Automatica 49 (2013) 1821–1829

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Robust static output feedback control synthesis for linear continuous systems with polytopic uncertainties[☆]

Jiuxiang Dong, Guang-Hong Yang [1]

College of Information Science and Engineering, Northeastern University, Shenyang, 110819, PR China
State Key Laboratory of Synthetical Automation for Process Industries, Northeastern University, Shenyang, 110819, PR China

A R T I C L E I N F O

Article history:
Received 11 October 2011
Received in revised form
27 December 2012
Accepted 28 January 2013
Available online 22 March 2013

Keywords:
Linear systems
Static output feedback
Polytopic uncertainty
Linear matrix inequalities (LMIs)

A B S T R A C T

This paper studies the static output feedback (SOF) control problem of continuous-time linear systems with polytopic uncertainties. Novel LMI conditions with a line search over a scalar variable for designing robust SOF controllers are proposed, where the uncertain output matrix of the considered system is allowed to be not of full row rank. In particular, it is shown that the new method can give less or at least the same conservative results than those methods by inserting a matrix equality constraint between system output matrix and Lyapunov matrix. Furthermore, the result is extended to the case of H_∞ control. Numerical examples are given to illustrate the effectiveness of the proposed method.

1. Introduction

In control theory and practice, one of the most important and challenging open problems is the synthesis of static output feedback (SOF) or reduced-order controllers. It has been proved to be a non-convex problem (Syrmos, Abdallah, Dorato, & Grigoriadis, 1997), which means that a convex sufficient and necessary condition for designing SOF or reduced-order controllers cannot be obtained. However, in contrast to other control schemes, the SOF or reduced-order controllers are with simpler structures and more easily realized in practice. Thus, much attention has been paid for obtaining less conservative conditions for SOF or reduced-order control synthesis, see the excellent survey paper (Syrmos et al., 1997) and the reference therein.

In recent years, various numerical algorithms for designing SOF controllers have been proposed. In these algorithms, two-step algorithms (Agulhari, Oliveira, & Peres, 2010; Mehdi, Boukas, & Bachelier, 2004; Peaucelle & Arzelier, 2001) and iterative algorithms (El Ghaoui, Oustry, & AitRami, 1997; Huang & Nguang, 2006; Leibfritz, 2001; Trofino, 2009) based on linear matrix inequality (LMI) are widely used. Moreover, a class of numerical tools based on nonsmooth optimization techniques is also a good choice for designing SOF controllers, see Apkarian and Noll (2006), Lewis (2007) and Yaesh and Shaked (2012). In contrast to the above mentioned approaches, although the convex conditions based on linear matrix inequalities (LMIs) are only sufficient, they can be solved by interior-point algorithms, which work very well in practice and are quite reliable like the methods for solving linear programs (Boyd & Vandenberghe, 2004). Therefore, various convex sufficient conditions for designing SOF controllers are proposed. By forcing a Lyapunov matrix to have a special structure (Ho & Lu, 2003; Lo & Lin, 2003) or inserting a linear matrix equality constraint on a Lyapunov matrix (Crusius & Trofino, 1999; De Souza & Trofino, 2000), sufficient LMI-based conditions for designing SOF stabilizing controllers are given. For exploiting more degrees of freedom in Lyapunov functions, special congruence transformations are adopted in Bara and Boutayeb (2005) and Prempain and Postlethwaite (2001) respectively for continuous- and discrete-time systems. A linear parameter dependent stabilization method for designing SOF controllers is proposed in Shaked (2003). By using Hit-and-Run methods, a mixed LMI/randomized method is proposed for SOF control synthesis in Arzelier, Gryazina, Peaucelle, and Polyak (2010).

☆ This work was supported in part by the Funds of National Science of China (Grant No. 60904010, No. 60974043, No. 61273148), the Program for New Century Excellent Talents in University (NCET-11-0072), the Fundamental Research Funds for the Central Universities (No. N110804001, No. N090404016), China Postdoctoral Science Foundation (No. 20100470074), China Postdoctoral Science Foundation Special Funded Project (No. 201104608), Creative Research Groups of China (No. 60821063), National 973 Program of China (Grant No. 2009CB320604), and the 111 Project (B08015), and the Nature Science of Foundation of Liaoning Province under Grant 201202063, the 985 fund and Postdoctoral Science Foundation of Northeastern University, China. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor James Lam under the direction of Editor Ian R. Petersen.

E-mail addresses: dongjiuxiang@ise.neu.edu.cn (J. Dong), yangguanghong@ise.neu.edu.cn (G.-H. Yang).

[1] Tel.: +86 13504182968; fax: +86 24 83681939.

0005-1098/$ – see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.automatica.2013.02.047

By using the properties of the null space of output matrices and introducing parameter-independent slack variables with a lower-triangular structure, sufficient conditions for designing SOF controllers are given in Dong and Yang (2007) and Dong and Yang (2008). By introducing a stabilizing delay, a static output feedback sliding mode controller is determined in Seuret, Edwards, Spurgeon, and Fridman (2009). Moreover, the SOF control synthesis problems for linear systems with an unknown state/input delay (Du, Lam, & Shu, 2010), positive linear systems (Ait Rami, 2011), Markovian jump linear systems (Shu, Lam, & Xiong, 2010), fragility issues (Peaucelle & Arzelier, 2005), mixed H_2/H_∞ control of discrete-time LPV systems (De Caigny, Camino, Oliveira, Peres, & Swevers, 2010) and so on, have been studied.

In this paper, new convex SOF control synthesis conditions with a line search over a scalar variable are proposed, where the uncertain output matrix of the considered linear system is not required to be of full row rank. In particular, it is proved that the new method can give less or at least the same conservative results than those methods by inserting a matrix equality constraint between system output matrix and Lyapunov matrix in Crusius and Trofino (1999). Furthermore, the result is extended to the case of H_∞ control.

The paper is organized as follows. Section 2 presents a system description and some preliminaries. Section 3 provides a robust static output feedback controller design method for guaranteeing the stability of the closed-loop systems. Further, the proposed method is extended to the case of H_∞ control. Four numerical examples are given to illustrate the effectiveness of the new proposed methods in Section 4. Concluding remarks are given in Section 5.

2. System description and problem statement

Consider a linear time-invariant system (1) with polytopic uncertainties described by state-space equations:

$$\begin{aligned}\dot{x}(t) &= \mathscr{A}x(t) + \mathscr{B}_1 w(t) + \mathscr{B}_2 u(t)\\ z(t) &= \mathscr{C}_1 x(t) + \mathscr{D}_{12} u(t)\\ y(t) &= \mathscr{C}_2 x(t) \end{aligned}\tag{1}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^p$ is the control input, $w(t) \in R^m$ is the disturbance, $y(t) \in R^r$ is the measured output, $z(t) \in R^q$ is the controlled output. The matrices $[\mathscr{A}]_{n\times n}$, $[\mathscr{B}_1]_{n\times m}$, $[\mathscr{B}_2]_{n\times p}$, $[\mathscr{C}_1]_{q\times n}$, $[\mathscr{C}_2]_{r\times n}$, $[\mathscr{D}_{12}]_{q\times p}$ belong to the following uncertainty polytope:

$$\begin{aligned}\Omega = \Bigg\{ &([\mathscr{A}]_{n\times n}, [\mathscr{B}_1]_{n\times m}, [\mathscr{B}_2]_{n\times p}, [\mathscr{C}_1]_{q\times n}, [\mathscr{C}_2]_{r\times n}, [\mathscr{D}_{12}]_{q\times p}) \,|\\ &\times ([\mathscr{A}]_{n\times n}, [\mathscr{B}_1]_{n\times m}, [\mathscr{B}_2]_{n\times p}, [\mathscr{C}_1]_{q\times n}, [\mathscr{C}_2]_{r\times n}, [\mathscr{D}_{12}]_{q\times p})\\ &= \sum_{i=1}^{N} \alpha_i ([A_i]_{n\times n}, [B_{1i}]_{n\times m}, [B_{2i}]_{n\times p}, [C_{1i}]_{q\times n},\\ &\times [C_{2i}]_{r\times n}, [D_{12i}]_{q\times p}), \alpha_i \geq 0, \sum_{i=1}^{N} \alpha_i = 1 \Bigg\}. \end{aligned}\tag{2}$$

Assume $\mathscr{C}_1^T \mathscr{D}_{12} = 0$, and the same assumption is also given on p. 401 in Zhou, Doyle, and Glover (1996).

In this paper, a static output feedback controller

$$u(t) = Ky(t) \tag{3}$$

will be designed, such that the resulting closed-loop system

$$\begin{aligned}\dot{x}(t) &= (\mathscr{A} + \mathscr{B}_2 K \mathscr{C}_2)x(t) + \mathscr{B}_1 w(t)\\ z(t) &= (\mathscr{C}_1 + \mathscr{D}_{12} K \mathscr{C}_2)x(t)\end{aligned}\tag{4}$$

is robustly stable or simultaneously meets the H_∞ performance bound requirement (5).

Definition 1. Suppose that the system (4) is asymptotically stable and satisfies,

$$\int_0^\infty z^T(t)z(t) < \gamma^2 \int_0^\infty w^T(t)w(t)dt \tag{5}$$

then H_∞ norm of the system (4) is said to be less than γ.

