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Dept. of Computer Science and Artificial Intelligence
E.T.S. de Ingeniería Informática. University of Granada
E-18071 Granada, SPAIN
Fax: +34 958 243317

A Consistency Based Procedure to Estimate Missing Pairwise Preference Values

S. Alonso [*] *F. Chiclana* [†] *F. Herrera*[*] *E. Herrera-Viedma*[*] *J. Alcalá-Fdez*[*]

Abstract

In this paper, we present a procedure to estimate missing preference values when dealing with pairwise comparison and heterogeneous information. The procedure attempts to estimate the missing information in an expert's incomplete fuzzy preference relation using only the preference values provided by that particular expert. Our procedure to estimate missing values can be applied to incomplete fuzzy, multiplicative, interval-valued and linguistic preference relations. Clearly, it would be desirable to maintain experts' consistency levels. We make use of the additive consistency property to measure the level of consistency, and use it to guide the procedure in the estimation of the missing values. Finally, conditions that guarantee the success of our procedure in the estimation of all the missing values of an incomplete preference relation are provided.

Keywords: Preference relations, missing values, incomplete information, additive consistency, multiplicative consistency.

1 Introduction

Decision-making procedures, which try to find the best alternative(s) from a feasible set, are increasingly being used in various different fields for evaluation, selection and prioritisation purposes. Obviously, the the comparison of different alternative actions according to their desirability in decision problems, in many cases, cannot be done using a single criterion or one person. Indeed, in the majority of decision making problems, procedures have been established to combine opinions about alternatives related to different points of view [5, 6]. These procedures are based on pair comparisons, in the sense that processes are linked to some degree of credibility of preference of one alternative over another. According to the nature of the information expressed for every pair of alternatives many different representation formats can be

[*]Sergio Alonso, Francisco Herrera, Enrique Herrera-Viedma and Jesús Alcalá-Fdez are with the Department of Computer Science and Artificial Intelligence, University of Granada, 18071 Granada, Spain. E-mail: {salonso, herrera, viedma, jalcala}@decsai.ugr.es

[†]Francisco Chiclana is with the Centre for Computational Intelligence, School of Computing, De Montfort University, Leicester LE1 9BH, UK. E-mail: chiclana@dmu.ac.uk

used to express preferences: *fuzzy preference relations* [2,7,14,19,15,25], *multiplicative preference relations* [11,17,20,21,22], *interval-valued preference relations* [4,12,23,27] and *linguistic preference relations* [8,28].

Since each expert is characterised by their own personal background and experience of the problem to be solved, experts' opinions may differ substantially (there are plenty of educational and cultural factors that influence an expert's preferences). This diversity of experts could lead to situations where some of them would not be able to efficiently express any kind of preference degree between two or more of the available options. Indeed, this may be due to an expert not possessing a precise or sufficient level of knowledge of part of the problem, or because that expert is unable to discriminate the degree to which some options are better than others. In these situations such an expert is forced to provide an *incomplete fuzzy preference relation* [26].

Usual procedures for group decision-making problems correct this lack of knowledge of a particular expert using the information provided by the rest of the experts together with some aggregation procedures [16]. These approaches have several disadvantages. Among them we can cite:

- The requirement of multiple experts in order to learn the missing value of a particular one.
- These procedures normally do not take into account the differences between experts' preferences, which could lead to the estimation of a missing value that would not naturally be compatible with the rest of the preference values given by that expert.
- Some of these missing information-retrieval procedures are interactive, that is, they need experts to collaborate in "real time", an option which is not always possible.

In this paper, we put forward a general procedure which attempts to find out the missing information in any of the above formats of incomplete preference relations: fuzzy, multiplicative, interval-valued and linguistic. Our proposal is quite different to the above procedures because the estimation of missing values in an expert's incomplete preference relation is done using only the preference values provided by that particular expert. By doing this, we assure that the reconstruction of the incomplete fuzzy preference relation is compatible with the rest of the information provided by that expert. In fact, the procedure we propose in this paper is guided by the expert's consistency which is measured taking into account only the provided preference values. Thus, an important objective in the design of our procedure is to maintain experts' consistency levels. In particular, we use the additive consistency property of a fuzzy preference relation [13], and its corresponding concept in the other preference relation formats, to define a consistency measure of the expert's information [1].

In order to do this, the paper is set out as follows. Section 2 presents the definitions and concepts on the four types of incomplete preference relations needed throughout the paper. Section 3 deals with the transitivity condition, and the consistency measure, for each one of the four different preference relations, to be used to guide the procedure in the estimation of the missing values. Both the estimation procedure, details of its implementation and examples of its application for each preference relation format are studied in Section 4. In Section 5, a brief discussion of the possible scenarios in which the procedure is successful in estimating all the missing values and sufficient conditions to guarantee this are provided. We also describes how to implement the additive reciprocity property of preference relation in the estimation procedure for those cases when all missing values cannot be estimated. Finally, our concluding remarks are pointed out in Section 6.

2 Preference Relations

The intensity of preference between any two alternatives of a set of feasible ones $\mathrm{X} = \{\mathrm{x}_1, ..., \mathrm{x}_n\}$, $(n \geq 2)$ may be adequately represented by means of a preference relation. Different types of preference relations can be used according to the domain used to evaluate the intensity of preference. This is expressed in the following definition:

Definition 1. *A preference relation P on a set of alternatives X is characterized by a function $\mu_P\colon X \times X \longrightarrow D$, where D is the domain of representation of preference degrees.*

When cardinality of X is small, the preference relation may be conveniently represented by an $n \times n$ matrix $P = (p_{ij})$, being $p_{ij} = \mu_P(\mathrm{x}_i, \mathrm{x}_j) \;\; \forall i, j \in \{1, \ldots, n\}$.

