Disjoint products and $\mathcal{S}$-partitions

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Abstract

Sum of disjoint products (SDP) algorithms analyze network reliability by producing partitions, and $\mathcal{S}$*-partitions* are certain partitions which generalize shellings. Existing SDP algorithms without multiple-variable inversion produce $\mathcal{S}$-partitions, so $\mathcal{S}$-partitions provide a theoretical framework in which these SDP algorithms may be studied. We introduce *MVI* $\mathcal{S}$*-partitions* which provide the same kind of theoretical framework for SDP algorithms with multiple-variable inversion, and we present some elementary theory of ordinary and MVI $\mathcal{S}$-partitions. We also present examples which show that (MVI) $\mathcal{S}$-partitions are not generally very economical: for these examples there are (MVI) interval partitions which require many fewer intervals than (MVI) $\mathcal{S}$-partitions. We conclude that existing SDP algorithms can be improved by modifying them to allow the production of non-$\mathcal{S}$-partitions, and we briefly discuss the structure of such improved SDP algorithms.

Keywords. Disjoint products, shellings, $\mathcal{S}$-partitions, network reliability.

1. Introduction

A *filter* on a finite set E is a family $\mathfrak{F}$ of subsets of E with the property that whenever $S \in \mathfrak{F}$ and $S \subseteq T \subseteq E$, $T \in \mathfrak{F}$ too. Filters are natural combinatorial models of network reliability problems: E represents the set of elements of a network, and $\mathfrak{F}$ represents the family of subsets of E whose successful operation is sufficient for the successful operation of the network. The assumption that $\mathfrak{F}$ is a filter reflects the natural assumption that a successfully operating network cannot be made to cease operation by enlarging the set of operational elements; this assumption is technically called *coherence*. We are concerned here with *SDP algorithms* (network reliability algorithms which produce partitions of filters), $\mathcal{S}$*-partitions* (a special kind of ordered partition of filters, generalizing shellings) and the relationship between them. We refer to [8] for a survey of SDP algorithms, and to [5] for an introduction to $\mathcal{S}$-partitions.

Our discussion is complicated by three facts. The first of these complicating facts is that filters do not actually appear in the literature of combinatorial geometry, where shellings and $\mathcal{S}$-partitions are studied; this literature is concerned with *complexes*, families which include all subsets of their elements rather than all supersets. This complication is essentially notational, because complementation serves to translate complexes into filters and vice versa. We use complexes in Section 3 when discussing $\mathcal{S}$-partitions because that seems most consistent with the combinatorial literature, and we use filters in the rest of the paper because that seems most consistent with the reliability literature. The second complicating fact is that there are two kinds of SDP algorithms, ordinary ones which utilize only single-variable inversion and *MVI-SDP* algorithms which also involve multiple-variable inversion. It is a simple matter to observe that the partitions produced by existing simple SDP algorithms are $\mathcal{S}$-partitions; we introduce *MVI* $\mathcal{S}$*-partitions* to provide a similar combinatorial framework for the partitions produced by existing MVI-SDP algorithms. The third complicating fact is that $\mathcal{S}$-partitions have received little attention in the literature, much less than shellings, and of course MVI $\mathcal{S}$-partitions have received none before this paper. We present some elementary theory of the two types of $\mathcal{S}$-partitions, including several generalizations of the Rearrangement Lemma of [4].

The central purpose of this paper, however, is not to study the combinatorics of the two types of $\mathcal{S}$-partitions or merely to establish the relationship between them and the two types of SDP algorithms; rather we use this relationship to study the performance of SDP algorithms. A convenient measure of performance is the number of terms the algorithms generate. If a reliability problem possesses a shelling, the shelling provides an optimal SDP analysis [1, 2]; for instance this is the case for an ordinary (all-terminal) reliability problem on a graph. $\mathcal{S}$-partitions are generalized shellings, and MVI $\mathcal{S}$-partitions in turn are generalized $\mathcal{S}$-partitions, so one might expect the fact that existing (MVI-) SDP algorithms produce (MVI) $\mathcal{S}$-partitions to indicate that these algorithms perform efficiently. However we provide examples for which existing (MVI-) SDP algorithm which produce (MVI) $\mathcal{S}$-partitions must produce highly non-optimal partitions, partitions which involve many more intervals than are actually necessary. Moreover, it is not difficult to understand why existing (MVI-) SDP algorithms are so ineffective for these examples. In the last section of the paper we discuss this ineffectiveness and briefly introduce improved SDP algorithms which can produce non-$\mathcal{S}$-partitions, and hence can be more effective than existing ones. We hope to provide a more detailed discussion of the implementation and performance of such algorithms after further study.

2. SDP algorithms

Recall that a *Boolean interval* is $[A, B] = \{S \mid A \subseteq S \subseteq B\}$. Many SDP algorithms produce a partition of a filter $\mathfrak{F}$ into Boolean intervals by following this common outline. If $M_1, ..., M_p$ are the minimal elements of $\mathfrak{F}$ then $\mathfrak{F}$ may be partitioned into the families $\mathfrak{F}_j = \{S \subseteq E \,|\, M_j \subseteq S$ but no $i < j$ has $M_i \subseteq S\}$, one for each $j \in \{1, ..., p\}$. Each $\mathfrak{F}_j$ may be partitioned into Boolean intervals by repeatedly using the fact that if $[A, B]$ is a Boolean interval and C is a set with $C - A = \{c_1, ..., c_k\} \neq \emptyset$ then $\{S \in [A, B] \,|\, C \not\subseteq S\}$ is either $[A, B]$ (if $C - A \not\subseteq B$) or else the union of the pairwise disjoint intervals $[A \cup \{c_1, ..., c_{k-1}\}, B - \{c_k\}]$, $[A \cup \{c_1, ..., c_{k-2}\}, B - \{c_{k-1}\}]$, ..., $[A \cup \{c_1\}, B - \{c_2\}]$, $[A, B - \{c_1\}]$. First this fact is applied with $A = M_j$, $B = E$ and $C =$ an M_{i_i} with $i_1 < j$ to find a partition of $\{S \in [M_j, E] \,|\, M_{i_1} \not\subseteq S\}$, then it is applied to each resulting interval $[A, B]$ using $C =$ an M_{i_2} with $i_1 \neq i_2 < j$, then it is applied to each resulting interval using $C =$ an M_{i_3} with $i_3 < j$ not equal to i_1 or the i_2 used in obtaining that interval, and so on; after $j - 1$ sets of applications one has partitioned $\mathfrak{F}_j$. Implementations of this common outline may differ in their choice of the original order $M_1, ..., M_p$, the orderings within the sets $C - A = \{c_1, ..., c_k\}$, or the orders $M_{i_1}, ..., M_{i_{j-1}}$ in which $M_1, ..., M_{j-1}$ are considered in constructing the various intervals $[A, B]$. These differences may significantly affect the resulting interval partitions [3, 8, 9, 10]. However it is clear that if the algorithm proceeds in an ordered manner, beginning with the list $M_1, ..., M_p$ and at each step inserting the replacement intervals $[A \cup \{c_1, ..., c_{k-1}\}, B - \{c_k\}]$, $[A \cup \{c_1, ..., c_{k-2}\}, B - \{c_{k-1}\}]$, ..., $[A \cup \{c_1\}, B - \{c_2\}]$, $[A, B - \{c_1\}]$ in order at the location of $[A, B]$, then the resulting list has these properties: if $X_1, ..., X_t$ are the minimal elements of the resulting Boolean intervals (in order) then every superset of X_i appears in an interval $[X_j, Y_j]$ with $j \leq i$, the minimal elements $M_1, ..., M_p$ of $\mathfrak{F}$ appear in order in the list $X_1, ..., X_t$, and if $X_i = M_j$ then $[X_1, Y_1] \cup ... \cup [X_i, Y_i] = \mathfrak{F}_1 \cup ... \cup \mathfrak{F}_j$. We will see later that this means the algorithm produces a special kind of $\mathcal{S}$-partition which we call *compatibly* *ordered*.

