

ON THE FINITISTIC GLOBAL DIMENSION CONJECTURE FOR ARTIN ALGEBRAS

Kiyoshi Igusa¹

Gordana G. Todorov²

ABSTRACT. We find a simple condition which implies finiteness of finitistic global dimension for artin algebras. As a consequence we obtain a short proof of the finitistic global dimension conjecture for radical cubed zero algebras. The same condition also holds for algebras of representation dimension less than or equal to three. Hence the finitistic dimension conjecture holds in that case as well.

Let Λ be an Artin algebra (an algebra of finite length over a commutative Artinian ring). Then the *finitistic global dimension conjecture* states that there exists a uniform bound called *findim* Λ for the finite projective dimensions (pd) of all f.g. (left) Λ -modules of finite pd. This conjecture would imply the Nakayama conjecture. Some of the known cases in which the finitistic global dimension conjecture holds are the radical cubed zero case [GZ] and the monomial relation case [GKK] (see also [IZ], [BFGZ]). The conjecture is also true in the case when the category of modules of finite pd is contravariantly finite in the category of all f.g. modules [AR]. However, the converse is not true [IST]. In this paper we give a short proof of the finitistic gl dim conjecture for all modules of radical square zero over any Artin algebra. This is a generalization of the radical cubed zero case since all syzygies have radical square zero in that case. A thorough overview of the state of the finitistic global dimension conjecture can be found in [Z-H].

As another consequence of the main theorem we prove the finitistic dimension conjecture for algebras with weak representation dimension at most 3, and consequently for algebras with representation dimension $\text{repdim}\Lambda \leq 3$. The notion of representation dimension was introduced by M. Auslander in his Queen Mary Notes [A1], and he and many others expect this dimension to be bounded by 3. O. Iyama showed that it is always finite [I], many classes of algebras are known to have $\text{repdim}\Lambda = 3$, the most recent class being subalgebras of algebras of finite representation type with the same radical [EHIS].

The proof of the main theorem is based on the following well-known elementary observation.

Lemma 1 (Fitting's Lemma). *a) Let M be a module over a Noetherian ring R and let $f : M \rightarrow M$ be an endomorphism of M . Then for any*

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f.g. submodule X of M there is an integer $\eta_f(X)$ so that f sends $f^m(X)$ isomorphically onto $f^{m+1}(X)$ for all $m \geq \eta_f(X)$.

b) If Y is a submodule of X then $\eta_f(Y) \leq \eta_f(X)$.

c) If R is an Artin algebra and $X = M$ there is a direct sum decomposition $X = Y \oplus Z$ so that $Z = \ker f^m$ and $Y = \text{im} f^m$ for all $m \geq \eta_f(X)$.

Let K_0 be the abelian group generated by all symbols $[M]$ where M is a f.g. Λ -module modulo the relations

a) $[C] = [A] + [B]$ if $C \approx A \oplus B$.

b) $[P] = 0$ if P is projective.

Then K_0 is the free abelian group generated by the isomorphism classes of indecomposable f.g. nonprojective Λ -modules. For any f.g. Λ -module M let $L[M] = [\Omega M]$ where ΩM is the first syzygy of M . Since Ω commutes with direct sum and takes projective modules to zero this gives a homomorphism $L : K_0 \rightarrow K_0$. For every f.g. Λ -module M let $\langle \text{add} M \rangle$ denote the subgroup of K_0 generated by all the indecomposable summands of M . Let

$$\phi(M) := \eta_L \langle \text{add} M \rangle.$$

Lemma 2. a) If M has finite projective dimension then $\phi(M) = \text{pd} M$.

b) If M is indecomposable with $\text{pd} M = \infty$ then $\phi(M) = 0$.

c) $\phi(A) \leq \phi(A \oplus B)$.

d) $\phi(kM) = \phi(M)$ if $k \geq 1$.

Proof. (c) follows from Lemma 1(b) since $\langle \text{add} A \rangle$ is a subgroup of $\langle \text{add}(A \oplus B) \rangle$. (d) follows from the fact that $\text{add } kM = \text{add } M$. (a) and (b) are easy exercises for the reader. \square

We need one more definition. For any f.g. Λ -modules M let

$$\psi(M) := \phi(M) + \sup\{\text{pd} X \mid \text{pd} X < \infty, X \text{ direct summand of } \Omega^{\phi(M)} M\}.$$

Lemma 3. a) $\psi(M) = \phi(M) = \text{pd} M$ whenever $\text{pd} M < \infty$.

b) $\psi(kM) = \psi(M)$ if $k \geq 1$.

c) $\psi(A) \leq \psi(A \oplus B)$.

d) If Z is a summand of $\Omega^n M$ where $n \leq \phi(M)$ and $\text{pd} Z < \infty$ then $\text{pd} Z + n \leq \psi(M)$.

Proof. (a) is easy and (b) follows from Lemma 2(d). (c) follows from (d) in the case when $M = A \oplus B$, $n = \phi(A)$ and from Lemma 2(c).

To prove (d) let $k = \phi(M) - n$. Then $\Omega^k Z$ is a direct summand of $\Omega^{\phi(M)} M$ so $\text{pd}(\Omega^k Z) + \phi(M) \leq \psi(M)$ by definition of $\psi(M)$. But $p + n = (p - k) + \phi(M) \leq \text{pd}(\Omega^k Z) + \phi(M)$. \square

Theorem 4. Suppose that $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence of f.g. Λ -modules so that C has finite projective dimension. Then $\text{pd} C \leq \psi(A \oplus B) + 1$.

