The Derivation of a Drag Coefficient Formula from Velocity-Voidage Correlations

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Abstract
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keywords: multiphase flow, fluid-solids drag, minimum fluidization, Richardson-Zaki equation

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Introduction

An important constitutive relation in any multiphase flow model is the formula for the fluid-particle drag force, which is often expressed in following form (eq. 2.9 in [1]):

$$F = \beta \, (v_f - v_s) \tag{1}$$

The factor β can be expressed in terms of a drag coefficient as

$$\beta = \frac{3}{4} \, \frac{C_D \, \varepsilon (1-\varepsilon) \rho_f}{d_p} \left| v_f - v_s \right| \tag{2}$$

The drag coefficient C_D is only a function of the particle Reynolds number and the void fraction and must be determined from experimental data.

One method is to derive a formula for C_D from empirical correlations for the pressure drop in packed beds. For example, Gidaspow [1] uses the Ergun equation [2], which is based on

$$C_D = \frac{200\ (1-\varepsilon)}{\varepsilon^2\ \mathrm{Re}} + \frac{7}{3\ \varepsilon} \tag{3}$$

pressure-drop data for packed beds with void fractions in the range of 0.4-0.6:

For values of the void fraction greater than 0.6, the error in the value of C_D calculated from the above equation increases with increasing void fraction. To correct this problem, Gidaspow [1] uses a Wen and Yu [3] correlation for void fractions greater than 0.8:

$$CD = \begin{cases} \dfrac{24}{\varepsilon\ \mathrm{Re}} \left(1 + 0.15(\varepsilon\ \mathrm{Re})^{0.687}\right) \varepsilon^{-2.65} & (\varepsilon\ \mathrm{Re}) < 1000 \\ 0.44\ \varepsilon^{-2.65} & (\varepsilon\ \mathrm{Re}) \geq 1000 \end{cases} \tag{4}$$

But such an approach makes C_D discontinuous at the switching void fraction of 0.8, with the magnitude of the discontinuity increasing with the Reynolds number.

An alternative method is to derive a formula for C_D from the Richardson-Zaki equation [4], which expresses the ratio of the terminal settling velocity of a multiparticle system to that of an isolated particle as a function of the void fraction:

$$V_r = \frac{V_t}{V_{ts}} = \varepsilon^{n-1} \tag{5}$$

The exponent is $n-1$, rather than n as usually written, because here we express the terminal velocity of the multiparticle system as the interstitial, rather than the superficial, velocity. The Richardson-Zaki exponent is given by

$$n = \begin{cases} 4.65 & Re_{ts} < 0.2 \\ 4.4\ Re_{ts}^{-0.03} & 0.2 > Re_{ts} < 1 \\ 4.4\ Re_{ts}^{-0.1} & 1 > Re_{ts} < 500 \\ 2.4 & Re_{ts} > 500 \end{cases} \tag{6}$$

Sinclair and Jackson [5], for example, uses the following formula based on the Richardson-Zaki equation

$$\beta = \frac{\rho_s\ g\ (1-\varepsilon)}{V_{ts}\ \varepsilon^n} \tag{7}$$

The difficulty with the above formula is that it depends upon the factor $\rho_s\ g$. The presence of such a factor is not justified because the drag force experienced by a particle placed in a flow field with a given Reynolds number and void fraction would not depend upon the particle density or the gravitational acceleration. The V_{ts} in the denominator of the formula, however, is proportional to $(\rho_s - \rho_g)\ g$ at Reynolds numbers less than 0.4 [6]. Therefore, the factor $\rho_s\ g$ gets cancelled at low Reynolds numbers (and for negligible gas density), making the formula acceptable for low Reynolds numbers. At higher Reynolds numbers, however, a complete cancellation does not occur. For Reynolds numbers greater than 500, the formula retains an undesirable dependence on a factor of $\sqrt{\rho_s \ g}$.

