Tableau calculus for preference-based conditional logics: PCL and its extensions

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We present a tableau calculus for some fundamental systems of propositional conditional logics. We consider the conditional logics that can be characterized by *preferential* semantics (i.e. possible world structures equipped with a family of preference relations). For these logics, we provide a uniform completeness proof of the axiomatization w.r.t. the semantics, and a uniform labelled tableau procedure.

Categories and Subject Descriptors: I.2.3 [**Artificial Intelligence**]: Deduction and Theorem Proving—*Deduction*; *Nonmonotonic reasoning and belief revision*; F.4.1 [**Mathematical Logic and Formal Languages**]: Mathematical Logic—*Computational Logic*; *Proof Theory*; *Modal Logic*

Additional Key Words and Phrases: Tableaux Calculi, Conditional Logics

1. INTRODUCTION

Conditional logics have been introduced by Stalnaker [Stalnaker 1968] and Lewis [Lewis 1973b] to formalize a kind of hypothetical reasoning (if ϕ were the case, then ψ would be the case) that cannot be captured by classical logic with its material implication. The basic idea was to elaborate a logic which admits two kinds of implication: classical implication (denoted by $\rightarrow$) and a weaker implication, namely conditional implication (denoted by $\Rightarrow$). Conditional implication lacks several properties of classical implication: monotonicity, transitivity, the property of contraposition. The weakest variant of conditional logic does not even admit reflexivity of conditional implication.

Conditional logics have been introduced and studied in order to formalize sentences like:

—Normally, if ϕ then ψ;

—ψ holds because of ϕ;

—Most of the time, if ϕ then ψ;

—If ϕ were the case, then ψ;

—If we learned ϕ, then we would conclude ψ.

There have been various applications of conditional logics for Philosophy and Epistemology, but also more recently in the context of Computer Science and Artificial Intelligence. Conditional logics have thus been applied to several areas such as knowledge base update [Grahne 1998], reasoning about prototypical properties [Ginsberg 1986], theory revision [Boutilier 1994; Giordano et al. 2002; 2005; Crocco et al. 1995], causality [Giordano and Schwind 2004; Galles and Pearl 1998; Lewis 1973a; 2000], nonmonotonic reasoning [Delgrande 1987; Lamarre 1992; Kraus et al. 1990], commonsense reasoning

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[Groeneboer and Delgrande 1988]. [Makinson 1993] discusses the similarities and differences between counterfactual reasoning, defeasible inference, belief revision and belief update.

Similarly to modal logics, the semantics of conditional logics can be defined in terms of possible world structures. The intuition is that a conditional $\phi \Rightarrow \psi$ is true in a world w just in case ψ is true in the ϕ-worlds (that is worlds satisfying ϕ) that are *most similar/most preferred/closest* to w. As there are different ways of formalizing the notion of "closest/preferred/most similar" worlds, it comes without surprise that there is not a standard semantics for conditional logics. This is a difference with (standard) modal logics, where there is a unique type of models, the usual Kripke models with an accessibility relation, and different modal systems are identified by classes of models characterized by the properties of the accessibility relation. In the case of conditional logics, there are at least three different types of semantics: the selection function semantics, the sphere semantics and the preferential semantics. We briefly describe them below.

Selection function semantics. In this semantics, models are equipped by a so-called selection function that, given a formula ϕ and a possible world w, selects a set of possible worlds $f(\phi, w)$. The idea is that f selects the ϕ-worlds which are closest/preferred/most similar to w. A conditional formula $\phi \Rightarrow \psi$ is true in a world w whenever ψ is true in all worlds satisfying ϕ which are most preferred with respect to w, i.e. in all worlds selected by $f(\phi, w)$. A further requirement is imposed, called normality: the set of worlds $f(\phi, w)$ does not depend on the syntactic form of ϕ, but only on the interpretation of ϕ, that is to say, on the set of worlds satisfying ϕ. There are two equivalent ways of expressing this requirement: (i) imposing that if ϕ and ϕ' hold in the same worlds, then $f(\phi, w) = f(\phi', w)$ and (ii) defining the function f to have as argument the set of worlds $[[\phi]]$ satisfying ϕ, rather than the formula ϕ itself. Without this condition, the logic would not have the substitution property on conditional formulas. This semantics is the most general one for conditional logics, the worlds in $f(\phi, w)$ are not even meant to satisfy ϕ. The selection function semantics characterizes all systems of conditional logic: from the weakest system **CK** to all its extensions by extra axioms. The extensions of **CK** are then captured by imposing further conditions on the selection function. For instance, the extension of **CK** by the well-known axiom (AC) (cumulativity)

$$(\phi \Rightarrow \psi) \wedge (\phi \Rightarrow \chi) \rightarrow (\phi \wedge \chi \Rightarrow \psi)$$

is semantically characterized by the class of models where the selection function satisfies the property:

if $f(\phi, w) \subseteq [[\chi]]$ then $f(\phi \wedge \chi, w) \subseteq f(\phi, w)$.

While it is usually an easy task to translate an axiom into a condition on the selection function, the conditions themselves (as the one above for AC) are not very informative. What is worse is the fact that this semantics does not allow to develop analytic deduction methods (except for some relatively-simple systems, see [Olivetti et al. 2007; Crocco and Fariñas del Cerro 1995; Artosi et al. 2002]). The problem is that the conditions on the selection function often make reference to the syntactic structure of a formula ϕ occurring in $f(\phi, w)$ (e.g. if the formula is a conjunction $\psi \wedge \chi$). On the other hand, by the normality principle, the syntactic form of ϕ should be irrelevant. These two requirements seem in conflict. The consequence is that it is very difficult to develop analytic proof systems from the selection function semantics. For many important extensions of **CK** (including almost all the systems examined in this work), analytic proof systems of this type have not been found, and it is unlikely that they will be.

Sphere Semantics. A sphere system associates to each world w a set S_w of nested sets of worlds, closed under union. Each nested set in S_w is called a sphere and the structure of S_w is thus onion-like. Intuitively, the spheres associated with a world represent a ranking of worlds in degrees of similarity with respect to the given world: a world w_1 is more similar or closer to w than w_2 if there is a sphere associated to w that contains w_1 but does not contain w_2. The idea is that a conditional $\phi \Rightarrow \psi$ is true in a world w if the worlds satisfying $\phi \wedge \psi$ are more preferred (i. e. belong to a smaller sphere) than the worlds satisfying $\phi \wedge \neg\psi$ (or if there is no world satisfying ϕ). This semantics, elaborated by Lewis [Lewis 1973b] captures in a good way some intuitions, but it only works for rather strong systems of conditional logics, such as Lewis **VC** and **VW**. De Swart [de Swart 1983], Gent [Gent 1992], and Lamarre [Lamarre 1993] give sequent/tableau calculi for conditional logics based on the sphere semantics.

Preferential semantics. This represents a generalization of the sphere semantics, and it has been studied by Burgess [Burgess 1981], Katsuno and Satoh [Katsuno and Satoh 1991], Grahne [Grahne 1998], Nejdl [Nejdl 1991], Friedman and Halpern [Friedman and Halpern 1994]. It has been developed also in a more restricted context (formalisation of default rules) by Kraus, Lehmann and Magidor [Kraus et al. 1990], and Boutilier [Boutilier 1994]. Instead of associating a sphere system to every world w, preferential semantics associates a preference relation $<_w$, that is a transitive and irreflexive relation among worlds: $x <_w y$ may be read as 'x is preferred to y with respect to w, or x is more similar to w than y is'. Given a world w we can then define that a conditional $\phi \Rightarrow \psi$ is true in w if ψ is true in all ϕ-worlds that are closest/most preferred with respect to w, that is that are minimal with respect to $<_w$. Denoting such a set by $Min_{<_w}([[\phi]])$, we can write that w satisfies $\phi \Rightarrow \psi$ if all worlds in $Min_{<_w}([[\phi]])$ satisfy ψ. We can see $Min_{<_w}([[\phi]])$ as a concrete way of defining a selection function $f(\phi, w)$. However the preferential semantics imposes some conditions on the selection function. That is to say, it is more restrictive than the selection function semantics, and it validates a certain number of axioms. Thus, it cannot characterize the weakest system **CK** and the like. It characterizes all systems which contain at least **CK** and the identity axiom

$$(ID)\phi \Rightarrow \phi$$

together with the 'or'- axiom

$$(CA)(\phi \Rightarrow \chi) \wedge (\psi \Rightarrow \chi) \rightarrow (\phi \vee \psi \Rightarrow \chi)$$

and the axiom (CSO)

$$(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi) \rightarrow (\phi \Rightarrow \chi) \leftrightarrow (\psi \Rightarrow \chi).$$

However, preferential logic is more general than the sphere semantics as it does not impose that worlds are always ranked with respect to a given world. In order to give a meaning to all conditionals, we assume that whenever ϕ-worlds exist, also closest/most preferred worlds with respect to any w do exist, that is $Min_{<_w}([[\phi]])$ is non-empty. This condition is the well-known Limit Assumption, that we explicitly assume[2].

The logic characterized by the preferential semantics will be our starting point. We will call it **PCL**, from *Preferential Conditional Logic* (and it coincides with the System_P in [Nejdl 1991]).

Independently from technical reasons, the preferential semantics is motivated at least by two kinds of applications of conditional logics: (i) the link between conditionals and the notion of belief update and (ii) the semantics of nonmonotonic consequence relation. For the former (see [Grahne 1998]), the semantics of update specifies that the worlds satisfying an updated belief set are the worlds where the update is true and that are closest to some world satisfying the initial belief set. For the latter (see [Kraus et al. 1990]), it can be seen that all systems of KLM logics correspond to the flat fragment of some conditional logics (namely Preferential **P** and Rational logic **R** correspond respectively to **PCL** and **VC**, studied by Lewis [Lewis 1973b]). The flat fragment of a conditional logic does not contain nested occurrences of the conditional operator (for instance, $p \Rightarrow r$ belongs to the flat fragment, whereas $p \Rightarrow (r \Rightarrow q)$ does not belong to the flat fragment). KLM logics are therefore less expressive than the logics we consider here.

Finally, the conditional logic of preferential structures has some relation with the logic of comparative concept similarity recently introduced in [Sheremet et al. 2005]. It turns out that considering arbitrary Min spaces (roughly speaking, spaces where the minimum of a set of distances always exists), the operator of comparative similarity is interdefinable with the conditional operator, where the latter is defined by a specific extension of **PCL** considered here.

Tableau calculi for preferential logics in general have never been provided. Partial results have been given by [Giordano et al. 2005] and [Giordano et al. 2006], that deal with KLM logics. The language of KLM logics being simpler than the language of **PCL**, the calculi proposed in [Giordano et al. 2005] and [Giordano et al. 2006] cannot be straightforwardly extended to the full language of nested conditionals.

To summarize, we have three semantics for conditional logic of increasing strength, or decreasing generality: selection function semantics, preferential semantics, sphere semantics.

In the present paper, we consider all conditional logics which can be characterized by *preferential* semantics. The semantics comes with two options:

1 The more general kind, that characterizes our basic logic **PCL**: each world w carries with it a set of accessible (or 'conceivable') worlds W_w which is a subset of the universe W; the preference relation $<_w$ then ranges on the set W_w.

2 The more restricted kind: we postulate that conceivable worlds are the same for all worlds and coincide with the universe, i.e: $\forall w\, W_w = W$.

