on I if all its nontrivial solutions y are rectifiable (resp., unrectifiable) oscillatory on I.

There are two essential examples for rectifiable and unrectifiable oscillatory linear problems, see [8]. The first one is a rectifiable oscillatory linear problem based on the famous Euler differential equation,

$$\begin{cases} y'' + \frac{\lambda}{x^2} y = 0 \quad \text{on } I, \ \lambda > 1/4, \\ \text{there are } c > 0 \text{ and } d \in I \text{ both depending on } y \text{ such that} \\ |y(x)| \le c\sqrt{x}, \ \text{for all } x \in (0, d). \end{cases}$$

The second one is an unrectifiable oscillatory linear problem,

$$\begin{cases} y'' + \frac{\lambda}{x^4} y = 0 \quad \text{on } I, \ \lambda > 0, \\ \text{there are } c > 0 \text{ and } d \in I \text{ both depending on } y \text{ such that} \\ |y(x)| \le cx, \ \text{for all } x \in (0, d). \end{cases}$$

Now, we are able to state the main results of the paper.

Theorem 1.4. *The following statements hold true:*
(i) *if* $\alpha = 2$*, then the linear equation* (4)*, that is* (1)*, is rectifiable oscillatory on* I*;*
(ii) *if* $\alpha \in (2, 4)$*, then the linear problem* (4), (5) *is rectifiable oscillatory on* I*;*
(iii) *if* $\alpha \ge 4$*, then the linear problem* (4), (5) *is unrectifiable oscillatory on* I *provided that it admits the existence of two linearly independent solutions.*

The related oscillation conjecture concerning the class of linear differential equations of Euler type has been already established in the author's paper [8]. The cases $\alpha = 2$ and $\alpha \in (2, 4)$ will be considered separately because of property for a sequence of consecutive zero points of solutions which appears in the second case but not in the first one. Therefore, to show rectifiable oscillations as $\alpha = 2$, we use the method of Minkowski content partially introduced in [7], and completely presented in Section 5 below. It is quite a different technique than in the case $\alpha \in (2, 4)$, which will be presented in Section 3 below. Finally, the unrectifiable oscillations of the main problem will be discussed in Section 4.

2. Oscillations for Equation (4)

First, we are going to show that each solution y of the equation (4) has at most a finite number of zero points on any subinterval $J \subset I$, $0 \notin \overline{J}$, and at the same time, it has an infinite number of zero points on the entire I, where $I = (0, b)$ and $b \in (0, 1)$. Also, we give an upper bound for the sequence of consecutive zero points of y on I.

Lemma 2.1. *Let* $y = y(x)$ *be a solution of the equation* (4)*.*
(i) *On any open interval* $J \subseteq I$*,* $0 \notin \overline{J}$*, there are at most a finite number of zero points of* y*.*
(ii) *There is a decreasing sequence* $a_k \in I$ *of consecutive zero points of* y *such that* $a_k \searrow 0$*, that is to say, for each* $k \in \mathbf{N}$*,*

$$a_{k+1} < a_k, \ y(a_k) = 0, \ y \ne 0 \ \text{on } (a_{k+1}, a_k) \ \text{ and } \ a_k \to 0 \text{ as } k \to \infty.$$

(iii) *Let* $\alpha > 2$*. Let* γ_0 *and* γ_1 *be two real numbers,* $0 < \gamma_0 < (\alpha - 2)/2 < \gamma_1$*. There are two positive constants* c_0 *and* c_1*, and a number* $k_0 \in \mathbf{N} \cup \{0\}$*, all* *depending only on* γ_0 *and* γ_1*, such that*

$$c_0\left(\frac{1}{k+k_0}\right)^{1/\gamma_0} \le a_k \le c_1\left(\frac{1}{k-k_0}\right)^{1/\gamma_1}, \text{ for all } k\in\mathbf{N},\ k>k_0. \tag{6}$$

The proofs of all conclusions of the previous lemma are based on a very well known Sturm's comparison principle. See in the Appendix of the paper.

Next, we present some elementary properties for the sequence of consecutive zero points of y as well as for the sequence of stationary points of y, where y is any solution of the equation (4). These properties will be frequently used in the next sections.

Lemma 2.2. *Let* $y=y(x)$ *be a solution of the equation* (4)*, and let* $a_k\in I$ *be a decreasing sequence of consecutive zero points of* y *such that* $a_k \searrow 0$.

(i) *For each* $k\in\mathbf{N}$*, the solution* y *is either convex or concave on* (a_{k+1},a_k)*. Moreover, for each* $k\in\mathbf{N}$ *there is exactly one stationary point* $s_k\in(a_{k+1},a_k)$ *such that*

$$y'(s_k)=0 \text{ and } (yy')(x)>0 \text{ on } (a_{k+1},s_k), \text{ and } (yy')(x)<0 \text{ on } (s_k,a_k).$$

(ii) *There is a constant* $c_0>0$ *which does not depend on* k *such that*

$$|a_k-a_{k+1}|\le c_0 a_k^{\alpha/2} \text{ for each } k\in\mathbf{N}. \tag{7}$$

(iii) *Let* $\alpha>2$*. There are a constant* $c_0>0$ *and a number* $k_0\in\mathbf{N}$ *such that*

$$1\le\frac{a_k}{a_{k+1}}\le c_0 \text{ for each } k\in\mathbf{N},\ k\ge k_0. \tag{8}$$

If $\alpha=2$*, then the previous assertion is not true, that is to say, there is at least one solution* y *of the equation* (4) *such that the sequence* a_k *of its consecutive zero points satisfies*

$$\frac{a_k}{a_{k+1}}\to+\infty \quad\text{as}\quad k\to+\infty.$$

(iv) *Let* $\alpha>2$*. There are a positive real constant* d_0*, and a number* $k_0\in\mathbf{N}$ *such that*

$$|y'(a_{k+1})|\le d_0 a_k^{-\alpha/2}|y(s_k)| \text{ for each } k\in\mathbf{N},\ k\ge k_0. \tag{9}$$

Proof. Let $y=y(x)$ be a solution of the equation (4), and let $a_k\in I$ be a decreasing sequence of consecutive zero points of y such that $a_k\searrow 0$.