In the late 1980s several authors introduced modified SDP algorithms which follow the same common outline but allow *multiple-variable inversion* (*MVI*); the first published account seems to appear in [7], and [8] provides a survey. An MVI-SDP algorithm generates a partition of a filter $\mathfrak{F}$ using families of sets which need not be Boolean intervals. Rather they are *MVI intervals*, logically represented by expressions $x_1...x_r\overline{y_{11}...y_{1s_1}}\,\overline{y_{21}...y_{2s_2}}...\overline{y_{u1}...y_{us_u}}$ in which the sets $X = \{x_1, ..., x_r\}$, $Y_1 = \{y_{11}, ..., y_{1s_1}\}$, ..., $Y_u = \{y_{u1}, ..., y_{us_u}\}$ are pairwise disjoint and $Y_1, ..., Y_u$ are nonempty. Here $\overline{y_{v1}...y_{vs_v}}$ represents the negation of the conjunction $y_{v1}...y_{vs_v}$, so the word $x_1...x_r\overline{y_{11}...y_{1s_1}}\,\overline{y_{21}...y_{2s_2}}...\overline{y_{u1}...y_{us_u}}$ represents $\{S \subseteq E \,|\, X \subseteq S$ and $Y_v \not\subseteq S$ for every $v \in \{1, ..., u\}\}$. We denote an MVI interval $[X; \{Y_1, ..., Y_u\}]$; it has a single minimal element X and $\prod_{v=1}^{u} s_v$ maximal elements $E - \{y_1, ..., y_u\}$, given by different choices of $y_v \in Y_v$. Observe that $[X; \emptyset] = [X, E]$ and if every $s_v = 1$ then $[X; \{Y_1, ..., Y_u\}]$ is the Boolean interval $[X, E - \{y_{11}, ..., y_{u1}\}]$. Clearly SDP algorithms which follow the outline given above, except for allowing MVI intervals, partition a filter $\mathfrak{F}$ into an ordered list of MVI intervals in such a way that if the minimal elements of the intervals are $X_1, ..., X_t$ (in order) then every superset of X_i appears in an interval $[X_j; \mathfrak{C}(X_j)]$ with $j \leq i$, the minimal elements $M_1, ..., M_p$ of $\mathfrak{F}$ appear in order in the list $X_1, ..., X_t$, and if $X_i = M_j$ then $[X_1; \mathfrak{C}(X_1)] \cup ... \cup [X_i; \mathfrak{C}(X_i)] = \mathfrak{F}_1 \cup ... \cup \mathfrak{F}_j$. We will see later that this means the partition is a compatibly ordered MVI $\mathcal{S}$-partition.

3. $\mathcal{S}$-partitions of complexes

A *complex* Δ on a finite set E is a family of subsets of E which contains all singleton subsets and has the property that whenever $S \in \Delta$ and $T \subseteq S \subseteq E$, $T \in \Delta$ too. The elements of Δ are *faces*, and the maximal elements are *facets*. An ordering $F_1, ..., F_t$ of some faces of a complex Δ is an *$\mathcal{S}$-partition* [5] if there are subsets $\mathfrak{R}(F_1) \subseteq F_1, ..., \mathfrak{R}(F_t) \subseteq F_t$ such that the Boolean intervals $[\mathfrak{R}(F_i), F_i]$ partition Δ and $\mathfrak{R}(F_i) \not\subseteq F_j$ for $i > j$; that is, every subset of F_j appears in exactly one interval $[\mathfrak{R}(F_i), F_i]$ with $i \leq j$ and in no interval $[\mathfrak{R}(F_i), F_i]$ with $i > j$. The subsets $\mathfrak{R}(F_1), ..., \mathfrak{R}(F_t)$ are the *restrictions* of $F_1, ..., F_t$. Observe that there is no freedom in choosing them: $\mathfrak{R}(F_1)$ must be $\emptyset$ and for $i > 1$, $\mathfrak{R}(F_i)$ must be the intersection of the subsets of F_i which are not contained in any F_j with $j < i$. For $F_1, ..., F_t$ to be an $\mathcal{S}$-partition it must be that for every $i > 1$, this intersection itself is not contained in any F_j with $j < i$.

Complementation serves to translate complexes into filters, and vice versa. Consequently an *$\mathcal{S}$-partition* of a filter $\mathfrak{F}$ may be defined to be a list $X_1, ..., X_t$ of some elements of $\mathfrak{F}$, together with supersets $\mathfrak{E}(X_1) \supseteq X_1, ..., \mathfrak{E}(X_t) \supseteq X_t$ (the *extensions* of $X_1, ..., X_t$) such that the Boolean intervals $[X_i, \mathfrak{E}(X_i)]$ partition $\mathfrak{F}$ and every superset of X_j appears in some interval $[X_i, \mathfrak{E}(X_i)]$ with $i \leq j$. Clearly SDP algorithms which follow the outline given above always produce $\mathcal{S}$-partitions of filters. Although our primary focus is on filters and network reliability, we use complexes in this section for the sake of compatibility with the combinatorial literature.

Here is a useful alternative definition of $\mathcal{S}$-partitions.

Proposition 3.1. *A list $F_1, ..., F_t$ is an $\mathcal{S}$-partition of a complex if and only if for every $j > 1$, the minimal elements of $\{F_j - F_1, ..., F_j - F_{j-1}\}$ are all singletons. Moreover, if $j > 1$ then $\Re(F_j)$ is precisely the union of these singletons.*

Proof. Suppose first that $F_1, ..., F_t$ is an $\mathcal{S}$-partition and $F_j - F_i$ is minimal among $F_j - F_1, ..., F_j - F_{j-1}$ but not a singleton. If $F_j - F_i = \emptyset$ then $\Re(F_j) \subseteq F_j \subseteq F_i$, an impossibility. If $|F_j - F_i| > 1$ then note that $\Re(F_j) \not\subseteq F_i$ implies that $F_j - F_i$ contains at least one element x of $\Re(F_j)$. The minimality of $F_j - F_i$ implies that $\{x\}$ does not contain any $F_j - F_k$ with $k < j$, i.e., for every $k < j$ there is a $y \neq x \in F_j - F_k$. Hence $F_j - \{x\} \not\subseteq F_k$ for every $k < j$. This is impossible, though, for $F_j - \{x\} \notin [\Re(F_j), F_j]$ and hence $F_j - \{x\} \in [\Re(F_k), F_k]$ for some $k < j$.