In order to give a comparison with the existing methods, the results in Crusius and Trofino (1999) are recalled as follows:

Lemma 2 (*Crusius & Trofino, 1999*). *Assume that there exists one i_0, such that C_{2i_0} is full row rank,*

(i): *If there exist matrices $W = W^T > 0$, M, R such that*

$$\mathrm{He}(A_i W + B_{2i} R C_{2j} + A_j W + B_{2j} R C_{2i}) < 0, \quad 1 \leq i \leq j \leq N \tag{6a}$$

$$M C_{2i} = C_{2i} W, \quad 1 \leq i \leq N \tag{6b}$$

then the controller (3) with $K = RM^{-1}$ stabilizes the system (1).

(ii): *For a given scalar $\gamma > 0$, if there exist matrices $W = W^T > 0$, M, R such that (6b) holds and satisfying*

$$\begin{bmatrix} \mathrm{He}(A_i W + B_{2i} R C_{2j}) & B_{1i} & W C_{1i}^T + C_{2j}^T R^T D_{12i}^T \\ B_{1i}^T & -\gamma^2 I & 0 \\ C_{1i} W + D_{12i} R C_{2j} & 0 & -I \end{bmatrix} + \begin{bmatrix} \mathrm{He}(A_j W + B_{2j} R C_{2i}) & B_{1j} & W C_{1j}^T + C_{2i}^T R^T D_{12j}^T \\ B_{1j}^T & -\gamma^2 I & 0 \\ C_{1j} W + D_{12j} R C_{2i} & 0 & -I \end{bmatrix} < 0, \quad 1 \leq i \leq j \leq N \tag{7}$$

then the system (1) is asymptotically stable via the SOF controller (3) with H_∞ norm less than γ, where the controller gain $K = RM^{-1}$.

Proof. By using the technique in Theorem 1 of Crusius and Trofino (1999), the proof is routine and omitted. □

Remark 3. Note that the equality constraint (6b) is imposed between output matrix and Lyapunov matrix, which might lead to a strict constraint if C_{2i}, $i = 1, \ldots, N$ are different. Moreover, in order to guarantee reversibility of the matrix variable M, one of C_{2i}, $1 \leq i \leq N$ has to be full row rank, which might not be satisfied for some uncertain systems. Therefore, this paper will explore new methods without the equality constraint and the full row rank constraint on C_{2i}. In particular, it will be proved that the new method can give less or at least the same conservative results than those methods by inserting a matrix equality constraint between system output matrix and Lyapunov matrix in Crusius and Trofino (1999).

3. Robust static output feedback control

In this section, a new LMI-based method for designing SOF controllers for guaranteeing stability is firstly presented, and it is proved that the new method can give less (or at least the same as) conservative results than Lemma 2(i). Subsequently, the result is extended to the H_∞ control case.

3.1. Static output feedback control synthesis

The following theorem gives a sufficient condition for designing SOF stabilizing controllers.

Theorem 4. *If there exist a symmetric matrix $Q(\alpha) > 0$ and matrices G, L, a scalar $\tau > 0$, satisfying the following matrix inequality,*

$$\begin{bmatrix} \mathrm{He}\,(A(\alpha)Q(\alpha) + B_2(\alpha)LTC_2(\alpha)) & * \\ C_2(\alpha)Q(\alpha) - GTC_2(\alpha) + \tau L^T B_2^T(\alpha) & -\tau G - \tau G^T \end{bmatrix} < 0 \quad (8)$$

where

$$T = \begin{cases} I, & C_{2i}, i = 1, 2, \ldots, N \\ & \text{are row non-full rank.} \\ (C_{2i_0}C_{2i_0}^T)^{-1}, & \exists \text{ some } i_0 \in \{1, 2, \ldots, N\}, \\ & \text{s.t. } C_{2i_0} \text{ is row full rank.} \end{cases} \quad (9)$$

then the system (1) with $w(k) = 0$ and

$$K = LG^{-1} \quad (10)$$

is robustly stable.

Proof. From $Q(\alpha) > 0$, we have that $Q(\alpha) > 0$ is invertible. Let vectors $x(t)$ and $\bar{x}(t)$ satisfy

$$\bar{x}(t) = Q^{-1}(\alpha)x(t). \quad (11)$$

For $x(t) \neq 0$, it follows that $\bar{x}(t) \neq 0$. Pre- and post-multiplying (8) with $\begin{bmatrix}\bar{x}^T(t) & \bar{x}^T(t)B_2(\alpha)K\end{bmatrix}$ and its transpose, then we have

$$\begin{aligned} &\bar{x}^T(t)A(\alpha)Q(\alpha)\bar{x}(t) + \bar{x}^T(t)Q(\alpha)A^T(\alpha)\bar{x}(t) + \bar{x}^T(t) \\ &\quad\times B_2(\alpha)LTC_2(\alpha)\bar{x}(t) + \bar{x}^T(t)C_2^T(\alpha)T^TL^TB_2^T(\alpha)\bar{x}(t) \\ &\quad + \bar{x}^T(t)Q(\alpha)C_2^T(\alpha)K^TB_2^T(\alpha)\bar{x}(t) \\ &\quad - \bar{x}^T(t)C_2^T(\alpha)T^TG^TK^TB_2^T(\alpha)\bar{x}(t) \\ &\quad + \tau\bar{x}^T(t)B_2(\alpha)LK^TB_2^T(\alpha)\bar{x}(t) + \bar{x}^T(t) \\ &\quad\times B_2(\alpha)KC_2(\alpha)Q(\alpha)\bar{x}(t) - \bar{x}^T(t)B_2(\alpha)KGTC_2(\alpha)\bar{x}(t) \\ &\quad + \tau\bar{x}^T(t)B_2(\alpha)KL^TB_2^T(\alpha)\bar{x}(t) - \tau\bar{x}^T(t)B_2(\alpha)KG \\ &\quad\times K^TB_2^T(\alpha)\bar{x}(t) - \tau\bar{x}^T(t)B_2(\alpha)KG^TK^TB_2^T(\alpha)\bar{x}(t) \\ &\quad < 0, \quad \text{for all } x(t) \neq 0 \end{aligned} \quad (12)$$

where K is the same as in (10), which implies that $L = KG$. Substituting KG for L, then (12) can be rewritten as follows:

$$\begin{aligned} &\bar{x}^T(t)A(\alpha)Q(\alpha)\bar{x}(t) + \bar{x}^T(t)Q(\alpha)A^T(\alpha)\bar{x}(t) + \bar{x}^T(t) \\ &\quad\times Q(\alpha)C_2^T(\alpha)K^TB_2^T(\alpha)\bar{x}(t) + \bar{x}^T(t)B_2(\alpha)KC_2(\alpha)Q(\alpha) \\ &\quad\times \bar{x}(t) < 0, \quad \text{for all } x(t) \neq 0 \end{aligned}$$

i.e., $2\bar{x}^T(t)\Big(A(\alpha) + B_2(\alpha)KC_2(\alpha)\Big)Q(\alpha)\bar{x}(t) < 0$, for all $x(t) \neq 0$, which can be rewritten as (13) from (11).

$$\begin{aligned} &2x^T(t)P(\alpha)\Big(A(\alpha) + B_2(\alpha)KC_2(\alpha)\Big)x(t) < 0, \\ &\quad \text{for all } x(t) \neq 0 \end{aligned} \quad (13)$$

where $P(\alpha) = (Q(\alpha))^{-1}$.

From $Q(\alpha) > 0$, we have $P(\alpha) > 0$ and choose Lyapunov function $V(t) = x^T(t)P(\alpha)x(t)$, then

$$\begin{aligned} \dot{V}(t) &= \dot{x}^T(t)P(\alpha)x(t) + x^T(t)P(\alpha)\dot{x}(t) \\ &= 2x^T(t)P(\alpha)\Big(A(\alpha) + B_2(\alpha)KC_2(\alpha)\Big)x(t). \end{aligned}$$

Combining it and (13), yields that $\dot{V}(t) < 0$ for all $x(t) \neq 0$. Therefore, we have that the closed-loop system (4) is asymptotically stable. Thus, the proof is complete. □

Remark 5. A condition for designing SOF controllers is given in Theorem 4 and we have to point that the condition is only sufficient. If we substitute a new matrix variable $W(\alpha)$ for $TC(\alpha)$ in Theorem 4, then a sufficient and necessary condition can be obtained for designing SOF controllers, where the condition and its proof are given in Appendix. Inhere, $W(\alpha)$ is chosen as $TC(\alpha)$ for obtaining LMI-based conditions and removing the equality constraint about Lyapunov matrix in Crusius and Trofino (1999).