Usual decision-making procedures assume that experts are capable of providing preference degrees between any pair of possible alternatives. However, this may not be always possible, which makes missing information a problem that has to be dealt with. A missing value in a fuzzy preference relation is not equivalent to a lack of preference of one alternative over another. A missing value can be the result of the incapacity of an expert to quantify the degree of preference of one alternative over another, in which case he/she may decide not to 'guess' it to maintain the consistency of the values already provided. It must be clear that when an expert is not able to express a particular value p_{ij}, because he doesn't have a clear idea of how better is the alternative x_i over the alternative x_j, this does not mean that he/she prefers both options with the same intensity.

In order to model these situations, in the following we introduce the concept of incomplete preference relation:

Definition 2. *A function $f\colon X \longrightarrow Y$ is* partial *when not every element in the set X necessarily maps to an element in the set Y. When every element from the set X maps to one element of the set Y then we have a* total *function.*

Definition 3. *A preference relation P on a set of alternatives X with a* partial *membership function is an* incomplete preference relation

As per this definition, a preference relation is complete when its membership function is a total one. Clearly, the usual definition of a preference relation (*definition 1*) includes both definitions of complete and incomplete preference relations. However, as there is no risk of confusion between a complete and an incomplete preference relation, in this paper we refer to the first type as simply preference relations.

In the following we introduce four different types of incomplete preference relations: incomplete fuzzy preference relations, incomplete multiplicative preference relations, incomplete interval-valued preference relations and incomplete linguistic preference relations.

2.1 Fuzzy Preference Relations

Fuzzy preference relations have been widely used to model preferences for decision-making problems. In this case, intensity of preference is usually measured using a difference scale $[0, 1]$ [2, 14, 19].

Definition 4. *A fuzzy preference relation P on a set of alternatives X is a fuzzy set on the product set $X \times X$, i.e., it is characterized by a membership function*

$$\mu_P\colon X \times X \longrightarrow [0,1]$$

Every value in the matrix P represents the preference degree or intensity of preference of the alternative x_i over x_j:

- $p_{ij} = 1/2$ indicates indifference between x_i and x_j ($\mathrm{x}_i \sim \mathrm{x}_j$)
- $p_{ij} = 1$ indicates that x_i is absolutely preferred to x_j
- $p_{ij} > 1/2$ indicates that x_i is preferred to x_j ($\mathrm{x}_i \succ \mathrm{x}_j$)

Based on this interpretation we have that $p_{ii} = 1/2 \ \forall i \in \{1, \ldots, n\}$ ($\mathrm{x}_i \sim \mathrm{x}_i$). An *incomplete fuzzy preference relation* P on a set of alternatives X is a fuzzy set on the product set X $\times$ X that is characterized by a *partial* membership function.

2.2 Multiplicative Preference Relations

In this case, the intensity of preference represents the ratio of the preference intensity between the alternatives. According to Miller's study [18], Saaty suggests measuring every value using a ratio scale, precisely the 1-9 scale [20, 21].

Definition 5. *A multiplicative preference relation A on a set of alternatives X is characterized by a function*

$$\mu_A\colon X \times X \longrightarrow [1/9, 9]$$

The following meanings are associated to numbers:

1	equally important
3	weakly more important
5	strongly more important
7	demonstrably or very strongly more important
9	absolutely more important
2,4,6,8	compromise between slightly differing judgments

If some values in the multiplicative preference relation are missing, then we have an *incomplete multiplicative preference relation.*

2.3 Interval-Valued Preference Relations

Interval-valued preference relations are used as an alternative to fuzzy preference relations when there exists a difficulty in expressing the preferences with exact numerical values, but there is enough information as to estimate the intervals [4, 12, 23, 27].

Definition 6. *An interval-valued preference relation P on a set of alternatives X is characterized by a membership function*

$$\mu_P\colon X \times X \longrightarrow \mathcal{P}[0,1]$$

where $\mathcal{P}[0,1] = \{[a,b],\ a,b \in [0,1],\ a \leq b\}$ *is the power set of* $[0,1]$.

An interval-valued preference relation P can be seen as two "independent" fuzzy preference relations, the first one PL corresponding to the left extremes of the intervals and the second one PR to the right extremes of the intervals, respectively:

$$P = (p_{ij}) = ([pl_{ij}, pr_{ij}]) \ \text{ with } \ PL = (pl_{ij}) \ \ PR = (pr_{ij}) \ \text{ and } \ pl_{ij} \leq pr_{ij} \ \ \forall i,j.$$

If some of the interval-values in the preference relation are not given we say we have an *incomplete interval-valued preference relation.*

2.4 Linguistic Preference Relations based on the 2-tuple Linguistic Model

There are situations where it could be very difficult for the experts to provide precise numerical or interval-valued preferences, and linguistic assessments are used instead [8, 28, 29]. In this paper we will make use of the 2-tuple linguistic model [9, 10] to express expert preferences. Different advantages of this representation to manage linguistic information over semantic and symbolic models were shown in [10]:

1. The linguistic domain can be treated as continuous, while in the symbolic model is treated as discrete.

2. The linguistic computational model based on linguistic 2-tuples carries out processes of computing with words easily and without loss of information.

This linguistic model takes as a basis the symbolic representation model and in addition defines the concept of symbolic translation to represent the linguistic information by means of a pair of values called linguistic 2-tuple, (s, α), where s is a linguistic term and α is a numeric value representing the symbolic translation.

Definition 7. *Let $\beta \in [0, g]$ be the result of an aggregation of the indexes of a set of labels assessed in a linguistic term set $S = \{s_0, s_1, \ldots, s_{g-1}, s_g\}$, i.e., the result of a symbolic aggregation operation. Let $i = round(\beta)$ and $\alpha = \beta - i$ be two values, such that, $i \in [0, g]$ and $\alpha \in [-0.5, 0.5)$, then α is called a symbolic translation.*

Based on the symbolic translation concept, a linguistic representation model which represents the linguistic information by means of 2-tuples (s_i, α_i), $s \in S$ and $\alpha_i \in [-0.5, 0.5)$ was developed. This model defines a set of transformation functions between linguistic terms and 2-tuples, and between numeric values and 2-tuples.

