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LOCAL COHOMOLOGY AND GORENSTEIN INJECTIVE DIMENSION OVER LOCAL HOMOMORPHISMS

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Abstract. Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of commutative noetherian local rings. Suppose that M is a finitely generated S-module. A generalization of Grothendieck's non-vanishing theorem is proved for M (i.e. the Krull dimension of M over R is the greatest integer i for which the ith local cohomology module of M with respect to $\mathfrak{m}$, $H^i_{\mathfrak{m}}(M)$, is non-zero). It is also proved that the Gorenstein injective dimension of M, if finite, is bounded below by dimension of M over R and is equal to the supremum of depth $R_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over the support of M as an R-module.

Introduction

Let R be a non-trivial commutative noetherian local ring. An R-module M is called *finite over a local homomorphism* if there exists a local homomorphism of noetherian local rings $R \to S$ such that M is a finite (that is finitely-generated) S-module and the S-action is compatible with the action of R. Studied by Apassov, Avramov, Christensen, Foxby, Iyengar, Miller, Sather-Wagstaff and others, cf. [1, 3, 5, 10, 13], homological properties of finite modules over (local) homomorphisms are shown to extend the those of finite modules.

It is well-known that over a local ring, injective dimension of a finite module is either infinite or equals the depth of the ring and is not less than the dimension of the module. This is known as the *Bass formula*.

A similar formula is also showed to be true for a module finite over a local homomorphism (cf. [5, 18, 14]).

In this paper, we deal with *Gorenstein injective* dimension of a module which is finite over a local homomorphism. Introduced by Enochs and Jenda [11, 12], Gorenstein injective dimension is the dual notion to the G-dimension, due to Auslander and Bridger [2].

In [14] and [19], a generalized Bass formula is proved for finite modules of finite Gorenstein injective dimension.

Theorem. Let R be a noetherian local ring and M be a finite module of finite Gorenstein injective dimension. Then

$$\dim_R M \leq \operatorname{Gid}_R(M) = \sup\{\operatorname{depth} R_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R(M)\}.$$

Our main goal is to extend this theorem to modules finite over a local homomorphism. To this end, we provide some preliminary results. The most interesting one among them, is a generalization of the Grothendieck's non-vanishing theorem (Theorem 1.3).

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Theorem. Let $(R,\mathfrak{m})$ be a noetherian local ring and M an R-module finite over a local homomorphism. Then $n = dim_R M$ is the greatest integer i for which the i-th local cohomology module $H^i_{\mathfrak{m}}(M)$ is non-zero.
Using the auxiliary results, we prove our main theorem (Theorem 2.3) with similar techniques as in [14] and [19]. As a corollary, we also prove that if a local ring R admits a module M finite over a local homomorphism with finite Gorenstein injective dimension and maximal dimension over R, then R is Cohen-Macaulay.
Convension. Throughout this notes all rings are assumed to be commutative, notherian and local.

1. Local cohomology over a local homomorphism

Let $(R,\mathfrak{m})$ be a noetherian local ring and M a finite R-module of the Krull dimension n. Then the Grothendieck's non-vanishing theorem says that the local cohomology module $H^n_{\mathfrak{m}}(M)$ does not vanish (cf. [6, 6.1.4]). In this section, we prove the same non-vanishing result for a module finite over a local homomorphism. First, we prove a couple of auxiliary lemmas.

The main tool we use in our proofs in this paper, is the so-called *Cohen factorization* of a homomorphism.
A local homomorphism $\varphi : R \to S$ is said to have a Cohen factorization if it can be represented as a composition $R \to R' \to S$ of local homomorphisms, where R' is a complete local ring, the map $R \to R'$ is flat with regular closed fiber $R'/\mathfrak{m}R'$ and $R' \to S$ is a surjective homomorphism. For any local homomorphism $\varphi : (R,\mathfrak{m}) \to (S,\mathfrak{n})$ its semi-completion $\varphi' : R \to S \to \hat{S}$ admits a Cohen factorization (cf. [4]).