First we show that if $y>0$ (resp., $y<0$) on the interval (a_{k+1},a_k), then with the help of (4) we get $y''<0$ (resp., $y''>0$) on (a_{k+1},a_k). It proves the conclusion (i) of this lemma.

Next, according to the conclusion (i), the condition $y>0$ on (a_{k+1},a_k) implies $y'>0$ on (a_{k+1},s_k) and $y'<0$ on (s_k,a_k). Let $y>0$ on (a_{k+1},a_k) (the opposite case $y<0$ on (a_{k+1},a_k) can be considered analogously). Multiplying the equation (4) by y' and integrating it over the interval (x,s_k) for all $x\in(a_{k+1},s_k)$ we obtain

$$(y')^2(x)=\int_x^{s_k}\frac{2}{t^\alpha}\left\{\frac14+\frac{\lambda}{|\ln t|^\beta}\right\}yy'dt\ge\frac{2c_1}{s_k^\alpha}\int_x^{s_k}yy'dt=\frac{c_1}{s_k^\alpha}\big(y^2(s_k)-y^2(x)\big),$$

where the constant $c_1 > 0$ does not depend on k. Hence we have

$$y'(x) \geq \frac{c_2}{s_k^{\alpha/2}} \sqrt{y^2(s_k) - y^2(x)},$$

that is to say,

$$\frac{d}{dx} \arcsin\left(\frac{y(x)}{y(s_k)}\right) \geq \frac{c_2}{s_k^{\alpha/2}}.$$

Since $f(t) \geq g(t)$ implies $\int f(t)dt \geq \int g(t)dt$, integrating the latter inequality over $[a_{k+1}, x]$ for all $x \in (a_{k+1}, s_k)$, it gives

$$arcsin\left(\frac{y(x)}{y(s_k)}\right) \geq \frac{c_2}{s_k^{\alpha/2}}(x - a_{k+1}).$$

In particular for $x = s_k$, we get

$$s_k - a_{k+1} \leq c_3 s_k^{\alpha/2} \leq c_3 a_k^{\alpha/2} \text{ for all } k \in \mathbf{N}.$$

Here the constant c_3 doesn't depend on k. Repeating the preceding procedure, where the interval (x, s_k) is replaced by (s_k, x) for all $x \in (s_k, a_k)$, we derive

$$a_k - s_k \leq c_4 a_k^{\alpha/2} \text{ for all } k \in \mathbf{N}.$$

The sum of two previous inequalities gives us the desired inequality (7).

Next, let $\alpha > 2$. Since a_k is a decreasing sequence of consecutive zero points of y, the left inequality in (8) is automatically satisfied. Multiplying the inequality (7) by $1/a_k$, where k tends to infinity, we get

$$\left|1 - \frac{a_{k+1}}{a_k}\right| \leq c_0 a_k^{\alpha/2-1} \to 0,$$

that is to say, $(a_{k+1}/a_k) \to 1$ as $k \to \infty$. This proves the right inequality in (8). However, if $\alpha = 2$, then the sequence $a_k = e^{-e^{k\pi/\rho}}$, $k \in \mathbf{N}$, is a sequence of consecutive zero points of the function $y(x) = \sqrt{x \ln x} \sin(\rho \ln \ln \frac{1}{x})$, which is a solution of the equation (4) as $\alpha = 2$. Now,

$$\frac{a_k}{a_{k+1}} = \left(e^{e^{k\pi/\rho}}\right)^{(e^{\pi/\rho}-1)} \to +\infty \quad \text{as} \quad k \to +\infty.$$

Next, it is easy to check that there are $x_{\alpha\beta} \in I$ and a constant $c_0 > 0$ such that

$$\frac{1}{x^\alpha}\left\{\frac{1}{4} + \frac{\lambda}{|\ln x|^\beta}\right\} \leq \frac{c_0}{x^\alpha} \text{ for all } x \in (0, x_{\alpha\beta}). \tag{10}$$

Since $a_k \searrow 0$, we observe that there is $k_0 \in \mathbf{N}$ such that $a_k \in (0, x_{\alpha\beta})$ for all $k \geq k_0$. Now, for all $k \geq k_0$, multiplying the equation (4) by y', integrating the obtained equality over the interval $[x, s_k]$, where $x \in [a_{k+1}, s_k]$ and $y'(s_k) = 0$, $yy'(x) > 0$ on (a_{k+1}, s_k) (see the conclusion (i) of this lemma), and using (10), we obtain

$$(y')^2(x) = -\int_x^{s_k} ((y')^2)' dt = -2\int_x^{s_k} y''y' dt = 2\lambda \int_x^{s_k} \frac{1}{t^\alpha}\left\{\frac{1}{4} + \frac{\lambda}{|\ln t|^\beta}\right\} yy' dt$$

$$\leq 2c_0\lambda \int_x^{s_k} \frac{yy'}{t^\alpha} dt \leq \frac{2c_0\lambda}{a_{k+1}^\alpha} \int_x^{s_k} yy' dt = \frac{c_0\lambda}{a_{k+1}^\alpha}(y^2(s_k) - y^2(x)).$$

From the above inequality, in particular for $x = a_{k+1}$, we obtain

$$(y')^2(a_{k+1}) \leq \frac{c_0\lambda}{a_{k+1}^\alpha} y^2(s_k).$$

Combining this inequality with (8), the desired inequality (9) immediately follows. □

Now, supposing the existence of two linearly independent solutions of the linear problem (4), (5), we complete our discussion on the elementary properties of the sequence of consecutive zero points of any solution y of the equation (4). In Section 4 below, the following property will be used to investigate some sufficient conditions for the unrectifiable oscillations of solutions y.