Definition 8. *Let $S = \{s_0, s_1, \ldots, s_{g-1}, s_g\}$ be a linguistic term set and $\beta \in [0, g]$ a value supporting the result of a symbolic aggregation operation, then the 2-tuple that expresses the equivalent information to β is obtained with the following function:*

$$\Delta : [0, g] \longrightarrow S \times [-0.5, 0.5)$$

$$\Delta(\beta) = \left\{ \begin{array}{cc} s_i, & i = round(\beta) \\ \alpha = \beta - i, & \alpha \in [-0.5, 0.5) \end{array} \right\}$$

where "round" is the usual round operation, s_i has the closest index label to "β" and "α" is the value of the symbolic translation.

There exist a function, Δ^{-1}, such that given a 2-tuple it returns its equivalent numerical value $\beta \in [0, g] \subset \mathbb{R}$:

$$\Delta^{-1} : S \times [-0.5, 0.5) \longrightarrow [0, g]$$

$$\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta$$

A linguistic term can be seen as a linguistic 2-tuple by adding to it the value 0 as symbolic translation, $s_i \in S \equiv (s_i, 0)$, and therefore, this linguistic model can be used to provide preference relations:

Definition 9. *A linguistic preference relation P on a set of alternatives X is a set of 2-tuples on the product set $X \times X$, i.e., it is characterized by a membership function*

$$\mu_P\colon X \times X \longrightarrow S \times [-0.5, 0.5)$$

If it is not possible to provide the 2-tuples for every pair of alternatives we will have an *incomplete linguistic preference relation.*

3 Transitivity and Consistency of Preference Relations

The definition of a preference relation does not imply any kind of consistency property. In fact, the values of a preference relation may be contradictory. Obviously, an inconsistent source of information is not as useful as a consistent one, and thus, it would be quite important to be able to *measure* the consistency of the information provided by experts for a particular problem.

Consistency is usually characterised by *transitivity*, which represents the idea that the preference value obtained by directly comparing two alternatives should be equal to or greater than the preference value between those two alternatives obtained using an indirect chain of alternatives. Clearly, different transitivity conditions can be used for different preference relations. In the following we will introduce the transitivity conditions that will be used in this paper to measure the consistency for each one of the above preference relations.

3.1 Additive and Multiplicative Transitivity

One of the properties suggested to model the concept of transitivity in the case of fuzzy preference relations is the *additive transitivity* property [24]

$$(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ik} - 0.5) \quad \forall i, j, k \in \{1, \ldots, n\}$$

or equivalently:

$$p_{ik} = p_{ij} + p_{jk} - 0.5 \quad \forall i, j, k \in \{1, \ldots, n\} \tag{1}$$

This kind of transitivity has the following interpretation [13]: suppose we do want to establish a ranking between three alternatives x_i, x_j and x_k, and that the information available about these alternatives suggests that we are in an indifference situation, that is, $\mathrm{x}_i \sim \mathrm{x}_j \sim \mathrm{x}_k$. In this case, when giving preferences this situation would be represented by $p_{ij} = p_{jk} = p_{ki} = 0.5$. Suppose now that we have a piece of information that says alternative $\mathrm{x}_i \prec \mathrm{x}_j$, that is $p_{ij} < 0.5$. It is clear that p_{jk} or p_{ki} have to change, otherwise there would be a contradiction, because we would have $\mathrm{x}_i \prec \mathrm{x}_j \sim \mathrm{x}_k \sim \mathrm{x}_i$. If we suppose that $p_{jk} = 0.5$ then we have the situation: x_j is preferred to x_i and there is no difference in preferring x_j to x_k. We must then conclude that x_k has to be preferred to x_i. Furthermore, as $\mathrm{x}_j \sim \mathrm{x}_k$ then $p_{ji} = p_{ki}$, and so $p_{ij} + p_{jk} + p_{ki} = p_{ij} + p_{jk} + p_{ji} = 1 + 0.5 = 1.5$. We have the same conclusion if $p_{ki} = 0.5$. In the case of being $p_{jk} < 0.5$, then we have that x_k is preferred to x_j and this to x_i, so x_k should be preferred to x_i. On the other hand, the value p_{ki} has to be equal or greater than p_{ji}, being equal only in the case of $p_{jk} = 0.5$ as we have already shown. Interpreting the value $p_{ji} - 0.5$ as the intensity of preference of alternative x_j over x_i, then it seems reasonable to suppose that the intensity of preference of x_k over x_i should be equal to the sum of the intensities of preferences when using and intermediate alternative x_j, that is, $p_{ki} - 0.5 = (p_{kj} - 0.5) + (p_{ji} - 0.5)$. The same reasoning can be applied in the case of $p_{jk} > 0.5$.

We consider a fuzzy preference relation to be "additive consistent" when for every three options in the problem $\mathrm{x}_i, \mathrm{x}_j, \mathrm{x}_k \in \mathrm{X}$ their associated preference degrees p_{ij}, p_{jk}, p_{ik} fulfil expression (1). An additive consistent fuzzy preference relation will be referred to as *consistent* throughout this paper, as this is the only transitivity property we are considering.

In [3] we studied the transformation function between (reciprocal) multiplicative preference relations with values in the interval scale $[1/9, 9]$ and (reciprocal) fuzzy preference relations with values in $[0, 1]$. This study can be summarized in the following proposition.