The requirement 2. is called (expectedly) *Universality* [Nute 1980; Lewis 1973b], that corresponds to a well-defined set of axioms. The consequence is that we have actually two families of conditional logics based on the preferential semantics: those where Universality

[2]A few authors [Lewis 1973b; Friedman and Halpern 1994; Boutilier 1994] have refused for theoretical reasons the Limit Assumption and have shown that redefining the conditional operator (in a much less natural way, as a matter of fact) one can avoid this assumption, obtaining equivalent systems, that is to say, systems with the same set of theorems.

is not assumed and the stronger systems with Universality. In both cases, stronger and stronger systems are obtained by imposing further conditions on the preference relation (e.g. connectedness, modularity, centering etc.).

In this work we consider systems of both kinds. Our first result is a uniform and direct proof of the completeness of the axiomatization of each system with respect to the preferential semantics with the Limit Assumption. Our result is a generalization of the seminal results presented by Burgess in [Burgess 1981]. Burgess provides a completeness proof for a similar logic but, as a difference, he does not accept the Limit Assumption. We will come back to this in Section 3.

Our second and main result is a uniform labelled tableau procedure for all systems under consideration. To the best of our knowledge, our tableau calculus is the first one to cover uniformly this spectrum of logics, since no tableau systems based on the preferential semantics in its generality have ever been studied. We notice that some tableau and sequent calculi have been presented for conditional logics weaker than **PCL** [Crocco and Fariñas del Cerro 1995; Artosi et al. 2002; Giordano et al. 2005; 2006]. Our tableau calculus gives a practical implementable decision procedure for these logics, and it allows us to obtain upper complexity results for the logics in a constructive way.

Intuitively, our tableau method is based on an analogy between the preferential semantics under the Limit Assumption and the modal logics of arithmetic provability (known as Goëdel-Löb **GL**): it turns out that the existence of minimal worlds enforced by the Limit Assumption can be captured by rules similar to the modal rules for **GL**. However, here we have to deal with arbitrary families of preference relations indexed by worlds so that the situation is more complicated than in modal logic **GL**. To account for the semantics in the calculus, we have to expand the syntax of formulas and we will have a hybrid language comprising pseudo-modalities indexed by worlds and some other relations and predicates on labels. The use of modal formulas to interpret the semantics of conditionals is not new: Boutilier [Boutilier 1994] introduces a bi-modal logic to define some conditional logics related to **PCL** and **VC**. However, there are two important differences between our approach and Boutilier's: first in Boutilier's semantics there is only one modality, rather than a family of modal operators indexed by worlds (as in our calculus). For this reason, Boutilier's logic is unable to properly represent nested conditionals (since $\beta \Rightarrow \gamma$ implies $\alpha \Rightarrow (\beta \Rightarrow \gamma)$). As a second difference, he does not accept the Limit Assumption, and thus uses a truth definition for conditionals which is different from ours (and is similar to the one adopted by Lewis [Lewis 1973b], Burgess [Burgess 1981] and Halpern and Friedman [Friedman and Halpern 1994]).

Our tableau method gives a decision procedure for all logics under consideration. Termination of the calculus is obtained by adding some loop-checking conditions on the tableau construction. Given the presence of multiple preference relations and pseudo-modalities, finding suitable conditions ensuring termination is not straightforward.

The structure of the paper is the following: in Section 2, we introduce conditional logics and the preferential semantics, and in Section 3 we give a general completeness result for all logics under consideration. In Section 4, we present our tableau calculus. In Section 5 we prove soundness, completeness of the calculus. Last, in Section 6 we give a set of restrictions that ensure the termination of our calculus and we remark on the complexity of our calculus.

2. THE CONDITIONAL LOGIC **PCL** AND ITS EXTENSIONS

2.1 The logic **PCL**

Let ATM be a set of propositional variables. We define the language $\mathcal{L}$ of logic **PCL** and its extensions. Formulas of $\mathcal{L}$ are built from propositional variables by means of the connectives $\neg, \wedge, \bot, \Rightarrow$; the last one $\Rightarrow$ is the conditional operator. $\vee, \rightarrow$ and $\leftrightarrow$ are defined as usual. Priorities among connectives are defined as usual; $\Rightarrow$ has the lowest priority (e.g. $\phi \Rightarrow \psi \rightarrow \chi$ must be read as $\phi \Rightarrow (\psi \rightarrow \chi)$).

Our models are structures of the form $(W, \{W_x\}_{x \in W}, \{<_x\}_{x \in W}, I)$. W is the set of possible worlds, I is an interpretation function that associates to each possible world the set of atoms that it satisfies. Furthermore, for each possible world x, W_x represents the set of worlds *accessible* from x, and $<_x$ is a preference relation: $y <_x z$ means that y is closer to x than z, or y is *preferred* to z w.r.t. $<_x$ [3].

DEFINITION 2.1 SEMANTICS OF **PCL**. *A* **PCL**-*model* M *has the form*

$$(W, \{W_x\}_{x \in W}, \{<_x\}_{x \in W}, I),$$

where:

—$W \neq \emptyset$ *is a set of items called worlds;*

—$\{W_x\}_{x \in W}$ *is a family of subsets of* W*. For each element* $x \in W$, $W_x \subseteq W$*;*

—*For each element* $x \in W$, $<_x \subseteq W_x \times W_x$ *is a binary irreflexive, transitive relation on* W_x*, satisfying the Limit Assumption below.*

—I *is a function* $W \longrightarrow Pow(ATM)$ *that associates to each world* $x \in W$ *the set of atoms satisfied by* x*.*

For $S \subseteq W$ *we define:*

$$Min_x(S) = \{w \in S \cap W_x \mid \neg \exists y \in S \cap W_x, \text{ such that } y <_x w\}.$$

We say that $Min_x(S)$ *is the set of* $<_x$*-minimal elements of* S*. Notice that* $Min_x(S) \subseteq W_x$*.*

We define the truth conditions of formulas with respect to worlds in a model M*, by the relation* $M, x \models \phi$*, as follows. For readability, we use* $[[\phi]]^M$ *(or* $[[\phi]]$*, when the model* M *is clear from the context) to denote* $\{y \in W \mid M, y \models \phi\}$*.*

(1) $M, x \models p$, *for* p *atomic, if* $p \in I(x)$*,*

(2) $M, x \not\models \bot$*,*

(3) $M, x \models \neg\phi$ *if* $M, x \not\models \phi$*,*

(4) $M, x \models \phi \wedge \psi$ *if* $M, x \models \phi$ *and* $M, x \models \psi$*.*

(5) $M, x \models \phi \Rightarrow \psi$ *if for all* $y \in Min_x([[\phi]]^M)$, $M, y \models \psi$*. We abbreviate* $Min_x([[\phi]]^M)$ *by* $Min_x(\phi)$*.*

We say that ϕ *is valid in a model* M *if* $M, x \models \phi$ *for every* $x \in W$*. We say that* ϕ *is* **PCL**-*valid (and write* $\models_{\mathbf{PCL}} \phi$*) if it is valid in every* **PCL**-*model.*

[3] An equivalent definition can be given, as in [Burgess 1981] and [Nejdl 1991], by considering structures equipped by a ternary relation R which corresponds to the non strict version of our parametrized preference relation. In this framework, W_x can be defined as $\{y : \exists z \text{ s.t. } Rxyz\}$.

In case where $[[\phi]]$ contains an infinitely descending $<_x$-chain of worlds, all conditionals $\phi \Rightarrow \psi$ are identically true, even if ϕ is not always false. In order to prevent this situation, we assume the following condition, called *Limit Assumption*:

Limit Assumption: for all $\phi \in \mathcal{L}$, if $[[\phi]] \cap W_x \neq \emptyset$, then $Min_x(\phi) \neq \emptyset$

The set of valid formulas according to the previous semantics is axiomatized by considering the axioms and rules given below. In the following rules, $\vdash$ refers to provability w.r.t. to the conditional logic being defined. For the time being, it stands for provability in **PCL** , when we will deal with the extensions of **PCL** (next section) it will stand for provability in the considered extension.

DEFINITION 2.2 AXIOM SYSTEM **PCL**. *The system* **PCL** *is defined by*

(TAUT) All classical tautologies and the Modus Ponens rule.
(ID) $\phi \Rightarrow \phi$
(CA) $(\phi \Rightarrow \chi) \wedge (\psi \Rightarrow \chi) \rightarrow (\phi \vee \psi \Rightarrow \chi)$
(CSO) $(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi) \rightarrow (\phi \Rightarrow \chi) \leftrightarrow (\psi \Rightarrow \chi)$
(RCEA) if $\vdash \phi \leftrightarrow \psi$ *then* $\vdash (\phi \Rightarrow \chi) \leftrightarrow (\psi \Rightarrow \chi)$
(RCK) if $\vdash (\phi_1 \wedge \ldots \wedge \phi_n) \rightarrow \chi$ *then* $\vdash (\psi \Rightarrow \phi_1 \wedge \ldots \wedge \psi \Rightarrow \phi_n) \rightarrow (\psi \Rightarrow \chi)$

(ID) states that A is conditionally implied by itself; (CA) states that a conditional conclusion of two separate premises is also a conditional conclusion of their disjunction; (CSO) states that two formulas which conditionally imply each other have the same conditional consequences; (RCEA) states that the syntactic form of the antecedent of a conditional formula is irrelevant; (RCK) states that the conditional distributes over classical implication.

Observe that the conditional operator is non monotonic, in the sense that from $\phi \Rightarrow \chi$ it does not follow that $\phi \wedge \psi \Rightarrow \chi$.

In the following, we will sometimes use the theorems of **PCL** described in the following proposition and remark.

PROPOSITION 2.3. *The following theorems are derivable from* **PCL**.

(DT) $(\phi \Rightarrow \chi) \rightarrow (\phi \vee \psi \Rightarrow \phi \rightarrow \chi)$,
(MOD) $(\phi \Rightarrow \bot) \rightarrow (\psi \Rightarrow \neg\phi)$,
(RT) $(\phi \wedge \chi \Rightarrow \psi) \wedge (\phi \Rightarrow \chi) \rightarrow (\phi \Rightarrow \psi)$.

PROOF.

(DT): if $\phi \Rightarrow \chi$, by (RCK) also $\phi \Rightarrow \phi \rightarrow \chi$, hence by (RCEA) also $((\phi \vee \psi) \wedge \phi) \Rightarrow \phi \rightarrow \chi$. But it can be shown by (ID) and (RCK) that also $((\phi \vee \psi) \wedge \neg\phi) \Rightarrow \phi \rightarrow \chi$. By (CA) and (RCEA) we conclude that $\phi \vee \psi \Rightarrow \phi \rightarrow \chi$.
(MOD): if $\phi \Rightarrow \bot$, then by (DT), also $\phi \vee \psi \Rightarrow \phi \rightarrow \bot$, hence by (RCK) $\phi \vee \psi \Rightarrow \neg\phi$. By (ID) and (RCK) $\phi \vee \psi \Rightarrow \psi$, and also $\psi \Rightarrow \phi \vee \psi$, we conclude by (CSO) that $\psi \Rightarrow \neg\phi$.
(RT): from $(\phi \Rightarrow \chi)$, by (ID) and (RCK), we also know that $(\phi \Rightarrow \phi \wedge \chi)$. By (ID) and (RCK) we also know that $(\phi \wedge \chi \Rightarrow \phi)$. From $(\phi \wedge \chi \Rightarrow \psi)$ and (CSO) we conclude that $(\phi \Rightarrow \psi)$.