Lemma 2.3. *Let $\alpha > 2$. Let us suppose that the linear problem* (4), (5) *admits the existence of two linearly independent solutions. Let $y = y(x)$ be a solution of the equation* (4)*, and let $a_k \in I$ be a decreasing sequence of consecutive zero points of y such that $a_k \searrow 0$. Then there are a real constant $c_0 > 0$, a sequence of real numbers $b_k \in (a_{k+1}, a_k)$, and a positive real number $\delta > \frac{\alpha-2}{2}$ such that*

$$|y(b_k)| \geq c_0 b_k^{\frac{\alpha}{2}-\delta} \quad \textit{for each} \quad k \in \mathbf{N}. \tag{11}$$

Proof. See in the Appendix of the paper. □

3. Rectifiable Oscillations for the Problem (4), (5)

In the preceding section, a decreasing sequence $a_k \in I$ of consecutive zero points of any solution y of the equation (4) is obtained satisfying some elementary properties given in Lemmas 2.1 and 2.2. In this section, we give a sufficient condition on the sequence a_k such that y is rectifiable oscillatory on the interval $I = (0, b)$, $b \in (0, 1)$, in the sense of Definition 1.3. At the end of this section, we give the proof of the conclusion (ii) of Theorem 1.4.

Lemma 3.1. *Let $y = y(x)$ be a real function defined on the interval I, $y \in C^2(I) \cap C(\bar{I})$ and let $a_k \in I$ be a decreasing sequence of consecutive zero points of y such that $a_k \searrow 0$. Let y be either convex or concave on (a_{k+1}, a_k) for each $k \in \mathbf{N}$. Then y is rectifiable oscillatory on I provided that the following series of real numbers is convergent,*

$$\sum_k (|y'(a_{k+1})| + |y'(a_k)|)(a_k - a_{k+1}). \tag{12}$$

This fact is clear from the geometric point of view. For the technical details concerning the proof of Lemma 3.1 see, for instance, [8].

In the next proposition we give useful example for both convergences in (12).

Proposition 3.2. *Let $\alpha \in (2,4)$. Let $y = y(x)$ be a solution of the linear problem* (4), (5), *and let $a_k \in I$ be a decreasing sequence of consecutive zero points of y such that $a_k \searrow 0$. Then the series of real numbers given in* (12) *is convergent.*

Proof. With the help of (8), (9), (7), (5), and (6), for each $k \in \mathbf{N}$, $k \geq k_0$, we obtain

$$\begin{aligned}(|y'(a_{k+1})| + |y'(a_k)|)(a_k - a_{k+1}) &\leq c_0 a_k^{-\alpha/2}\big(|y(s_k)| + |y(s_{k-1})|\big)(a_k - a_{k+1}) \\ &\leq c_1\big(|y(s_k)| + |y(s_{k-1})|\big) \leq c_2(s_k^\delta + s_{k-1}^\delta) \leq c_3 a_k^\delta \leq c_4 \left(\frac{1}{k-k_0}\right)^{\frac{\delta}{\gamma_1}},\end{aligned}$$

where γ_1 is taken to satisfy $(\alpha - 2)/2 < \gamma_1 < \delta$. It implies the convergence of the series

$$\sum_k (|y'(a_{k+1})| + |y'(a_k)|)(a_k - a_{k+1}).$$

Thus, the proposition is shown. □

Proof of the conclusion (ii) *of Theorem* 1.4. The existence of a sequence a_k of consecutive zero points of the solution y follows from the conclusion (ii) of Lemma 2.1. According to the conclusion (i) of Lemma 2.2 and Proposition 3.2, we are able to apply Lemma 3.1 to all solutions y of the linear problem (4), (5), from which immediately follows the conclusion (ii) of Theorem 1.4. □

4. Unrectifiable Oscillations for the Problem (4), (5)

In this section, we give a sufficient condition on the sequence of consecutive zero points of any solution y of the linear problem (4), (5) such that y is unrectifiable oscillatory on the interval $I = (0, b)$, $b \in (0, 1)$, in the sense of Definition 1.3. The conclusion (iii) of Theorem 1.4 will be proved at the end of this section.

Lemma 4.1. *Let $y = y(x)$ be a real function, $y \in C(\overline{I})$, and let $a_k \in I$ be a decreasing sequence of consecutive zero points of y such that $a_k \searrow 0$. If there is a sequence of real numbers $b_k \in (a_{k+1}, a_k)$ such that the series*

$$\sum_k |y(b_k)| \tag{13}$$

is divergent, then y is unrectifiable oscillatory on I.

Proof. Let $y = y(x)$ be a real function, $y \in C(\overline{I})$, and let $a_k \in I$ be a decreasing sequence of consecutive zero points of y such that $a_k \searrow 0$. We will prove the oposite claim of this lemma.