Proposition 1. *Suppose that we have a set of alternatives, $X = \{x_1, \ldots, x_n\}$, and associated* *with it a multiplicative reciprocal preference relation* $A = (a_{ij})$*, with* $a_{ij} \in [1/9, 9]$ *and* $a_{ij} \cdot a_{ji} = 1 \quad \forall i, j$*. Then, the corresponding fuzzy reciprocal preference relation,* $P = (p_{ij})$*, associated to* A*, with* $p_{ij} \in [0, 1]$ *and* $p_{ij} + p_{ji} = 1 \quad \forall i, j$ *is given as follows:*

$$p_{ij} = f(a_{ij}) = \frac{1}{2}(1 + log_9\ a_{ij})$$

The above transformation function is bijective and, therefore, allows us to transpose concepts that have been defined for fuzzy preference relations to multiplicative preference relations. Indeed, the additive transitivity condition for fuzzy preference relations corresponds to the following *multiplicative transitivity* condition for multiplicative preference relations [13]:

$$a_{ik} = a_{ij} \cdot a_{jk} \quad \forall i, j, k. \tag{2}$$

Expression (2) coincides with the original consistency property for multiplicative preference relations defined by Saaty in [20]. This result supports the choice of the additive transitivity property to model consistency of fuzzy preference relations.

A multiplicative preference relation will be considered consistent when for every three alternatives x_i, x_j and x_k their associated preference values verify (2).

3.2 Extending the Additive Transitivity Property to the Interval-Valued and Linguistic Cases

Additive transitivity property can be used to define a consistency property for both interval-valued and preference relations and linguistic preference relations based on the 2-tuple linguistic model.

Definition 10. *An interval-valued preference relation* P *is additive consistent if both left and right interval preference relations, (PL,PR), are additive consistent, i.e.*

$$pl_{ik} = pl_{ij} + pl_{jk} - 0.5 \quad and \quad pr_{ik} = pr_{ij} + pr_{jk} - 0.5 \ \ \forall i, j, k \in \{1, \ldots, n\}$$

Using function Δ^{-1} to transform 2-tuple values into numerical values in $[0, g]$, we adapt the definition of additive transitivity to linguistic preference relations:

Definition 11. *A linguistic preference relation will be considered consistent if for every three alternatives* x_i*,* x_j *and* x_k*, the following condition holds*

$$p_{ik} = \Delta(\Delta^{-1}(p_{ij}) + \Delta^{-1}(p_{jk}) - \frac{g}{2}) \ \ \forall i, j, k \in \{1, \ldots, n\} \tag{3}$$

3.3 Consistency Measures for Preference Relations

The transitivity conditions presented in the previous sections allow to find out whether or not a preference relation is consistent. However, they do not directly offer the possibility of measuring the "level of inconsistency". In [1] we defined a consistency measure for fuzzy preference relations based on the additive transitivity property for fuzzy preference relations. In this section we will extend this consistency measure to multiplicative, interval-valued and linguistic preference relations. This measures will be used to guide the iterative procedure to estimate missing values in incomplete preference relations.

For fuzzy preference relations, expression (1) can be used to estimate the value of a given preference degree using other preference degrees. In fact,

$$cp_{ik}^{j} = p_{ij} + p_{jk} - 0.5 \tag{4}$$

where cp_{ik}^{j} means the calculated value of p_{ik} via j, that is, the value that should take p_{ik} according to the values p_{ij} and p_{jk}. Obviously, when the information provided in a fuzzy preference relation is completely consistent then cp_{ik}^{j}, $\forall j \in \{1, \ldots, n\}$ and p_{ik} coincide. However, the information given in fuzzy preference relations does not usually fulfil (1). In such cases, the value

$$\varepsilon p_{ik} = \frac{\sum_{\substack{j=1 \\ j \neq i,k}}^{n} t_{ik}^{j}}{n-2} \quad where \;\; t_{ik}^{j} = \left| cp_{ik}^{j} - p_{ik} \right|$$

can be used to measure the error expressed in a preference degree between two options or alternatives. This error can also be interpreted as the consistency level between the preference degree p_{ik} and the rest of the preference values of the fuzzy preference relation. Clearly, if $\varepsilon p_{ik} = 0$ then there is no inconsistency at all, and the higher the value of εp_{ik} the more inconsistent p_{ik} is with respect to the rest of the information.

The *consistency level* of a fuzzy preference relation P is defined as follows:

$$CL_P = \frac{\sum_{\substack{i,k=1 \\ i \neq k}}^{n} \varepsilon p_{ik}}{n^2 - n} \tag{5}$$

If $CL_P = 0$ then the preference relation P is fully (additive) consistent, otherwise, the higher CL_P the more inconsistent P is.

The measurement of consistency of multiplicative preference relations follows a similar process to the above. Indeed, the values ca_{ik}^{j} are obtained using the expression 2):

$$ca_{ik}^{j} = a_{ij} \cdot a_{jk} \tag{6}$$

and the error between a_{ik} and ca_{ik}^{j} is defined as the following ratio:

$$\varepsilon a_{ik} = \frac{\sum_{\substack{j=1 \\ j \neq i,k}}^{n} t_{ik}^{j}}{n-2} \quad where \; t_{ik}^{j} = \left\{ max \left(\frac{ca_{ik}^{j}}{a_{ik}}, \frac{a_{ik}}{ca_{ik}^{j}} \right) \right\}$$

Clearly, if $\varepsilon a_{ik} = 1$ then the preference degree a_{ik} is consistent with the rest of information in the multiplicative preference relation. Otherwise, the higher εa_{ik}, the more inconsistent a_{ik} is with respect to the rest of the information. The consistency level of a multiplicative preference relation is defined as follows:

$$CL_A = \frac{\sum_{\substack{i,k=1 \\ i \neq k}}^{n} \varepsilon a_{ik}}{n^2 - n} \tag{7}$$

If $CL_A = 1$ then the multiplicative preference relation is fully (multiplicative) consistent, otherwise, the the higher CL_A the more inconsistent A is.

The consistency level of an interval-valued preference relations is measured using the corresponding consistency levels of both PL and PR:

$$CL_P = (CL_{PL}, CL_{PR}) = \left(\frac{\sum_{\substack{i,k=1 \\ i \neq k}}^{n} \varepsilon pl_{ik}}{n^2 - n}, \frac{\sum_{\substack{i,k=1 \\ i \neq k}}^{n} \varepsilon pr_{ik}}{n^2 - n}\right)$$

When $CL_P = (0, 0)$ the interval-valued preference relation is completely consistent.