Remark 6. Note that the condition of Theorem 4 is a matrix inequality with the parameters α_i, $1 \le i \le N$, which cannot be directly used for designing SOF controllers. The matrices $A(\alpha)$, $B_2(\alpha)$, $C_2(\alpha)$ are the same as in (2), we can choose the matrix $Q(\alpha)$ as a polynomial function of α, for example, $Q(\alpha) = \sum_{i_1=1}^{N} \cdots \sum_{i_M=1}^{N} (\prod_{j=1}^{M} \alpha_{i_j}) Q_{i_1\cdots i_j}$, then less conservative results can be obtained by increasing M. In this paper, $Q(\alpha)$ is chosen as a linear function of α, i.e., $Q(\alpha) = \sum_{i=1}^{N} \alpha_i Q_i$, then the inequality (8) becomes

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i\alpha_j \begin{bmatrix} \mathrm{He}\left(A_iQ_j + B_{2i}LTC_{2j}\right) & * \\ C_{2i}Q_j - GTC_{2i} + \tau L^TB_{2i}^T & -\tau G - \tau G^T \end{bmatrix} < 0 \quad (14)$$

where the parameters α_i, $1 \le i \le N$ satisfy $0 \le \alpha_i \le 1$, $\sum_{i=1}^{N} \alpha_i = 1$. In the following theorem, a simple technique is applied to convert (14) into a set of LMIs. It should be noted that the techniques in Oliveira and Peres (2005), Ramos and Peres (2002) and Yang and Dong (2008), are also applicable to (14) for obtaining less conservative conditions, but the computational complexity will increase.

Remark 7. Motivated by the work in de Oliveira, Geromel, and Hsu (1999) and Peaucelle, Arzelier, Bachelier, and Bernussou (2000), some methods with extended LMI characterizations have been proposed for robust control problems. In particular, a general, projection lemma is proposed in Pipeleers, Demeulenaerea, Sweversa, and Vandenbergheb (2009) and reproduces the known extended LMIs and completes some currently missing results. Inspired by these works, we introduce a matrix variable G by a special change of variables in Theorem 4, further, an LMI condition with a line search over a scalar variable for designing robust SOF controllers is proposed. Note that many extended LMI characterizations are covered by some excellent designs of U and V with some special structures in Pipeleers et al. (2009). The condition of Theorem 4 is not a particular case of some of the results in Pipeleers et al. (2009), but it is also obtained based on projection lemma by the design of the matrices $U = \begin{bmatrix}-K^TB_2^T(\alpha) & I\end{bmatrix}$, $V = \begin{bmatrix}TC_2(\alpha) & \tau I\end{bmatrix}$, $X = -G$ and $Z = \begin{bmatrix} A(\alpha)Q(\alpha) + Q(\alpha)A^T(\alpha) & (C_2(\alpha)Q(\alpha))^T \\ C_2(\alpha)Q(\alpha) & 0 \end{bmatrix}$ (the notations U, V, X, Z are the same as in the projection lemma of Pipeleers et al., 2009).

Theorem 8. *If there exist symmetric matrices $Q_i > 0$, $1 \le i \le N$ and matrices G, L, a scalar $\tau > 0$ satisfying*

$$\begin{aligned} &\begin{bmatrix} \mathrm{He}\left(A_iQ_j + B_{2i}LTC_{2j}\right) & * \\ C_{2i}Q_j - GTC_{2i} + \tau L^TB_{2i}^T & -\tau G - \tau G^T \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathrm{He}\left(A_jQ_i + B_{2j}LTC_{2i}\right) & * \\ C_{2j}Q_i - GTC_{2j} + \tau L^TB_{2j}^T & -\tau G - \tau G^T \end{bmatrix} \\ &\quad < 0, \quad 1 \le i \le j \le N \end{aligned} \quad (15)$$

then the system (1) with $w(k) = 0$ and the gain (10) is robustly stable.

Proof. Multiplying (15) by $\alpha_i\alpha_j$ for $1 \le i < j \le N$ and summing them, then we have

$$\sum_{1 \le i < j \le N} \alpha_i\alpha_j(H_{ij} + H_{ji}) < 0 \quad (16)$$

where

$$H_{ij} = \begin{bmatrix} \mathrm{He}\left(A_iQ_j + B_{2i}LTC_{2j}\right) & * \\ C_{2i}Q_j - GTC_{2i} + \tau L^TB_{2i}^T & -\tau G - \tau G^T \end{bmatrix}.$$

Multiplying (15) by $\frac{1}{2}\alpha_i^2$ for $1 \le i \le N$ and summing them, then we can obtain that

$$\sum_{1\le i\le N} \alpha_i^2 H_{ii} < 0. \tag{17}$$

From (16) and (17), it yields that $\sum_{i=1}^N \sum_{j=1}^N \alpha_i\alpha_j H_{ij} < 0$, i.e., (14) holds, which implies that (8) holds. Then we have that the system (1) is robust stable by Theorem 4. Thus, the proof is complete. □

Remark 9. Note that Theorem 8 is a set of LMIs with a line search over a scalar variable τ, then Theorem 8 is no longer convex. Because τ is a scalar variable, a constructive numerical procedure can be given. The procedure always achieves a reasonable solution provided τ is initialized with a sufficiently large value and the search is carefully performed (for instance with small enough steps near the optimum). Some methods for a line search can be found in Bernussou, Geromel, and de Oliveira (1999); Sato (2011). Inhere, the method in Sato (2011) is used, i.e., the line search for τ in Theorem 8 may be conducted with 100 points linearly gridded over a logarithmic scale in the interval $[10^{-5}, 10^5]$.

Remark 10. There exist many convex methods (or with a line search) for designing SOF controllers in the existing literature, for example, sufficient conditions with equality constraint on Lyapunov matrix in Crusius and Trofino (1999) and De Souza and Trofino (2000), the methods by using the properties of the null space of output matrices in Dong and Yang (2007) and Dong and Yang (2008), linear parameter dependent (LPD) stabilization method in Shaked (2003) (which is a method with several line searches), and so on. Moreover, some non-convex algorithms are also proposed, for example, the iterative LMI algorithm in Cao, Lam, and Sun (1998), the cone complementarity linearization algorithm in El Ghaoui et al. (1997), convex–concave decompositions algorithm in Dinh, Gumussoy, Michiels, and Diehl (2012) and so on.

The comparisons with the existing methods are given by testing numerical examples in Section 4. In particular, it is shown in Theorems 11 and 14 that the new methods can give less or at least the same conservative results than Lemma 2(the existing methods in Crusius and Trofino (1999), where a linear matrix equality constraint is imposed on a Lyapunov matrix).

Theorem 11. *If the condition of Lemma 2(i) holds, then the condition of Theorem 8 holds.*

Proof. If the condition of Lemma 2(i) holds, then there exists some $i_0 \in \{1, 2, \ldots, N\}$ such that C_{2i_0} is full row rank and

$$GTC_{2i} = C_{2i}W, \quad i = 1, \ldots, N \tag{18}$$

where $G = MT^{-1}$ and T is the same as in (9).

For $i = i_0$, right-multiply (18) by $C_{2i_0}^T$, then it yields

$$GTC_{2i_0}C_{2i_0}^T = C_{2i_0}WC_{2i_0}^T. \tag{19}$$

Because C_{2i_0} is row full rank, $C_{2i_0}C_{2i_0}^T$ is invertible, then $TC_{2i_0}C_{2i_0}^T = I$ from (9). It can be obtained from $W > 0$ and (19) that

$$G = C_{2i_0}WC_{2i_0}^T > 0. \tag{20}$$

On the other hand, for a simple description, we denote $H_{ij} = A_iW + WA_i^T + B_{2i}RC_{2j} + C_{2j}^TR^TB_{2i}^T$, then (6a) can be rewritten as follows:

$$H_{ij} + H_{ji} < 0, \quad 1 \le i \le j \le N.$$

Then there exists a positive scalar τ^* such that

$$H_{ij} + H_{ji} + \tau^* I < 0, \quad 1 \le i \le j \le N. \tag{21}$$

Note that $G + G^T > 0$ by virtue of (20), then there exists a positive scalar τ, satisfying

$$\frac{1}{2}\tau(B_{2i} + B_{2j})RT^{-1}(G + G^T)^{-1}(RT^{-1})^T(B_{2i} + B_{2j})^T < \tau^* I,\quad 1 \le i \le j \le N.$$

Combining it and (20)–(21), then we have

$$\frac{1}{2}\tau(B_{2i} + B_{2j})RT^{-1}(G + G^T)^{-1}(RT^{-1})^T(B_{2i} + B_{2j})^T + H_{ij} + H_{ji} < 0, \quad 1 \le i \le j \le N \quad G + G^T > 0$$

where G is the same as in (20).

By the Schur complement lemma, the above inequality is equivalent to

$$\begin{bmatrix} H_{ij} & \tau B_{2i}RT^{-1} \\ \tau(RT^{-1})^TB_{2i}^T & -\tau G - \tau G^T \end{bmatrix} + \begin{bmatrix} H_{ji} & \tau B_{2j}RT^{-1} \\ \tau(RT^{-1})^TB_{2j}^T & -\tau G - \tau G^T \end{bmatrix} < 0, \quad 1 \le i \le j \le N.$$

Combining it and (6b), then we have

$$\begin{bmatrix} H_{ij} & * \\ C_{2i}W - MC_{2i} + \tau(RT^{-1})^TB_{2i}^T & -\tau G - \tau G^T \end{bmatrix} + \begin{bmatrix} H_{ji} & * \\ C_{2j}W - MC_{2j} + \tau(RT^{-1})^TB_{2j}^T & -\tau G - \tau G^T \end{bmatrix} < 0, \quad 1 \le i \le j \le N. \tag{22}$$

Choose $Q_i = W$, $1 \le i \le N$, $L = RT^{-1}$ and consider (20), then (22) can be rewritten as follows:

$$\begin{bmatrix} \mathrm{He}(A_iQ_j + B_{2i}LTC_{2j}) & * \\ C_{2i}Q_j - GTC_{2i} + \tau L^TB_{2i}^T & -\tau G - \tau G^T \end{bmatrix} + \begin{bmatrix} \mathrm{He}(A_jQ_i + B_{2j}LTC_{2i}) & * \\ C_{2j}Q_i - GTC_{2j} + \tau L^TB_{2j}^T & -\tau G - \tau G^T \end{bmatrix} < 0, \quad 1 \le i \le j \le N \tag{23}$$

i.e., (15) holds, which implies that the condition of Theorem 8 holds. Thus the proof is complete. □

3.2. H_∞ SOF control synthesis

In this subsection, assume that the external disturbance $w(t) \not\equiv 0$, and a method for designing H_∞ SOF controllers is proposed by extending the above results.