□

REMARK 2.4. *The following theorems are derivable in* **PCL** . *The proof of these theorems is along the same lines as the proof for the flat fragment that can be found in [Kraus et al. 1990].*

1 $\vdash (\phi \vee \psi \Rightarrow \phi) \wedge (\psi \Rightarrow \chi) \rightarrow (\phi \Rightarrow (\psi \rightarrow \chi))$

2 $\vdash (\phi \vee \psi \Rightarrow \phi) \wedge (\psi \vee \chi \Rightarrow \psi) \rightarrow (\phi \Rightarrow (\chi \rightarrow \psi))$

3 $\vdash (\phi \vee \psi \Rightarrow \phi) \wedge (\psi \vee \chi \Rightarrow \psi) \rightarrow (\phi \vee \chi \Rightarrow \phi)$

4 $\vdash (\phi \vee \psi \Rightarrow \phi) \rightarrow (\phi \vee \psi \Rightarrow \chi) \leftrightarrow (\phi \Rightarrow \chi)$.

2.2 Extensions of PCL

Different logics can be constructed depending on the assumptions we make about $<_x$. These additional assumptions characterize extensions of **PCL** . We consider here a set of extensions of **PCL** that frequently occur in conditional logic. These extensions contain, besides **PCL** , one or more of the following well-known axioms.

(MP) $((\phi \Rightarrow \psi) \wedge \phi) \rightarrow \psi$,
its corresponding semantic property is (S-MP):
$\forall x \in W, x \in W_x$, and $\forall y \in W_x \quad \neg(y <_x x)$.

(CS) $(\phi \wedge \psi) \rightarrow (\phi \Rightarrow \psi)$,
its corresponding semantic property is (S-CS):
$\forall x \in W, \forall y \in W_x \quad x <_x y \vee x = y$.

(CV) $(\phi \Rightarrow \psi) \wedge \neg(\phi \Rightarrow \neg\chi) \rightarrow (\phi \wedge \chi \Rightarrow \psi)$,
its corresponding semantic property is (S-CV):
$\forall x \in W, \forall y, z, u \in W_x \quad y <_x z \rightarrow (y <_x u \vee u <_x z)$.

(CEM) $(\phi \Rightarrow \psi) \vee (\phi \Rightarrow \neg\psi)$,
its corresponding semantic property is (S-CEM):
$\forall x \in W, \forall y, z \in W_x \quad y = z \vee y <_x z \vee z <_x y$.

(A0)+ (A1) + (A2) :
(A0): $(\neg\phi \Rightarrow \bot) \rightarrow \phi$;
(A1): $(\neg\phi \Rightarrow \bot) \rightarrow \neg(\neg\phi \Rightarrow \bot) \Rightarrow \bot$;
(A2): $\neg(\phi \Rightarrow \bot) \rightarrow (\phi \Rightarrow \bot) \Rightarrow \bot$,
the overall semantic property corresponding to these axioms is Universality, i. e.
$\forall x \in W, W_x = W$.

These axioms and semantic conditions have been variously considered in the literature on conditional logics; they are variously related to each other (for instance, given the axioms and rules of **PCL** , (CEM) implies (CV), see for instance [Nute 1980]). We give some intuitive justification for the semantic conditions.

(S-MP) and (S-CS) express a connection between the real world and a conditional. In both cases the truth value of a formula in the real world has an influence on the conditional containing that formula (and vice versa). According to (S-MP) no world is preferred (with respect to $<_x$) to the 'real world', x. According to (S-CS), which is stronger than (S-MP), the real world x is preferred to all other possible worlds with respect to its associated relation $<_x$.

Condition (S-CV) makes $<_x$ a *modular* (or ranked) relation. This condition is required by some important extensions of **PCL** . It is contained, for instance in Lewis's logics **VC** and **VW** [Lewis 1973b].

The property (S-CEM) requires that given two different worlds y and z, either y is strictly preferred to z or z is strictly preferred to y w.r.t. $<_x$. In turn, this entails that for each formula ϕ, $Min_x(\phi)$ contains at most one world. This property is the well-known Uniqueness Assumption that characterizes Stalnaker's logic **C2** [Stalnaker 1968; Nute 1980].

Axioms $\{$(A0)+ (A1) + (A2)$\}$ must be considered together. They characterize the property of Universality which requires that all worlds are 'accessible' from every world. In order to get an intuitive idea of these axioms, notice that we could define an "internal" necessity modality operator $\Box$ (that must not be confused with the modality $\Box_x$ used in the calculus and introduced later in the paper) by means of the conditional implication in the following way:

$$\Box\phi \leftrightarrow (\neg\phi \Rightarrow \bot)$$

Notice that the right part of the equivalence is true in a world x just in case $Min_x(\neg\phi) = \emptyset$, i. e. by the Limit Assumption, in case $W_x \cap [[\neg\phi]] = \emptyset$. By the intuitive meaning of W_x, this corresponds to the case in which there is no conceivable world satisfying $\neg\phi$, i. e. in all conceivable worlds ϕ holds. The $\Box$ modality just defined has hence the usual meaning: $\Box\phi$ is satisfiable in a world x just in case for all $y \in W_x$, $y \models \phi$, i. e. ϕ holds in all conceivable worlds. With this interpretation, axioms $\{$(A0)+ (A1) + (A2)$\}$ become nothing more than the usual S5-axioms of modal logic.

(T) $\Box\phi \rightarrow \phi$ (A0)

(S4) $\Box\phi \rightarrow \Box\Box\phi$ (A1)

(S5) $\diamond\phi \rightarrow \Box \diamond \phi$ (A2)

As for modal logics, it can be shown that these axioms characterize *Universal* models.

Notice that in presence of Universality, for all $x \in W$, it holds that $W_x = W$. In this case, W_x becomes redundant and the definition of model could be simplified by considering triples of the form $(W, \{<_x\}_{x \in W}, I)$. However, for uniformity reasons, also in this case we will consider models as defined in Definition 2.1.

In the following, we only consider systems containing either all axioms (A0) + (A1) + (A2) or none of them. We call a system containing (A0) + (A1) + (A2) a system with Universality.

For $\Sigma \subseteq$ { (MP), (CS), (CV), (CEM), (A0), (A1), (A2)}, such that either $\{(A0), (A1), (A2)\} \subseteq \Sigma$ or $\{(A0), (A1), (A2)\} \cap \Sigma = \emptyset$, we will use the following notions:

DEFINITION 2.5 Σ-MODEL. *A* Σ-model *is a* **PCL** *model that satisfies the semantic conditions corresponding to the axioms in* Σ*. If* $\Sigma = \emptyset$*, a* Σ*-model is simply a* **PCL** *model, as in Definition 2.1.*

DEFINITION 2.6 $\Sigma-$VALID. *ϕ is valid in a $\Sigma-$model M if $M, x \models \phi$ for every $x \in W$. ϕ is Σ-valid (and write $\models_\Sigma \phi$) if it is valid in every Σ-model. If $\Sigma = \emptyset$, a formula is Σ-valid if it is* **PCL** *-valid, as defined in 2.1.*

DEFINITION 2.7 DERIVABLE IN **PCL**$\cup\Sigma$. *A formula ϕ is derivable in* $\mathbf{PCL} \cup \Sigma$ *(denoted by $\vdash_\Sigma \phi$) if it is derivable using the axioms and rules contained in* **PCL** *and in Σ. If $\Sigma = \emptyset$, a formula is derivable in* $\mathbf{PCL} \cup \Sigma$ *if it is derivable in* **PCL** *(i.e. $\vdash_\Sigma$ is $\vdash_{\mathbf{PCL}}$).*

When the logic is clear from the context, we will simply write $\vdash \phi$ and $\models \phi$ (instead of $\vdash_\Sigma \phi$ or $\vdash_{\mathbf{PCL}} \phi$, and $\models_\Sigma \phi$ or $\models_{\mathbf{PCL}} \phi$) .

3. SOUNDNESS AND COMPLETENESS OF PCL AND ITS EXTENSIONS

We show the following soundness and completeness theorem for the logics $\mathbf{PCL} \cup \Sigma$, where $\Sigma \subseteq \{$ (MP), (CS), (CV), (CEM), (A0), (A1), (A2)$\}$ such that either $\{(A0), (A1), (A2)\} \subseteq \Sigma$ or $\{(A0), (A1), (A2)\} \cap \Sigma = \emptyset$. Notice that Σ can also be empty: in this case the logic is **PCL**. The first theorem can be easily proven by showing that axioms are valid in the corresponding semantics and that rules preserve validity.

THEOREM 3.1 SOUNDNESS OF $\mathbf{PCL} \cup \Sigma$. *If a formula is derivable in $\mathbf{PCL} \cup \Sigma$, then it is $\Sigma-$valid.*

THEOREM 3.2 COMPLETENESS OF $\mathbf{PCL} \cup \Sigma$. *If a formula is $\Sigma-$valid, then it is derivable in $\mathbf{PCL} \cup \Sigma$.*

The completeness is shown by the construction of a canonical model. We show that $\mathbf{PCL} \cup \Sigma$ is complete with respect to the semantics by showing that for any set of formulas Γ, if Γ is consistent with respect to $\mathbf{PCL} \cup \Sigma$, then Γ is satisfiable in a Σ-model.
To this purpose, we first introduce some definitions.

DEFINITION 3.3.

(*1*) *A set of formulas Γ is called* inconsistent *with respect to $\mathbf{PCL} \cup \Sigma$ iff there is a finite subset of Γ, $\{\gamma_1, \ldots \gamma_n\}$ such that $\vdash_\Sigma \neg\gamma_1 \vee \neg\gamma_2 \vee \ldots \neg\gamma_n$. Γ is called* consistent *if Γ is not inconsistent. If an (in)consistent Γ contains only one formula γ, we say that γ is (in)consistent.*

(*2*) *A set of formulas Γ is called* maximal consistent *iff it is consistent and if for any formula γ not in Γ, $\Gamma \cup \{\gamma\}$ is inconsistent.*

We will use properties of maximal consistent sets, the proofs of which can be found in most textbooks of logic (see e.g. [Shoenfield 1967]). In particular:

REMARK 3.4. *Every consistent set of formulas is contained in a maximal consistent set of formulas.*

REMARK 3.5. *Let w be a maximal consistent set of formulas and ϕ, ψ formulas in $L_\Rightarrow$. Then w has the following properties:*

(*1*) *If $\vdash_\Sigma \phi \rightarrow \psi$ and $\phi \in w$, then $\psi \in w$*

(*2*) *If from $\phi \in w$ we infer $\psi \in w$, then $\phi \rightarrow \psi \in w$.*

(*3*) *$\phi \wedge \psi \in w$ iff $\phi \in w$ and $\psi \in w$*

(*4*) *$\phi \notin w$ iff $\neg\phi \in w$*

Let ATM be the set of atoms of the language, L the propositional (classical) language and $L_\Rightarrow$ the propositional conditional language. Let U be the set of all maximal consistent sets of formulas, defined as usual.

DEFINITION 3.6. *Let $w \in U$ be a complete formula set, and $\phi, \psi, \ldots \in L_\Rightarrow$ formulas. We define:*

1. $\phi \leq_w \psi$ *iff* $\phi \vee \psi \Rightarrow \phi \in w$

2. $w^\phi = \{\psi \mid \phi \Rightarrow \psi \in w\}$

w^ϕ may be inconsistent. Formula $(\bot \Rightarrow p) \wedge (\bot \Rightarrow \neg p)$ is satisfiable.

REMARK 3.7. *Let $w, w' \in U$ be a maximal consistent set of formulas, and ϕ, ψ, χ $L_{\Rightarrow}$ formulas. Then we have:*

1. *For all $w \in U$, $\phi \vee \psi \leq_w \phi$*
2. *If $w^{\phi} \subseteq w'$ then $\phi \in w'$*
3. *$\leq_w$ is reflexive and transitive*
4. *If $\phi \leq_w \psi$ and $w^{\phi} \subseteq w'$ and $\psi \in w'$ then $w^{\psi} \subseteq w'$*
5. *If $\phi \leq_w \psi \leq_w \chi$ and $w^{\phi} \subseteq w'$ and $\chi \in w'$ then $w^{\psi} \subseteq w'$*
6. *If $\phi \leq_w \psi$ then $w^{\phi} = w^{\phi \vee \psi}$*

PROOF.