For linguistic preference relations, we use expression (5)in conjunction with (3)to define its consistency level. In this case, the definition of cp_{ik}^j, εp_{ik} and CL_P are:

$$cp_{ik}^j = \Delta(\Delta^{-1}(p_{ij}) + \Delta^{-1}(p_{jk}) - \frac{g}{2})$$

$$\varepsilon p_{ik} = \frac{\sum_{\substack{j=1 \\ j \neq i,k}}^{n} t_{ik}^j}{n - 2} \quad where \quad t_{ik}^j = \left|\Delta^{-1}(cp_{ik}^j) - \Delta^{-1}(p_{ik})\right|$$

and

$$CL_P = \frac{\sum_{\substack{i,k=1 \\ i \neq k}}^{n} \varepsilon p_{ik}}{n^2 - n}$$

When $\varepsilon p_{ik} = 0$ the preference degree p_{ik} is consistent with respect to the rest of information in the preference relation. The linguistic preference relation is consistent when $CL_P = 0$.

4 Estimation of Missing Values in Preference Relations

As aforementioned, missing information is a problem that we have to deal with because usual decision-making procedures assume that experts are able to provide preference degrees between any pair of possible alternatives, which may not not the case. This section is devoted to the presentation of an iterative procedure to estimate missing values of incomplete preference relations and the sufficient conditions to guarantee the successful estimation of all the missing values. Firstly, we will describe the general procedure and later on we will point out the implementation details for each type of preference relation. Appropriate examples will be used illustrate the application of the iterative procedure.

4.1 General Procedure

In an incomplete preference relations there exists at least a pair of alternatives (x_i, x_j) for which p_{ij} is not known. We will use throughout this paper the letter x to represent these unknown preference values, i.e. $p_{ij} = x$. We also introduce the following sets:

$$B = \{(i,j) \mid i,j \in \{1,\ldots,n\} \wedge i \neq j\}$$

$$MV = \{(i,j) \mid p_{ij} = x, (i,j) \in B\}$$

$$EV = B \setminus MV$$

MV is the set of pairs of alternatives for which the preference degree of the first alternative over the second one is unknown or missing; EV is the set of pairs of alternatives for which the expert provides preference values. Note that we do not take into account the preference value of one alternative over itself, as $\mathrm{x}_i \sim \mathrm{x}_i$ is always assumed.

The above expressions for CL_P and CL_A cannot be used for incomplete preference relations. Therefore, we need to extend them to include those cases when the preference relations are incomplete. The necessary changes to the first one are provided:

$$H_{ik} = \{j \mid (i,j),(j,k) \in EV\} \, \forall i \neq k$$

$$\varepsilon p_{ik} = \frac{\sum_{j \in H_{ik}} t_{ik}^{j}}{\# H_{ik}}$$

$$CE_P = \{(i,k) \in EV \mid \exists j : (i,j),(j,k) \in EV\}$$

$$CL_P = \frac{\sum_{(i,k) \in CE_P} \varepsilon p_{ik}}{\# CE_P}$$

We call CE_P the *computable error* set because it contains all the elements for which we can compute every εp_{ik}. Clearly, this redefinition of CL_P is an extension of expression (5) for fuzzy preference relations and the corresponding ones for interval-valued and linguistic preference relations. Indeed, when a fuzzy preference relation is complete, both CE_P and B coincide and thus $\# CE_P = n^2 - n$.

To develop the iterative procedure to estimate missing values two different tasks have to be carried out: (A) to establish the elements that can be estimated in each iteration of the procedure, and (B) to produce the particular expression that will be used to estimate a particular missing value.

A) Elements to be estimated in every iteration of the procedure

A missing value p_{ik} can be estimated if there exist at least one j so that p_{ij} and p_{jk} are known (they where provided in the initial incomplete preference relation or they have been estimated in a previous iteration of the procedure). Therefore, the subset of missing values MV that can be estimated in iteration h is denoted by EMV_h (*estimated missing values*) and defined as follows:

$$EMV_h = \left\{ (i,k) \in MV \setminus \bigcup_{l=0}^{h-1} EMV_l \;\middle|\; \exists j : (i,j),(j,k) \in EV \cup \left(\bigcup_{l=0}^{h-1} EMV_l \right) \right\}$$

with $EMV_0 = \emptyset$.

When $EMV_{maxIter} = \emptyset$ with $maxIter > 0$ the procedure stops because there will not be any more missing values to be estimated. Furthermore, if $\bigcup_{l=0}^{maxIter} EMV_l = MV$ then all missing values are estimated and consequently the procedure is said to be successful in the completion of the fuzzy preference relation.

B) Expression to estimate a particular missing value

In iteration h, to estimate a particular value p_{ik} with $(i,k) \in EMV_h$, the function in figure 1 is applied.

```
function estimate_p(i,k)
```

1. $I_{ik} = \left\{ j \;\middle|\; (i,j),(j,k) \in EV \cup \left(\bigcup_{l=0}^{h-1} EMV_l \right) \right\}$
2. Calculate $cp'_{ik} = \dfrac{\sum_{j \in I_{ik}} cp_{ik}^j}{\# I_{ik}}$
3. Apply a small random transformation to the estimated cp'_{ik} to maintain the consistency level

```
end function
```

Fig. 1: Function to estimate a particular missing value p_{ik}

Summarizing, a missing value p_{ik} can be estimated when there is at least one chained pair of known preference values (p_{ij}, p_{jk}) that allow the application the expression to calculate cp_{ik}^j, in which case the average of the values obtained using it, cp'_{ik}, is calculated. The final estimation will be obtained applying a small random transformation to the estimated cp'_{ik} to maintain the consistency level of the preference relation. This transformation is dependant on the type of preference relation.