Lemma 1.1. *Let $\varphi : (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local homomorphism. If M is an S-module, then*

$$Supp_R(M) = \{\varphi^{-1}(\mathfrak{q}) \mid \mathfrak{q} \in Supp_S(M)\} \text{ and}$$
$$Supp_S(M) = \{\mathfrak{q} \in Spec(S) \mid \varphi^{-1}(\mathfrak{q}) \in Supp_R(M)\}.$$

Proof. Suppose that $\mathfrak{q} \in \operatorname{Spec}(S)$ and set $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. Therefore φ induces a (faithfully) flat local homomorphism $R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ and the following isomorphisms hold.

$$\begin{aligned} M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}} &\cong (M \otimes_R R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}} \\ &\cong M \otimes_R S_{\mathfrak{q}} \\ &\cong M \otimes_R (S \otimes_S S_{\mathfrak{q}}) \\ &\cong M_{\mathfrak{q}} \otimes_R S \end{aligned}$$

If $\mathfrak{q} \in \operatorname{Supp}_S(M)$ then, φ being faithfully flat, $M_{\mathfrak{q}} \otimes_R S$ and thus $M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$ are non-zero. Hence $\mathfrak{p} = \varphi^{-1}(\mathfrak{q}) \in \operatorname{Supp}_R(M)$.

On the other hand, if $\mathfrak{p} \in \operatorname{Supp}_R(M)$, using faithfully flatness of φ we can find a prime $\mathfrak{q} \in \operatorname{Spec}(S)$ such that $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$ and the above isomorphisms give $M_{\mathfrak{q}} \otimes_R S \neq 0$ and hence $\mathfrak{q} \in \operatorname{Supp}_S(M)$. □

Lemma 1.2. *Let $\varphi : (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local homomorphism and M a non-zero S-module. Then*

$$dim_S M \geq dim_R M + dim_S S/\mathfrak{m}S.$$

Proof. First note that since φ is flat, every prime ideal of S which contains $\mathfrak{m}S$ minimally, contracts to $\mathfrak{m}$ via φ. Let $\mathfrak{q} \in \operatorname{Spec}(S)$ be one such prime. By Lemma 1.1, $\mathfrak{q} \in \operatorname{Supp}_S(M)$.
Suppose that $\dim_R M = n$ and

$$\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{m}$$

is a maximal chain of prime ideals in $\operatorname{Supp}_R(M)$. Since φ is flat and $\varphi^{-1}(\mathfrak{q}) = \mathfrak{m}$ we get a strict chain in $\operatorname{Spec}(S)$,

$$\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_n = \mathfrak{q}$$

such that $\varphi^{-1}(\mathfrak{q}_i) = \mathfrak{p}_i$ for $0 \leq i \leq n$. Using Lemma 1.1 again, this is a chain in $\operatorname{Supp}_S(M)$ which gives the inequalities

$$\dim_S M \geq \dim_{S_\mathfrak{q}} M_\mathfrak{q} + \dim_S S/\mathfrak{q} \geq n + \dim_S S/\mathfrak{q}.$$

In particular, we can choose $\mathfrak{q}$ such that $\dim_S S/\mathfrak{m}S = \dim_S S/\mathfrak{q}$ to get the desired inequality. □

Now we are ready to prove the main theorem of this section.

Theorem 1.3. *Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism and M a non-zero finite S-module. If $n = \dim_R M$ then*

$$H^n_\mathfrak{m}(M) \neq 0.$$

Proof. First suppose that φ is a flat homomorphism.
As in the proof of Lemma 1.2, we pick a prime $\mathfrak{q} \in \operatorname{Supp}_S(M)$ which contains $\mathfrak{m}S$ minimally. Therefore, the induced homomorphism $R \to S_\mathfrak{q}$ is faithfully flat and has zero-dimensional fiber $S_\mathfrak{q}/\mathfrak{m}S_\mathfrak{q}$. Furthermore, the inequality $\dim_{S_\mathfrak{q}} M_\mathfrak{q} \geq \dim_R M$ holds.