1 This immediately follows from definition of $\leq_w$ and from (ID).

2 From definition of w^{ϕ} and (ID).

3 Reflexivity follows from definition of $\leq_w$ together with the fact that by (ID) and (RCK) $\phi \vee \phi \Rightarrow \phi \in w$. Transitivity follows from Remark 2.4, 3.

4 Let be $\chi \in w^{\psi}$, then $\psi \Rightarrow \chi \in w$. Since $\phi \vee \psi \Rightarrow \phi \in w$ (by (ID) and (RCK)) we conclude by Remark 2.4, 1 that $(\phi \Rightarrow (\psi \rightarrow \chi) \in w$ (by the completeness of w). Therefore $\psi \rightarrow \chi \in w'$. Since $\psi \in w'$, we conclude that $\chi \in w'$.

5 By Remark 2.4, 1, we have that $\phi \Rightarrow (\chi \rightarrow \psi) \in w$. Therefore, $\chi \rightarrow \psi \in w'$ and since $\chi \in w'$ we conclude that $\psi \in w'$. The lemma follows by the last point 2.

6 Let $\chi \in w^{\phi}$, i.e. $\phi \Rightarrow \chi \in w$. Then, since by (ID) and (RCK) $\phi \Rightarrow \phi \vee \psi \in w$ by (CSO) $\phi \vee \psi \Rightarrow \chi \in w$, i.e. $\chi \in w^{\phi \vee \psi}$.

□

We are now ready to prove Theorem 3.2:

PROOF. We distinguish four cases.

Case 1: $\{$(CS), (CEM), (A0), (A1), (A2)$\} \cap \Sigma = \emptyset$

DEFINITION 3.8 CANONICAL MODEL. *We let*

$$M = \langle W, \{<_{(w,\gamma)}\}_{(w,\gamma) \in W}, \{W_{(w,\gamma)}\}_{(w,\gamma) \in W}, I \rangle,$$

where:

—*W* = $\{(w, \gamma)$*:* $\gamma \in w$*, for* $w \in U$*, and* $\gamma \in \mathcal{L}_{\Rightarrow}\}$
—$I(w, \gamma) = w \cap ATM$.
—$(w', \phi) <_{(w,\gamma)} (w'', \psi)$ *iff:*
  —$w^{\phi} \subseteq w'$, and
  —either $w^{\psi} \not\subseteq w''$ or $w^{\psi} \subseteq w''$, $w^{\phi} \not\subseteq w''$, $\phi \leq_w \psi$*, and* $\psi \notin w'$
—$W_{(w,\gamma)} = \{(w', \phi)$: *for all* $\psi \in \mathcal{L}_{\Rightarrow}$*, if* $\psi \Rightarrow \perp \in w$ *then* $\neg\psi \in w'\}$

Since the formula γ in (w, γ) does not play any role in the definitions of $<_{(w,\gamma)}$ and $W_{(w,\gamma)}$, we will write for short $<_w$ and W_w respectively.
Given I, the satisfiability relation $\models$ is defined in the usual way.
We prove the following facts:

Fact 1: The relation $<_w$ is irreflexive and transitive. Irreflexivity immediately follows from definition of $<_w$. As far as transitivity is concerned, let (i): $(w', \phi) <_w (w'', \psi)$, and (ii): $(w'', \psi) <_w (w''', \chi)$. We show that (iii): $(w', \phi) <_w (w''', \chi)$. From (i) and (ii), by Definition 3.8, we know that (iv): $w^\phi \subseteq w'$, $w^\psi \subseteq w''$, $\phi \leq_w \psi$ and $\psi \notin w'$.
We consider (ii) and distinguish two cases.
1. $w^\chi \not\subseteq w'''$: the result (iii) immediately follows from definition of $<_w$ in Definition 3.8.
2. $w^\chi \subseteq w'''$: from (ii), and Definition 3.8, we also know that (v) $\psi \leq_w \chi$, $\chi \notin w''$, and $w^\psi \not\subseteq w'''$. By transitivity of $\leq_w$ and (iv) also $\phi \leq_w \chi$. In order to prove (iii) we still have to prove that (a) $\chi \notin w'$ and (b) $w^\phi \not\subseteq w'''$. (a): suppose $\chi \in w'$. By Remark 3.7 we get that $w^\psi \subseteq w'$, hence by Remark 3.7.2, $\psi \in w'$, which contradicts (iv). (b): suppose $w^\phi \subseteq w'''$. By Remark 3.7, also $w^\psi \subseteq w'''$, which contradicts (v). We have thus shown that $\phi \leq_w \chi$, that $\chi \notin w'$ and $w^\phi \not\subseteq w'''$, hence by Definition 3.8, that (iii): $(w', \phi) <_w (w''', \chi)$.

Fact 2: Let $|\phi| = \{(w', \psi) : \phi \in w'\}$.
We let $Min_w(|\phi|) = \{(w', \psi) : (w', \psi) \in |\phi|$ and there is no $(w'', \chi) \in |\phi|$ such that $(w'', \chi) <_w (w', \psi)\}$. We prove that $Min_w(|\phi|) \subseteq |\chi|$ iff $\phi \Rightarrow \chi \in w$.
$\Leftarrow$. Let $(w', \psi) \in Min_w(|\phi|)$. We can prove the following facts. (i) $\phi \in (w', \psi)$. Obvious. (ii) $w^\psi \subseteq w'$. Indeed, suppose by absurd that $w^\psi \not\subseteq w'$. Consider (w'', ϕ) with $w^\phi \subseteq w''$ (this world exists: since $(w', \psi) \in W_w$, $\phi \in w'$, hence by definition of W_w, $\phi \Rightarrow \bot \notin w$, hence w^ϕ is consistent). Clearly, $(w'', \phi) \in |\phi|$. Furthermore, by definition of $<_w$, $(w'', \phi) <_w (w', \psi)$, which contradicts the hypothesis that $(w', \psi) \in Min_w(|\phi|)$. (iii) $w^\phi \subseteq w'$. Suppose $w^\phi \not\subseteq w'$. Then, $\neg((\phi \vee \psi) \Rightarrow \psi) \in w$ (otherwise $\psi \leq_w \phi$ and by (ii) and Remark 3.7,4 $w^\phi \subseteq w'$). Hence in W there is $(w'', \phi \vee \psi)$ with $w^{\phi\vee\psi} \subseteq w''$ and $\neg\psi \in w''$. It can be easily shown that $(w'', \phi \vee \psi) \in |\phi|$, and $(w'', \phi \vee \psi) <_w (w', \psi)$, thus contradicting the hypothesis that $(w', \psi) \in Min_w(|\phi|)$. We can therefore conclude that $w^\phi \subseteq w'$. It follows that if $\phi \Rightarrow \chi \in w$, then $Min_w(|\phi|) \subseteq |\chi|$.
$\Rightarrow$. Let $Min_w(|\phi|) \subseteq |\chi|$, and suppose for a contradiction that (i) $\phi \Rightarrow \chi \notin w$. By (i) and by definition of W, there is (w', ϕ) in W with $w^\phi \subseteq w'$ and $\chi \notin w'$. Furthermore, $(w', \phi) \in W_w$: indeed, by (MOD) for all ψ s.t. $\psi \Rightarrow \bot \in w$, also $\phi \Rightarrow \neg\psi \in w$, and since $w^\phi \subseteq w'$, it follows that for all ψ, if $\psi \Rightarrow \bot \in w$, then $\neg\psi \in w'$. By definition of W_w, $(w', \phi) \in W_w$. It can be easily shown, by definition of $<_w$, that $(w', \phi) \in Min_w(|\phi|)$, which contradicts the assumption that $Min_w(|\phi|) \subseteq |\chi|$. We conclude that if $Min_w(|\phi|) \subseteq |\chi|$, then $\phi \Rightarrow \chi \in w$.

Fact 3: For all formulas ϕ, $(w, \gamma) \models \phi$ iff $(w, \gamma) \in |\phi|$. We reason by induction on the complexity of ϕ. If ϕ is an atom, the property follows by definition of I. If ϕ is a boolean combination of formulas, the proof easily follows by the inductive step. Consider the case in which $\phi = \psi \Rightarrow \chi$. By inductive hypothesis, $|\psi| = [[\psi]]$ and $|\chi| = [[\chi]]$. Since it can be easily shown that $Min_w(\psi) = Min_w(|\psi|)$, the property immediately follows from Fact 2 above.

Fact 4: The relation $<_{(w)}$ satisfies the Limit Assumption: for all formula ϕ, if $[[\phi]] \cap W_w \neq \emptyset$, then $Min_{<_w}(\phi) \neq \emptyset$.
Let $[[\phi]] \cap W_w \neq \emptyset$. By definition of W_w it follows that $\phi \Rightarrow \bot \notin w$, hence w^ϕ is consistent. Consider (w', ϕ) with $w^\phi \subseteq w'$. Clearly, $(w', \phi) \in |\phi|$, and by definition of $<_w$, $(w', \phi) \in Min_w(|\phi|)$. By Fact 3 we can easily show that $(w', \phi) \in Min_w(\phi)$, hence $Min_w(\phi) \neq \emptyset$.

Furthermore, we show that:

Fact 5: If $(CV) \in \Sigma$, then M satisfies (S-CV) (i.e. modularity). Let $(w', \phi) <_w (w'', \psi)$. Then by definition of $<_w$ in Definition 3.8, $w^\phi \subseteq w'$, and either (a) $w^\psi \subseteq w''$ or (b) $w^\psi \not\subseteq w''$. If (b), the property follows from the observation that given any (w''', χ) if $w^\chi \not\subseteq w'''$, then by definition of $<_w$ in Definition 3.8, $(w', \phi) <_w (w''', \chi)$, whereas if $w^\chi \subseteq w'''$, then $(w''', \chi) <_w (w'', \psi)$.

If (a), by definition of $<_w$, the following facts also hold: (i)$\phi \leq_w \psi$, i.e. $(\phi \vee \psi) \Rightarrow \phi \in w$, (ii) $\psi \notin w'$, and (iii) $w^\phi \not\subseteq w''$. Furthermore, from (i) and by Remark 3.7.6 it follows that: (iv)$w^{\phi \vee \psi} = w^\phi$. From (iii) and (iv) it follows that: (v) $w^{\phi \vee \psi} \not\subseteq w''$. From (v), it follows that (vi): $\phi \vee \psi \Rightarrow \neg\psi$ (otherwise by (CV) and (RCEA), for all χ, if $\phi \vee \psi \Rightarrow \chi \in w$, then $\psi \Rightarrow \chi \in w$, i.e. $w^{\phi \vee \psi} \subseteq w^\psi$, hence $w^{\phi \vee \psi} \subseteq w''$, against(v)).

With these facts at hand, consider now any (w''', χ). If $w^\chi \not\subseteq w'''$, then $(w', \phi) <_w (w''', \chi)$ by definition of $<_w$. If $w^\chi \subseteq w'''$, we distinguish two cases.