Conversely, suppose $F_1, ..., F_t$ is a list with the property that for every $j > 1$ all the inclusion-minimal elements of $\{F_j - F_1, ..., F_j - F_{j-1}\}$ are singletons. Let $\Re(F_1) = \emptyset$ and for $j > 1$ let $\Re(F_j) = \{$ elements of the minimal $F_j - F_i$ with $i < j$ $\}$. Note that if $i < j$ and $S \in [\Re(F_j), F_j]$ then S contains an element of $F_j - F_i$, so $S \notin [\Re(F_i), F_i]$; hence $[\Re(F_j), F_j] \cap [\Re(F_i), F_i] = \emptyset$. Also, if $S \subseteq F_j$ and $S \notin [\Re(F_j), F_j]$ then $\Re(F_j) \not\subseteq S$ and hence there is an $i < j$ with $(F_j - F_i) \cap S = \emptyset$; it follows that $S \subseteq F_i$. If $S \notin [\Re(F_i), F_i]$ then the same argument shows that $S \subseteq F_h$ for some $h < i$. Continuing in this vein, we must eventually find a $g < j$ with $S \in [\Re(F_g), F_g]$. ■

Corollary 3.2. *Suppose $F_1, ..., F_t$ is an $\mathcal{S}$-partition, $j > 1$ and $x \in F_j - (\cup_{i<j} F_i)$. Then $\Re(F_j) = \{x\}$.*

Proof. The hypothesis implies that $\{x\}$ is a subset of $F_j - F_i$ for every $i < j$. Some such $F_j - F_i$ is a singleton and hence is equal to $\{x\}$; then that $F_j - F_i$ is the only minimal one. ■

An $\mathcal{S}$-partition $F_1, ..., F_t$ of Δ is called *full* [5] or a *shelling* [4] if only facets appear in the list. We refer the reader to the references of these two papers for the beautiful and deep theory of shellings of *pure* complexes (those whose facets are all of the same cardinality), and to more recent papers citing these two for subsequent work on shellings of non-pure complexes.

Every complex admits $\mathcal{S}$-partitions; for instance there are always the trivial $\mathcal{S}$-partitions in which $F_1, ..., F_t$ are all the faces of Δ, listed in any order of nondecreasing cardinality. These trivial examples may give the impression that $\mathcal{S}$-partitions are generally less interesting or less important than shellings. One reason we do not agree with this impression is that every $\mathcal{S}$-partition of a complex Δ gives rise to a shelling of a related complex Δ' through the following simple construction. Given a list $F_1, ..., F_t$ of subsets of E such that $F_i \not\subseteq F_j$ for $i > j$, let $e'_1, ..., e'_{t-1} \notin E$, and let $F'_1 = F_1 \cup \{e'_1, ..., e'_{t-1}\}$, $F'_2 = F_2 \cup \{e'_2, ..., e'_{t-1}\}$, ..., $F'_{t-1} = F_{t-1} \cup \{e'_{t-1}\}$, $F'_t = F_t$. Let Δ be the complex of subsets of E which are contained in some F_i, and let Δ' be the complex of subsets of $E' = E \cup \{e'_1, ..., e'_{t-1}\}$ which are contained in some F'_i. Then it is easily seen that $F_1, ..., F_t$ is an $\mathcal{S}$-partition of Δ if and only if $F'_1, ..., F'_t$ is a shelling of Δ'. (The elements $e'_1, ..., e'_{t-1}$ are adjoined to E to guarantee that no F'_j contains another, so there is a more economical version of the construction which introduces a new element e'_j only if there is an $i > j$ with $F_j \subseteq F_i$.)

Chari [5] observes that the unshellable complex with facets $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, and $\{4,5\}$ may be partitioned into the four intervals $[\emptyset, \{4,5\}]$, $[\{1\}, \{1,2\}]$, $[\{3\}, \{1,3\}]$, and $[\{2\}, \{2,3\}]$. This shows that it is possible that no $\mathcal{S}$-partition of a complex Δ

is a minimum interval partition, i.e., a partition involving the smallest possible number of Boolean intervals. We proceed to present examples for which $\mathcal{S}$-partitions are arbitrarily far from being minimum interval partitions.

Let $n > 1$ be an integer, and let $E = \{a_1, ..., a_n, b_1, ..., b_n, c_1, ..., c_n\}$. Let $A = \{a_1, ..., a_n\}$, $B = \{b_1, ..., b_n\}$ and $C = \{c_1, ..., c_n\}$, and consider the complex Δ on E whose facets are $A \cup B$, $B \cup C$ and $A \cup C$. Δ may be partitioned into the $3n + 1$ Boolean intervals $[\{a_1\}, A \cup B]$, $[\{a_2\}, (A \cup B) - \{a_1\}]$, ..., $[\{a_n\}, (A \cup B) - \{a_1, ..., a_{n-1}\}]$, $[\{b_1\}, B \cup C]$, $[\{b_2\}, (B \cup C) - \{b_1\}]$, ..., $[\{b_n\}, (B \cup C) - \{b_1, ..., b_{n-1}\}]$, $[\{c_1\}, A \cup C]$, $[\{c_2\}, (A \cup C) - \{c_1\}]$, ..., $[\{c_n\}, (A \cup C) - \{c_1, ..., c_{n-1}\}]$ and $[\emptyset, \emptyset]$.

Suppose $F_1, ..., F_t$ is an $\mathcal{S}$-partition of Δ; we may presume that A, B, C are named in such a way that $0 = |F_1 \cap C| \leq |F_1 \cap B| \leq |F_1 \cap A| \neq 0$. Suppose $\frac{n}{2} \leq |F_1 \cap B|$ and consider any $c_i \in C$. Let $j(i)$ be the least j with $c_i \in F_j$; then Corollary 3.2 implies that $\{c_i\} = \mathfrak{R}(F_{j(i)})$. Suppose $F_{j(i)} \subseteq A \cup C$ and consider any $b_k \in F_1 \cap B$. The set $\{b_k, c_i\}$ must appear in some interval $[\mathfrak{R}(F_a), F_a]$. It cannot happen that $a = 1$ or $a = j(i)$, because $F_1 \subseteq A \cup B$ and $F_{j(i)} \subseteq A \cup C$. It follows that $\{b_k, c_i\} = \mathfrak{R}(F_a)$, for $\emptyset, \{b_k\} \in [\mathfrak{R}(F_1), F_1]$ and $\{c_i\} \in [\mathfrak{R}(F_{j(i)}, F_{j(i)}]$ and hence no proper subset of $\{b_k, c_i\}$ can appear in $[\mathfrak{R}(F_a), F_a]$. Similarly, if $F_{j(i)} \subseteq B \cup C$ then every set $\{a_k, c_i\}$ with $a_k \in F_1 \cap A$ must be $\mathfrak{R}(F_a)$ for some a. Whereas $\frac{n}{2} \leq |F_1 \cap B| \leq |F_1 \cap A|$, we conclude that there are at least $n(\frac{n}{2})$ values of a such that $\mathfrak{R}(F_a)$ has two elements. Corollary 3.2 implies that there are at least n values of a such that $\mathfrak{R}(F_a)$ is a singleton, one for each $c_i \in C$; considering F_1, we conclude that $t \geq 1 + n + \frac{n^2}{2}$.