On the other hand, the isomorphism $\mathrm{H}^i_\mathfrak{m}(M_\mathfrak{q}) \cong \mathrm{H}^i_{\mathfrak{q}S_\mathfrak{q}}(M_\mathfrak{q})$ holds for every $i \geq 0$. Since $M_\mathfrak{q}$ is a finite $S_\mathfrak{q}$-module, $\mathrm{H}^d_{\mathfrak{q}S_\mathfrak{q}}(M_\mathfrak{q}) \neq 0$ for $d = \dim_{S_\mathfrak{q}} M_\mathfrak{q}$ and therefore we get $\dim_{S_\mathfrak{q}} M_\mathfrak{q} \leq \dim_R M_\mathfrak{q}$.

Note that $\operatorname{Supp}_R(M_\mathfrak{q}) \subseteq \operatorname{Supp}_R(M)$ and hence

$$\dim_{S_\mathfrak{q}} M_\mathfrak{q} \leq \dim_R M_\mathfrak{q} \leq \dim_R M \leq \dim_{S_\mathfrak{q}} M_\mathfrak{q}.$$

Thus the equality $\dim_{S_\mathfrak{q}} M_\mathfrak{q} = \dim_R M = n$ holds.

Finally, the following isomorphisms give the desired non-vanishing in this case.

$$\mathrm{H}^n_\mathfrak{m}(M_\mathfrak{q}) \cong \mathrm{H}^n_{\mathfrak{m}S_\mathfrak{q}}(M_\mathfrak{q}) \cong \mathrm{H}^n_\mathfrak{m}(M) \otimes_R S_\mathfrak{q}$$

In general, $R \to S \to \hat{S}$ admits a Cohen factorization $R \to R' \to \hat{S}$. Since M is finite over S, so is $\hat{M} = M \otimes_S \hat{S}$ as an $\hat{S}$-module and thus as an R'-module. It is also easy to see that $\operatorname{Supp}_R(\hat{M}) = \operatorname{Supp}_R(M)$ and consequently we have $\dim_R \hat{M} = \dim_R M = n$. So using the first part of the proof, we get

$$\mathrm{H}^n_\mathfrak{m}(\hat{M}) \neq 0.$$

Using the isomorphisms

$$\mathrm{H}^n_\mathfrak{m}(\hat{M}) \cong \mathrm{H}^n_{\mathfrak{m}\hat{S}}(\hat{M}) \cong \mathrm{H}^n_{\mathfrak{m}S}(M) \otimes_S \hat{S}$$

we conclude that $\mathrm{H}^n_{\mathfrak{m}S}(M) \cong \mathrm{H}^n_\mathfrak{m}(M)$ is non-zero.

□

2. A Generalized Bass Formula

This section is dedicated to proving a Bass-type formula for a module which is finite over a local homomorphism and has finite Gorenstein injective dimension.

Definition 2.1. *An R-module G is said to be* Gorenstein injective *if and only if there exists an exact complex of injective R-modules,*

$$I = \cdots \to I_2 \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow I_{-2} \longrightarrow \cdots$$

such that the complex $Hom_R(J, I)$ is exact for every injective R-module J and G is the kernel in degree 0 of I. The Gorenstein injective dimension *of an R-module M, $Gid_R(M)$, is defined to be the infemum of integers n such that there exists an exact sequence*

$$0 \to M \to G_0 \to G_{-1} \to \cdots \to G_{-n} \to 0$$

with all G_i's Gorenstein injective.