Another example of the use of the Richardson-Zaki equation is the following formula derived by Gibilaro et al. [7]:

$$C_D = C_{Ds}(Re_{ts}) \left(\frac{\varepsilon \ |v_f - v_s|}{V_{ts}} \right)^{\frac{(4.8-2n)}{n}} \varepsilon^{-3.8} \qquad (8)$$

To derive the above expression, they assumed that C_D has a voidage dependency of $\varepsilon^{-3.8}$. There is no need for such an assumption, as will be shown in this paper. Also the above formula incorrectly depends upon V_{ts} and, hence, upon the particle density and the gravitational constant.

The objective of this paper is to derive a formula for the multiparticle drag coefficient C_D from a Richardson-Zaki type velocity-voidage correlation and a formula for the single-particle drag coefficient. The formula will be based on two parameters only, the Reynolds number and the void fraction.

Multiparticle drag coefficient

The single-particle drag coefficient is defined as

$$F_s = C_{Ds} \ \frac{\pi \ d_p^2}{4} \frac{\rho_f \ (v_f - v_s)^2}{2} \qquad (9)$$

From a dimensional analysis it can be shown that C_{Ds} is only a function of the Reynolds number Re_s. Correlations for C_{Ds} have been developed from experimental data and theoretical analysis and are well-established, for example see [8]. Here we use the following simple formula given by Dalla Valle [9]:

$$C_{Ds} = \left[0.63 + \frac{4.8}{\sqrt{Re_s}} \right]^2 \tag{10}$$

Under terminal settling conditions, the drag force on a particle is equal to its buoyant weight, and the momentum balance is given by

$$C_{Ds} \frac{\pi\ d_p^2}{4} \frac{\rho_f\ V_{ts}^2}{2} = \frac{\pi\ d_p^3}{6} (\rho_s - \rho_f)\ g \tag{11}$$

which can be written in a dimensionless form as

$$\frac{3}{4} C_{Ds}\ Re_{ts}^2 = Ar \tag{12}$$

The multiparticle drag coefficient C_D is defined in a similar manner, as shown by eq. (2). C_D is a function of the void fraction in addition to the Reynolds number. Under terminal settling conditions, the momentum balance is given by

$$\frac{3}{4} C_D\ Re_t^2 = Ar \tag{13}$$

which, for example, is a dimensionless form of eq. 2.17 in [1] with the friction and the solids pressure terms ignored.

From eqs. (12) and (13) we get

$$C_D(Re_t, \varepsilon) = \left(\frac{Re_{ts}}{Re_t}\right)^2 C_{Ds}(Re_{ts}) \tag{14}$$

Although eqs. (12) and (13) were written for a particular value of the magnitude of the drag force -- the buoyant weight of a particle -- the magnitude of the drag force does not explicitly appear in eq. (14). Therefore, we claim that eq. (14) can be used for calculating any magnitude of the drag force, or equivalently C_D, by dropping the subscript t for the terminal settling condition. This amounts to changing the question from "What is the Re_t of a multiparticle system of void fraction ε, consisting of particles of known Re_{ts}?" to "What is the Re_{ts} of certain (fictitious) particles that will be under terminal settling conditions for the given ε and Re?" The validity of the method, therefore, hinges only on the uniqueness of the inversion of the velocity voidage equation

$$V_r(Re_{ts}, \varepsilon) = V_r\left(\frac{Re_t}{V_r}, \varepsilon\right) = V_r(Re_t, \varepsilon) \tag{15}$$

which is demonstrated for the Richardson-Zaki [4] and the Garside and Al-Dibouni [10] equations in this study. Thus, replacing Re_t by Re and Re_{ts} by Re_s and substituting

$$Re_s = Re \; / \; V_r \tag{16}$$

in eq. (14), we get

$$C_D(\mathrm{Re}, \varepsilon) = \frac{C_{Ds}(\mathrm{Re}/V_r)}{V_r^2} \tag{17}$$

which is a formula for calculating C_D from the velocity-voidage correlation V_r and the single-particle drag coefficient C_{Ds} and, as desired, Re and ε are the only parameters needed.