Case (a): $\phi \vee \psi \vee \chi \Rightarrow \neg(\psi \vee \chi) \in w$. By (RCK) and propositional reasoning, we also have that (vii) $\phi \vee \psi \vee \chi \Rightarrow \neg\chi$, and (viii) $\phi \vee \psi \vee \chi \Rightarrow \neg\psi$. From (viii), by (ID) and (RCK), we derive that $\phi \vee \psi \vee \chi \Rightarrow \phi \vee \chi$. From this, together with (vii) and Remark 2.4,4 we derive that (ix) $\phi \vee \chi \Rightarrow \neg\chi$. From (ID) we have $\phi \vee \chi \Rightarrow \phi \vee \chi \in w$, hence from (ix) and (RCK), we derive that (x) $\phi \vee \chi \Rightarrow \phi$, hence $\phi \leq_w \chi$. Furthermore, from (x) and Remark 2.4, 4, also $\phi \Rightarrow \neg\chi \in w$, hence (xi) $\chi \notin w'$, and by consistency of w''', (xii) $w^\phi \not\subseteq w'''$. By definition of $<_w$ it follows that $(w', \phi) <_w (w''', \chi)$.

Case (b): $\neg(\phi \vee \psi \vee \chi \Rightarrow \neg(\psi \vee \chi)) \in w$. In this case, from (vi) above and (DT) we have that $\phi \vee \psi \vee \chi \Rightarrow \phi \vee \psi \rightarrow \neg\psi \in w$. By (CV), $((\phi \vee \psi \vee \chi) \wedge (\psi \vee \chi)) \Rightarrow \phi \vee \psi \rightarrow \neg\psi \in w$, hence by (RCEA), also $\psi \vee \chi \Rightarrow \phi \vee \psi \rightarrow \neg\psi$, and by (RCK) $\psi \vee \chi \Rightarrow \neg\psi \in w$. By reasoning analogously to what done just above in order to show points (x)-(xii), we conclude that $(w''', \chi) <_w (w'', \psi)$.

Fact 6: If $(MP) \in \Sigma$, then M satisfies (S-MP). First of all, by (MP) we know that if $\psi \Rightarrow \bot \in w$, also $\psi \rightarrow \bot \in w$, hence by consistency and maximality of w, $\neg\psi \in w$, and from definition of $W_{(w,\gamma)}$, $(w, \gamma) \in W_{(w,\gamma)}$. We now show that for no (w', ϕ) $((w', \phi) <_{(w,\gamma)} (w, \gamma))$ holds. For a contradiction, suppose $(w', \phi) <_{(w,\gamma)} (w, \gamma)$. Then by definition of $<_{(w,\gamma)}$ in Definition 3.8, either (a) $w^\gamma \not\subseteq w$ or, among other facts, (b) (i) $\phi \leq_w \gamma$, i.e. $\phi \vee \gamma \Rightarrow \phi$ and (ii) $w^\phi \not\subseteq w$. (a) is impossible: given (MP), and $\gamma \in w$, also $w^\gamma \subseteq w$. Consider (b). By (i) and Remark 3.7.6 it follows that $w^\phi = w^{\phi \vee \gamma}$. From (ii) we conclude that $w^{\phi \vee \gamma} \not\subseteq w$. However this contradicts the facts that: $\gamma \in w$, hence $\phi \vee \gamma \in w$, hence by (MP) $w^{\phi \vee \gamma} \subseteq w$. We conclude that it cannot be that $(w', \phi) <_{(w,\gamma)} (w, \gamma)$, hence $\neg((w', \phi) <_{(w,\gamma)} (w, \gamma))$ holds.

Fact 7: $\mathbf{PCL} \cup \Sigma$ *is complete w.r.t the semantics.* From the facts above.

Case 2: $\{$(CS)$\} \cap \Sigma \neq \emptyset$, and $\{$(CEM), (A0), (A1), (A2)$\} \cap \Sigma = \emptyset$. Let M be the model obtained by the same construction used in Definition 3.8, starting from maximal sets of formulas consistent w.r.t. **PCL** $+\Sigma$. Notice that even if (CS) $\in \Sigma$, the model M does not satisfy (S-CS) but something weaker, namely that either $(w, \gamma) <_{(w,\gamma)} (w', \phi)$ or $w = w'$ (but possibly not $(w, \gamma) = (w', \phi)$).

In order to obtain a model satisfying (CS), we need to strengthen the definition of $\{<'_{(w,\gamma)}\}_{(w,\gamma) \in W'}$. The resulting model is M' defined as follows.

DEFINITION 3.9 CANONICAL MODEL IF (CS) $\in \Sigma$. *We let*

$$M' = \langle W, \{<'_{(w,\gamma)}\}_{(w,\gamma)\in W}, \{W_{(w,\gamma)}\}_{(w,\gamma)\in W}, I\rangle,$$

where W, $\{W_{(w,\gamma)}\}_{(w,\gamma)\in W}$*, and* I *are defined as in* M*, and* $\{<'_{(w,\gamma)}\}_{(w,\gamma)\in W}$ *is defined as follows.*

(*1*) *if* $(w', \phi) <_{(w,\gamma)} (w'', \psi)$ *in* M*, then* $(w', \phi) <'_{(w,\gamma)} (w'', \psi)$.

(*2*) *for all* ϕ*,* $(w, \gamma) <'_{(w,\gamma)} (w, \phi)$.

We can prove that M' is a **PCL** -model satisfying (S-CS) that preserves the properties of M by showing the following facts:

Fact 1: (S-CS) holds. We show that for all (w, γ) and all (w', ϕ), either $(w, \gamma) <_{(w,\gamma)} (w', \phi)$ or $(w, \gamma) = (w, \phi)$. First of all notice that since $\gamma \in w$, by (CS), $w^\gamma = w$, i. e. $w^\gamma \subseteq w$. If $w^\phi \not\subseteq w'$, by definition of $<_w$, $(w, \gamma) <_{(w,\gamma)} (w', \phi)$.
Let $w^\phi \subseteq w'$. We reason as follows. Since both $\gamma \in w$ and $\phi \vee \gamma \in w$, by (CS) it follows that $\phi \vee \gamma \Rightarrow \gamma \in w$. We distinguish two cases: (a) $\phi \in w$; (b) $\phi \notin w$. (b): in order to prove that $(w, \gamma) <_{(w,\gamma)} (w', \phi)$, we still have to prove that $w^\gamma \not\subseteq w'$. This holds since $w^\gamma = w$, hence by maximality of w, if $w^\gamma \subseteq w'$, then $w = w'$ but this is impossible, since $\phi \in w'$ whereas $\phi \notin w$. In case (a) holds, by (CS) $w^\phi = w$ and by maximality of w, $w' = w$. By definition of $<'_{(w,\gamma)}$, in this last case either $(w, \gamma) = (w', \phi)$ or $(w, \gamma) <'_{(w,\gamma)} (w', \phi)$.

Fact 2: $<'_{(w,\gamma)}$ *is irreflexive*. This follows from the irreflexivity of $<_{(w,\gamma)}$ in M together with the observation that for all (w, ϕ) (having the same first element than (w, γ)), only $(w, \gamma) <'_{(w,\gamma)} (w, \phi)$ holds.

Fact 3: $<'_{(w,\gamma)}$ *is transitive*. This follows from the fact that $<_{(w,\gamma)}$ is transitive together with the observation that the only possible extra case is: (i) $(w, \gamma) <'_{(w,\gamma)} (w, \phi)$ for some (w, ϕ) with the first element equal to the first element of (w, γ), and $(w, \phi) <'_{(w,\gamma)} (w'', \psi)$ for some (w'', ψ). We show that $(w, \gamma) <'_{(w,\gamma)} (w'', \psi)$. If $w^\psi \not\subseteq w''$ then $(w, \gamma) <_{(w,\gamma)} (w'', \psi)$ (since by (CS) $w^\gamma \subseteq w$), i.e. $(w, \gamma) <'_{(w,\gamma)} (w'', \psi)$. If $w^\psi \subseteq w''$ then we reason as follows. First of all, by (CS) (since both $\gamma \in w$ and $\gamma \vee \psi \in w$), $\gamma \vee \psi \Rightarrow \gamma \in w$. Second, by definition of $<_{(w,\gamma)}$ in Definition 3.8, from (ii), we derive that $\psi \notin w$. From (ii), always by definition of $<_{(w,\gamma)}$ in Definition 3.8, we also know that $w^\phi \not\subseteq w''$. In presence of (CS), since both $\phi \in w$ and $\gamma \in w$, we know that $w^\phi = w = w^\gamma$, hence $w^\phi \not\subseteq w''$. We can therefore conclude by definition of $<_{(w,\gamma)}$ in Definition 3.8 that $(w, \gamma) <_{(w,\gamma)} (w'', \psi)$, hence $(w, \gamma) <'_{(w,\gamma)} (w'', \psi)$.

Fact 4: $<'_{(w,\gamma)}$ *satisfies the Limit Assumption*. If $[[\phi]] \neq \emptyset$, then by the Limit Assumption in M there is $(w', \phi) \in Min_{(w,\gamma)}(\phi)$. If in M' $(w', \phi) \notin Min_{(w,\gamma)}(\phi)$, this can only be because $(w, \gamma) \in [[\phi]]$, and the relation $(w, \gamma) <'_{(w,\gamma)} (w', \phi)$ has been inserted in M'. In this case, it can be easily shown that by (S-CS) $(w, \gamma) \in Min_{(w,\gamma)}(\phi)$, hence $Min_{(w,\gamma)}(\phi) \neq \emptyset$.

Fact 5: For all formulas ϕ*,* ϕ *is satisfiable in* M *iff it is satisfiable in* M'. By induction on the complexity of ϕ. If ϕ is an atom, the property immediately follows since the valuation function in M and M' is the same. If ϕ is a boolean combination of formulas, the property easily follows by the inductive step. Let $\phi = \psi \Rightarrow \chi$. We show that for all $(w, \gamma), Min_{(w,\gamma)}(\psi) \subseteq [[\chi]]$ in M iff $Min_{(w,\gamma)}(\psi) \subseteq [[\chi]]$ in M'. The $\Rightarrow$ direction immediately follows, since for all (w', μ) if $(w', \mu) \in Min_{(w,\gamma)}(\psi)$ in M', then also $(w', \mu) \in Min_{(w,\gamma)}(\psi)$ in M. For the $\Leftarrow$ direction we reason as follows. Let $Min_{(w,\gamma)}(\psi) \subseteq [[\chi]]$ in M', and $Min_{(w,\gamma)}(\psi) \not\subseteq [[\chi]]$ in M. Then there is (w', μ) s.t. (i) $(w', \mu) \notin [[\chi]]$, (ii) $(w', \mu) \in Min_{(w,\gamma)}(\psi)$ in M and (iii) $(w', \mu) \notin Min_{(w,\gamma)}(\psi)$ in M'. If this is the case, then there must be a world (w'', μ') in $[[\psi]]$ s.t. $(w'', \mu') <'_{(w,\gamma)} (w', \mu)$ in M', whereas it does not hold that $(w'', \mu') <_{(w,\gamma)} (w', \mu)$ in M. This can only happen in case $(w'', \mu') = (w, \gamma)$ and $w = w'$ (indeed, for all other worlds $<$ and $<'$ behave in the same way). Since $w = w'$, by (i), also $(w, \gamma) \notin [[\chi]]$. Furthermore, since M' satisfies (S-CS), it can be easily proven that $(w, \gamma) \in Min_{(w,\gamma)}(\psi)$ in M', hence $Min_{(w,\gamma)}(\psi) \not\subseteq [[\chi]]$ in M', which contradicts the hypothesis. We can conclude that if $Min_{(w,\gamma)}(\psi) \subseteq [[\chi]]$ in M', then also $Min_{(w,\gamma)}(\psi) \not\subseteq [[\chi]]$ in M.

Fact 6: M' satisfies (S-MP). Since (S-MP) is a consequence of (S-CS).