Theorem 12. *For a given scalar $\gamma > 0$, if there exist a symmetric matrix $Q(\alpha) > 0$ and matrices G, L, a scalar $\tau > 0$ satisfying the following matrix inequality,*

$$\begin{bmatrix} \mathrm{He}\left(\bar{A}(\alpha)\bar{Q}(\alpha) + \bar{B}_2(\alpha)LT\bar{C}_2(\alpha)\right) & * & * & * \\ \bar{C}_2(\alpha)\bar{Q}(\alpha) - GT\bar{C}_2(\alpha) + \tau L^T\bar{B}_2^T(\alpha) & -\tau G - \tau G^T & * & * \\ \bar{B}_1^T(\alpha) & 0 & -\gamma^2 I & * \\ \bar{C}_1(\alpha)\bar{Q}(\alpha) & 0 & 0 & -I \end{bmatrix} < 0 \tag{24}$$

where

$$\bar{A}(\alpha) = \begin{bmatrix} A(\alpha) & 0 \\ 0 & -\frac{1}{2}I \end{bmatrix}, \quad \bar{B}_2(\alpha) = \begin{bmatrix} B_2(\alpha) \\ D_{12}(\alpha) \end{bmatrix},$$

$$\bar{C}_2(\alpha) = \begin{bmatrix} C_2(\alpha) & 0 \end{bmatrix}, \quad \bar{B}_1(\alpha) = \begin{bmatrix} B_1(\alpha) \\ 0 \end{bmatrix},$$

$$\bar{C}_1(\alpha) = \begin{bmatrix} C_1(\alpha) & 0 \end{bmatrix}, \quad \bar{Q}(\alpha) = \begin{bmatrix} Q(\alpha) & 0 \\ 0 & I \end{bmatrix}$$

then the system (1) is asymptotically stable via the SOF controller (3) with H_∞ norm less than or equal to γ, where the controller gain K is the same as in (10).

Proof. Applying Schur complement lemma to (24), then we have Eq. (25) is given in Box I.

Let $\begin{bmatrix}\bar{\bar{x}}(t)\\ w(t)\end{bmatrix} \neq 0$, pre- and post-multiplying the above inequality with $\begin{bmatrix}\bar{\bar{x}}^T(t) & \bar{\bar{x}}^T(t)\bar{B}_2(\alpha)K & w^T(t)\end{bmatrix}$ and its transpose, it yields from $L = KG$ that

$$\begin{aligned}&2\bar{\bar{x}}^T\left(\bar{A}(\alpha)\bar{Q}(\alpha) + \bar{B}_2(\alpha)LT\bar{C}_2(\alpha)\right)\bar{\bar{x}} + \bar{\bar{x}}^T\bar{Q}(\alpha)\bar{C}_1^T(\alpha)\\ &\quad\times \bar{C}_1(\alpha)\bar{Q}(\alpha)\bar{\bar{x}} + 2\bar{\bar{x}}^T\bar{B}_2(\alpha)K\bar{C}_2(\alpha)\bar{Q}(\alpha)\bar{\bar{x}} - 2\bar{\bar{x}}^T\bar{B}_2(\alpha)\\ &\quad\times KGT\bar{C}_2(\alpha)\bar{\bar{x}} + 2\tau\bar{\bar{x}}^T\bar{B}_2(\alpha)KL^T\bar{B}_2^T(\alpha)\bar{\bar{x}} - 2\tau\bar{\bar{x}}^T\bar{B}_2(\alpha)\\ &\quad\times KG^TK^T\bar{B}_2^T(\alpha)\bar{\bar{x}} + 2w^T\bar{B}_1^T(\alpha)\bar{\bar{x}} - \gamma^2 w^T w\\ &= 2\bar{\bar{x}}^T\bar{A}(\alpha)\bar{Q}(\alpha)\bar{\bar{x}} + \bar{\bar{x}}^T\bar{Q}(\alpha)\bar{C}_1^T(\alpha)\bar{C}_1(\alpha)\bar{Q}(\alpha)\bar{\bar{x}}\\ &\quad+ 2\bar{\bar{x}}^T\bar{B}_2(\alpha)K\bar{C}_2(\alpha)\bar{Q}(\alpha)\bar{\bar{x}} + 2\bar{\bar{x}}^T\bar{B}_1(\alpha)w - \gamma^2 w^T w\\ &< 0, \quad \text{for all} \begin{bmatrix}\bar{\bar{x}}(t)\\ w(t)\end{bmatrix} \neq 0. \end{aligned} \tag{26}$$

Since $Q(\alpha) > 0$, $\bar{Q}(\alpha)$ is invertible, let $P(\alpha) = (Q(\alpha))^{-1}$, $\bar{P}(\alpha) = \text{diag}\begin{bmatrix}P(\alpha) & I\end{bmatrix} = (\bar{Q}(\alpha))^{-1}$. Therefore, (26) can be rewritten as follows:

$$\begin{aligned}&\tilde{x}^T(t)\Big(\bar{P}(\alpha)\big(\bar{A}(\alpha) + \bar{B}_2(\alpha)K\bar{C}_2(\alpha)\big)\\ &\quad+ \big(\bar{A}(\alpha) + \bar{B}_2(\alpha)K\bar{C}_2(\alpha)\big)^T\bar{P}(\alpha) + \bar{C}_1^T(\alpha)\bar{C}_1(\alpha)\Big)\tilde{x}(t)\\ &\quad+ \tilde{x}^T\bar{P}(\alpha)\bar{B}_1(\alpha)w + w^T\bar{B}_1^T(\alpha)\bar{P}(\alpha)\tilde{x} - \gamma^2 w^T w < 0,\\ &\qquad \text{for all} \begin{bmatrix}\tilde{x}(t)\\ w(t)\end{bmatrix} \neq 0 \end{aligned} \tag{27}$$

where

$$\tilde{x}(t) = \bar{Q}(\alpha)\bar{\bar{x}}(t). \tag{28}$$

(27) is equivalent to

$$\begin{bmatrix}\text{He}\left(\bar{P}(\alpha)\left(\bar{A}(\alpha) + \bar{B}_2(\alpha)K\bar{C}_2(\alpha)\right)\right) + \bar{C}_1^T(\alpha)\bar{C}_1(\alpha) & *\\ \bar{B}_1^T(\alpha)\bar{P}(\alpha) & -\gamma^2 I\end{bmatrix} < 0$$

which can be rewritten as follows:

$$\begin{bmatrix}\text{He}\left(P(\alpha)\left(A(\alpha) + B_2(\alpha)KC_2(\alpha)\right)\right) + C_1^T(\alpha)C_1(\alpha) & * & *\\ D_{12}(\alpha)KC_2(\alpha) & -I & *\\ B_1^T(\alpha)P(\alpha) & 0 & -\gamma^2 I\end{bmatrix} < 0.$$

Applying Schur complement to the above inequality, then we can obtain

$$\begin{bmatrix}\psi(\alpha) & P(\alpha)B_1(\alpha)\\ B_1^T(\alpha)P(\alpha) & -\gamma^2 I\end{bmatrix} < 0 \tag{29}$$

where $\psi(\alpha) = \text{He}\left(P(\alpha)\left(A(\alpha) + B_2(\alpha)KC_2(\alpha)\right)\right) + C_1^T(\alpha)C_1(\alpha) + C_2^T(\alpha)K^TD_{12}^T(\alpha)D_{12}(\alpha)KC_2(\alpha)$.

From $C_1^T(\alpha)D_{12}(\alpha) = 0$, it follows that $\psi(\alpha) = \text{He}(P(\alpha)A(\alpha) + P(\alpha)B_2(\alpha)KC_2(\alpha)) + (C_1(\alpha) + D_{12}(\alpha)KC_2(\alpha))^T(C_1(\alpha) + D_{12}(\alpha)KC_2(\alpha))$.