To determine C_D from the Richardson-Zaki equation [4] with this method, a numerical procedure, as shown in Table I, is required. First, V_r is calculated iteratively, as shown by steps

Table 1 Calculation of C_D from Richardson-Zaki Equation

1. Guess a value for V_r, say 1.
2. Calculate Re_s from eq. (16).
3. Calculate n from eq. (6).
4. Calculate V_r from eq. (5).
5. Check for convergence. If not converged, update V_r and go to step 2.
6. Calculate C_D from eq. (17) and eq. (10).

2 through 5 in the table. A successive substitution method converges to a unique solution for V_r within a tolerance of 10^{-5} usually under 10 iterations. After obtaining a converged value for V_r, C_D can be calculated from eq. (17) and a suitable formula for C_{Ds}, e.g., eq. (10).

An analytical formula for V_r and, hence, for C_D can be derived, from the following velocity-voidage correlation proposed by Garside and Al-Dibouni [10]:

$$\frac{V_r - A}{B - V_r} = 0.06\,Re_s \tag{18}$$

where

$$A = \varepsilon^{4.14} \tag{19}$$

and

$$B = \begin{cases} 0.8\,\varepsilon^{1.28} & \varepsilon \le 0.85 \\ \varepsilon^{2.65} & \varepsilon > 0.85 \end{cases} \tag{20}$$

Substituting $Re_s = Re/V_r$ in eq. (18) and solving for V_r we get

$$V_r = 0.5\left[A - 0.06Re + \sqrt{0.0036\,Re^2 + 0.12Re(2B - A) + A^2}\right] \tag{21}$$

Eqs (10), (17), and (21) give the desired formula for C_D.

Figure 1 shows a plot of C_D as a function of Re for three different values of ε. C_D calculated from the Garside and Al-Dibouni equation, the Richardson and Zaki equation, and the Ergun equation are shown. The Garside and Al-Dibouni equation is always in reasonable agreement with the Richardson and Zaki equation. At a void fraction of 0.6, all three of the correlations are in good agreement. However, as mentioned, the Ergun equation deviates significantly from the other two equations at a void fraction of 0.9.

Minimum fluidization velocity

From the Garside and Al-Dibouni formula for C_D, an explicit formula for the minimum fluidization velocity is derived as follows. Substituting eq. (10) in eq. (12) and solving for the Reynolds number we get

$$Re_{ts} = \left[\frac{\sqrt{4.8^2 + 2.52 \sqrt{\frac{4Ar}{3}}} - 4.8}{1.26} \right]^2 \tag{22}$$

which is the Reynolds number based on the terminal settling velocity of a single-particle. Since the right-hand side of eq. (22) is only a function of Ar, we will call it Ar^*. Substituting eq. (22) in eq. (18) and solving for V_r we get

$$V_r = \left[\frac{A + 0.06\ B\ Ar^*}{1 + 0.06\ Ar^*} \right] \tag{23}$$

Now using the identity $Re_t = Re_{ts}\ V_r$ and eq. (22), we get the following formula for the Reynolds number at minimum fluidization condition:

$$Re_t = Ar^* \left[\frac{A + 0.06\ B\ Ar^*}{1 + 0.06\ Ar^*} \right] \tag{24}$$

The Reynolds number calculated from eq. (24) is compared with experimental data in Fig. 2. The data are for spherical particles or sand, covering a wide range of conditions usually encountered in fluidized beds: void fraction, 0.36 - 0.48; temperature, 298 - 1123 K; pressure, 100 - 3500 kPa; particle diameter, 125 - 6350 μm; particle density, 1100 - 7840 kg/m3. Four data points are for a water fluidized bed; all others are for air or nitrogen fluidized beds. The agreement between the theory and the experiment is very good for Reynolds numbers larger than 10. For smaller Reynolds numbers, however, the theory systematically over predicts the Reynolds number.