Fact 7: if $(CV) \subseteq \Sigma$, then $<'_{(w,\gamma)}$ is modular. This follows from the fact that $<_{(w,\gamma)}$ in M is modular. Moreover, the only extra relation we have to consider here is $(w, \gamma) <_{(w,\gamma)} (w', \phi)$. In this case the modularity is satisfied since by (S-CS) for any (w'', ϕ), $(w, \gamma) <_{(w,\gamma)} (w'', \psi)$.

Fact 8: $\mathbf{PCL} \cup \Sigma$ is complete w.r.t the semantics. From the facts above.

Case 3: $\{$(CEM)$\} \cap \Sigma \neq \emptyset$, and $\{$ (A0), (A1), (A2)$\} \cap \Sigma = \emptyset$. Let M be the canonical model built by using the same construction of Definition 3.8 above, starting from sets of formulas that are maximal and consistent w.r.t. **PCL** $+\Sigma$. If $(CS) \in \Sigma$, then consider the model M' defined in Case 2 above. Whether $(CS) \in \Sigma$ or not, we will refer to the starting model by M. Notice that even if (CEM) $\in \Sigma$, this model does not satisfy (S-CEM) but something weaker. In order to obtain a model that satisfies (S-CEM), we build M' which is equivalent to M and satisfies (S-CEM), as follows. The only difference between M and M' is in the definition of the relations $<'_{(w,\gamma)}$.

DEFINITION 3.10 CANONICAL MODEL IF (CEM) $\in \Sigma$. *We let*

$$M' = \langle W, \{<'_{(w,\gamma)}\}_{(w,\gamma)\in W}, \{W_{(w,\gamma)}\}_{(w,\gamma)\in W}, I\rangle,$$

where W, $\{W_{(w,\gamma)}\}_{(w,\gamma)\in W}$, and I are defined as in M. $<'_{(w,\gamma)}$ is defined as follows.

1 *if $(w', \phi) <_{(w,\gamma)} (w'', \psi)$, then $(w', \phi) <'_{(w,\gamma)} (w'', \psi)$;*

2 *let $\pi_{(w,\gamma)}$ be an enumeration of the set of worlds $S_{(w,\gamma)} = \{(w', \phi)$ s.t. $w^{\phi} \not\subseteq w\}$. For all (w', ϕ) and (w'', ψ) in $S_{(w,\gamma)}$, we let $(w', \phi) <'_{(w,\gamma)} (w'', \psi)$ iff $\pi_{(w,\gamma)}(w', \phi) < \pi_{(w,\gamma)}(w'', \psi)$;*

3 *for each $(w', \phi) \in W$ such that $w^{\phi} \subseteq w'$, we let $[(w', \phi)]_{(w,\gamma)}$ be the set of worlds (w', ψ) such that $w^{\psi} \subseteq w'$ having the same first element than (w', ϕ). For all $[(w', \phi)]_{(w,\gamma)}$, we let $\pi'_{([(w',\phi)],(w,\gamma))}$ be an enumeration over $[(w', \phi)]_{(w,\gamma)}$. If $(w, \gamma) \in S'$, then the first element of the enumeration will be (w, γ) itself. We let $(w', \phi) <'_{(w,\gamma)} (w', \psi)$ iff $\pi'_{([(w',\phi)],(w,\gamma))}(w', \phi) < \pi'_{([(w',\phi)],(w,\gamma))}(w', \psi)$.*

We can easily show the following facts in order to prove that M' is a **PCL** -model, that it is equivalent to M, that it satisfies (S-CEM), and that it preserves the properties of $<'$.

Fact 1: $<'$ is irreflexive. Consider the definition of $<'_{(w,\gamma)}$ above. $(w',\phi) <'_{(w,\gamma)} (w',\phi)$ is not introduced by step 1, since $<$ is irreflexive; it is not introduced by step 2 either, since $\pi_{(w,\gamma)}$ assigns a different number to each world (w',ϕ); it is not introduced by step 3 for the same reason.

Fact 2: $<'$ is transitive. Let (a) $(w',\phi) <'_{(w,\gamma)} (w'',\psi)$, and (b) $(w'',\psi) <'_{(w,\gamma)} (w''',\chi)$. If both relations have been introduced at step 1, then transitivity follows by transitivity of $<$ in M. If both relations have been inserted at point 2 or 3, then transitivity immediately follows from the total ordering established by the enumerations. Since the set of worlds considered at point 2 are disjoint from the sets of worlds considered at point 3, it cannot happen that one relation is introduced by step 2 and the other by step 3. We are left with the possible combinations of step 1 and either step 2 or step 3.

Let us consider steps 1 and 2. Notice that by definition of $<_{(w,\gamma)}$ in Definition 3.8, it cannot be that (a) has been introduced at step 2, and (b) at step 1 (indeed, if $w^\psi \not\subseteq w''$, then in M it never holds that $(w'',\psi) <_{(w,\gamma)} (w''',\chi)$). If (a) has been introduced at step 1 and (b) has been introduced at step 2, then $w^\phi \subseteq w'$, and $w^\chi \not\subseteq w'''$. By definition of $<_{(w,\gamma)}$ in Definition 3.8, $(w',\phi) <_{(w,\gamma)} (w''',\chi)$, hence also $(w',\phi) <'_{(w,\gamma)} (w''',\chi)$.

Let us consider steps 1 and 3. If (a) has been introduced at step 1 and (b) at step 3, we reason as follows. By Definition 3.8, we know that (c): $\phi \vee \psi \Rightarrow \phi \in w$, (d) $\psi \notin w'$ and (e) $w^\phi \not\subseteq w''$. Furthermore, we know that (f) $w'' = w'''$, that $w^\psi \subseteq w''$, and $w^\chi \subseteq w'''$. By maximality of w^ψ and w^χ given (CEM), we know that $w^\psi = w''$, $w^\chi = w'''$, hence, given (f), that (g) $w^\psi = w^\chi$. We want to show that (c') $\phi \vee \chi \Rightarrow \phi \in w$, (d') $\chi \notin w'$ and (e') $w^\phi \not\subseteq w'''$, from which we can conclude by Definition 3.8 and step 1 that $(w',\phi) <'_{(w,\gamma)} (w''',\chi)$.

First of all we show that given (CEM) and (c) (i.e. $\phi \vee \psi \Rightarrow \phi \in w$), then also (h) $\phi\vee\chi\vee\psi \Rightarrow \phi \in w$ holds, otherwise by (CEM) $\phi\vee\chi\vee\psi \Rightarrow \neg\phi \in w$, which contradicts (c). Indeed, from $\phi \vee \chi \vee \psi \Rightarrow \neg\phi \in w$, we derive that $\phi \vee \chi \vee \psi \Rightarrow \neg(\phi \vee \psi) \in w$ (by (RCK) together with the fact that (DT) and (c) entail $\phi \vee \chi \vee \psi \Rightarrow ((\phi \vee \psi) \rightarrow \phi) \in w$). By (ID) and (RCK) it follows that $\phi \vee \chi \vee \psi \Rightarrow \chi \in w$ and by Remark 2.4,6 that $\chi \Rightarrow \neg(\phi \vee \psi) \in w$, hence $\chi \Rightarrow \neg\psi \in w$. However this is impossible given (g) (since by (ID) $\psi \Rightarrow \psi \in w$). Hence we conclude that (h) holds.

We are now ready to show that (c'), (d') and (e') hold. For a contradiction, first suppose that (c') does not hold, i.e. $(\phi\vee\chi) \Rightarrow \phi \notin w$, which given (CEM) means that $(\phi\vee\chi) \Rightarrow \neg\phi \in w$. This is impossible, given (h) (i.e. $\phi\vee\chi\vee\psi \Rightarrow \phi \in w$). Indeed, by (RCK), from (h) it follows that $\phi\vee\chi\vee\psi \Rightarrow \phi\vee\chi \in w$, and by Remark 2.4,6, from $(\phi\vee\chi) \Rightarrow \neg\phi \in w$ we would conclude $\phi \vee \chi \vee \psi \Rightarrow \neg\phi \in w$, which contradicts (h). Hence (c') must hold. Furthermore, (d'), namely that $\chi \notin w'$, holds, otherwise by (c'), the fact that by (g) $\psi \Rightarrow \chi \in w$, and Remark 2.4, 3 we would have that $\phi \Rightarrow \chi \rightarrow \psi \in w$. In turn, this would entail that $\chi \rightarrow \psi \in w$, hence if $\chi \in w'$ also $\psi \in w'$, which contradicts that by (d) $\psi \notin w'$. Furthermore, by (CEM) either $\phi \Rightarrow \chi \in w$ or $\phi \Rightarrow \neg\chi \in w$. $\phi \Rightarrow \chi \in w$ is impossible since $w^\phi \subseteq w'$, $\chi \notin w'$ and w' is consistent. Hence $\phi \Rightarrow \neg\chi \in w$, and by consistency of w''', since $\chi \in w'''$, $w^\phi \not\subseteq w'''$, hence (e') holds, and the result follows.

If (a) has been introduced at step 3 and (b) at step 1, we reason as follows. By (a) we know that (c) $w^\phi \subseteq w'$, that (d) $w' = w''$, and by maximality of w^ϕ and w^ψ, together with (d), that (e) $w^\phi = w^\psi$. If $w^\chi \not\subseteq w'''$, the result follows by definition of $<_{(w,\gamma)}$ in Definition 3.8 and step 1. If $w^\chi \subseteq w'''$, then by (b) we know that (f) $\psi \vee \chi \Rightarrow \psi \in w$, (g) $\chi \notin w''$ and (h) $w^\psi \not\subseteq w'''$. We want to show that (f') $\phi \vee \chi \Rightarrow \phi \in w$, (g') $\chi \notin w'$

and (h') $w^\phi \not\subseteq w'''$, from which we conclude that $(w', \phi) <'_{(w,\gamma)} (w''', \chi)$, by Definition 3.8 and point 1. From (g) and (d) we immediately conclude that (g'), and from (h) and (e) we immediately conclude that (h'). We are still left to prove (f'). This can be proven by reasoning analogously to what done above when proving (c'), and the result follows.

Fact 3: $<'_{(w,\gamma)}$ satisfies the Limit Assumption. By the Limit Assumption of $<_{(w,\gamma)}$ in M together with fact (a) of point (iv) below.

Fact 4: For all (w, γ), for all ϕ, $M', (w, \gamma) \models \phi$ iff $M, (w, \gamma) \models \phi$. If ϕ is an atom this is obvious. If ϕ is a boolean combination of formulas the result easily follows by the inductive step. Let $\phi = \psi \Rightarrow \chi$. We can show that (a) $Min_{(w,\gamma)}(\psi) \subseteq [[\chi]]$ in M iff $Min_{(w,\gamma)}(\psi) \subseteq [[\chi]]$ in M'. The only if direction immediately follows by inductive hypothesis, since it can be easily proven that for all (w', δ), if $(w', \delta) \in Min_{(w,\gamma)}(\psi)$ in M', then $(w', \delta) \in Min_{(w,\gamma)}(\psi)$ in M.