Pre- and post-multiplying (29) with $\begin{bmatrix}x^T(t) & w^T(t)\end{bmatrix} \neq 0$ and its transpose, then we have

$$\begin{aligned}&2x^T(t)P(\alpha)\big(A(\alpha) + B_2(\alpha)KC_2(\alpha)\big)x(t) + x^T(t)\\ &\quad\times \big(C_1(\alpha) + D_{12}(\alpha)KC_2(\alpha)\big)^T\big(C_1(\alpha) + D_{12}(\alpha)KC_2(\alpha)\big)\\ &\quad\times x(t) + 2x^T(t)P(\alpha)B_1(\alpha)w(t) - \gamma^2 w^T(t)w(t) < 0. \end{aligned} \tag{30}$$

Choose

$$V(t) = x^T(t)P(\alpha)x(t)$$

as Lyapunov function, then (30) can be rewritten as follows:

$$\dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0. \tag{31}$$

Integrating both sides of this inequality yields

$$\begin{aligned}&\int_0^\infty \dot{V}(t) + \int_0^\infty z^T(t)z(t) - \gamma^2\int_0^\infty w^T(t)w(t)\\ &\quad= V(\infty) - V(0) + \int_0^\infty z^T(t)z(t) - \gamma^2\int_0^\infty w^T(t)w(t)\\ &\quad< 0.\end{aligned}$$

Using the fact that $x(0) = 0$ and $V(\infty) \geq 0$, we obtain

$$\int_0^\infty z^T(t)z(t)dt \leq \gamma^2\int_0^\infty w^T(t)w(t)dt.$$

Hence, (5) holds and the H_∞ performance is fulfilled.

If the disturbance $w(t) \equiv 0$, then from (31), we have $\dot{V}(t) < 0$. Hence, based on the Lyapunov theorem, the closed-loop system (4) is asymptotically stable when $w(t) \equiv 0$. Thus, the proof is complete. □

Applying the same technique as Theorem 8, we can obtain Theorem 13 from Theorem 12.

Theorem 13. *For a given scalar $\gamma > 0$, if there exist symmetric matrices $Q_i > 0$, $1 \leq i \leq N$ and matrices G, L, a scalar $\tau > 0$ satisfying*

$$\begin{aligned}&\begin{bmatrix}\text{He}\left(\bar{A}_i\bar{Q}_j + \bar{B}_{2i}LT\bar{C}_{2j}\right) & * & * & *\\ \bar{C}_{2i}\bar{Q}_j - GT\bar{C}_{2i} + \tau L^T\bar{B}_{2i}^T & -\tau G - \tau G^T & * & *\\ \bar{B}_{1i}^T & 0 & -\gamma^2 I & *\\ \bar{C}_{1i}\bar{Q}_j & 0 & 0 & -I\end{bmatrix}\\ &\quad+ \begin{bmatrix}\text{He}\left(\bar{A}_j\bar{Q}_i + \bar{B}_{2j}LT\bar{C}_{2i}\right) & * & * & *\\ \bar{C}_{2j}\bar{Q}_i - GT\bar{C}_{2j} + \tau L^T\bar{B}_{2j}^T & -\tau G - \tau G^T & * & *\\ \bar{B}_{1j}^T & 0 & -\gamma^2 I & *\\ \bar{C}_{1j}\bar{Q}_i & 0 & 0 & -I\end{bmatrix}\\ &\quad< 0, \quad 1 \leq i \leq j \leq N, \end{aligned} \tag{32}$$

where

$$\begin{aligned}&\bar{A}_i = \begin{bmatrix}A_i & 0\\ 0 & -\frac{1}{2}I\end{bmatrix}, \quad \bar{B}_{2i} = \begin{bmatrix}B_{2i}\\ D_{12i}\end{bmatrix}, \quad \bar{C}_{2i} = \begin{bmatrix}C_{2i} & 0\end{bmatrix}\\ &\bar{B}_{1i} = \begin{bmatrix}B_{1i}\\ 0\end{bmatrix}, \quad \bar{C}_{1i} = \begin{bmatrix}C_{1i} & 0\end{bmatrix}, \quad \bar{Q}_i = \begin{bmatrix}Q_i & 0\\ 0 & I\end{bmatrix}\end{aligned} \tag{33}$$

$i = 1, \ldots, N$, then the system (1) is asymptotically stable via the SOF controller (3) with H_∞ norm less than or equal to γ, where the controller gain K is the same as in (10).

Proof. From the proof of Theorem 12, the proof is routine and omitted. □

As follows, it is proved that the condition of Theorem 13 is more relaxed than that of Lemma 2(ii), where the equality constraints on Lyapunov matrix and system output matrix are required.

$$\begin{bmatrix} \mathrm{He}\left(\bar{A}(\alpha)\bar{Q}(\alpha)+\bar{B}_2(\alpha)TL\bar{C}_2(\alpha)\right)+\bar{Q}(\alpha)\bar{C}_1^T(\alpha)\bar{C}_1(\alpha)\bar{Q}(\alpha) & * & * \\ \bar{C}_2(\alpha)\bar{Q}(\alpha)-GT\bar{C}_2(\alpha)+\tau L^T\bar{B}_2^T(\alpha) & -\tau G-\tau G^T & * \\ \bar{B}_1^T(\alpha) & 0 & -\gamma^2 I \end{bmatrix}<0. \quad (25)$$

Box I.

Theorem 14. *If the condition of Lemma 2(ii) holds, then the condition of Theorem 13 holds.*

Proof. If the condition of Lemma 2(ii) holds, then choose $G = MT^{-1}$, further by the same proof in the first part of Theorem 11, we have that $G+G^T>0$.

Now, applying Schur complement to (7), then we have

$$\begin{bmatrix} \phi_{ij} & B_{1i}+B_{1j} \\ (B_{1i}+B_{1j})^T & -2\gamma^2 I \end{bmatrix}<0, \quad 1\le i\le j\le N \quad (34)$$

with

$$\begin{aligned}\phi_{ij} &= \mathrm{He}(A_iW+B_{2i}RC_{2j}+A_jW+B_{2j}RC_{2i}) \\ &\quad+\frac{1}{2}\Big((C_{1i}+C_{1j})W+D_{12i}RC_{2j}+D_{12j}RC_{2i}\Big)^T \\ &\quad\times\Big((C_{1i}+C_{1j})W+D_{12i}RC_{2j}+D_{12j}RC_{2i}\Big).\end{aligned}$$

Because $C_2(\alpha)^TD_{12}(\alpha)=0$, then $C_{2i}^TD_{12j}=0$, which implies that

$$\begin{aligned}\phi_{ij} &= \mathrm{He}(A_iW+B_{2i}RC_{2j}+A_jW+B_{2j}RC_{2i}) \\ &\quad+\frac{1}{2}W(C_{1i}+C_{1j})^T(C_{1i}+C_{1j})W \\ &\quad+\frac{1}{2}(D_{12i}RC_{2j}+D_{12j}RC_{2i})^T(D_{12i}RC_{2j}+D_{12j}RC_{2i}).\end{aligned}$$

Then applying the Schur complement to (34), it follows that

$$\begin{bmatrix} \mathrm{He}(A_iW+A_jW+B_{2i}RC_{2j}+B_{2j}RC_{2i}) & * & * & * \\ D_{12i}RC_{2j}+D_{12j}RC_{2i} & -2I & * & * \\ (B_{1i}+B_{1j})^T & 0 & -2\gamma^2 I & * \\ C_{1i}W+C_{1j}W & 0 & 0 & -2I \end{bmatrix}$$
$$<0, \quad 1\le i\le j\le N$$

which can be rewritten as follows:

$$\begin{bmatrix} \mathrm{He}(\bar{A}_i\bar{W}+\bar{B}_{2i}R\bar{C}_{2j}) & \bar{B}_{1i} & (\bar{C}_{1i}\bar{W})^T \\ \bar{B}_{1i}^T & -\gamma^2 I & 0 \\ \bar{C}_{1i}\bar{W} & 0 & -I \end{bmatrix} + \begin{bmatrix} \mathrm{He}(\bar{A}_j\bar{W}+\bar{B}_{2j}R\bar{C}_{2i}) & \bar{B}_{1j} & (\bar{C}_{1j}\bar{W})^T \\ \bar{B}_{1j}^T & -\gamma^2 I & 0 \\ \bar{C}_{1j}\bar{W} & 0 & -I \end{bmatrix}$$
$$<0, \quad 1\le i\le j\le N \quad (35)$$

where $\bar{A}_i, \bar{B}_{1i}, \bar{B}_{2i}, \bar{C}_{1i}, \bar{C}_{2i}$ are the same as in (33), and $\bar{W} = \mathrm{diag}\begin{bmatrix} W & I\end{bmatrix}$.