Summary

Based on a correlation proposed by Garside and Al-Dibouni [10], an analytical formula for the multiparticle drag coefficient is

$$C_D(Re,\ \varepsilon) = \left(\frac{0.63}{V_r} + \frac{4.8}{\sqrt{V_r\ Re}} \right)^2 \tag{25}$$

where V_r is given by

$$V_r = 0.5 \left[A - \ 0.06 Re\ +\ \sqrt{0.0036\, Re^2\ +\ 0.12 Re(2B - A)\ +\ A^2} \right] \tag{26}$$

$$A = \varepsilon^{4.14} \quad (27)$$

$$B = \begin{cases} 0.8\ \varepsilon^{1.28} & \varepsilon \le 0.85 \\ \varepsilon^{2.65} & \varepsilon > 0.85 \end{cases} \quad (28)$$

The above formula compares favorably with the Ergun equation [2] in the void fraction range of 0.5-0.6 and correctly reduces to a formula for the single-particle drag coefficient, when the void fraction becomes 1.0. The derivative $\frac{\partial\ C_D}{\partial\ \mathrm{Re}}$ is a continuous function of Re. C_D and its derivative with respect to ε are continuous, except at $\varepsilon = 0.85$ where C_D is continuous (rounded off to three significant figures), but its derivative is discontinuous. The minimum fluidization velocities calculated from the formula compares well with experimental data, especially for Reynolds numbers greater than 10.

LIST OF SYMBOLS

A	A function of void fraction defined by eq. (19)
Ar	Archimedes number, $d_p^3\ \rho_f\ (\rho_s - \rho_f)\ g / \mu_f^2$
Ar^*	A function of Ar defined by the right hand side of eq. (22)
B	A function of void fraction defined by eq. (20)
C_D	Multiparticle drag coefficient
C_{Ds}	Single-particle drag coefficient
d_p	Particle diameter, m
F	The drag force per unit volume in a two-phase system, N/m^3

F_s The drag force on an isolated particle, N

g gravitational acceleration, m/s^2

Re Reynolds number for a multiparticle system, $d_p\ \rho_f\ |v_f - v_s|\ /\ \mu_f$

Re_s Reynolds number for a single-particle, $d_p\ \rho_f\ |v_f - v_s|\ /\ \mu_f$

Re_t Reynolds number for a multiparticle system under terminal settling conditions, $d_p\ \rho_f\ V_t\ /\ \mu_f$

Re_{ts} Reynolds number for a single-particle under terminal settling conditions, $d_p\ \rho_f\ V_{ts}\ /\ \mu_f$

v_f Fluid velocity (interstitial), m/s

v_s Solids velocity, m/s

V_r The ratio of the terminal settling velocity of a multiparticle system to that of an isolated single particle

V_t $|v_f - v_s|$ for a multiparticle system under terminal settling conditions, m/s

V_{ts} $|v_f - v_s|$ for an isolated, single particle under terminal settling conditions, m/s