Suppose now that $\frac{n}{2} > |F_1 \cap B|$. Given any a_i, b_j and c_k such that $b_j \notin F_1$, consider the three pairs $\{a_i, b_j\}$, $\{a_i, c_k\}$, and $\{b_j, c_k\}$. No two may appear together in any one interval $[\mathfrak{R}(F_a), F_a]$, because no F_a may intersect all three sets A, B, C. Also, none of the pairs may appear in the first interval $[\mathfrak{R}(F_a), F_a]$ containing one of $\{a_i\}, \{b_j\}, \{c_k\}$, because Corollary 3.2 and the hypothesis $b_j \notin F_1$ together imply that this first F_a only contains one of a_i, b_j, c_k. It follows that at least one of these pairs is $\mathfrak{R}(F_a)$ for some a. There are more than $n^2(\frac{n}{2})$ such triples, and no more than n of them may give rise to the same pair $\mathfrak{R}(F_a)$. We conclude that there are at least $\frac{n^2}{2}$ values of a with $|\mathfrak{R}(F_a)| = 2$. As before, this implies that $t \geq 1 + n + \frac{n^2}{2}$.

We conclude that for these examples, the ratio

$$\frac{\text{minimum number of Boolean intervals in an } \mathcal{S}\text{-partition of } \Delta}{\text{minimum number of Boolean intervals in an interval partition of } \Delta} \geq \frac{1 + n + \binom{n^2}{2}}{3n+1}.$$

Of course, the filters complementary to these complexes satisfy a corresponding inequality.

Before proceeding we observe that not only $\mathcal{S}$-partitions are subject to the lower bound $1 + n + \frac{n^2}{2} \leq t$. The lower bound also applies to any Boolean interval partition $[X_1, Y_1], ..., [X_t, Y_t]$ of Δ which is compatibly ordered in the sense that the facets $F_1 = A \cup B$, $F_2 = A \cup C$ and $F_3 = B \cup C$ appear in order in the list $Y_1, ..., Y_t$ and if $Y_i = F_j$ then $[X_1, Y_1] \cup ... \cup [X_i, Y_i] = \{$elements of Δ contained in one of $F_1, ..., F_j\}$. To verify this, observe first that if $Y_i = A \cup B$ then $[X_1, Y_1] \cup ... \cup [X_i, Y_i] = [\emptyset, A \cup B]$, so we may as well assume that $[X_1, Y_1] = [\emptyset, A \cup B]$. Now if $Y_i = A \cup C$ then $[X_2, Y_2] \cup ... \cup [X_i, Y_i] = \{$subsets of $A \cup C$ which are not contained in $A \cup B\}$ has n minimal elements, namely the singletons $\{c_k\}$, so there must be at least n such intervals. Finally, the remaining intervals partition $\{$subsets of $B \cup C$ which are not contained in $A \cup B$ or $A \cup C\}$, and this set has n^2 minimal elements, namely the pairs

$\{b_j, c_k\}$; hence there must be at least n^2 of these remaining intervals. Consequently such a partition must actually satisfy $1 + n + n^2 \leq t$.

4. Rearrangements

In this section we present several generalizations of the Rearrangement Lemma for Shellings [4]. These lemmas are not directly useful in our discussion of SDP algorithms, but they provide some insight into the structure of the partitions the algorithms produce.

First we generalize the definition of an $\mathcal{S}$-partition of a filter $\mathfrak{F}$ by defining a list $X_1, ..., X_t$ of elements of $\mathfrak{F}$ to be an *MVI $\mathcal{S}$-partition* if for each i there is an *extension* $\mathfrak{E}(X_i) = \{Y_{i1}, ..., Y_{iu_i}\}$ such that $Y_{i1}, ..., Y_{iu_i}$ are pairwise disjoint nonempty subsets of $E - X_i$, the MVI intervals $[X_i; \{Y_{i1}, ..., Y_{iu_i}\}]$ partition $\mathfrak{F}$, and for each j every superset of X_j appears in an MVI interval $[X_i; \{Y_{i1}, ..., Y_{iu_i}\}]$ with $i \leq j$. As with ordinary $\mathcal{S}$-partitions, there is no freedom in choosing the extensions $\mathfrak{E}(X_i)$.

Proposition 4.1. *Let $X_1, ..., X_t$ be a sequence of subsets of a finite set E and let $\mathfrak{F} = \{S \subseteq E \mid \text{some } X_i \subseteq S\}$. Then $X_1, ..., X_t$ is an MVI $\mathcal{S}$-partition of $\mathfrak{F}$ if and only if for every $j > 1$ the inclusion-minimal elements of $\{X_1 - X_j, ..., X_{j-1} - X_j\}$ are nonempty and pairwise disjoint. Moreover, if $j > 1$ then $\mathfrak{E}(X_j)$ is the set of inclusion-minimal elements of $\{X_1 - X_j, ..., X_{j-1} - X_j\}$.*

Proof. Suppose $X_1, ..., X_t$ is an MVI $\mathcal{S}$-partition with extensions $\mathfrak{E}(X_j)$. Suppose $i < j$ and $X_i - X_j$ does not contain any Y_{jv}. Then $(X_i - X_j) \cup X_j = X_i \cup X_j$ also does not contain any Y_{jv}, so $X_i \cup X_j \in [X_j; \mathfrak{E}(X_j)]$. This is impossible, though, for $X_i \cup X_j$ is a superset of X_i and hence must appear in some $[X_k; \mathfrak{E}(X_k)]$ with $k \leq i$;

hence $X_i - X_j$ contains some Y_{jv}. If $v \in \{1, ..., u_j\}$ then $X_j \cup Y_{jv} \notin [X_j; \mathfrak{E}(X_j)]$, so there is an $i < j$ with $X_j \cup Y_{jv} \in [X_i; \mathfrak{E}(X_i)]$. It follows that $X_i - X_j \subseteq Y_{jv}$. If $X_i - X_j$ is a proper subset of Y_{jv} then $X_i \cup X_j = (X_i - X_j) \cup X_i \in [X_j; \mathfrak{E}(X_j)]$, an impossibility because $X_i \cup X_j$ is a superset of X_i and $i > j$; hence $X_i - X_j = Y_{jv}$.

This shows that $\mathfrak{E}(X_j)$ is the set of inclusion-minimal elements of $\{X_1 - X_j, ..., X_{j-1} - X_j\}$; the elements of $\mathfrak{E}(X_j)$ are nonempty and pairwise disjoint by definition.

Conversely, suppose that for every $j > 1$ the inclusion-minimal elements of $\{X_1 - X_j, ..., X_{j-1} - X_j\}$ are nonempty and pairwise disjoint. Let $\mathfrak{E}(X_1) = \emptyset$ and for $j > 1$ let $\mathfrak{E}(X_j)$ be the set of inclusion-minimal relative complements $X_i - X_j$ with $i < j$. If S is a superset of X_j which is not an element of $[X_j; \mathfrak{E}(X_j)]$ then S must contain $X_i - X_j$ for some $i < j$, and hence $X_i \subseteq S$. If S is not an element of $[X_i; \mathfrak{E}(X_i)]$ then the same argument shows that S must be a superset of an X_h with $h < i$; continuing in this vein we must ultimately find a $g < j$ with $S \in [X_g; \mathfrak{E}(X_g)]$. Also, if $i < j$ then $[X_i; \mathfrak{E}(X_i)]$ and $[X_j; \mathfrak{E}(X_j)]$ must be disjoint, because X_i contains an element of $\mathfrak{E}(X_j)$. Consequently the intervals $[X_j; \mathfrak{E}(X_j)]$ partition $\mathfrak{F}$. ■

There are two generalizations of the Rearrangement Lemma to MVI $\mathcal{S}$-partitions.