From (35) and $G+G^T>0$, we have that there exists a scalar $\tau>0$, such that

$$\begin{bmatrix} \mathrm{He}(\bar{A}_i\bar{W}+\bar{B}_{2i}R\bar{C}_{2j}) & \bar{B}_{1i} & (\bar{C}_{1i}\bar{W})^T \\ \bar{B}_{1i}^T & -\gamma^2 I & 0 \\ \bar{C}_{1i}\bar{W} & 0 & -I \end{bmatrix} + \begin{bmatrix} \mathrm{He}(\bar{A}_j\bar{W}+\bar{B}_{2j}R\bar{C}_{2i}) & \bar{B}_{1j} & (\bar{C}_{1j}\bar{W})^T \\ \bar{B}_{1j}^T & -\gamma^2 I & 0 \\ \bar{C}_{1j}\bar{W} & 0 & -I \end{bmatrix}$$
$$+\begin{bmatrix} \frac{\tau}{2}(\bar{B}_{2i}+\bar{B}_{2j})RT^{-1}(G+G^T)^{-1}(RT^{-1})^T(\bar{B}_{2i}+\bar{B}_{2j})^T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$<0, \quad 1\le i\le j\le N.$$

Applying the Schur complement to the above inequality, then yields

$$\begin{bmatrix} \mathrm{He}(\bar{A}_i\bar{W}+\bar{B}_{2i}R\bar{C}_{2j}) & \tau\bar{B}_{2i}RT^{-1} & \bar{B}_{1i} & (\bar{C}_{1i}\bar{W})^T \\ \tau(RT^{-1})^T\bar{B}_{2i}^T & -\tau G-\tau G^T & 0 & 0 \\ \bar{B}_{1i}^T & 0 & -\gamma^2 I & 0 \\ \bar{C}_{1i}\bar{W} & 0 & 0 & -I \end{bmatrix} + \begin{bmatrix} \mathrm{He}(\bar{A}_j\bar{W}+\bar{B}_{2j}R\bar{C}_{2i}) & \tau\bar{B}_{2j}RT^{-1} & \bar{B}_{1j} & (\bar{C}_{1j}\bar{W})^T \\ \tau(RT^{-1})^T\bar{B}_{2j}^T & -\tau G-\tau G^T & 0 & 0 \\ \bar{B}_{1j}^T & 0 & -\gamma^2 I & 0 \\ \bar{C}_{1j}\bar{W} & 0 & 0 & -I \end{bmatrix}$$
$$<0, \quad 1\le i\le j\le N.$$

Choose $Q_i = W$, $1\le i\le N$, $L=RT^{-1}$, then the above inequality can be rewritten as follows:

$$\begin{bmatrix} \mathrm{He}(\bar{A}_i\bar{Q}_j+\bar{B}_{2i}LT\bar{C}_{2j}) & \tau\bar{B}_{2i}L & \bar{B}_{1i} & (\bar{C}_{1i}\bar{Q}_j)^T \\ \tau L^T\bar{B}_{2i}^T & -\tau G-\tau G^T & 0 & 0 \\ \bar{B}_{1i}^T & 0 & -\gamma^2 I & 0 \\ \bar{C}_{1i}\bar{Q}_j & 0 & 0 & -I \end{bmatrix} + \begin{bmatrix} \mathrm{He}(\bar{A}_j\bar{Q}_i+\bar{B}_{2j}LT\bar{C}_{2i}) & \tau\bar{B}_{2j}L & \bar{B}_{1j} & (\bar{C}_{1j}\bar{Q}_i)^T \\ \tau L^T\bar{B}_{2j}^T & -\tau G-\tau G^T & 0 & 0 \\ \bar{B}_{1j}^T & 0 & -\gamma^2 I & 0 \\ \bar{C}_{1j}\bar{Q}_i & 0 & 0 & -I \end{bmatrix}$$
$$<0, \quad 1\le i\le j\le N.$$

Combining it and (6b), we can obtain that

$$\begin{bmatrix} \mathrm{He}(\bar{A}_i\bar{Q}_j+\bar{B}_{2i}LT\bar{C}_{2j}) & * & * & * \\ \bar{C}_{2i}\bar{Q}_j-GT\bar{C}_{2i}+\tau L^T\bar{B}_{2i}^T & -\tau G-\tau G^T & * & * \\ \bar{B}_{1i}^T & 0 & -\gamma^2 I & * \\ \bar{C}_{1i}\bar{Q}_j & 0 & 0 & -I \end{bmatrix} + \begin{bmatrix} \mathrm{He}(\bar{A}_j\bar{Q}_i+\bar{B}_{2j}LT\bar{C}_{2i}) & * & * & * \\ \bar{C}_{2j}\bar{Q}_i-GT\bar{C}_{2j}+\tau L^T\bar{B}_{2j}^T & -\tau G-\tau G^T & * & * \\ \bar{B}_{1j}^T & 0 & -\gamma^2 I & * \\ \bar{C}_{1j}\bar{Q}_i & 0 & 0 & -I \end{bmatrix}$$
$$<0, \quad 1\le i\le j\le N$$

i.e., (32) holds, which implies that the condition of Theorem 13 holds. Thus, the proof is complete. □

4. Example

In this section, several examples will be given for illustrating the effectiveness of the proposed method. The implementations are done in Matlab 7.9.0 (2009b) running on a PC Desktop Intel(R) Core i5 and 4 GB RAM. We use the LMI toolbox in Matlab 7.9.0 (2009b).

Example 15. Consider a continuous-time system which belongs to the 2-polytopic convex polyhedron in the form of (2) with $w(t)=0$ and

$$A_1=\begin{bmatrix} -1 & 4 & 0 \\ 0 & 0 & 1 \\ a & 6 & -1 \end{bmatrix}, \qquad A_2=\begin{bmatrix} -1 & 1 & 0 \\ 0 & -5 & 1 \\ 10 & 1 & -1 \end{bmatrix},$$

$$B_{21}=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Table 1
Stabilization interval.

	Theorem 8	Lemma 2(i)	Corollary 1 of Crusius and Trofino (1999)	Methods in Arzelier et al. (2010); Dong and Yang (2007); Trofino (2009)
a	[3.6 82.2]	Infeasible	[8.7 11.5]	Non-applicable

Table 2
The number of the feasible examples.

	MA	MB	MC	MD
Number	75	24	70	87

$$B_{22}=\begin{bmatrix}0\\0\\1\end{bmatrix},\qquad C_{21}=\begin{bmatrix}1&1&0\\0&1&0\end{bmatrix},\qquad C_{22}=\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}.$$

Note that the output matrix $C(\alpha)=\alpha_1C_{21}+\alpha_2C_{22}$ is not always of full row rank. For the case of $\alpha_1=0,\alpha_2=1$, the output matrix $C(\alpha)=C_{22}$, which implies that the second sensor for measuring output variables is failed. The methods in Arzelier et al. (2010); Dong and Yang (2007); Trofino (2009) are not applicable. However, Theorem 8, Lemma 2(i) and Corollary 1 of Crusius and Trofino (1999) can be used for designing SOF controllers. Now the stabilization intervals of a are computed by Theorem 8 and Corollary 1 of Crusius and Trofino (1999) and the computational results are given in Table 1.

It can be seen from Table 1 that the new method can give less conservative results.

Example 16. 100 stability critical systems $(A;B;C)$'s of 5th order systems are generated randomly.

We use the existing method in Cao et al. (1998); Crusius and Trofino (1999); El Ghaoui et al. (1997) and Theorem 8 to design PI controllers for these generated systems. By augmenting the system description to include the integral of the measured output, the PI control problem becomes one of finding a SOF control (He & Wang, 2006; Yaesh & Shaked, 1997). Theorem 8, Lemma 2(i), the iterative LMI (ILMI) method in Cao et al. (1998) and the cone complementarity linearization (CCL) algorithm in El Ghaoui et al. (1997) are applied to these generated examples to testing proposed method. For a simple description, Theorem 8, Lemma 2(i), the iterative LMI (ILMI) method in Cao et al. (1998) and the cone complementarity linearization (CCL) algorithm (MD) in El Ghaoui et al. (1997) are respectively represented as MA, MB, MC and MD.

The number of the feasible examples by the different methods are shown in Table 2. It can be found from Table 2 that Theorem 8 (MA) may stabilize more linear systems in these examples by PI controllers than Lemma 2(i) (MB) and ILMI method in Cao et al. (1998) (MC). The CCL algorithm (MD) in El Ghaoui et al. (1997) can stabilize the most linear systems in these examples. By the computational results, it can be seen that the CCL algorithm (MD) are with less conservatism. Note that the CCL algorithm is non-convex, can only be applied to linear determinate systems, cannot be extended to the case of H_∞ control. However, Theorem 8 is convex with a line search and can be used for linear uncertain systems and it is extended to the case of H_∞ control.

Example 17. Consider a continuous-time system which belongs to the 2-polytopic convex polyhedron in the form of (2) with

$$A_1=\begin{bmatrix}-0.9896&17.41&96.15\\0.2648&-0.8512&-11.39\\0&0&-30\end{bmatrix}$$

Table 3
Controller gain and H_∞ performance.

	Lemma 2(ii) Crusius and Trofino (1999)	Theorem 4 of Shaked (2003)	Theorem 13 with $\tau=0.05$
γ	9.7315	6.8028	2.3267
K	[0.5558 5.0823]	[0.0536 0.6384]	[0.4474 4.1860]

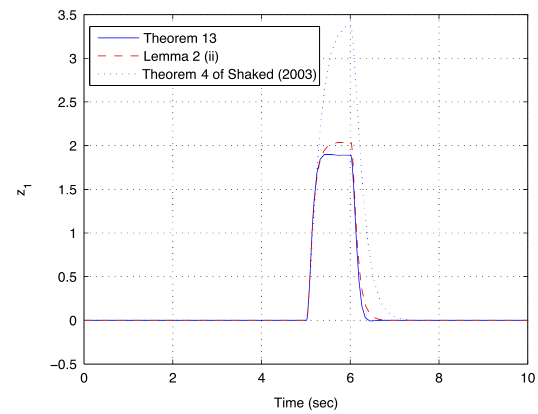

Fig. 1. The trajectory of $z_1(t)$.

$$A_2=\begin{bmatrix}-1.702&50.72&263.5\\0.2201&-1.418&-31.99\\0&0&-30\end{bmatrix},\qquad B_{11}=B_{12}=\begin{bmatrix}0\\1\\1\end{bmatrix},$$

$$B_{21}=\begin{bmatrix}-97.78\\0\\30\end{bmatrix},\qquad B_{22}=\begin{bmatrix}-85.09\\0\\30\end{bmatrix}$$

$$C_{11}=C_{12}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\qquad C_{21}=C_{22}=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$$

$$D_{121}=0,\qquad D_{122}=0.$$

Theorem 4 of Shaked (2003), Lemma 2(ii) (the method in Crusius and Trofino (1999)) and Theorem 13 are applicable for designing H_∞ SOF controllers (note that our earlier work in Dong and Yang (2007) cannot be used for H_∞ control synthesis). The obtained optimal H_∞ performance indices are shown in Table 3.