Greek symbols

β A coefficient defined by eq. (1), $kg/(m^3 \cdot s)$

ε Void fraction

μ_f Fluid viscosity, $Pa \cdot s$

ρ_f Fluid density, kg/m^3

ρ_s Solids density, kg/m^3

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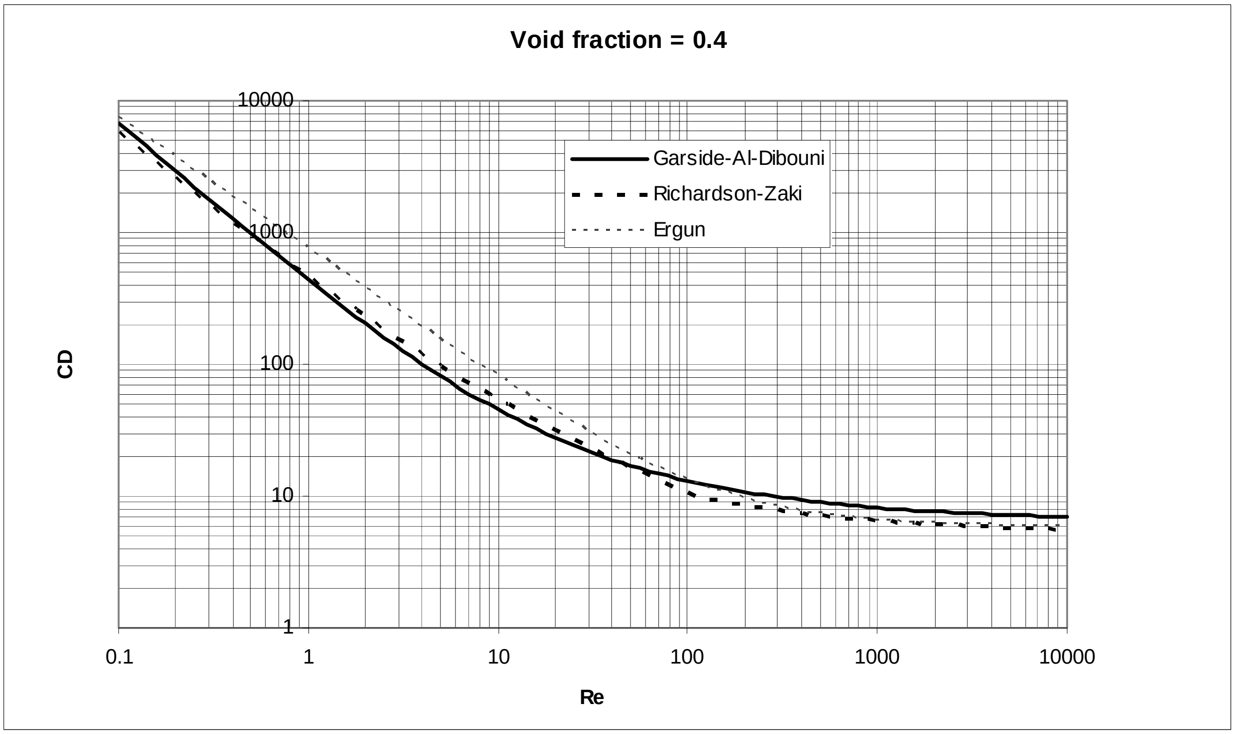