For the if direction, suppose $Min_{(w,\gamma)}(\psi) \subseteq [[\chi]]$ in M' and $Min_{(w,\gamma)}(\psi) \not\subseteq [[\chi]]$ in M. Then there is (w', δ) s.t. (b) $(w', \delta) \notin [[\chi]]$, (c) $(w', \delta) \in Min_{(w,\gamma)}(\psi)$ in M, and (d) $(w', \delta) \notin Min_{(w,\gamma)}(\psi)$ in M'. However, this is impossible. Indeed, it can be easily shown that by (b) $w^\delta \subseteq w'$. By (c) and (d), we can infer that there is $(w'', \mu) \in [[\psi]]$ s.t. $(w'', \mu) <'_{(w,\gamma)} (w', \delta)$ without $(w'', \mu) <_{(w,\gamma)} (w', \delta)$. This means that the relation $(w'', \mu) <'_{(w,\gamma)} (w', \delta)$ has been inserted at step 3, hence $w'' = w'$. We can assume without loss of generality that (w'', μ) is minimal w.r.t. $\pi'_{([(w',\delta)],(w,\gamma))}$. We can show that (e) $(w'', \mu) \in Min_{(w,\gamma)}(\psi)$ in M'. Suppose it was not, then there would be $(w''', \epsilon) \in [[\psi]]$ s.t. $(w''', \epsilon) <'_{(w,\gamma)} (w'', \mu)$. First notice that by transitivity of $<'$, $(w''', \epsilon) <'_{(w,\gamma)} (w', \delta)$. Second, notice that by minimality of (w'', μ) w.r.t. $\pi'_{([(w',\delta)],(w,\gamma))}$, $(w''', \epsilon) \notin [(w', \delta)]_{(w,\gamma)}$. This means that the relation can only have been inserted by point 1, hence also $(w''', \epsilon) <_{(w,\gamma)} (w', \delta)$ in M, which contradicts $(w', \delta) \in Min_{(w,\gamma)}(\psi)$ in M. Hence, (e) holds. Since $w'' = w'$, and $(w', \delta) \notin [[\chi]]$ also $(w'', \mu) \notin [[\chi]]$, and from (e) we derive that $Min_{(w,\gamma)}(\psi) \not\subseteq [[\chi]]$ in M'. Contradiction. We hence have shown that if $Min_{(w,\gamma)}(\psi) \subseteq [[\chi]]$ in M' also $Min_{(w,\gamma)}(\psi) \subseteq [[\chi]]$ in M, and the result follows.

Fact 5: $<'_{(w,\gamma)}$ in M' satisfies $(S - CEM)$. Consider (w', ϕ) and (w'', ψ). If $w^\phi \not\subseteq w'$ and $w^\psi \not\subseteq w''$, then if $\pi_{(w,\gamma)}(w', \phi) < \pi_{(w,\gamma)}(w'', \psi)$, then $(w', \phi) <'_{(w,\gamma)} (w'', \psi)$, otherwise $(w'', \psi) <'_{(w,\gamma)} (w', \phi)$.

If $w^\phi \subseteq w'$ and $w^\psi \not\subseteq w''$, then $(w', \phi) <'_{(w,\gamma)} (w'', \psi)$ by point 1 and definition of $<$ in 3.8; if $w^\phi \not\subseteq w'$ and $w^\psi \subseteq w''$, then $(w'', \psi) <'_{(w,\gamma)} (w', \phi)$.

Let us consider the case in which $w^\phi \subseteq w'$ and $w^\psi \subseteq w''$. We distinguish two cases: (a)$\phi \vee \psi \Rightarrow \neg\psi \in w$ (b)$\phi \vee \psi \Rightarrow \psi \in w$. (a): it can be easily shown that $(w', \phi) <'_{(w,\gamma)} (w'', \psi)$. (b): we distinguish two other cases: (c)$\phi \vee \psi \Rightarrow \neg\phi \in w$ (d)$\phi \vee \psi \Rightarrow \phi \in w$. (c): we can easily show that $(w'', \psi) <_{(w,\gamma)} (w', \phi)$. (d): $w^\phi = w^{\phi \vee \psi} = w^\psi$, and by maximality of w^ϕ in case (CEM) holds $w' = w''$. In this case, a relation is introduced at step 3. Hence either $(w', \phi) = (w'', \psi)$ or $(w', \phi) <'_{(w,\gamma)} (w'', \psi)$ or $(w'', \psi) <'_{(w,\gamma)} (w', \phi)$.

Fact 6: $<'$ is modular. Since it is total (by $(S - CEM)$).

Fact 7: If in M $(S-MP)$ holds then it holds also in M'. Consider (w, γ). By (MP), since $\gamma \in w$, also $w^\gamma \subseteq w$. By definition of $W_{(w,\gamma)}$ and by consistency and maximality of w, it follows that $(w, \gamma) \in W_{(w,\gamma)}$. Consider any (w', ϕ). The relation $(w', \phi) <_{(w,\gamma)} (w, \gamma)$

is not introduced at step 1, since $<$ in M satisfies $(S-MP)$. It is not introduced at step 2. It is not introduced at step 3 either, since (w, γ) is minimal w.r.t $\pi'_{(w,\gamma)}$.

Fact 8: If $(CS) \in \Sigma$, *then* M' *satisfies* $(S-CS)$. Indeed, $(S-CS)$ holds in M and if $(w, \gamma) <_{(w,\gamma)} (w', \phi)$ in M, then also $(w, \gamma) <'_{(w,\gamma)} (w', \phi)$ in M'.

Fact 9: $\mathbf{PCL} \cup \Sigma$ *is complete w.r.t the semantics*. From the facts above.

Case 4: $\{$*(A0), (A1), (A2)*$\} \cap \Sigma \neq \emptyset$.

Let M be a model built by one of the constructions in the case above corresponding to the axioms in Σ. Even if (A0),(A1),(A2) belong to Σ, the resulting model may not entail the Universality.

Let $M, (w_0, \psi) \models \Gamma$ for some possible world (w_0, ψ). Starting from this model we can build a model M' satisfying Γ and in which the Universality holds. From this follows the completeness of the logic w.r.t. models satisfying Universality.

Before we proceed, notice that by (A0),(A1), (A2) we can prove the following facts:

Fact 1: $(w_0, \psi) \in W_{(w_0,\psi)}$. If it were not so, then for some ϕ, $\phi \in w_0$ and $\phi \Rightarrow \perp \in w_0$. But this is impossible, given $(A0)$.

Fact 2: If $(w', \gamma) \in W_{(w_0,\psi)}$, *then* $W_{(w_0,\psi)} = W_{(w',\gamma)}$. $\Rightarrow$ Suppose there were (w'', ϕ) such that (a) $(w'', \phi) \in W_{(w_0,\psi)}$ and (b) $(w'', \phi) \notin W_{(w',\gamma)}$, then for some χ, $\chi \in w''$ and $\chi \Rightarrow \perp \in w'$, whereas (c) $\neg(\chi \Rightarrow \perp) \in w_0$. However, since $(w', \gamma) \in W_{(w_0,\psi)}$ by hypothesis, from $\chi \Rightarrow \perp \in w'$ it follows that $\neg((\chi \Rightarrow \perp) \Rightarrow \perp) \in w_0$, which contradicts (c), by (A2). $\Leftarrow$ Suppose now (a) $(w'', \phi) \in W_{(w',\gamma)}$ and (b) $(w'', \phi) \notin W_{(w_0,\psi)}$. Then for some χ, $\chi \in w''$, (c) $\chi \Rightarrow \perp \in w_0$, whereas $\neg(\chi \Rightarrow \perp) \in w'$. However, since $(w', \gamma) \in W_{(w_0,\psi)}$, $\neg(\neg(\chi \Rightarrow \perp) \Rightarrow \perp) \in w_0$, which contradicts (c), given (A1).

With these facts at hand, we build M' as follows. The only difference between M' and M is in the definition of the set of possible worlds W:

DEFINITION 3.11 CANONICAL UNIVERSAL MODEL. *We let*

$$M' = \langle W', \{W_{(w,\gamma)}\}_{(w,\gamma) \in W'}, \{<_{(w,\gamma)}\}_{(w,\gamma) \in W'}, I \rangle,$$

where:

(*1*) $W' = W \cap W_{(w_0,\psi)}$

We can show the following Facts:

Fact 3: $(w_0, \psi) \in W'$. By Fact 1 above.

Fact 4: For each $(w, \gamma) \in W'$, $W_{(w,\gamma)} = W'$, *i.e. Universality holds*. By Fact 2 above.

Fact 5: Since $<_{(w,\gamma)}$ *has not changed, it is still irreflexive, transitive, satisfies the Limit Assumption, it satisfies* $(S-MP)$ *if (MP)* $\in \Sigma$, *it satisfies (S-CS) if (CS)* $\in \Sigma$, *it satisfies the modularity if (CV)* $\in \Sigma$, *it satisfies (S-CEM) if (CEM)*$\in \Sigma$.

Fact 6: For all ϕ, *for all* $(w, \gamma) \in W'$, $M, (w, \gamma) \models \phi$ *iff* $M', (w, \gamma) \models \phi$. By induction on the complexity of ϕ. If ϕ is an atom, obvious. If ϕ is a boolean combination of formulas, this follows by the inductive step. If ϕ is a conditional $\psi \Rightarrow \chi$, notice that in M $Min_{(w,\gamma)}(\psi) \subseteq W_{(w,\gamma)}$, by definition of $Min_{(w,\gamma)}(\psi)$. In turn, by Fact 2 above, $W_{(w,\gamma)} \subseteq W'$, hence $Min_{(w,\gamma)}(\psi) \subseteq W'$ and $Min_{(w,\gamma)}(\psi)$ in M' coincides with $Min_{(w,\gamma)}(\psi)$ in M. The result follows by inductive hypothesis.

Fact 7: $M', (w_0, \psi) \models \Gamma$. As a consequence of the point above, for all ϕ, if $M, (w_0, \psi) \models \phi$, also $M', (w_0, \psi) \models \phi$. Hence, from $M', (w_0, \psi) \models \Gamma$, it follows that $M', (w_0, \psi) \models \Gamma$.

Fact 8: $\mathbf{PCL} \cup \Sigma$ *is complete w.r.t the semantics*. We have shown that if Γ is consistent w.r.t. $\mathbf{PCL} \cup \Sigma$ where $(A0), (A1), (A2)$ are in Σ, then it is satisfiable in a **PCL** -model satisfying Universality. Hence, $\mathbf{PCL} \cup \Sigma$ is complete w.r.t the semantics.

□

We have given uniform and modular proof of completeness for **PCL** and its extensions. A result of such generality does not seem to be known in the literature. In order to relate our completeness proof to existing work, we recall some of the most important work on conditional logic systems and their completeness.

Completeness proofs for the more general selection function semantics [Nute 1980] are close to those for modal logics, and the definition of the canonical model is straightforward.

Burgess [Burgess 1981] gives a completeness proof for a conditional logic, called $\mathcal{S}$ which has the same theorems than **PCL** but a different semantics. As a difference with respect to our approach, Burgess considers a semantics without the Limit Assumption. Furthermore, he considers only systems without Universality. He then considers extensions $\mathcal{S}$ corresponding to additional properties of the preference relation, namely connectivity (**CV**), nonvacuity ($\neg(\top \Rightarrow \bot)$), **MP** and **CS** (centering). Notice that nonvacuity is derivable in **PCL** with Universality (by A0).

Similarly to Burgess, Friedman and Halpern [Friedman and Halpern 1994] consider the same families of systems we consider, but they show the completeness of the systems with respect to the preferential semantics without the Limit Assumption (with a different definition of the semantics of the conditional operator).

Lewis [Lewis 1973b] proves the completeness of his logic with respect to sphere semantics, which is less general than preferential semantics, since it characterizes systems which contain (at least) axiom CV.

Grahne [Grahne 1998] presents a system of conditional logic augmented by an update operator. The purpose of his work is the logical modelling of an update operator following the postulates by Katsuno and Satoh [Katsuno and Satoh 1991] and allowing for the Ramsey rule. He obtains a completeness proof with respect to the selection function semantics for a logic stronger than **PCL** (including CV). He also establishes a correspondence between the selection function semantics and the sphere semantics and then between the latter and the preferential semantics. However, this correspondence only holds for logics including CV that have a sphere semantics. By this reason, Grahne's completeness proof cannot be adapted to handle **PCL** .