The *iterative estimating procedure pseudo-code* is presented figure 2. Again, as it will be seen in the following sections, both the *initializations* and *post-processing operations* steps are dependant on the type of the preference relation.

```
0. Initializations
1. EMV_0 = ∅
2. h = 1
3. while EMV_h ≠ ∅ {
4.     for every (i, k) ∈ EMV_h {
5.         function estimate_p(i,k)
6.     }
7.     h++
8. }
9. Post-processing operations
```

Fig. 2: General Procedure

4.2 Fuzzy Preference Relations: Implementation Details

The estimation in a fuzzy preference relation of p_{ik} is obtained by adding a random value $z \in [-CL_P, CL_P]$ to the average value cp'_{ik}. This is done in order to maintain the consistency level of the expert, and is subject to the condition of being the final estimated value in the range of fuzzy preference values $[0, 1]$. That is:

3. $p_{ik} = cp'_{ik} + z$ with $z \in [-CL_P, CL_P]$ randomly selected, subject to $0 \leq cp'_{ik} + z \leq 1$.

There is no need to implement any special initialization nor any kind of post-processing operations in this case. The following example illustrate the application of the above procedure:

Example 1. *Suppose that an expert provide the following incomplete fuzzy preference relation P over a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$:*

$$P = \begin{pmatrix} - & x & 0.4 & x \\ x & - & 0.7 & 0.85 \\ x & 0.4 & - & 0.75 \\ 0.3 & x & x & - \end{pmatrix}$$

In this case we have:

$$CE_P = \{(2,4),(3,4)\} \quad \Rightarrow \quad \epsilon p^1_{24} = 0.1 \quad ; \quad \epsilon p^1_{34} = 0 \quad \Rightarrow \quad CL_P = (\epsilon p^1_{24} + \epsilon p^1_{34})/2 = 0.05$$

With the application of only two iterations of our procedure all the missing values are successfully estimated.

$$\begin{pmatrix} - & x & 0.4 & x \\ x & - & 0.7 & 0.85 \\ x & 0.4 & - & 0.75 \\ 0.3 & x & x & - \end{pmatrix} \rightarrow \begin{pmatrix} - & 0.32 & 0.4 & 0.61 \\ 0.68 & - & 0.7 & 0.85 \\ 0.5 & 0.4 & - & 0.75 \\ 0.3 & x & 0.24 & - \end{pmatrix} \rightarrow \begin{pmatrix} - & 0.32 & 0.4 & 0.61 \\ 0.68 & - & 0.7 & 0.85 \\ 0.5 & 0.4 & - & 0.75 \\ 0.3 & 0.17 & 0.24 & - \end{pmatrix}$$

4.3 Multiplicative Preference Relations: Implementation Details

For an incomplete multiplicative preference relation, in order to calculate a cp^j_{ik} value in the second step of estimate_p(i,k), expression (6) has to be used instead of (4). The third step in estimate_p(i,k) in this case is:

3. $p_{ik} = cp'_{ik} \cdot z$ with $z \in [1/CL_P, CL_P]$ randomly selected, subject to $1/9 \leq cp'_{ik} \cdot z \leq 9$.

Example 2. *Suppose that we have the following incomplete multiplicative preference relation over a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$:*

$$A = \begin{pmatrix} - & 0.80 & 1.55 & 1 \\ 1.25 & - & x & 3.74 \\ 0.65 & x & - & 1.93 \\ 1 & 0.33 & 0.52 & - \end{pmatrix}$$

The complete multiplicative preference relation obtained after just 1 iteration is:

$$\begin{pmatrix} - & 0.80 & 1.55 & 1 \\ 1.25 & - & x & 3.74 \\ 0.65 & x & - & 1.93 \\ 1 & 0.33 & 0.52 & - \end{pmatrix} \rightarrow \begin{pmatrix} - & 0.80 & 1.55 & 1 \\ 1.25 & - & 1.85 & 3.74 \\ 0.65 & 0.59 & - & 1.93 \\ 1 & 0.33 & 0.52 & - \end{pmatrix}$$

4.4 Interval-Valued Preference Relations: Implementation Details

Interval-valued preference relations need some initializations steps in order to create the extreme interval fuzzy preference relations, PL and PR, and post-processing operations to unify the completed extreme interval fuzzy preference relations into a complete interval-valued preference relation.

Other changes have to be made to the estimate_p(i,k) function to adapt it to this kind of preference relation as they are shown in figure 3:

function estimate_p(i,k)

1. $I_{ik} = \left\{ j \;\middle|\; (i,j),(j,k) \in EV \cup \left(\bigcup_{l=0}^{h-1} EMV_l \right) \right\}$

2. Calculate $cp'_{ik} = (cpl'_{ik}, cpr'_{ik}) = (\frac{\sum_{j \in I_{ik}} cpl^j_{ik}}{\#I_{ik}}, \frac{\sum_{j \in I_{ik}} cpr^j_{ik}}{\#I_{ik}})$

3. $p_{ik} = cp'_{ik} + (zl, zr)$ with $zl \in [-CL_{PL}, CL_{PL}]$, $zr \in [-CL_{PR}, CL_{PR}]$ randomly selected, subject to $0 \leq cpl'_{ik} + zl \leq cpr'_{ik} + zr \leq 1$.

end function

Fig. 3: Function to estimate a particular missing value p_{ik} in the interval-valued case

Example 3. *Suppose a set of four alternatives* $X = \{x_1, x_2, x_3, x_4\}$, *and the following incomplete interval-valued preference relation*

$$P = \begin{pmatrix} - & (0.45, 0.60) & (0.55, 0.75) & (0.30, 0.40) \\ (0.40, 0.55) & - & (0.45, 0.80) & x \\ (0.25, 0.45) & (0.20, 0.55) & - & x \\ (0.60, 0.70) & x & x & - \end{pmatrix}$$