Figure 1a

Figure 1. Comparison of multiparticle drag coefficients

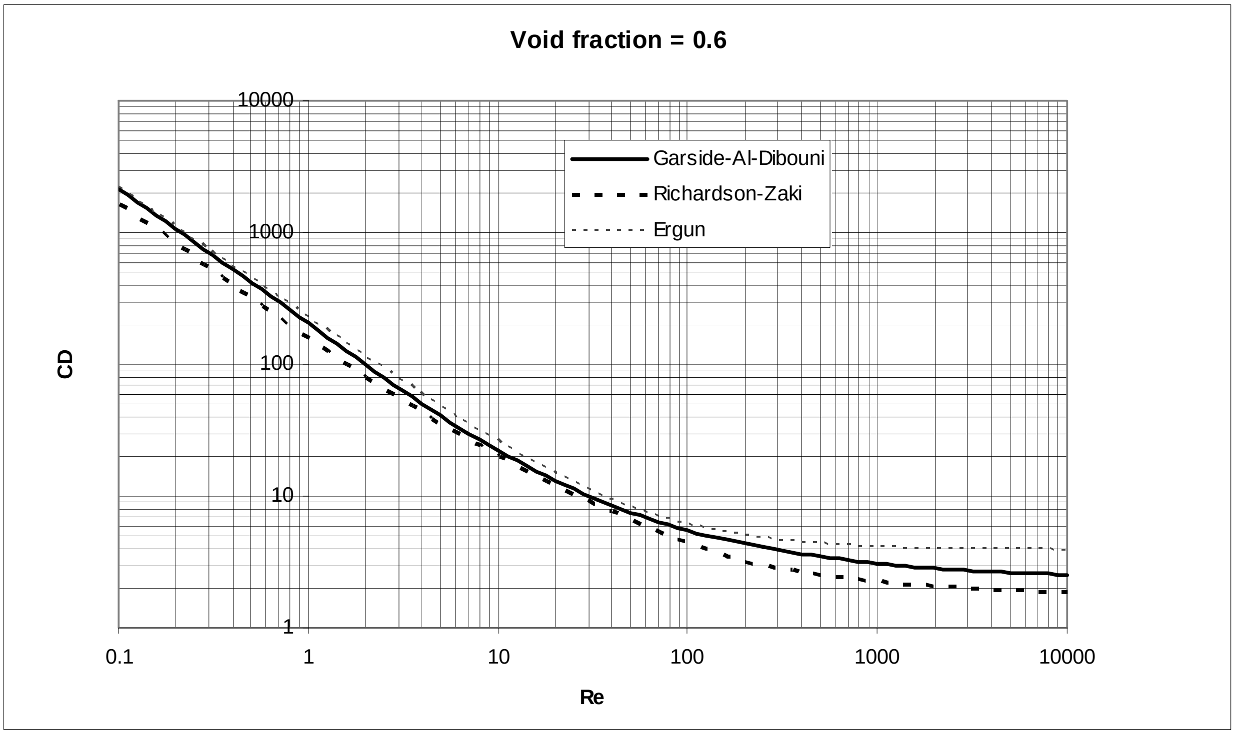

Figure 1b

Figure 1. Comparison of multiparticle drag coefficients

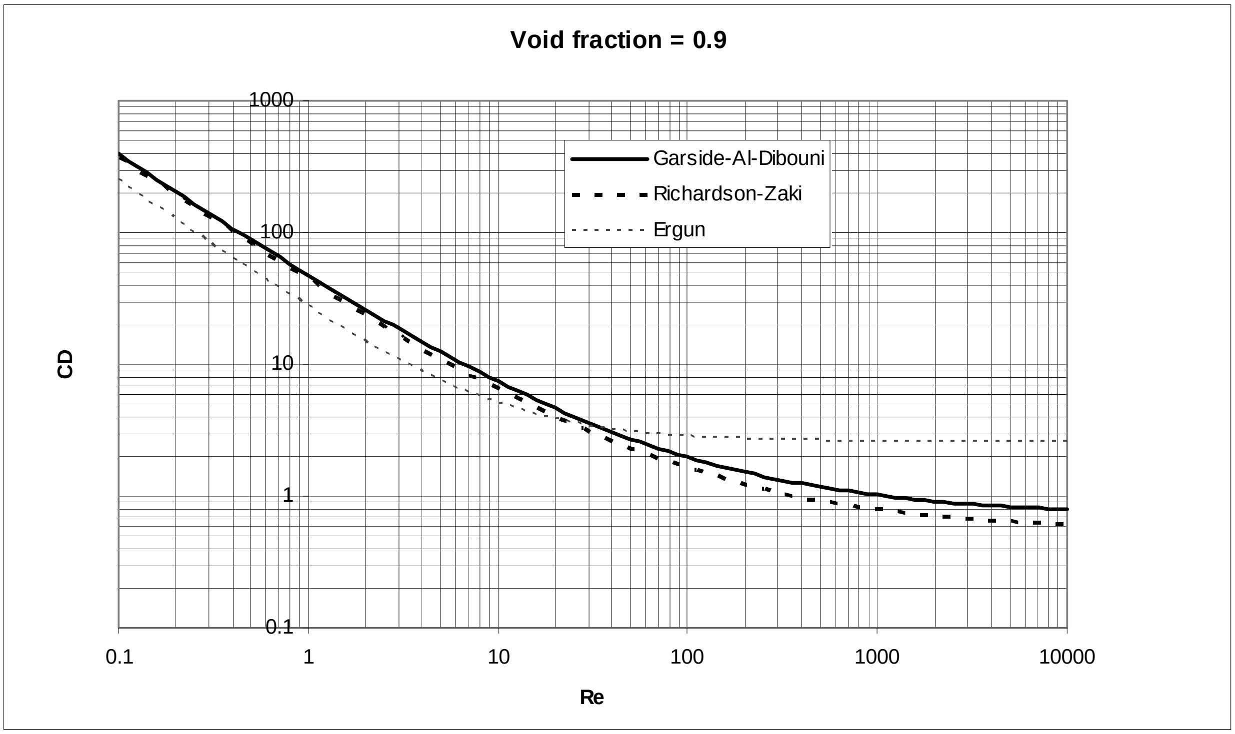