The Transposition Lemma for MVI $\mathcal{S}$-Partitions. *Suppose $X_1, ..., X_t$ is an MVI $\mathcal{S}$-partition and $a < t$. Then $X_1, ..., X_{a-1}, X_{a+1}, X_a, X_{a+2}, ..., X_t$ is not an MVI $\mathcal{S}$-partition with the same extension function $\mathfrak{E}$ if and only if $X_{a+1} \cup X_a \in [X_a; \mathfrak{E}(X_a)]$. Moreover, if this is the case then $X_a - X_{a+1} \in \mathfrak{E}(X_{a+1})$.*

Proof. Let $\mathfrak{E}(X_{a+1}) = \{Y_{(a+1)1}, ..., Y_{(a+1)u_{a+1}}\}$. If $X_1, ..., X_{a-1}, X_{a+1}, X_a, X_{a+2}, ..., X_t$ is not an MVI $\mathcal{S}$-partition with the same extensions as $X_1, ..., X_t$, there must be a subset S of E which contains X_{a+1} and appears in $[X_a; \mathfrak{E}(X_a)]$. S cannot also appear in $[X_{a+1}; \mathfrak{E}(X_{a+1})]$, for the intervals are disjoint; hence there must be a $v \in \{1, ..., u_{a+1}\}$ such that $Y_{(a+1)v} \subseteq S$.

Consider $X_{a+1} \cup Y_{(a+1)v}$. It contains X_{a+1} and does not appear in $[X_{a+1}; \mathfrak{E}(X_{a+1})]$, so $X_{a+1} \cup Y_{(a+1)v} \in [X_i; \mathfrak{E}(X_i)]$ for some $i \leq a$. If $i < a$ then $S \supseteq X_i$ and hence S appears in $[X_j; \mathfrak{E}(X_j)]$ for some $j \leq i$, violating the disjointness of $[X_j; \mathfrak{E}(X_j)]$ and $[X_a; \mathfrak{E}(X_a)]$. It follows that $i = a$, and hence that $Y_{(a+1)v} \supseteq X_a - X_{a+1}$. Note that $X_a \subseteq X_{a+1} \cup X_a \subseteq X_{a+1} \cup Y_{(a+1)v}$ and $X_a, X_{a+1} \cup Y_{(a+1)v} \in [X_a; \mathfrak{E}(X_a)]$, so $X_{a+1} \cup X_a \in [X_a; \mathfrak{E}(X_a)]$. If $X_a - X_{a+1}$ were a proper subset of $Y_{(a+1)v}$ then $X_{a+1} \cup X_a = X_{a+1} \cup (X_a - X_{a+1})$ would be an element of $[X_{a+1}; \mathfrak{E}(X_{a+1})]$; this would violate the disjointness of $[X_{a+1}; \mathfrak{E}(X_{a+1})]$ and $[X_a; \mathfrak{E}(X_a)]$. Hence $X_a - X_{a+1} = Y_{(a+1)v}$.

Conversely, if $X_a \cup X_{a+1} \in [X_a; \mathfrak{E}(X_a)]$ then clearly $X_1, ..., X_{a-1}, X_{a+1}, X_a, X_{a+2}, ..., X_t$ is not an MVI $\mathcal{S}$-partition with the same extensions as $X_1, ..., X_t$, because $X_a \cup X_{a+1}$ is a superset of X_{a+1} which appears in an MVI interval listed after $[X_{a+1}; \mathfrak{E}(X_{a+1})]$. ■

The Rearrangement Lemma for MVI $\mathcal{S}$-Partitions. *An MVI $\mathcal{S}$-partition $X_1, ..., X_t$ may be rearranged as an MVI $\mathcal{S}$-partition $X'_1, ..., X'_t$ with the same extension function, such that for each a either $|X'_a| \leq \left|X'_{a+1}\right|$ or $X'_a \cup X'_{a+1} \in [X'_a; \mathfrak{E}(X'_a)]$.*

Proof. Let $b = b(X_1, ..., X_t)$ be the least a with $|X_a| > |X_{a+1}|$ and $X_a \cup X_{a+1} \notin [X_a; \mathfrak{E}(X_a)]$. If there is no such a then let $b(X_1, ..., X_t) = t$ and observe that the proposition is satisfied without any rearrangement.

Suppose $b = t - 1$. The Transposition Lemma implies that $X_1, ..., X_{t-2}, X_t, X_{t-1}$ is an MVI $\mathcal{S}$-partition with the same extensions as $X_1, ..., X_t$. If $|X_{t-2}| \leq |X_t|$ or

$X_t \cup X_{t-2} \in [X_{t-2}; \mathfrak{E}(X_{t-2})]$ then this rearrangement satisfies the proposition, because $|X_{t-1}| > |X_t|$. Otherwise the Transposition Lemma implies that $X_1, ..., X_{t-3}, X_t, X_{t-2}, X_{t-1}$ is an MVI $\mathcal{S}$-partition with the same extensions as $X_1, ..., X_t$. Continuing in this vein, if we do not find a rearrangement that satisfies the proposition we will eventually conclude that $X_1, X_t, X_2, ..., X_{t-1}$ is an MVI $\mathcal{S}$-partition with the same extensions as $X_1, ..., X_t$. This rearrangement satisfies the proposition because $X_1 \cup X_t \in [X_1; \mathfrak{E}(X_1)] = [X_1, E]$.

The proof proceeds by induction on $t - b > 1$. There must be a $k < b$ such that $X_1, ..., X_k, X_{b+1}, X_{k+1}, ..., X_b, X_{b+2}, ..., X_t$ is an MVI $\mathcal{S}$-partition with the same extensions as $X_1, ..., X_t$ and either $|X_k| \leq |X_{b+1}|$ or $X_{b+1} \cup X_k \in [X_k; \mathfrak{E}(X_k)]$, for $k = 1$ will be such a k if there is no greater such k. Then $b(X_1, ..., X_k, X_{b+1}, X_{k+1}, ..., X_b, X_{b+2}, ..., X_t) > b$ and the inductive hypothesis applies. ■

Comparing Propositions 3.1 and 4.1, we see that $\mathcal{S}$-partitions of filters are simply MVI $\mathcal{S}$-partitions for which the extensions $\mathfrak{E}(X_j)$ contain only singletons. It follows that the Transposition and Rearrangement Lemmas for MVI $\mathcal{S}$-partitions apply directly to $\mathcal{S}$-partitions, because they are concerned with rearrangements which do not alter the extensions. We deduce the following results regarding rearrangements of $\mathcal{S}$-partitions of complexes.