In this direction, let $G(y)$ be a rectifiable curve in $\mathbf{R}^2$ and let $b_k \in (a_{k+1}, a_k)$ be an arbitrarily given sequence of real numbers. Next, for arbitrarily given $n \in \mathbf{N}$, we can take the following dissection of the interval $\overline{I} = [0, b]$: $0 < a_n < b_n < a_{n-1} < b_{n-1} < \cdots < b$. Let us remark that from $y(a_k) = 0$, for each $k \in \mathbf{N}$,

$y \in C(\bar{I})$ and $a_k \searrow 0$, it immediately follows that $y(0) = 0$. Now, according to Definition 1.2, we have

$$\sum_{k=1}^{n} |y(b_k)| = \sum_{k=1}^{n} \sqrt{(y(b_k) - y(a_k))^2} \leq \|(0, y(0)) - (a_n, y(a_n))\|_2 + \sum_{k=1}^{n} \|(a_k, y(a_k)) - (b_k, y(b_k))\|_2 \leq \sup \sum_{i=1}^{m} \|(t_i, y(t_i)) - (t_{i-1}, y(t_{i-1}))\|_2 = \mathit{length}(G(y)),$$

where the supremum is taken over all dissections $0 = t_0 < t_1 < \cdots < t_m = b$ of the interval $\bar{I}$. Thus we have proved that if $G(y)$ is a rectifiable curve in $\mathbf{R}^2$, then the series of real numbers given in (13) is convergent for any sequence $b_k \in (a_{k+1}, a_k)$. It is the same to say: if there is a sequence $b_k \in (a_{k+1}, a_k)$ such that the series in (13) is divergent, then $G(y)$ is an unrectifiable curve in $\mathbf{R}^2$, which proves this lemma. □

Now we give a usefully example for the divergence in (13).

Proposition 4.2. *Let $\alpha \geq 4$. Suppose that the linear problem* (4), (5) *admits the existence of two linearly independent solutions. Let $y = y(x)$ be a solution of the equation* (4), *and let $a_k \in I$ be a decreasing sequence of consecutive zero points of y such that $a_k \searrow 0$. There is a sequence of real numbers $b_k \in (a_{k+1}, a_k)$ such that the series appearing in* (13) *is divergent.*

Proof. Let δ and b_k be as in Lemma 2.3. Since $\alpha \geq 4$, the inequality $\delta > (\alpha - 2)/2$ implies $\delta > 1$ and thus it is satisfied: $\frac{\alpha}{2} - \delta < \frac{\alpha-2}{2}$. Therefore, γ_0 can be choosen such that

$$\frac{\alpha}{2} - \delta \leq \gamma_0 < \frac{\alpha - 2}{2} \quad \text{and} \quad \gamma_0 > 0. \tag{14}$$

Now, with the help of (6), (8), and (11), we obtain

$$|y(b_k)| \geq c_0 b_k^{\frac{\alpha}{2}-\delta} \geq c_1 a_k^{\frac{\alpha}{2}-\delta} \geq c_2 \left(\frac{1}{k + k_0}\right)^{(\frac{\alpha}{2}-\delta)\frac{1}{\gamma_0}}.$$

From the left inequality in (14), we have $(\frac{\alpha}{2} - \delta)\frac{1}{\gamma_0} \leq 1$, which together with the previous inequality implies the desired conclusion of this proposition. □

Proof of the conclusion (iii) *of Theorem* 1.4. The existence of the sequence a_k of consecutive zero points of the solution y follows from the conclusion (ii) of Lemma 2.1. According to Proposition 4.2, we are able to apply Lemma 4.1 to all solutions y of the linear problem (4), (5), from which immediately follows the conclusion (iii) of Theorem 1.4. □

5. Rectifiable Oscillations for the Equation (1)

In this section, we will give the proof of the conclusion (i) of Theorem 1.4. In this case, for $\alpha = 2$, the assertion (8) is not true and therefore we are not able to use the argumentation from Section 3. Hence we introduce the notion of Minkowski content.

Definition 5.1. For a real function $y = y(x)$ and $\varepsilon > 0$, let $G_\varepsilon(y)$ denote the ε – neighborhood of the graph $G(y)$ and let $|G_\varepsilon(y)|$ denote the Lebesgue measure of $G_\varepsilon(y)$. *The upper Minkowski content* of $G(y)$ denoted by $M^1(G(y))$ is defined by

$$M^1(G(y)) = \limsup_{\varepsilon\to 0}(2\varepsilon)^{-1}|G_\varepsilon(y)|.$$

Referring to [1, Ch. 5.2], [3, Ch. 5.5] and [16, Ch. 9], we know that if the graph $G(y)$ is a rectifiable curve in $\mathbf{R}^2$, then $length(G(y)) = M^1(G(y))$. However, this result is not helpful here because the condition of rectifiability of the graph $G(y)$ is not presumed and it must be shown. Therefore we will use the following complementary claim.

Lemma 5.2. *Let $y = y(x)$ be a real continuous function defined on the interval $[a, b]$ and let $y|_{[a+\varepsilon,b]}$ denote the function restriction of y on the interval $[a + \varepsilon, b]$, where $\varepsilon \in (0, b - a)$. We assume that the graph $G(y|_{[a+\varepsilon,b]})$ is a rectifiable curve in $\mathbf{R}^2$ for all $\varepsilon > 0$. If $M^1(G(y)) < \infty$, then the graph $G(y)$ is also a rectifiable curve in $\mathbf{R}^2$. Moreover,*

$$\lim_{\varepsilon\to 0} length(G(y|_{[a+\varepsilon,b]})) = length(G(y)) \le M^1(G(y)).$$

The proof of this lemma is given in the author's paper [7]. Here this lemma will be used in the following way.