Proposition 2.2. *Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism and M a finite S-module. If $\mathfrak{p} \in Supp_R(M)$ then $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \neq 0$.*

Proof. First suppose that φ admits a Cohen factorization $R \to R' \to S$. Since $R' \to S$ is surjective, M is also finite as an R'-module. Thus we can assume that φ is flat. By Lemma 1.1, there exists a $\mathfrak{q} \in \mathrm{Supp}_S(M)$ such that $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$.
By NAK we have $M_\mathfrak{q}/\mathfrak{p}M_\mathfrak{q} \neq 0$. Since $M_\mathfrak{q}/\mathfrak{p}M_\mathfrak{q} \cong (T^{-1}M/\mathfrak{p}T^{-1}M)_{T^{-1}\mathfrak{q}}$, where $T = \varphi(R - \mathfrak{p})$, we get $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \cong T^{-1}M/\mathfrak{p}T^{-1}M \neq 0$.
In general $R \to S \to \hat{S}$ admits a Cohen factorization. If $\mathfrak{p} \in \mathrm{Supp}_R(M)$ then $\mathfrak{p} \in \mathrm{Supp}_R(M \otimes_S \hat{S})$ as well and then, setting $\hat{M} = M \otimes_S \hat{S}$, we have $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \otimes_S \hat{S} \cong \hat{M}_\mathfrak{p}/\mathfrak{p}\hat{M}_\mathfrak{p} \neq 0$ and we are done. □

Theorem 2.3. *Let R be a noetherian local ring and M be an R-module finite over a local homomorphism with $Gid_R(M) < \infty$. Then*

$$dim_R M \leq Gid_R(M) = \sup\{depth\, R_\mathfrak{p} \mid \mathfrak{p} \in Supp_R(M)\}.$$

Proof. To prove the first inequality we use the proof of [15, 3.4] to see that $\mathrm{H}^i_\mathfrak{m}(M) = 0$ for $i > \mathrm{Gid}_R(M)$. Thus, by Theorem 1.3, we have

$$\dim_R M \leq \mathrm{Gid}_R(M).$$

To prove the formula, we use [9, 2.18], to get an exact sequence

$$0 \to K \to L \to M \to 0$$

with K Gorenstein injective and $\mathrm{inj.dim}_R(L) = \mathrm{Gid}_R(M)$. By [7], we have

$$\mathrm{inj.dim}_R(L) = \sup\{\mathrm{depth}\, R_\mathfrak{p} - \mathrm{width}_{R_\mathfrak{p}} L_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R(L)\},$$

where $\mathrm{width}_{R_\mathfrak{p}} M_\mathfrak{p} = \inf\{\, i \,| \mathrm{Tor}_i^{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p}) \neq 0\}$.

For any $\mathfrak{p} \in \mathrm{Supp}_R(M)$, we get an exact sequence $L_\mathfrak{p}/\mathfrak{p}L_\mathfrak{p} \to M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \to 0$ which gives $L_\mathfrak{p}/\mathfrak{p}L_\mathfrak{p} \neq 0$, by Proposition 2.2. Therefore we have

$$\begin{gathered}\sup\{\mathrm{depth}\, R_\mathfrak{p} - \mathrm{width}_{R_\mathfrak{p}} L_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R(L)\} \geq \\ \sup\{\mathrm{depth}\, R_\mathfrak{p} - \mathrm{width}_{R_\mathfrak{p}} L_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R(M)\} = \\ \sup\{\mathrm{depth}\, R_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R(M)\} \geq 0.\end{gathered}$$

This proves the desired formula for M a Gorenstein injective R-module.