The Transposition Lemma for $\mathcal{S}$-Partitions. *Suppose $F_1, ..., F_t$ is an $\mathcal{S}$-partition of a complex and $a < t$. Then $F_1, ..., F_{a-1}, F_{a+1}, F_a, F_{a+2}, ..., F_t$ is not an $\mathcal{S}$-partition with the same restriction function $\mathfrak{R}$ if and only if $F_{a+1} \cap F_a \in [\mathfrak{R}(F_a), F_a]$. Moreover if this is the case then $F_{a+1} - F_a \subseteq \mathfrak{R}(F_{a+1})$ and $|F_{a+1} - F_a| = 1$.*

The Rearrangement Lemma for $\mathcal{S}$-Partitions. *An $\mathcal{S}$-partition $F_1, ..., F_t$ of* *a complex may be rearranged as an $\mathcal{S}$-partition $F_1', ..., F_t'$ with the same restriction function, such that for each a either $|F_a'| \geq |F_{a+1}'|$ or $F_a' = F_{a+1}' - \{x\}$ for some $x \in F_{a+1}'$.*

Proof. The Rearrangement Lemma for MVI $\mathcal{S}$-partitions provides a rearrangement $F_1', ..., F_t'$ with the same restriction function, such that for each a either $|F_a'| \geq |F_{a+1}'|$ or $F_a' \cap F_{a+1}' \in [\Re(F_a'), F_a']$. If $|F_a'| < |F_{a+1}'|$ then $F_a' \cap F_{a+1}' \in [\Re(F_a'), F_a']$ and the Transposition Lemma for $\mathcal{S}$-partitions tells us that $F_{a+1}' - F_a' \subseteq \Re(F_{a+1}')$ and $|F_{a+1}' - F_a'| = 1$. Together, $|F_a'| < |F_{a+1}'|$ and $|F_{a+1}' - F_a'| = 1$ imply $F_a' = F_{a+1}' - \{x\}$ for some $x \in F_{a+1}'$. ■

The Rearrangement Lemma for $\mathcal{S}$-Partitions is a direct generalization of the Rearrangement Lemma for Shellings [4]: if $F_1, ..., F_t$ are all facets then no F_j may contain another and hence $F_a' = F_{a+1}' - \{x\}$ is impossible.

5. Some examples of MVI partitions

Let $n \geq 1$, and let b_n be the number of partitions of an n-element set into subsets of cardinality ≤ 2. The first few of these numbers are $b_1 = 1$, $b_2 = 2$, $b_3 = 4$, $b_4 = 10$, $b_5 = 26$, $b_6 = 76$, and $b_7 = 232$; in general $b_{n+2} = b_{n+1} + (n+1)b_n$.

Lemma 5.1.

$$b_{n+2} > n \sum_{i=1}^{n} b_i.$$

Proof. The inequality is clearly true for the listed values of b_n. If $n \geq 6$ then

$$b_{n+2} = b_{n+1} + (n+1)b_n = b_n + nb_{n-1} + (n+1)b_n = nb_n + nb_{n-1} + 2b_n$$

and hence by induction

$$b_{n+2} > nb_n + nb_{n-1} + 2(n-2)\sum_{i=1}^{n-2} b_i > nb_n + nb_{n-1} + n\sum_{i=1}^{n-2} b_i.$$

■

Let $E = \{e_{\{i,j\}} \,|\, 1 \leq i \neq j \leq n\}$, and let $\mathfrak{F}$ be the filter on E with minimal elements $M_1, ..., M_n$ given by $M_j = \{e_{\{i,j\}} \,|\, i \neq j\}$. We claim that if $2 \leq p \leq n-1$ then it is possible to partition $\mathfrak{F}_p = \{S \subseteq E \,|\, S$ contains M_p but does not contain any of $M_1, ..., M_{p-1}\}$ into b_{p-1} MVI intervals; the partition we present is actually part of the output of an MVI-SDP algorithm of the type described in Section 2. We order the subsets of $\{1, ..., p-1\}$ of cardinality ≤ 2 lexicographically, i.e., if $1 \leq a < b < c \leq p-1$ then $\{a\} < \{a,b\} < \{a,c\} < \{b\} < \{b,c\} < \{c\}$. Then we order the partitions of $\{1, ..., p-1\}$ into sets of cardinality ≤ 2 lexicographically, i.e., if π and π' are partitions of $\{1, ..., p-1\}$ into sets of cardinality ≤ 2 and a is the smallest index which appears in different sets in π and π' then $\pi < \pi'$ or $\pi > \pi'$ according to the sets in π and π' which contain a. For instance, if $p = 8$ then $\{\{1,3\},\{2,5\},\{4\},\{6,7\}\} > \{\{1,3\},\{2,4\},\{5,6\},\{7\}\}$ because the smallest index which appears in different sets in the two partitions is $a = 2$, and $\{2,5\} > \{2,4\}$ lexicographically.

Suppose π is a partition of $\{1, ..., p-1\}$ into sets of cardinality ≤ 2, and let P_π be the set of all pairs $\{a,b\}$ for which there is a partition π' of $\{1, ..., p-1\}$ into sets of cardinality ≤ 2 such that $\pi < \pi'$, a is the smallest index which appears in different sets in π and π', and $\{a,b\} \in \pi'$. That is, if we regard π as the result of a sequence of choices made in lexicographic order then P_π is the set of pairs $\{a,b\}$ which could have been chosen but were not, and would have resulted in a lexicographically greater partition. Let $X_\pi = M_p \cup \{e_{\{a,b\}} \,|\, \{a,b\} \in P_\pi\}$ and $\mathfrak{E}(X_\pi) = \{\{e_{\{a,b\}}\} \,|\, \{a,b\} \in \pi\} \cup \{M_a - X_\pi \,|\, \{a\} \in \pi\}$; obviously the MVI interval $[X_\pi; \mathfrak{E}(X_\pi)]$ is contained in $\mathfrak{F}_p$.

If $S \subseteq E$ contains M_p but does not contain any of $M_1, ..., M_{p-1}$ then there are certainly partitions π of $\{1, ..., p-1\}$ into sets of cardinality ≤ 2 such that $e_{\{a,b\}} \notin S$ for all $\{a,b\} \in \pi$, for instance $\{\{1\}, \{2\}, \{3\}, ..., \{p-1\}\}$. Let $\pi(S)$ be the lexicographically greatest partition of $\{1, ..., p-1\}$ into sets of cardinality ≤ 2 such that $e_{\{a,b\}} \notin S$ for all $\{a,b\} \in \pi(S)$, and suppose there is an $\{a,b\} \in P_{\pi(S)}$ with $e_{\{a,b\}} \notin S$. According to the definition of $P_{\pi(S)}$, there is a partition π' of $\{1, ..., p-1\}$ into sets of cardinality ≤ 2 such that $\pi(S) < \pi'$, a is the smallest index which appears in different sets in $\pi(S)$ and π', and $\{a,b\} \in \pi'$. If π'' is obtained from π' by replacing every pair $\{c,d\} \in \pi' - \pi - \{\{a,b\}\}$ with the two singletons $\{c\}$ and $\{d\}$ then π'' is lexicographically greater than $\pi(S)$ and $\{a,b\}$ is the only pair in π'' which is not also an element of $\pi(S)$. Hence $e_{\{c,d\}} \notin S$ for all $\{c,d\} \in \pi''$, contradicting the choice of $\pi(S)$. This contradiction shows that there is no $\{a,b\} \in P_{\pi(S)}$ with $e_{\{a,b\}} \notin S$, and hence $X_{\pi(S)} \subseteq S$. Recalling that $e_{\{a,b\}} \notin S$ for all $\{a,b\} \in \pi(S)$ and that S does not contain any of $M_1, ..., M_{p-1}$, we conclude that $S \in [X_{\pi(S)}; \mathfrak{E}(X_{\pi(S)})]$.