Corollary 5.3. *Let $a_k \in (a, b]$ be a decreasing sequence of real numbers satisfying $a_k \searrow a$ and let $a_1 = b$. Let $y = y(x)$ be a function such that $y \in C([a, b]) \cap C^2(a, b)$ and let y be either convex or concave on (a_{k+1}, a_k) for each $k \in \mathbf{N}$. If $M^1(G(y)) < \infty$, then the graph $G(y)$ is a rectifiable curve in $\mathbf{R}^2$.*

Proof. Since $a_k \searrow a$ as $k \to \infty$, for any $\varepsilon > 0$ there is $k \in \mathbf{N}$ such that

$$[a + \varepsilon, b] \subseteq [a_{k+1}, b] = \bigcup_{j=1}^{k} [a_{j+1}, a_j]. \tag{15}$$

The graph $G(y|_{[a_{j+1},a_j]})$ is a rectifiable curve in $\mathbf{R}^2$ for each $j \in \mathbf{N}$ because y is either convex or concave on (a_{j+1}, a_j). Indeed, if, for instance, y is concave on (a_{j+1}, a_j), then there is a constant $M_j > 0$ such that $|y'(x)| \le M_j = \max\{|y'(a_{j+1})|, |y'(a_j)|\}$ for all $x \in (a_{j+1}, a_j)$. Therefore

$$length(G(y|_{[a_{j+1},a_j]})) = \int_{a_{j+1}}^{a_j} \sqrt{1 + (y'(x))^2}dx \le (a_j - a_{j+1})\sqrt{1 + M_j^2}.$$

It implies that $G(y|_{\cup_{j=1}^{k}[a_{j+1},a_j]})$ is also a rectifiable curve in $\mathbf{R}^2$. Therefore, with the help of (15) and $G(y|_{[a+\varepsilon,b]}) \subseteq G(y|_{\cup_{j=1}^{k}[a_{j+1},a_j]})$, we have that for all $\varepsilon > 0$ the graph $G(y|_{[a+\varepsilon,b]})$ is a rectifiable curve in $\mathbf{R}^2$ which, together with the assumption that $M^1(G(y)) < \infty$ and with Corollary 5.3 shows us that the graph $G(y)$ is a rectifiable curve in $\mathbf{R}^2$. □

The following general result plays an essential role in the proof of rectifiable oscillations for the equation (1).

Lemma 5.4. *Let $\tilde{\omega}(t)$ be a continuously increasing and concave function on $[a,b]$ such that $\tilde{\omega}(a) = 0$. Further, let $a_1 = b$ and let $a_k \in (a,b]$ be a decreasing sequence of real numbers satisfying $a_k \searrow a$ and:*

$$\begin{cases} \text{there is } \varepsilon_1 > 0 \text{ such that for all } \varepsilon \in (0,\varepsilon_1) \text{ there is } m(\varepsilon) \in \mathbf{N} \\ \text{such that } a_{j-1} - a_j > 4\varepsilon \text{ for each } j \le m(\varepsilon), \text{ where } m(\varepsilon) \to \infty \text{ as } \varepsilon \to 0. \end{cases} \tag{16}$$

Let $y = y(x)$ be a function such that $y \in C([a,b]) \cap C^2(a,b)$ and let y be either convex or concave on (a_{k+1}, a_k), for each $k \in \mathbf{N}$, and let $-\tilde{\omega}(t) \le y(t) \le \tilde{\omega}(t)$ for all $t \in [a,b]$. Then we have

$$\begin{aligned} M^1(G(y)) \le \limsup_{\varepsilon\to 0} &\left[\frac{1}{\varepsilon}(a_{m(\varepsilon)} - a)\tilde{\omega}(a_{m(\varepsilon)}) + (\pi+4)\varepsilon\, m(\varepsilon)\right] \\ &+ (\pi+6)\left[(b-a) + \sum_{j=1}^{\infty}\tilde{\omega}(a_j)\right]. \end{aligned} \tag{17}$$

Proof. Referring to the proof of [6, Lemma 2.2], where one should assume that $\tilde{\theta}(t) = -\tilde{\omega}(t)$ (see also [9, Lemma 5.3]), we have

$$\begin{aligned} |G_\varepsilon(y|_{[a,a_1]})| \le{}& 2(a_{m(\varepsilon)} - a + 2\varepsilon)(\tilde{\omega}(a_{m(\varepsilon)}) + \varepsilon) + 2(\pi+4)\varepsilon^2 m(\varepsilon) \\ &+ 2(\pi+6)\varepsilon\sum_{j=2}^{m(\varepsilon)}[\tilde{\omega}(a_{j-1}) + (a_{j-1} - a_j)], \end{aligned} \tag{18}$$

where $a_1 = b$. Since $m(\varepsilon) \to \infty$, $a_{m(\varepsilon)} \searrow a$, and $\tilde{\omega}(a) = 0$, and since

$$\sum_{j=2}^{m(\varepsilon)}(a_{j-1}-a_j) = (a_1-a_2)+(a_2-a_3)+\cdots+(a_{m(\varepsilon)-1}-a_{m(\varepsilon)}) = a_1 - a_{m(\varepsilon)} \searrow b-a,$$

the right-hand side in the inequality (18) can be simplified in the following way:

$$\begin{aligned} |G_\varepsilon(y)| \le{}& 2(a_{m(\varepsilon)} - a)\tilde{\omega}(a_{m(\varepsilon)}) + 2(\pi+4)\varepsilon^2 m(\varepsilon) \\ &+ 2(\pi+6)\varepsilon\left[(b-a) + \sum_{j=1}^{\infty}\tilde{\omega}(a_j)\right]. \end{aligned}$$

Multiplying this inequality by $(2\varepsilon)^{-1}$ and taking $\limsup_{\varepsilon\to 0}$, we prove the inequality (17). □

As a consequence of the preceding lemma we formulate the following statement.