From Table 3, it can be seen that the obtained optimal H_∞ performance bound by Theorem 13 is smaller than those by Lemma 2(ii) and Theorem 4 of Shaked (2003).

Now the simulations are performed by using the obtained controller gains under assuming that the initial condition $x(0)=0$, and the exogenous disturbance input $w(t)=\begin{cases}1, & 5\le t\le 6\\ 0, & \text{others}\end{cases}$. The response curve of the output $z_1(t)$ and the square root of ratio of the regulated output energy to the disturbance input noise energy are respectively depicted in Figs. 1 and 2. From the figures, it can also be seen that the controller designed by Theorem 13 achieves the best H_∞ performance.

Example 18. The data from the COMP$l_e$ib library (Leibfritz & Lipinski, 2003) is used for testing H_∞ control algorithm in Theorem 13. COMP$l_e$ib library consists of more than 120 examples collected from the engineering literature and real-life (engineering) applications. These examples can be considered as the Benchmark Examples, see Leibfritz and Lipinski (2003).

The numerical results are computed by Theorem 13, Lemma 2 (ii) (the method in Crusius and Trofino (1999)), Theorem 4 of Shaked (2003), the method of Dinh et al. (2012), HIFOO (Gumussoy, Henrion, Millstone, & Overton, 2009), and PENBMI (Henrion, Loefberg, Kocvara, & Stingl, 2005). These methods are respectively represented as M1, M2, M3, M4, M5 and M6 for a simple description.

Table 4
H_∞ synthesis benchmarks on COMPl$_e$ib plants.

Problem	M2	M3	M4	M5	M6	M1	τ in M1	Time of M1 (s)
EB1	–	6.4095	3.1225	39.9526	2.0276	1.3780	3.1623×10^{-5}	30.5294
EB2	–	2.4487	2.0201	39.9547	0.8148	0.8149	1×10^{-5}	0.8148
DIS1	11.3735	4.5584	4.1716	–	4.1943	2.7335	0.4786	40.1256
DIS2	1.1865	0.9895	1.0548	1.7423	1.1546	1.1069	1×10^{-5}	35.9156
DIS3	1.1545	2.5785	1.0816	–	1.1382	1.3569	1×10^{-5}	54.0038
DIS4	0.9315	1.7483	0.7465	–	0.7498	0.9036	1×10^{-5}	77.5972
AC2	–	0.1192	0.1115	–	0.1140	0.2076	1	42.2145
AC3	12.9870	11.7428	4.7021	–	3.4859	3.4488	0.3311	29.0162
AC6	14.8442	12.5441	4.1140	–	4.1954	5.8944	0.1445	92.9142
AC7	–	2.3944	0.0651	0.3810	0.0548	0.1687	0.1	35.3317
AC8	–	2.7881	2.0050	–	3.0520	1.9943	0.2630	43.6520

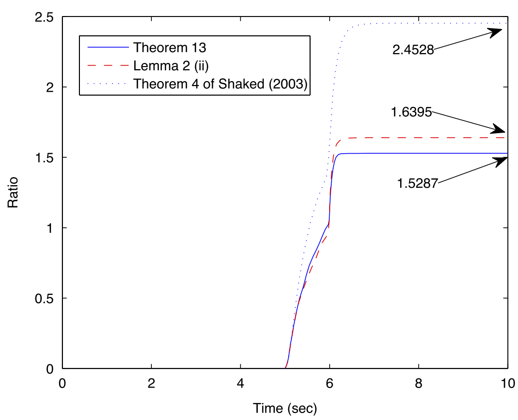

Fig. 2. $\sqrt{\int_0^\infty z^T(t)z(t)dt / \int_0^\infty w^T(t)w(t)dt}$.

Among these methods, Lemma 2(ii) (M2) is convex with a matrix equality constraint; Theorem 4 of Shaked (2003) (M3) is convex with several line searches; HIFOO (Gumussoy et al., 2009) (M4) is an open-source Matlab package for fixed-order controller design by using a hybrid algorithm for nonsmooth, non-convex optimization; PENBMI (Henrion et al., 2005) (M5) is a commercial software for solving optimization problems with quadratic objective and BMI constraints; The method of Dinh et al. (2012) (M6) combines convex–concave decomposition and linearization approaches for solving BMIs. The numerical computational results by using the above-mentioned algorithms (M1–M6) are shown in Table 4.

It can be seen from Table 4 that the optimal values by Theorem 13 (M1) are less than or equal to the ones by Lemma 2(ii) (M2). In contrast to the non-convex methods (M3–M6), the new method can give better performances than Theorem 4 of Shaked (2003) (M3) and PENBMI (M5) in most of the examples, and gives similar results to HIFOO (M4) and convex–concave decomposition and linearization approaches (M6). In particular, Theorem 13 gives the best results for examples EB1 and DIS1.

5. Conclusion

In this paper, the problem of designing static output feedback controllers for continuous-time linear systems has been studied. Sufficient conditions for designing static output feedback controllers have been given in terms of solutions to a set of linear matrix inequalities with a line search over a scalar variable, and the results are also extended to H_∞ static output feedback controller design. In contrast to the existing results, the new proposed approach is applicable for linear polytopic systems, whose uncertain output matrices are not required to be full row rank. In particular, it has been proved that the new proposed conditions are more relaxed than the existing ones with equality constraints between output matrix and Lyapunov matrix. The numerical examples have shown the effectiveness of the new design methods.

Appendix

Theorem 19. *If there exist a symmetric matrix $Q(\alpha) > 0$ and matrices $W(\alpha)$, G, L, a scalar $\tau > 0$, satisfying the following matrix inequality,*

$$\begin{bmatrix} \mathrm{He}\,(A(\alpha)Q(\alpha) + B_2(\alpha)LW(\alpha)) & * \\ C_2(\alpha)Q(\alpha) - GW(\alpha) + \tau L^T B_2^T(\alpha) & -\tau G - \tau G^T \end{bmatrix} < 0 \qquad (36)$$

if and only if the system (1) with $w(k) = 0$ and

$K = LG^{-1}$

is robustly stable.

Proof. The sufficiency can be obtained from the proof of Theorem 4. The necessary part is given as follows:

If the system (1) with $w(k) = 0$ is robustly stable, then from Lyapunov theory, there exists a symmetric matrix $P(\alpha) > 0$, such that

$$\mathrm{He}\,(P(\alpha)A(\alpha) + P(\alpha)B_2(\alpha)KC_2(\alpha)) < 0$$

which is equivalent to

$$\mathrm{He}\,(A(\alpha)Q(\alpha) + B_2(\alpha)KC_2(\alpha)Q(\alpha)) < 0 \qquad (37)$$

with $Q(\alpha) = P^{-1}(\alpha)$.

Choose a matrix G satisfying $G + G^T > 0$, which implies that G is invertible. Let $L = KG$, $W(\alpha) = G^{-1}C_2(\alpha)Q(\alpha)$, then (37) can be rewritten as follows:

$$\mathrm{He}\,(A(\alpha)Q(\alpha) + B_2(\alpha)LW(\alpha)) < 0$$

then there exists a positive scalar τ, such that

$$\mathrm{He}\,(A(\alpha)Q(\alpha) + B_2(\alpha)LW(\alpha)) + \tau B_2(\alpha)L(G + G^T)^{-1}L^T B_2^T(\alpha) < 0.$$

Applying the Schur complement lemma to the above inequality, then it yields that

$$\begin{bmatrix} \mathrm{He}\,(A(\alpha)Q(\alpha) + B_2(\alpha)LW(\alpha)) & \tau B_2(\alpha)L \\ \tau L^T B_2^T(\alpha) & -\tau G - \tau G^T \end{bmatrix} < 0.$$

Combining it and $W(\alpha) = G^{-1}C_2(\alpha)Q(\alpha)$, then we have that (36) holds. Thus, the necessity is proved. □

References

Agulhari, C.M., Oliveira, R. C. L. F., & Peres, P. L. D. (2010). Static output feedback control of polytopic systems using polynomial Lyapunov functions. In *Proceedings of the 49th IEEE conference on decision and control* (pp. 6894–6901).
Ait Rami, Mustapha (2011). Solvability of static output-feedback stabilization for LTI positive systems. *Systems & Control Letters*, *60*(9), 704–708.
Apkarian, P., & Noll, D. (2006). Controller design via nonsmooth multidirectional search. *SIAM Journal on Control and Optimization*, *44*(6), 1923–1949.
Arzelier, D., Gryazina, E. N., Peaucelle, D., & Polyak, B. T. (2010). Mixed LMI/randomized methods for static output feedback control design. In *American control conference* (pp. 4683–4688).