For any $\mathfrak{p} \in \operatorname{Supp}_R(L) - \operatorname{Supp}_R(M)$, we get $L_\mathfrak{p} \cong K_\mathfrak{p}$. Since K is Gorenstein injective, there exists an exact sequence

$$\cdots \to I_1 \to I_0 \to L_\mathfrak{p} \to 0,$$

where for each $\ell \geq 0$, I_ℓ is an injective $R_\mathfrak{p}$-module.
Set $K_\ell = \ker(I_{\ell-1} \to I_\ell)$. Then for any $R_\mathfrak{p}$-module T with $\operatorname{proj.dim}_{R_\mathfrak{p}}(T) = t < \infty$ we have $\operatorname{Ext}^i_{R_\mathfrak{p}}(T, L_\mathfrak{p}) \cong Ext^{i+t}_{R_\mathfrak{p}}(T, K_t) = 0$ for $i > 0$. Thus, by [8, 5.3(c)], we get

$$\operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} L_\mathfrak{p} \leq 0.$$

Therefore, if $\operatorname{Gid}_R(M) = \operatorname{inj.dim}_R(L) > 0$ then we have

$$\begin{gathered} \operatorname{Gid}_R(M) = \operatorname{inj.dim}_R(L) = \\ \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} L_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R(L)\} = \\ \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} L_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R(M)\} = \\ \sup\{\operatorname{depth} R_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R(M)\}. \end{gathered}$$

□

The following corollary of Theorem 2.3, gives a sufficient condition for Cohen-Macaulayness of a noetherian local ring. This result generalizes [17, 3.5] as well as [19, 1.3].

Corollary 2.4. *Let R be a local ring and M an R-module finite over a local homomorphism. If M has finite Gorenstein dimension and maximal Krull dimension over R (i.e. $Gid_R(M) < \infty$ and $\dim_R M = \dim R$), then R is Cohen-Macaulay.*

Proof. By 2.3, there is a prime ideal $\mathfrak{p} \in \operatorname{Supp}_R(M)$ such that

$$\dim R = \dim_R M \leq \operatorname{Gid}_R(M) = \operatorname{depth} R_\mathfrak{p} \leq \dim R_\mathfrak{p} \leq \dim R.$$

Therefore, $\mathfrak{p}$ must be the maximal ideal of R and thus R is Cohen-Macaulay . □

Corollary 2.5. *Let R be an almost Cohen-Macaulay local ring (i.e. $\dim(R) - \operatorname{depth}(R) \leq 1$) and let M be an R-module finite over a local homomorphism. If M has finite Gorenstein injective dimension over R then $Gid_R(M) = \operatorname{depth} R$.*

Proof. It is enough to use Theorem 2.3 and the fact that over an almost Cohen-Macaulay ring R the inequality $\operatorname{depth} R_\mathfrak{p} \leq \operatorname{depth} R_\mathfrak{q}$ holds for $\mathfrak{p} \subseteq \mathfrak{q}$ in $\operatorname{Spec}(R)$. □

Note that Corollary 2.5 can be considered as a generalized Bass formula for Gorenstein injective dimension.

We conclude this paper with a change of ring result for Gorenstein injective dimension .

Corollary 2.6. *Let $(R, \mathfrak{m})$ be a local ring and M be an R-module finite over a local homomorphism. If $x \in \mathfrak{m}$ is an R- and M-regular element, then*

$$Gid_{R/xR}(M/xM) \leq Gid_R(M) - 1.$$

Furthermore, the equality holds when R is almost Cohen-Macaulay and $Gid_R(M) < \infty$.

Proof. If M has finite Gorenstein injective dimension over R then the proof of [16, Lemma 2] shows that $\mathrm{Gid}_{R/xR}(M/xM) < \infty$, too . By 2.2, for any $\mathfrak{p} \in \mathrm{Supp}_R(M)$ with $x \in \mathfrak{p}$, the $R_\mathfrak{p}$-module $M_\mathfrak{p}/xM_\mathfrak{p}$ is non-zero. Therefore,

$$\mathrm{Supp}_{R/xR}(M/xM) = \{\mathfrak{p}/xR | \mathfrak{p} \in \mathrm{Supp}_R(M) \text{ with } x \in \mathfrak{p}\}.$$

Thus Theorem 2.3 gives the desired inequality. In the case of an almost Cohen-Macaulay base ring the equality is a consequence of 2.5. □

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