Corollary 5.5. *Let $\tilde{\omega}(t)$ be a continuously increasing and concave function on $[a,b]$ such that $\tilde{\omega}(a)=0$. Further, let $a_1=b$ and let $a_k\in(a,b]$ be a decreasing sequence of real numbers satisfying $a_k\searrow a$ as well as* (16). *We suppose that there are positive constants c_1, c_2 and c_3, and $\varepsilon_1>0$ satisfying*

$$\sum_{j=1}^{\infty}\tilde{\omega}(a_j)\le c_1,\quad (a_{m(\varepsilon)}-a)\tilde{\omega}(a_{m(\varepsilon)})\le c_2\varepsilon,\quad \text{and}\quad \varepsilon\, m(\varepsilon)\le c_3, \tag{19}$$

for all $\varepsilon\in(0,\varepsilon_1)$. Let $y=y(x)$ be a function such that $y\in C([a,b])\cap C^2(a,b)$ and let y be either convex or concave on (a_{k+1},a_k) for each $k\in\mathbf{N}$ and let $-\tilde{\omega}(t)\le y(t)\le\tilde{\omega}(t)$ for all $t\in[a,b]$. Then the graph $G(y)$ is a rectifiable curve in $\mathbf{R}^2$.

Proof. Putting all terms from (19) into the right-hand side of the inequality (17), we derive the existence of a constant c such that $M^1(G(y))\le c<\infty$. Now, by means of Corollary 5.3, we observe that the graph $G(y)$ is a rectifiable curve in $\mathbf{R}^2$. □

Next, we give a useful example for the function $\tilde{\omega}(t)$, for the sequence a_k and for the function y which satisfy all assumptions of Corollary 5.5.

Proposition 5.6. *Let the function $\tilde{\omega}(t)$, the sequence a_k and the function $y(x)$ be given by*

$$\begin{cases} \tilde{\omega}(t)=\sqrt{t\ln(1/t)} \text{ and } a_k=e^{-e^{k\pi/\rho}},\ k\in\mathbf{N},\ \text{where } \rho=\sqrt{\lambda-\frac14}, \\ \text{and } y(x)=\sqrt{x\ln(1/x)}\sin(\rho\ln\ln(1/x)). \end{cases} \tag{20}$$

Then the data $\tilde{\omega}(t)$, a_k and $y(x)$ satisfy all assumptions appearing in Corollary 5.5 *on the interval $[a,b]=[0,e^{-1}]$ with respect to $\varepsilon_1>0$ and $m(\varepsilon)\in\mathbf{N}$, both given by*

$$\begin{cases} \varepsilon_1=\frac{T_0}{4}e^{-e^{2\pi/\rho}},\ \text{where } T_0=1-\left(e^{-e^{\pi/\rho}}\right)^{(e^{\pi/\rho}-1)},\ \text{and} \\ \frac{\rho}{2\pi}\ln\left(\ln\frac{T_0}{4\varepsilon}\right)\le m(\varepsilon)\le\frac{\rho}{\pi}\ln\left(\ln\frac{T_0}{4\varepsilon}\right). \end{cases}$$

Proof. It is clear that $T_0>0$ and that there is $m(\varepsilon)\in\mathbf{N}$ for all $\varepsilon\in(0,\varepsilon_1)$. This is a consequence of the fact

$$\frac{\rho}{\pi}\ln\left(\ln\frac{T_0}{4\varepsilon}\right)-\frac{\rho}{2\pi}\ln\left(\ln\frac{T_0}{4\varepsilon}\right)=\frac{\rho}{2\pi}\ln\left(\ln\frac{T_0}{4\varepsilon}\right)\ge 1,\quad\text{for all } \varepsilon\in(0,\varepsilon_1).$$

Also, it is elementary to check that $(d\tilde{\omega}/dt)>0$ and $(d^2\tilde{\omega}/dt^2)<0$ on $(0,e^{-1})$ as well as that the sequence a_k is decreasing on $(0,e^{-1})$ and $a_k\searrow 0$, and that the function y is either convex or concave on (a_{k+1},a_k) for each $k\in\mathbf{N}$.

Next, we claim that the sequence a_k satisfies the assumption (16). Indeed, for all $k\in\{1,\dots,m(\varepsilon)\}$ and $\varepsilon\in(0,\varepsilon_1)$,

$$\begin{aligned} a_k-a_{k+1}&=e^{-e^{k\pi/\rho}}-e^{-e^{(k+1)\pi/\rho}}=e^{-e^{k\pi/\rho}}\left[1-\left(e^{-e^{k\pi/\rho}}\right)^{(e^{\pi/\rho}-1)}\right] \\ &\ge T_0e^{-e^{k\pi/\rho}}\ge T_0e^{-e^{m(\varepsilon)\pi/\rho}}=T_0e^{-e^{\ln\ln\frac{T_0}{4\varepsilon}}}=4\varepsilon. \end{aligned}$$

Since $\ln x < e^{c_0 x}$ for all $x > 0$ and any $c_0 > 0$, there exists a constant $c > 0$ such that

$$\varepsilon m(\varepsilon) = \varepsilon \frac{\pi}{\rho} \ln \big(\ln \frac{T_0}{4\varepsilon} \big) < c, \quad \text{for all } \varepsilon \in (0, \varepsilon_1).$$

Furthermore, it is clear that

$$\sum \tilde{\omega}(a_k) = \sum \sqrt{-a_k \ln a_k} = \sum e^{-\frac{1}{2} e^{k\pi/\rho}} e^{\frac{1}{2} k\pi/\rho} = \sum \frac{e^{\frac{1}{2} k\pi/\rho}}{e^{\frac{1}{2} e^{k\pi/\rho}}}$$
$$\leq 2 \sum \frac{e^{\frac{1}{2} k\pi/\rho}}{e^{k\pi/\rho}} = 2 \sum \left(\frac{1}{e^{\frac{1}{2}\pi/\rho}} \right)^k < \infty.$$