In this case, the our procedure algorithm is capable of estimating the missing values in just one iteration:

$$\begin{pmatrix} - & (0.45, 0.60) & (0.55, 0.75) & (0.30, 0.40) \\ (0.40, 0.55) & - & (0.45, 0.80) & x \\ (0.25, 0.45) & (0.20, 0.55) & - & x \\ (0.60, 0.70) & x & x & - \end{pmatrix} \rightarrow \begin{pmatrix} - & (0.45, 0.60) & (0.55, 0.75) & (0.30, 0.40) \\ (0.40, 0.55) & - & (0.45, 0.80) & (0.21, 0.43) \\ (0.25, 0.45) & (0.20, 0.55) & - & (0.12, 0.33) \\ (0.60, 0.70) & (0.54, 0.78) & (0.64, 0.91) & - \end{pmatrix}$$

4.5 Linguistic Preference Relations: Implementation Details

In the initialization step for linguistic preference relations we apply the transformation function Δ^{-1} to obtain a numeric preference relation. As a post-processing operation the completed numeric preference relation is transformed back to a linguistic preference relation by the application of the inverse of the previous transformation function, Δ. In this case, we also need to adapt the estimate_p(i,k) function:

2. Calculate $cp'_{ik} = \frac{\sum_{j \in I_{ik}} \Delta^{-1}(cp^j_{ik})}{\#I_{ik}}$

3. $p_{ik} = \Delta(cp'_{ik} + z)$ with $z \in [-CL_P, CL_P]$ randomly selected, subject to $0 \leq cp'_{ik} + z \leq g$.

Example 4. *Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of four alternatives and $S = \{MW, W, E, B, MB\}$ the set of linguistic labels used to provide preferences, with the following meaning:*

MW = Much Worse W = Worse E = Equally Preferred B = Better MB = Much Better

Suppose the following incomplete linguistic preference relation

$$P = \begin{pmatrix} - & x & W & x \\ x & - & x & W \\ B & x & - & W \\ x & B & B & - \end{pmatrix}$$

Note that the expert did not provide any α values, which is a common practice when expressing preferences with linguistic terms. In these cases, we set $\alpha = 0$

$$P = \begin{pmatrix} - & x & (W,0) & x \\ x & - & x & (W,0) \\ (B,0) & x & - & (W,0) \\ x & (B,0) & (B,0) & - \end{pmatrix}$$

Firstly, we transform the preferences given to the continuous domain $[0,4]$ using the transformation function Δ^{-1}:

$$P = \begin{pmatrix} - & x & (W,0) & x \\ x & - & x & (W,0) \\ (B,0) & x & - & (W,0) \\ x & (B,0) & (B,0) & - \end{pmatrix} \xrightarrow{\Delta^{-1}} \begin{pmatrix} - & x & 1 & x \\ x & - & x & 1 \\ 3 & x & - & 1 \\ x & 3 & 3 & - \end{pmatrix}$$

Applying the estimation procedure we have:

$$\begin{pmatrix} - & x & 1 & x \\ x & - & x & 1 \\ 3 & x & - & 1 \\ x & 3 & 3 & - \end{pmatrix} \rightarrow \begin{pmatrix} - & x & 1 & 0 \\ x & - & 2 & 1 \\ 3 & 2 & - & 1 \\ 4 & 3 & 3 & - \end{pmatrix} \rightarrow \begin{pmatrix} - & 1.25 & 1 & 0 \\ 1.75 & - & 2 & 1 \\ 3 & 2 & - & 1 \\ 4 & 3 & 3 & - \end{pmatrix}$$

Finally, the application of the inverse of the previous transformation function, Δ, produces the final 2-tuple linguistic preference relation:

$$\begin{pmatrix} - & 1.25 & 1 & 0 \\ 1.75 & - & 2 & 1 \\ 3 & 2 & - & 1 \\ 4 & 3 & 3 & - \end{pmatrix} \xrightarrow{\Delta} \begin{pmatrix} - & (W,0.25) & (W,0) & (MW,0) \\ (B,-0.25) & - & (E,0) & (W,0) \\ (B,0) & (E,0) & - & (W,0) \\ (MB,0) & (B,0) & (B,0) & - \end{pmatrix}$$

5 Sufficient Conditions to Estimate all Missing Values

In this section we provide sufficient conditions to assure that all missing values can be successfully estimated using our procedure. However, there may be situations where not all missing values can be learnt. In these cases, we propose the use of the reciprocity property in our procedure.

5.1 Sufficient Conditions

To establish conditions that guarantee that all the missing values of an incomplete preference relation can be estimated is of great importance. In the following, we provide sufficient conditions that guarantee the success of the above procedure:

1. If there exists a value j such that for all $i \in \{1, 2, \ldots, n\}$ both pairs (i, j) and (j, k) do not belong to MV, then all missing information can be estimated in the first iteration of the procedure ($EMV_1 = MV$). Indeed, in this case for every $p_{ik} \in MV$, at least the pair of preference values (p_{ij}, p_{jk}) can be used to estimate it.

2. In [13], a different sufficient condition that guarantees the estimation of all missing values was given. This condition states that any incomplete fuzzy preference relation can be converted into a complete one when the set of $n-1$ values $\{p_{12}, p_{23}, \ldots, p_{n-1n}\}$ is known.

3. A more general condition than the previous one is that of having when a set of $n-1$ non-leading diagonal preference values, where each one of the alternatives is compared at least once. This general case includes the one when a complete row or column of preference values is known.

In these two last cases the additive reciprocity property is also assumed. However, there may be cases where all missing information cannot be estimated using the proposed learning procedure, which is illustrated in the following example.