Figure 1c

Figure 1. Comparison of multiparticle drag coefficients

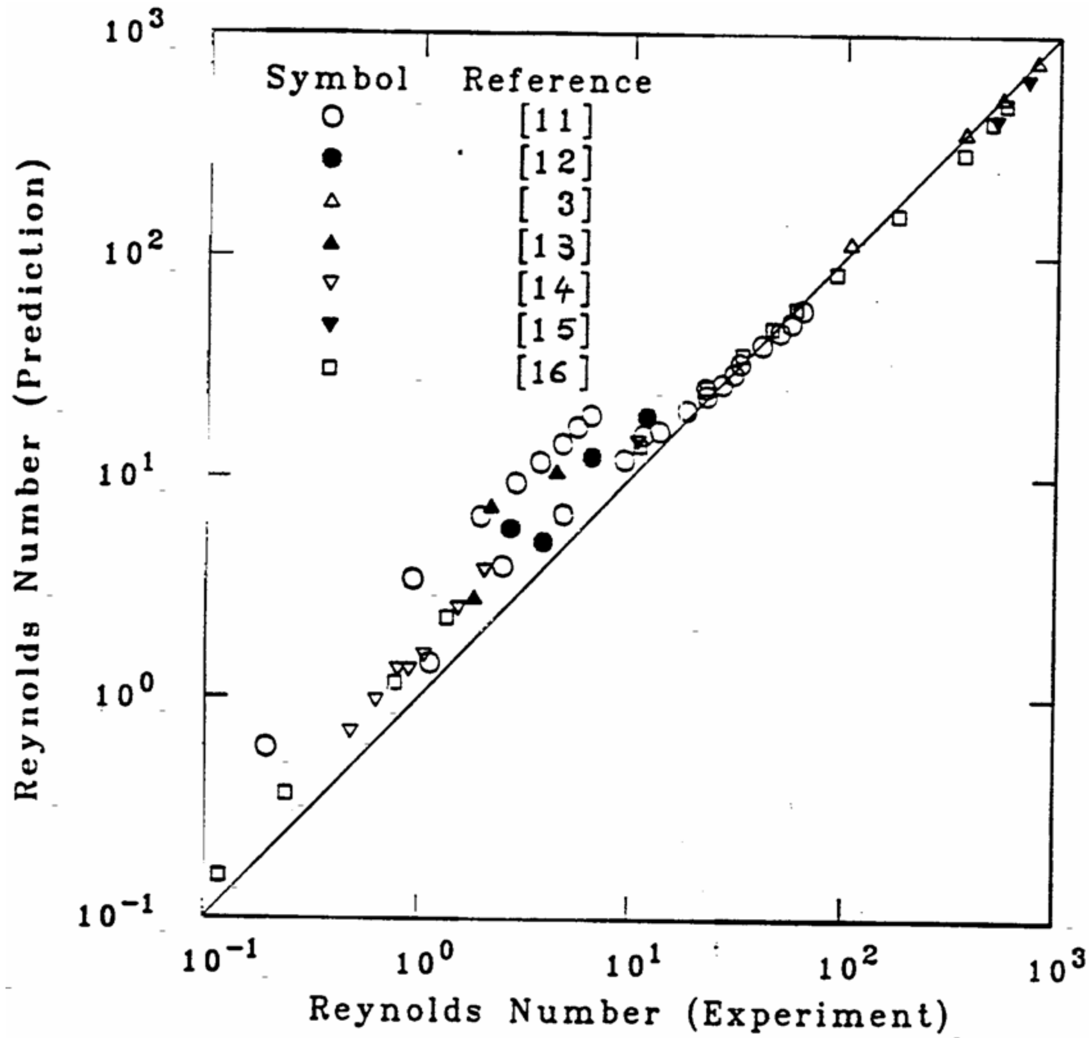

Figure 2 Experimental and predicted Reynolds numbers at minimum fluidization conditions.