Bara, G. I., & Boutayeb, M. (2005). Static output feedback stabilization with H_∞ performance for linear discrete-time systems. *IEEE Transactions on Automatic Control*, *50*(2), 250–254.
Bernussou, J., Geromel, J. C., & de Oliveira, M. C. (1999). On strict positive real systems design: guaranteed cost and robustness issues. *Systems & Control Letters*, *36*, 135–141.
Boyd, S., & Vandenberghe, L. (2004). *Convex optimization*. Cambridge University Press.
Cao, Y.-Y., Lam, J., & Sun, Y.-X. (1998). Static output feedback stabilization: an ILMI approach. *Automatica*, *34*(12), 1641–1645.
Crusius, C. A. R., & Trofino, A. (1999). Sufficient LMI conditions for output feedback control problems. *IEEE Transactions on Automatic Control*, *44*(5), 1053–1057.
De Caigny, J., Camino, J. F., Oliveira, R. C. L., Peres, P. L. D., & Swevers, J. (2010). Gain-scheduled H_2 and H_∞ control of discrete-time polytopic time-varying systems. *IET Control Theory & Applications*, *4*(3), 362–380.
de Oliveira, M. C., Geromel, J. C., & Hsu, L. (1999). LMI characterization of structural and robust stability: the discrete-time case. *Linear Algebra and its Applications*, *296*, 27–38.
De Souza, C. E., & Trofino, A. (2000). An LMI approach to stabilization of linear discrete-time periodic systems. *International Journal of Control*, *73*(8), 696–703.
Dinh, Q. T., Gumussoy, S., Michiels, W., & Diehl, M. (2012). Combining convex concave decompositions and linearization approaches for solving bmis, with application to static output feedback. *IEEE Transactions on Automatic Control*, *57*(6), 1377–1390.
Dong, J., & Yang, G.-H. (2007). Static output feedback control synthesis for linear systems with time-invariant parametric uncertainties. *IEEE Transactions on Automatic Control*, *52*(10), 1930–1936.
Dong, J., & Yang, G.-H. (2008). Robust static output feedback control for linear discrete-time systems with time-varying uncertainties. *Systems & Control Letters*, *57*(2), 123–131.
Du, Baozhu, Lam, James, & Shu, Zhan (2010). Stabilization for state/input delay systems via static and integral output feedback. *Automatica*, *46*(12), 2000–2007.
El Ghaoui, L., Oustry, F., & AitRami, M. (1997). A cone complementarity linearization algorithm for static output-feedback and related problems. *IEEE Transactions on Automatic Control*, *42*(8), 1171–1176.
Gumussoy, S., Henrion, D., Millstone, M., & Overton, M. L. (2009). Multiobjective robust control with HIFOO 2.0. In *Proc. IFAC symp. robust control design*. Haifa, Israel.
He, Y., & Wang, Q.-G. (2006). An improved ILMI method for static output feedback control with application to multivariable PID control. *IEEE Transactions on Automatic Control*, *51*(10), 1678–1683.
Henrion, D., Loefberg, J., Kocvara, M., & Stingl, M. (2005). Solving polynomial static output feedback problems with PENBMI. In *Proc. joint IEEE conf. decision control and Eur. control conf.* (pp. 7581–7586). Sevilla, Spain.
Ho, Daniel W. C., & Lu, Guoping (2003). Robust stabilization for a class of discrete-time non-linear systems via output feedback: the unified LMI approach. *International Journal of Control*, *76*(2), 105–115.
Huang, D., & Nguang, Sing Kiong (2006). Robust H_∞ static output feedback control of fuzzy systems: an ILMI approach. *IEEE Transactions on Systems, Man and Cybernetics, Part B*, *36*(1), 216–222.
Leibfritz, F. (2001). An LMI-based algorithm for designing suboptimal static H_2/H_∞ output feedback controllers. *Siam Journal on Control and Optimization*, *39*(6), 1711–1735.
Leibfritz, F., & Lipinski, W. (2003). Description of the benchmark examples in COMPl$_e$ib 1.0. *Tech. rep.* Univ. Trier, Dept. Math., Trier, Germany.
Lewis, A. S. (2007). Nonsmooth optimization and robust control. *Annual Reviews in Control*, *31*(2), 167–177.
Lo, Ji-Chang, & Lin, Min-Long (2003). Robust H_∞ nonlinear control via fuzzy static output feedback. *IEEE Transactions on Circuits and Systems I: Regular Papers*, *50*(11), 1494–1502.
Mehdi, D., Boukas, E. K., & Bachelier, O. (2004). Static output feedback design for uncertain linear discrete time systems. *IMA Journal of Mathematical Control and Information*, *21*(1), 1.
Oliveira, Ricardo C. L. F., & Peres, Pedro L. D. (2005). Stability of polytopes of matrices via affine parameter-dependent Lyapunov functions: asymptotically exact LMI conditions. *Linear Algebra and its Applications*, *405*, 209–228.
Peaucelle, D., & Arzelier, D. (2001). An efficient numerical solution for H_2 static output feedback synthesis. In *Proc. Eur. control conf.* (pp. 3800–3805).
Peaucelle, D., & Arzelier, D. (2005). Ellipsoidal sets for resilient and robust static output-feedback. *IEEE Transactions on Automatic Control*, *50*(6), 899–904.
Peaucelle, D., Arzelier, D., Bachelier, O., & Bernussou, J. (2000). A new robust d-stability condition for real convex polytopic uncertainty. *Systems & Control Letters*, *40*(1), 21–30.
Pipeleers, G., Demeulenaerea, B., Sweversa, J., & Vandenbergheb, L. (2009). Extended LMI characterizations for stability and performance of linear systems. *Systems & Control Letters*, *58*, 510–518.
Prempain, E., & Postlethwaite, I. (2001). Static output feedback stabilisation with H_∞ performance for a class of plants. *Systems & Control Letters*, *43*(3), 159–166.
Ramos, D. C. W., & Peres, P. L. D. (2002). An LMI condition for the robust stability of uncertain continuous-time linear systems. *IEEE Transactions on Automatic Control*, *47*(4), 675–678.
Sato, Masayuki (2011). Gain-scheduled output-feedback controllers depending solely on scheduling parameters via parameter-dependent Lyapunov functions. *Automatica*, *47*, 2786–2790.
Seuret, A., Edwards, C., Spurgeon, S. K., & Fridman, E. (2009). Static output feedback sliding mode control design via an artificial stabilizing delay. *IEEE Transactions on Automatic Control*, *54*(2), 256–265.
Shaked, U. (2003). An LPD approach to robust H_2 and H_∞ static output-feedback design. *IEEE Transactions on Automatic Control*, *48*(5), 866–872.
Shu, Zhan, Lam, James, & Xiong, Junlin (2010). Static output-feedback stabilization of discrete-time Markovian jump linear systems: a system augmentation approach. *Automatica*, *46*(4), 687–694.
Syrmos, V. L., Abdallah, C. T., Dorato, P., & Grigoriadis, K. (1997). Static output feedback—a survey. *Automatica*, *33*(2), 125–137.
Trofino, A. (2009). Sufficient LMI conditions for the design of static and reduced order controllers. In *Proceedings of the 48th IEEE conference on decision and control & the 28th Chinese control conference* (pp. 6668–6673).
Yaesh, I., & Shaked, U. (1997). Minimum entropy static output-feedback control with an H_∞-norm performance bound. *IEEE Transactions on Automatic Control*, *42*(6), 853–858.
Yaesh, I., & Shaked, U. (2012). H_∞ optimization with pole constraints of static output-feedback controllers—a non-smooth optimization approach. *IEEE Transactions on Control Systems Technology*, *20*(4), 1066–1072.
Yang, G.-H., & Dong, J. (2008). Robust stability of polytopic systems via affine parameter-dependent Lyapunov functions. *SIAM Journal on Control and Optimization*, *47*(5), 2642–2662.
Zhou, K., Doyle, J. C., & Glover, K. (1996). *Robust and optimal control*. Upper Saddle River, NJ, USA: Prentice-Hall, Inc.

Jiuxiang Dong received the B.S. degree in Mathematics and Applied Mathematics and the M.S. degree in Applied Mathematics, both from Liaoning Normal University, Liaoning, China, in 2001 and 2004, respectively, and the Ph.D. degree in Navigation Guidance and Control from Northeastern University, Shenyang, in 2009. He is currently with the College of Information Science and Engineering and State Key Laboratory of Synthetical Automation for Process Industries, Northeastern University. His research interests include fuzzy control,robust control, and fault-tolerant control.

Guang-Hong Yang received the B.S. and M.S. degrees from Northeast University of Technology, China, in 1983 and 1986, respectively, and the Ph.D. degree in Control Engineering from Northeastern University, China (formerly Northeast University of Technology), in 1994. He was a Lecturer/Associate Professor with Northeastern University from 1986 to 1995. He joined the Nanyang Technological University in 1996 as a Postdoctoral Fellow. From 2001 to 2005, he was a Research Scientist/Senior Research Scientist with the National University of Singapore. He is currently a Professor at the College of Information Science and Engineering, Northeastern University. His current research interests include fault-tolerant control, fault detection and isolation, nonlinear control systems design, and robust control. Dr. Yang is an Associate Editor for the IEEE Transactions on Fuzzy Systems, and the International Journal of Systems Science (IJSS).