Suppose $S \in \mathfrak{F}_p$ and $\pi' \neq \pi(S)$. If $\pi' < \pi(S)$ then there must be a pair $\{a,b\} \in \pi(S) \cap P_{\pi'}$. Then $e_{\{a,b\}} \notin S$, so $X_{\pi'} \nsubseteq S$. If $\pi' > \pi(S)$ then there must be a pair $\{a,b\} \in \pi'$ such that $e_{\{a,b\}} \in S$; then $\{\{a,b\}\} \in \mathfrak{E}(X_{\pi'})$. Either way, $S \notin [X_{\pi'}; \mathfrak{E}(X_{\pi'})]$. We conclude that the MVI intervals $[X_\pi; \mathfrak{E}(X_\pi)]$ partition $\mathfrak{F}_p$, verifying our claim.

It follows directly from our claim that if $2 \leq p \leq n-1$ and $j_1, ..., j_p \in \{1, ..., n\}$ are pairwise distinct then it is possible to partition $\{S \subseteq E \,|\, S$ contains M_{j_p} but does not contain any of $M_{j_1},...,M_{j_{p-1}}\}$ into b_{p-1} MVI intervals. More generally, if $2 \le p \le q \le n-1$ and $j_1,...,j_q \in \{1,...,n\}$ then it is possible to partition $\{S \subseteq E \,|\, S$ contains all of $M_{j_p},...,M_{j_q}$ but does not contain any of $M_{j_1},...,M_{j_{p-1}}\}$ into at most b_{p-1} MVI intervals: simply partition $\{S \subseteq E \,|\, S$ contains M_{jp} but does not contain any of $M_{j_1},...,M_{j_{p-1}}\}$ into b_{p-1} MVI intervals, replace each X_π by $X'_\pi = X_\pi \cup M_{j_{p+1}} \cup ... \cup M_{j_q}$, replace each $M_a - X_\pi$ in $\mathfrak{E}(X_\pi)$ by $M_a - X'_\pi$, and remove any of the resulting intervals that are empty.

To use this analysis in constructing an MVI partition of $\mathfrak{F}$ we will require the following lemma, whose proof we leave to the reader.

Lemma 5.2. *The disjunction $E_1 \vee E_2 \vee E_3 \vee E_4 \vee E_5 \vee E_6 \vee E_7$ of seven statements is logically equivalent to the disjunction*

$$
\begin{aligned}
&(E_1 \wedge E_2 \wedge E_3 \wedge E_4 \wedge E_5 \wedge E_6 \wedge E_7)\\
&\bigvee \left(\bigvee_{a=1}^{7} (E_a \wedge E_{a+1} \wedge E_{a+2} \wedge E_{a+3} \wedge E_{a+5} \wedge \bar{E}_{a+6}) \right)\\
&\bigvee \left(\bigvee_{a=1}^{7} (E_a \wedge \bar{E}_{a+1} \wedge E_{a+3} \wedge \bar{E}_{a+4} \wedge E_{a+5}) \right)\\
&\bigvee \left(\bigvee_{a=1}^{7} (E_a \wedge \bar{E}_{a+1} \wedge \bar{E}_{a+2} \wedge E_{a+3} \wedge E_{a+4}) \right)\\
&\bigvee \left(\bigvee_{a=1}^{7} (E_a \wedge \bar{E}_{a+1} \wedge \bar{E}_{a+2} \wedge \bar{E}_{a+3}) \right)
\end{aligned}
$$

of 29 disjoint statements, where all indices involving a are considered modulo 7. That is, every assignment of truth values to $E_1,...,E_7$ which involves at least one "true" will satisfy exactly one of the 29 given statements.

If $n > 7$ then Lemma 5.2 may be used to construct $n+22$ disjoint statements whose disjunction is logically equivalent to $E_1 \vee ... \vee E_n$. Twenty-nine of these statements are the conjunctions of the terms from Lemma 5.2 with $\bar{E}_8 \wedge ... \wedge \bar{E}_n$, and the remaining $n-7$ are $E_8 \wedge \bar{E}_9 \wedge ... \wedge \bar{E}_n$, $E_9 \wedge \bar{E}_{10} \wedge ... \wedge \bar{E}_n$, ..., $E_{n-2} \wedge \bar{E}_{n-1} \wedge \bar{E}_n$, $E_{n-1} \wedge \bar{E}_n$, and E_n. It follows that if $n \geq 7$ then $\mathfrak{F}$ may be partitioned into the following families of sets, with indices involving a considered modulo 7.

(i) the family of subsets of E which contain no M_j with $j > 7$ and contain $M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5 \cup M_6 \cup M_7$

(ii) for each $a \in \{1, 2, 3, 4, 5, 6, 7\}$, the family of subsets of E which contain no M_j with $j > 7$, contain $M_a, M_{a+1}, M_{a+2}, M_{a+3}$ and M_{a+5} but do not contain M_{a+6}

(iii) for each $a \in \{1, 2, 3, 4, 5, 6, 7\}$, the family of subsets of E which contain no M_j with $j > 7$, contain M_a, M_{a+3} and M_{a+5} but do not contain M_{a+1} or M_{a+4}

(iv) for each $a \in \{1, 2, 3, 4, 5, 6, 7\}$, the family of subsets of E which contain no M_j with $j > 7$, contain M_a, M_{a+3} and M_{a+4} but do not contain M_{a+1} or M_{a+2}

(v) for each $a \in \{1, 2, 3, 4, 5, 6, 7\}$, the family of subsets of E which contain no M_j with $j > 7$, contain M_a but do not contain M_{a+1}, M_{a+2} or M_{a+3}

(vi) the family of subsets of E which contain M_8 but no M_j with $j > 8$,

(vii) the family of subsets of E which contain M_9 but no M_j with $j > 9$,...,

(n-4) the family of subsets of E which contain M_{n-2} but not M_{n-1} or M_n,

(n-3) the family of subsets of E which contain M_{n-1} but not M_n,

and (n-2) the family of subsets of E which contain M_n.

Applying our earlier analysis to this partition of $\mathfrak{F}$, we conclude that $\mathfrak{F}$ may be partitioned as a union of $b_{n-7}+7b_{n-6}+7b_{n-5}+7b_{n-5}+7b_{n-4}+b_{n-8}+b_{n-9}+...+b_2+b_1+1$ MVI intervals. According to Lemma 5.1, this is fewer than $\frac{14\, b_{n-2}}{(n-4)}$ MVI intervals.