Also, for all $\varepsilon \in (0, \varepsilon_1)$,

$$a_{m(\varepsilon)} \tilde{\omega}(a_{m(\varepsilon)}) = (a_{m(\varepsilon)})^{3/2} \sqrt{-\ln a_{m(\varepsilon)}} = \big(e^{-e^{m(\varepsilon)\pi/\rho}} \big)^{3/2} e^{\frac{1}{2} m(\varepsilon)\pi/\rho}$$
$$= \big(e^{-e^{\ln \ln \frac{T_0}{4\varepsilon}}} \big)^{3/2} e^{\frac{1}{2} \ln \ln \frac{T_0}{4\varepsilon}} = c_1 \varepsilon^{3/2} \sqrt{\ln \frac{T_0}{4\varepsilon}} \leq c_2 \varepsilon^{3/2} \varepsilon^{-1/2} = c_2 \varepsilon.$$

Thus it is proved that the data $\tilde{\omega}(t)$, a_k and $y(x)$ given in (20) satisfy all assumptions appearing in Corollary 5.5 on the interval $[a, b] = [0, e^{-1}]$. ☐

Corollary 5.7. *Let* $y = y(x) = \sqrt{x \ln(1/x)} \sin(\rho \ln \ln(1/x))$. *Then the graph* $G(y)$ *is a rectifiable curve in* $\mathbf{R}^2$.

Proof. By Proposition 5.6, the function y satisfies all assumptions appearing in Corollary 5.5 on the interval $[a, b] = [0, e^{-1}]$. Therefore applying Corollary 5.5, we observe that the graph $G(y|_{[0,e^{-1}]})$ is a rectifiable curve in $\mathbf{R}^2$. If $\overline{I} \subseteq [0, e^{-1}]$, we have that the graph $G(y)$ is also a rectifiable curve in $\mathbf{R}^2$. In the other hand, if $[0, e^{-1}] \subseteq \overline{I}$, then since the derivative $y'(x)$ is bounded on $\overline{I} \setminus [0, e^{-1})$, it is easy to check that y is an absolutely continuous function (of bounded variation) on the interval $\overline{I} \setminus [0, e^{-1})$ and therefore the graph $G(y|_{\overline{I} \setminus [0,e^{-1})})$ is a rectifiable curve in $\mathbf{R}^2$. Since $G(y|_{[0,e^{-1}]}) \cup G(y|_{\overline{I} \setminus [0,e^{-1})})) = G(y)$, the graph $G(y)$ is also a rectifiable curve in $\mathbf{R}^2$. ☐

Lemma 5.8. *Let* $y_1 = y_1(x)$ *and* $y_2 = y_2(x)$ *be two functions defined on* $[a, b]$ *such that* $y_1, y_2 \in C([a, b]) \cap C^2(a, b)$. *If the graphs* $G(y_1)$ *and* $G(y_2)$ *are two rectifiable curves in* $\mathbf{R}^2$, *then for any* $c_1, c_2 \in \mathbf{R}$ *the graph* $G(c_1 y_1 + c_2 y_2)$ *is also a rectifiable curve in* $\mathbf{R}^2$.

Proof. Obviously, there are two positive constants c_3 and c_4 depending only on c_1 and c_2 such that

$$\int_a^b \sqrt{1 + [(c_1 y_1 + c_2 y_2)']^2} dx \leq \int_a^b \sqrt{1 + 2c_1^2 (y_1')^2 + 2c_2^2 (y_2')^2} dx$$
$$\leq c_3 \int_a^b \sqrt{1 + (y_1')^2} dx + c_4 \int_a^b \sqrt{1 + (y_2')^2} dx < \infty.$$

Since $y_1, y_2 \in C([a,b]) \cap C^2(a,b)$, we immediately obtain that the graph $G(c_1y_1 + c_2y_2)$ is a rectifiable curve in $\mathbf{R}^2$. □

Proof of the conclusion (i) *of Theorem* 1.4. From Corollary 5.7, we have that the graph $G(y_2)$ of the function $y_2(x) = \sqrt{x\ln(1/x)}\sin(\rho\ln\ln(1/x))$ is a rectifiable curve in $\mathbf{R}^2$. Analogously to Proposition 5.6 and Corollary 5.7, we can show that the graph $G(y_1)$ of the function $y_1(x) = \sqrt{x\ln(1/x)}\cos(\rho\ln\ln(1/x))$ is a rectifiable curve in $\mathbf{R}^2$. Now, let $y = c_1y_1 + c_2y_2$ be a solution of the equation (1) given in (2). By Lemma 5.8, we conclude that the graph $G(y)$ is also a rectifiable curve in $\mathbf{R}^2$. This implies together with the conclusion (ii) of Lemma 2.1 that y is rectifiable oscillatory on the interval I. According to Definition 1.3, the equation (1) is rectifiable oscillatory on the interval I. □

6. Appendix

At the end of the paper, we will prove some elementary results which were presented in Section 2. The proofs are based on the well known Sturm's comparison principle, see, for instance, [15].

Proof of the conclusion (ii) *of Lemma* 2.1. Let y be a solution of the equation (4) and let $J_0 \subseteq I$ be an open interval, $0 \in \overline{J_0}$.

If $\alpha = 2$, we have that $\beta = 2$ and $\lambda > 1/4$ and so the equation (4) becomes the Riemann–Weber version of the Euler differential equation, that is to say, the equation (1). Therefore, in this case, we need to show that the equation (1) satisfies the conclusion (ii) of Lemma 2.1. In this context, let us remark that all solutions of (1) have explicit forms given by (2). One of them, $y_1 = y_1(x) = \sqrt{x\ln x}\sin(\rho\ln\ln\frac{1}{x})$ has a sequence of all zero points given by $a_k = e^{-e^{k\pi/\rho}}$, $k \in \mathbf{N}$. Since $a_k \searrow 0$, the function y_1 has an infinite number of zero points on J_0. Moreover, the same is true for any other solution y of (1) because of the Sturm's comparison principle. Thus, it is shown that each solution of the equation (1) has an infinite number of zero points on J_0.