Example 5. *Suppose an expert provides the following incomplete preference relation over a set of five different alternatives,* $X = \{x_1, x_2, x_3, x_4, x_5\}$,

$$P = \begin{pmatrix} - & e & e & x & x \\ e & - & x & e & x \\ x & x & - & x & x \\ e & x & x & - & e \\ x & x & e & e & - \end{pmatrix}$$

where x *means "a missing value" and* e *means "a value is known". We note that the actual values of the known preference values or even the type of preference relation are not relevant for the purpose of this example.*

At the beginning of our iterative procedure we have

$$EMV_1 = \{(1,4),(2,3),(2,5),(4,2),(4,3),(5,1)\}$$

and the following table shows all the pairs of alternatives that are available to estimate each one of the above missing values:

Missing value (i,k)	**Pairs of values to estimate** p_{ik}
$(1,4)$	$(1,2),(2,4)$
$(2,3)$	$(2,1),(1,3)$
$(2,5)$	$(2,4),(4,5)$
$(4,2)$	$(4,1),(1,2)$
$(4,3)$	$(4,1),(1,3);(4,5),(5,3)$
$(5,1)$	$(5,4),(4,1)$

Tab. 1: Pairs of values that permit the estimation of missing values in iteration 1

In iteration 2, the estimated values of iteration 1 are added to the values expressed directly by the expert to construct the set EMV_2*. In our case we have* $EMV_2 = \{(1,5),(5,2)\}$ *and the following table:*

Missing value (i,k)	**Pairs of values to estimate** p_{ik}
$(1,5)$	$(1,2),(2,5);(1,4),(4,5)$
$(5,2)$	$(5,1),(1,2);(5,4),(4,2)$

Tab. 2: Pairs of values that permit the estimation of missing values in iteration 2

The incomplete fuzzy preference relation obtained is:

$$P = \begin{pmatrix} - & e & e & 1 & 2 \\ e & - & 1 & e & 1 \\ x & x & - & x & x \\ e & 1 & 1 & - & e \\ 1 & 2 & e & e & - \end{pmatrix}$$

where numbers 1 and 2 indicate the steps in which missing the missing values were estimated.

In iteration 3, $EMV_3 = \emptyset$*, and thus, the procedure ends and fails in the completion of the preference relation.*

The reason of this failure is that the expert did not provide any preference degree of the alternative x_3 over the rest of the alternatives. Fortunately, this kind of situation is not very common in real problems, and therefore the procedure will usually be successful in estimating all missing values. Clearly, if additive reciprocity is also assumed (this is a direct consequence of the additive transitivity property) then the chances of succeeding in the estimation of all the missing values would increase, as we will show in what follows.

5.2 Implementation of Reciprocity Property in the Estimation Procedure

In most studies, preference relations are usually assumed reciprocal. In particular, *additive reciprocity*, $p_{ij} + p_{ji} = 1 \;\; \forall i, j$, is used in many decision models as one of the properties that fuzzy preference relations have to verify [14]. For multiplicative preference relations, the multiplicative reciprocity condition is defined as follows: $a_{ij} \cdot a_{ji} = 1 \;\; \forall i, j$ [20,21]. For interval-valued preference relations and linguistic preference relations, the reciprocity condition can be rewritten as: $(pl_{ij} + pl_{ji}, pr_{ij} + pr_{ji}) = (1, 1) \;\; \forall i, j$ and $\Delta(p_{ij}) + \Delta(p_{ji}) = g \;\; \forall i, j$, respectively.

The iterative procedure presented in previous sections does not imply any kind of reciprocity, and estimates missing values in preference relations when this condition is not satisfied. Furthermore, the procedure itself does not assure that the estimated values will fulfil the reciprocity property. However, some of the missing values that were not possible to estimate could be easily estimated under reciprocity. Indeed, in the previous example all p_{3k} values could have been directly estimated assuming additive reciprocity. In the following, a description of how to implement the additive reciprocity for fuzzy preference relations in the above procedure, and the changes needed to assure that the estimated values fulfil this property are provided. Clearly, a similar implementation is to apply for the other types of preference relations by using the corresponding reciprocity condition.

A first step of the procedure would consist of checking that the incomplete preference relation given by the expert fulfils the reciprocity property that we are considering, i.e., $p_{ik} + p_{ki} = 1 \;\; \forall (i, k), (k, i) \in EV$. Next, those missing values with a known reciprocal one are computed, i.e., $p_{ki} \leftarrow 1 - p_{ik} \;\; \forall (k, i) \in MV$ and $(i, k) \in EV$.

The following steps of the procedure are as described in *subsection 4.1*, but restricted to the estimation of missing values above the leading diagonal of the incomplete preference relation, i.e., p_{ik} with $i < k$. Using the notation $EMV_h^{\uparrow}$ for the set of estimated missing values above the leading diagonal in iteration h, the pseudo-code of the procedure is presented in figure 4:

6 Conclusions

We have looked at the issue of incomplete preference relations, that is, preference relation with some of its values missing or not known. We have proposed an iterative procedure to estimate missing preference values in different types of incomplete preference relations: fuzzy, multiplicative, interval-valued and linguistic preference relations. Our proposal attempts to estimate the missing information in an expert's incomplete fuzzy preference relation using only the preference values provided by that particular expert. By doing this, we assure that the reconstruction of the incomplete fuzzy preference relation is compatible with the rest of the information provided by that expert. Because an important objective in the design of our procedure was to maintain experts' consistency levels, the procedure is guided by the expert's consistency, and this is measured taking into account only the available preference values. In particular, in this paper we made use of the additive consistency property. We have also

```
Initializations
Check that reciprocity property is satisfied ∀(i,k),(k,i) ∈ EV
for every (k,i) ∈ MV, (i,k) ∈ EV {
    Compute p_ki by using reciprocity property
}
EMV_0^↑ = ∅
h = 1
while EMV_h^↑ ≠ ∅ {
    for every (i,k) ∈ EMV_h^↑ {
        function estimate_p(i,k)
        Use reciprocity to estimate values under the leading diagonal
    }
    h++
}
Post-processing operations
```

Fig. 4: General Procedure with Reciprocity Property

provided implementation details of the procedure for the aforementioned four different types of preference relations.

We have shown that in many cases all the missing values on incomplete fuzzy preference relations can be estimated using the proposed iterative procedure. Moreover, if extra conditions are imposed, such as reciprocity, the procedure becomes more efficient, allowing the estimation of some missing values that were not possible to obtain with the original procedure. However, there may still be cases in which not every missing value in an incomplete fuzzy preference relation can be estimated with this procedure. This is a problem that was not covered in this paper, being an issue for further research in the near future.

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