Remark 5. This is a generalization of the well known formula $\text{pd} C \leq \text{pd}(A \oplus B) + 1$. Y. Wang [W] pointed out that $\text{pd} B \leq \psi(A \oplus \Omega C) + 1$.

Proof. Since C has finite pd we must have $\Omega^n A \approx \Omega^n B$ for some $n \geq 0$. Take n to be minimal so that $[\Omega^n A] = L^n[A] = L^n[B] = [\Omega^n B]$. Then $n \leq \text{pd}C$ and since $[A], [B] \in \langle \text{add}(A \oplus B) \rangle$ we also have $n \leq \phi(A \oplus B)$. The n -th syzygies of our short exact sequence then give an exact sequence of the form

$$0 \rightarrow X \oplus P \rightarrow X \oplus Q \rightarrow M \rightarrow 0$$

where P, Q are projective and $M = \Omega^n C$. Let $f : X \rightarrow X$ be the $X - X$ component of the map $X \oplus P \rightarrow X \oplus Q$, i.e. the composition $X \rightarrow X \oplus P \rightarrow X \oplus Q \rightarrow X$. Then by Lemma 1(c) $X = Y \oplus Z$ so that any sufficiently large iterate f^m of f has kernel Z and image Y . This means there is another exact sequence:

$$0 \rightarrow Z \oplus P \rightarrow Z \oplus Q \rightarrow M \rightarrow 0 \quad (1)$$

where the $Z - Z$ component of the map $Z \oplus P \rightarrow Z \oplus Q$ is a nilpotent endomorphism g of Z .

We claim that Z has finite projective dimension. To see this suppose that $\text{Ext}^k(Z, S) \neq 0$ for some simple module S . Then the endomorphism of $\text{Ext}^k(Z, S)$ induced by g must be nilpotent and therefore cannot be surjective. By the exactness of the sequence:

$$\text{Ext}^k(Z \oplus Q, S) \rightarrow \text{Ext}^k(Z \oplus P, S) \rightarrow \text{Ext}^{k+1}(M, S)$$

we see that $\text{Ext}^{k+1}(M, S) \neq 0$.

Since Z is a direct summand of $\Omega^n(A)$, $\text{pd}Z + n \leq \psi(A) \leq \psi(A \oplus B)$ by Lemma 3(c) and (d) and $\text{pd}M \leq \text{pd}Z + 1$ by the exactness of (1). Therefore $\text{pd}C = \text{pd}M + n \leq \text{pd}Z + n + 1 \leq \psi(A \oplus B) + 1$. \square

Corollary 6. *If M is a f.g. Λ -module with Loewy length 2 and finite projective dimension then*

$$\text{pd}M \leq \psi(\Lambda/\text{rad}\Lambda \oplus \Lambda/(\text{rad}\Lambda)^2) + 1.$$

Proof. There is a short exact sequence $0 \rightarrow A \rightarrow P/\text{rad}^2 P \rightarrow M \rightarrow 0$ where P is the projective cover of M and A is semisimple and $\psi(A \oplus P/\text{rad}^2 P) \leq \psi(\Lambda/\text{rad}\Lambda \oplus \Lambda/(\text{rad}\Lambda)^2)$ by Lemma 3. \square

Corollary 7. *Suppose that Λ is an artin algebra with $(\text{rad}\Lambda)^3 = 0$ then*

$$\text{findim}\Lambda \leq \psi(\Lambda/\text{rad}\Lambda \oplus \Lambda/(\text{rad}\Lambda)^2) + 2$$

where $\text{findim}\Lambda = \sup\{\text{pd}M \mid \text{pd}M < \infty\}$.

Proof. ΩM has Loewy length 2 for any Λ -module M . \square

The following Corollary gives another class of algebras for which the finitistic dimension conjecture holds.

Corollary 8. *Let $\Lambda = \text{End}_\Gamma(P)^{\text{op}}$, where P is a projective module over an artin algebra Γ with $\text{gldim}\Gamma \leq 3$. Then $\text{findim}\Lambda \leq \psi((P, \Gamma)) + 3$ where $(P, \Gamma) = \text{Hom}_\Gamma(P, \Gamma)$ is considered as a Λ -module.*

Proof. We use a construction due to Auslander [A2]. Any Λ -module X has a Λ -projective presentation $(P, P_1) \rightarrow (P, P_0) \rightarrow X \rightarrow 0$ with P_0, P_1 in $\text{add}(P)$. The associated sequence of projective Γ -modules $0 \rightarrow P_3 \rightarrow P_2 \rightarrow P_1 \rightarrow P_0$ induces an exact sequence of Λ -modules $0 \rightarrow (P, P_3) \rightarrow (P, P_2) \rightarrow (P, P_1) \rightarrow (P, P_0) \rightarrow X \rightarrow 0$. Hence $\text{pd}X \leq \text{pd}(\Omega^2 X) + 2 = \text{pd}(\text{coker}((P, P_3) \rightarrow (P, P_2))) + 2 \leq \psi((P, P_3) \oplus (P, P_2)) + 3 \leq \psi((P, \Gamma)) + 3$. \square

Corollary 9. *If $\text{repdim}\Lambda \leq 3$ then $\text{findim}\Lambda < \infty$.*

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