If $\alpha > 2$, then there is $x_0 \in I$ which depends on the parameters α, β and λ such that:

$$\frac{1}{x^2}\left\{\frac{1}{4} + \frac{1/3}{|\ln x|^2}\right\} < \frac{1}{x^\alpha}\left\{\frac{1}{4} + \frac{\lambda}{|\ln x|^\beta}\right\}, \text{ for all } x \in (0, x_0).$$

Now, by Sturm's comparison principle applied to the equations (1) for $\lambda = 1/3$ and (4) for $\lambda > 0$, the solution y of the equation (4) also has an infinite number of zero points on J_0.

Thus, in all cases for α, the conclusion (ii) of Lemma 2.1 is proved. □

Proof of the conclusion (iii) *of Lemma* 2.1. Let y be a solution of the equation (4) and let $a_k \in I$ be a decreasing sequence of consecutive zero points of y such that $a_k \searrow 0$. It easy to check that for any $\gamma > (\alpha - 2)/2$, there is $x_0 \in I$ depending only on α and γ such that

$$\frac{1}{x^\alpha}\left\{\frac{1}{4} + \frac{\lambda}{|\ln x|^\beta}\right\} < \frac{\gamma^2}{x^{2\gamma+2}} + \frac{1-\gamma^2}{4x^2} \quad \text{for all } x \in (0, x_0). \tag{21}$$

Let us remark that the sequence $b_k = (1/k\pi)^{1/\gamma}$, $k \in \mathbf{N}$, contains all zero points of the function $z = z(x) = x^{(\gamma+1)/2}\sin(1/x^{\gamma})$ satisfying the equation

$$z'' + \left(\frac{\gamma^2}{x^{2\gamma+2}} + \frac{1-\gamma^2}{4x^2}\right) z = 0. \tag{22}$$

Since $a_k \searrow 0$ and $b_k \searrow 0$ as $k \to \infty$, it is clear that there is $k_0 \in \mathbf{N}$ such that $a_k \in (0, x_0)$ and $b_k \in (0, x_0)$ for all $k > k_0$. Now, according to (21) and Sturm's comparison principle applied to the equations (4) and (22), we obtain that between two consecutive zero points a_{k_0+1} and a_{k_0} of y there is at least one b_k, denoted by b_{i_0}, $i_0 \in \mathbf{N}$, such that $a_{k_0+1} < b_{i_0} < a_{k_0}$. Repeating this procedure for all pairs of consecutive zero points a_{k_0+j} and a_{k_0+j-1}, we get $a_{k_0+j} < b_{i_0+j-1}$ for each $j \geq 1$. Putting $k = k_0 + j$, we derive

$$a_k < b_{k-k_0+i_0-1} \leq b_{k-k_0} = \left(\frac{1}{(k-k_0)\pi}\right)^{1/\gamma} \quad \text{for each } k > k_0,$$

which proves the right-hand inequality in (6). The left-hand inequality in (6) follows analogously. □

Proof of Lemma 2.3. It is known that for any two linearly independent solutions y and z of the equation (4), there is a positive constant c such that its Wronskian

$$W(y,z)(x) = (y'z - z'y)(x) = c \neq 0, \text{ for all } x \in I. \tag{23}$$

Next, let y be a solution of the equation (4), and let $a_k \in I$ be a decreasing sequence of consecutive zero points of y such that $a_k \searrow 0$. Also, let y_1 and y_2 be two linearly independent solutions of the problem (4), (5). It is clear that the solution y is linearly independent with respect either to y_1 or to y_2. For instance, let y and y_1 be two linearly independent solutions.

By Sturm's comparison principle, there is a sequence $b_k \in (a_{k+1}, a_k)$ such that $y_1(b_k) = 0$ for each $k \in \mathbf{N}$. Now, from the equality (23) applied to y and y_1 for $x = b_k$ we obtain

$$|y_1'(b_k)||y(b_k)| = |y'(b_k)y_1(b_k) - y_1'(b_k)y(b_k)| = c \neq 0 \text{ for each } k \in \mathbf{N}. \tag{24}$$

On the other hand, by (5), (8) and (9) applied to the solutions y_1 and to the sequence b_k of its zero points, where $s_k \in (b_{k+1}, b_k)$ is a sequence of stationary points of y_1 for each $k \geq k_0$,

$$\begin{aligned} |y_1'(b_{k+1})| &\leq d_0 b_k^{-\alpha/2}|y_1(s_k)| \leq d_1 b_k^{-\alpha/2} s_k^{\delta} \leq d_1 b_k^{-\alpha/2} b_k^{\delta} \\ &= d_1 b_k^{\delta-\alpha/2} \leq d_2 b_{k+1}^{\delta-\alpha/2}. \end{aligned} \tag{25}$$

Combining (24) and (25), we have that the sequence $b_k \in (a_{k+1}, a_k)$ satisfies

$$|y(b_k)| = \frac{c}{|y_1'(b_k)|} \geq \frac{c}{d_3 b_k^{\delta-\alpha/2}} \geq c_0 b_k^{\alpha/2-\delta} \quad \text{for each } k \in \mathbf{N}.$$

This proves the desired inequality (11). □

Remark (added in proofs). In the meantime, the problem of rectifiable oscillations have been enlarged to some linear and nonlinear second-order differential equations of different types, see: M. K. Kwong, M. Pašić, and J. S. W. Wong,

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