Suppose now that $[X_1; \mathfrak{E}(X_1)], ..., [X_t; \mathfrak{E}(X_t)]$ is a compatibly ordered MVI interval partition of $\mathfrak{F}$, i.e., a partition of $\mathfrak{F}$ into MVI intervals such that $M_1, ..., M_n$ appear in order in the list $X_1, ..., X_t$ and if $X_i = M_j$ then $[X_1; \mathfrak{E}(X_1)] \cup ... \cup [X_i; \mathfrak{E}(X_i)] = \mathfrak{F}_1 \cup ... \cup \mathfrak{F}_j$. ($X_1, ..., X_t$ may be an MVI $\mathcal{S}$-partition, but it need not be.) Let π be a partition of $\{1, ..., n-2\}$ into sets of cardinality ≤ 2, and let $S_\pi = E - \{e_{\{i,j\}} \mid \{i,j\} \in \pi\} - \{e_{\{i,n\}} \mid \{i\} \in \pi\}$. Then S_π contains M_{n-1} but not any other M_j with $j < n$, so S_π appears in an MVI interval $[X_k; \mathfrak{E}(X_k)]$ with X_k appearing between M_{n-2} and M_{n-1} in the list $X_1, ..., X_t$. If $\{i,j\} \in \pi$ then $e_{\{i,j\}}$ is the only element of $M_i - S_\pi$ or $M_j - S_\pi$. No element of $[X_k; \mathfrak{E}(X_k)]$ may contain M_i or M_j, for by hypothesis every element of $\mathfrak{F}$ which contains M_i or M_j appears in an interval $[X_a; \mathfrak{E}(X_a)]$ with X_a preceding M_{n-2} in the list $X_1, ..., X_t$. Hence there are $Y_v, Y_w \in \mathfrak{E}(X_k)$ such that $Y_v \subseteq M_i - X_k$ and $Y_w \subseteq M_j - X_k$. $S_\pi \in [X_k; \mathfrak{E}(X_k)]$, so S_π cannot contain Y_v or Y_w; $e_{\{i,j\}}$ is the only element of $E - S_\pi$ which could lie in either Y_v or Y_w. The elements of $\mathfrak{E}(X_k)$ are pairwise disjoint, and Y_v and Y_w are not disjoint, so it must be that $Y_v = Y_w$; then $Y_v = Y_w \subseteq M_i \cap M_j = \{e_{\{i,j\}}\}$, so $Y_v = Y_w = \{e_{\{i,j\}}\}$.

This shows that if $\{i,j\} \in \pi$ and $S_\pi \in [X_k; \mathfrak{E}(X_k)]$ then $\{e_{\{i,j\}}\} \in \mathfrak{E}(X_k)$. Conversely, if $S_\pi \in [X_k; \mathfrak{E}(X_k)]$, $i,j < n-1$ and $\{e_{\{i,j\}}\} \in \mathfrak{E}(X_k)$ then $e_{\{i,j\}} \notin S_\pi$, so it must be that $\{i,j\} \in \pi$; consequently $[X_k; \mathfrak{E}(X_k)]$ cannot contain any $S_{\pi'} \neq S_\pi$. It follows that there must be at least b_{n-2} different X_k in $X_1, ..., X_t$ which appear between M_{n-2} and M_{n-1}, so certainly $t \geq b_{n-2}$. We conclude that the ratio

$$\frac{\text{minimum number of MVI intervals in a compatibly ordered MVI partition of } \mathfrak{F}}{\text{minimum number of MVI intervals in an MVI interval partition of } \mathfrak{F}} > \frac{n-4}{14}.$$

We would guess that also every MVI $\mathcal{S}$-partition of $\mathfrak{F}$ involves at least b_{n-2} intervals, but we have been able to verify the lower bound only for compatibly ordered MVI partitions. Existing MVI-SDP algorithms always produce compatibly ordered MVI partitions, so despite our inability to extend the lower bound to arbitrary MVI $\mathcal{S}$-partitions we may conclude that existing MVI-SDP algorithms always create highly non-optimal partitions of these examples.

6. Conclusion: improving SDP algorithms

SDP algorithms which follow the outline discussed in Section 2 are based on a simple premise: if a filter $\mathfrak{F}$ has minimal elements $M_1, ..., M_p$ then a partition of $\mathfrak{F}$ may be constructed by partitioning the families $\mathfrak{F}_j = \{$ sets which contain M_j but not any of $M_1, ..., M_{j-1}\}$ individually. In logical symbolism, the algorithms are based on the premise that a disjunction $E_1 \vee ... \vee E_p$ is equivalent to the disjunction of disjoint statements $E_1 \vee (E_2 \wedge \bar{E}_1) \vee (E_3 \wedge \bar{E}_2 \wedge \bar{E}_1) \vee ... \vee (E_p \wedge \bar{E}_{p-1} \wedge \bar{E}_{p-2} \wedge ... \wedge \bar{E}_2 \wedge \bar{E}_1)$. It is precisely the choice of this *standard* disjoint disjunction as the basis for existing (MVI-) SDP algorithms that causes these algorithms to produce compatibly ordered (MVI) $\mathcal{S}$-partitions. There are other disjoint disjunctions equivalent to $E_1 \vee ... \vee E_p$, and SDP algorithms using them would not produce compatibly ordered (MVI) $\mathcal{S}$-partitions.

Consider the examples of Section 3. When translated by complementation, they are simply filters whose minimum Boolean interval partitions follow the disjoint disjunction $(E_1 \wedge E_2 \wedge E_3) \vee (E_1 \wedge \bar{E}_2) \vee (E_2 \wedge \bar{E}_3) \vee (E_3 \wedge \bar{E}_1)$ rather than the standard disjoint disjunction $E_1 \vee (E_2 \wedge \bar{E}_1) \vee (E_3 \wedge \bar{E}_2 \wedge \bar{E}_1)$. It may seem strange that the former disjunction, which has four terms, can produce smaller interval partitions than the standard disjoint disjunction, which has only three terms, but that disjunction is more compatible with the structure of these examples because it is symmetric.

The examples of Section 5 are also symmetric. The disjoint disjunction of Lemma 5.2 gives rise to a disjoint disjunction of n statements as outlined in the discussion following the lemma; this resulting disjoint disjunction is symmetric with respect to cyclic permutations of the first seven statements and has no term involving more than $n-4$ negations. It is this upper bound on the maximum number of negations per term which produces MVI interval partitions with fewer intervals than compatibly ordered MVI $\mathcal{S}$-partitions based on the standard disjoint disjunction. The reader might guess that disjoint disjunctions which are more highly symmetric may have still fewer negations per term, and hence produce partitions with even fewer intervals than the one presented in Section 5. After much work we have been able to partially verify this guess, using the following generalization of Lemma 5.2.

Proposition 6.1. *For every odd prime $p \leq 23$ there is a disjoint disjunction which is equivalent to $E_1 \vee ... \vee E_p$, symmetric with respect to cyclic permutations of $E_1, ..., E_p$, and has no term involving more than $\frac{p-1}{2}$ negations.*

These cyclically symmetric disjoint disjunctions are quite large: the one mentioned in Lemma 5.2 involves only 29 terms, but the other examples we have found involve hundreds of terms ($p = 11$ or 13), thousands of terms ($p = 17$ or 19), even hundreds of thousands of terms ($p = 23$).

In sum, the performance of existing SDP and MVI-SDP algorithms can be bettered by algorithms which construct (MVI) interval partitions using a variety of disjoint disjunctions, not just the standard one, and then compare the results. We do not know whether these disjunctions have been studied systematically, and the details of such algorithms remain to be developed.

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