Kraus, Lehmann and Magidor [Kraus et al. 1990] prove completeness for the flat fragment of **PCL** and of **PCL** + CV (corresponding to their systems **P** and **R**, as recalled above). Observe that the completeness proof for the flat case is considerably simpler than the case of **PCL** , since it comprises only one preference relation rather than a family of preference relations parametrized to worlds.

Nejdl in [Nejdl 1991] investigates a number of conditional logics that we study in this work. He adopts the preferential semantics, rephrased in terms of structures equipped with a ternary relation. He does not present however any new completeness result, as his purpose is mainly to clarify the relations among the systems found in the literature with respect to alternative axiomatizations. Except for the weakest logic CK (included for exaustiveness), the range of systems he considers largely overlaps with ours. More precisely, his starting point is our logic PCL (named System_P), then he considers its extensions with one or more of the following conditions: centering (weak and strong corresponding to MP and CS), modularity (CV), totality (CEM). Moreover he examines systems with the additional property of Absoluteness (for each x, y, $<_x = <_y$) that we do not consider here; observe that this property makes nested conditional collapse, simplifying dramatically the logics. On the opposite, he does not take into account the property of Universality as we do. For all systems under consideration he presents alternative axiomatizations and characteristic theorems.

4. TABLEAU CALCULUS FOR **PCL** AND ITS EXTENSIONS

Our calculus makes use of labels to represent possible worlds.
In order to test the satisfiability of a formula ϕ, we start the tableau by $x : \phi$, for an initial label x. We then verify whether the tableau so obtained is closed. If it is, ϕ is not satisfiable, otherwise it is satisfiable (and, in Theorem 5.8, we show how to extract a model from the open tableau). In order to test the satisfiability of a finite set of formulas Γ, we start the tableau by labelling all its formulas by the initial label x (which is the same than considering the conjunction of all the elements of Γ).

As usual, a *tableau* is a tree. Its *branches* are sets of tableau formulas. These formulas are either the labelled formulas of the initial set Γ or they are obtained from previous tableau formulas by the application of tableau rules.

Tableau rules encode the semantics of the formulas. It is well known how this works for classical logic. Let us look at the conditional formulas $\phi \Rightarrow \psi$ and $\neg(\phi \Rightarrow \psi)$ under preferential semantics. Then we have:
$M, x \models \phi \Rightarrow \psi$ iff for all y, if $y \in Min_x(\phi)$ then $y \models \psi$

Let us look closer at the definition of $Min_x(\phi)$. The following conditions are equivalent:

(i) $y \in Min_x(\phi)$
(ii) $y \models \phi$ and $y \in W_x$ and $\neg\exists z$ s.t. $(z \models \phi \wedge z \in W_x \wedge z <_x y)$
(iii) $y \models \phi$ and $y \in W_x$ and $\forall z((z <_x y) \wedge z \in W_x \rightarrow z \models \neg\phi)$

At this point, we observe that the second part of (iii), namely $\forall z((z <_x y) \wedge z \in W_x \rightarrow z \models \neg\phi)$ can be understood as the definition of a modality indexed by world x and characterized by the preference relation $<_x$. Let us call this modality $\Box_x$. $\Box_x$ is then defined by:

$$M, y \models \Box_x \phi \text{ iff } \forall z \in W_x \text{ if } z <_x y \text{ then } M, z \models \phi$$

In order to represent the assertion $z \in W_x$, we introduce another modality indexed by x, namely V_x, whose meaning is

$$M, y \models V_x \text{ iff } y \in W_x$$

From (iii) above, by using this definition, we get:

$$(*) y \in Min_x(\phi) \text{ iff } M, y \models V_x \wedge \phi \wedge \Box_x \neg\phi$$

In the following, we will refer to formulas of the kind $\Box_x \neg\phi$ or V_x as *pseudo-formulas*.

DEFINITION 4.1 TABLEAU FORMULAS. *Our tableau formulas are of the following kinds:*

(a) $x : \phi$*, where* ϕ *is a formula or a pseudo-formula*

(b) $x <_y z$

(c) $x = y$

where x*,* y *and* z *are labels.*

The formulas of the form (a) are called *world formulas*, the formulas of the form (b) are called *relation formulas*, the formulas of the form (c) are called *equalities*. Formulas of the form (a) or (b) are used in all tableau rules whereas the equalities only occur within the rules introduced to deal with axioms (CS) or (CEM).

In order to develop the tableau rule for the positive conditional formula $\phi \Rightarrow \psi$ we use pseudo-formulas. We have $M, x \models \phi \Rightarrow \psi$ if for all y, $y \notin Min_x(\phi)$ or $y \models \psi$. Using $(*)$, we then obtain for all y

$$y \models \neg\phi \vee \neg\Box_x\neg\phi \vee \neg V_x \vee \psi$$

This disjunction has four disjuncts and each disjunct will produce a branch in the tableau. Hence, we obtain the following tableau rule (T⇒) for the positive conditional

$$\frac{x : \phi \Rightarrow \psi}{y : \neg\phi \;\mid\; y : \neg\Box_x\neg\phi \;\mid y : \psi \;\mid\; y : \neg V_x}$$

Obviously, the tableau rule for the negative conditional $\neg(\phi \Rightarrow \psi)$ can be obtained in the same way from the corresponding satisfiability condition.

$M, x \models \neg(\phi \Rightarrow \psi)$ if there is y such that $y \in Min_x(\phi)$ and $y \not\models \psi$, i.e. if there is y such that:

$$y \models \phi \wedge \Box_x\neg\phi \wedge V_x \wedge \neg\psi$$

This conjunction leads to the following tableau rule (F⇒) for the negative conditional

$$\frac{x : \neg(\phi \Rightarrow \psi)}{\begin{array}{c} y : \phi \\ y : \Box_x\neg\phi \\ y : \neg\psi \\ y : V_x \end{array}}$$

where y is a new parameter in the branch.

Let us derive the two rules for the pseudo modalities in the same way. According to their definition, we have:

$M, z \models \Box_x\phi$ iff for every $y \in W_x$ if $y <_x z$ then $M, y \models \phi$

From this, we derive : if $M, z \models \Box_x\phi$, and $y <_x z$ and $y \in W_x$, then $M, y \models \phi$ leading to the tableau rule:

$$\frac{\begin{array}{c} z : \Box_x\phi \\ y <_x z \\ y : V_x \end{array}}{y : \phi}$$

On the other hand, we have that $M, z \models \neg\Box_x\phi$ iff there is $y \in W_x$ such that $y <_x z$ and $M, y \models \neg\phi$. The rule we use in the tableau calculus is slightly different since it takes into account the minimality imposed by the Limit Assumption: if $M, z \models \neg\Box_x\phi$, by (*) $z \notin Min(\neg\phi)$ and by the Limit Assumption we know that there is an y such that $y <_x z$, and $y \in Min_x(\neg\phi)$, i. e. by (*) $M, y \models \neg\phi$, $M, y \models \Box\phi$, $y \models V_x$. The resulting rule is the following (called (F□)). This rule does the same job as the corresponding rule (due to Fitting [Fitting 1983]) for modal system **GL**, the extension of K4 by Löb axiom $\Box(\Box\phi \rightarrow \phi) \rightarrow \Box\phi$. Hence the tableau rule for $\neg\Box_x$ has the form

$$\frac{z : \neg\Box_x\phi}{\begin{array}{c} y <_x z \\ y : \neg\phi \\ y : \Box_x\phi \\ y : V_x \end{array}}$$

where y is new. In order to simplify the calculus, we avoid to explicitly introduce $y : V_x$ in a branch in which we also have $y <_x z$ or $z <_x y$. In these cases, we implicitly assume that $y : V_x$ (and $z : V_x$). This is captured by condition (iii) in the definition of closed branch (definition 4.3 below). The above rules for the modality $\Box_x$ hence become the simplified tableau rules:

(T□):

$$\frac{\begin{array}{c} z : \Box_x\phi \\ y <_x z \end{array}}{y : \phi}$$

and (F□):

$$\frac{z : \neg\Box_x\phi}{\begin{array}{c} y <_x z \\ y : \neg\phi \\ y : \Box_x\phi \end{array}}$$

Figure 1 shows all the tableau rules for **PCL** . Following the terminology of [Goré 1999], we refer to the rules generating new worlds as *dynamic rules*, and to all other rules as *static rules*. Observe that all the rules we introduced are static except for $(F \Rightarrow)$ and $(F\Box)$ which are dynamic.

In order to deal with the extensions of **PCL** , we need to add some extra rules, shown in Figure 2. These rules straightforwardly capture the semantic properties of the corresponding axioms. We provide one extra rule for each extra axiom, with the exception of (MP) for which we introduce two rules. In order to obtain a calculus for extensions of **PCL** containing more than one axiom, it will be enough to consider the calculus containing all the rules introduced for the extra axioms. Hence, for instance, in order to obtain a calculus for the logic obtained by adding to **PCL** axioms (CV) and (MP) below, we will have to consider all the rules introduced for **PCL** + (R-CV) + (R-MP).

Notice that the rules (R-CS) and (R-CEM) introduce in the tableau some equalities of the kind $x = y$. In the presence of these rules, we need some extra rules to deal with the equality. The rules for equality are shown in Figure 3. Hence, for instance, in order to deal

$$(T\wedge)\ \frac{x:\phi\wedge\psi}{x:\phi, x:\psi} \qquad (F\wedge)\ \frac{x:\neg(\phi\wedge\psi)}{x:\neg\phi \mid x:\neg\psi}$$

$$(NEG)\ \frac{x:\neg\neg\psi}{x:\psi}$$

$$(T\Rightarrow)(*)\ \frac{x:\phi\Rightarrow\psi}{y:\neg\phi \mid y:\neg\Box_x\neg\phi \mid y:\psi \mid y:\neg V_x} \qquad (F\Rightarrow)(**)\ \frac{x:\neg(\phi\Rightarrow\psi)}{y:\phi, y:\Box_x\neg\phi, y:\neg\psi, y:V_x}$$

$$(T\Box)(*)\ \frac{z:\Box_x\phi, y<_x z}{y:\phi} \qquad (F\Box)(**)\ \frac{z:\neg\Box_x\phi}{y<_x z, y:\neg\phi, y:\Box_x\phi,}$$

$$(Trans)\ \frac{y<_x z\ ,\ z<_x u}{y<_x u}$$

(*) y is a label occurring in the branch.
(**) y is a new label not occurring in the branch.

Fig. 1. Tableau rules for **PCL**

$$(R-MP)\ \frac{y<_x x}{x:\bot}\ \frac{x:\neg V_x}{x:\bot}$$

$$(R-CS)\ \frac{}{x<_x y \mid x=y \mid y:\neg V_x}\ old\ x,y$$

$$(R-CV)\ \frac{y<_x z}{u:\neg V_x \mid y<_x u \mid u<_x z}\ old\ u$$

$$(R-CEM)\ \frac{}{y:\neg V_x \mid z:\neg V_x \mid y=z \mid y<_x z \mid z<_x y}\ old\ x,y,z$$

$$(R-UNIV)\ \frac{y:\neg V_x}{x:\bot}$$

Fig. 2. Rules for the extensions of **PCL** .

with Stalnaker's logic **C2**, containing **PCL** plus axioms (CV), (MP) and (CEM) we have to consider the rules for **PCL** + (R-CV) + (R-MP) + (R-CEM) + all the rules for equality ((E-R) + (E-T) + (E-$<$) + (E-$\neg$ V) + (E-ϕ)).

The following is a formal definition of a tableau for **PCL** and its extensions.

DEFINITION 4.2 Σ-TABLEAU. *Let* $\Sigma \subseteq \{$*(R-MP), (R-CS), (R-CV), (R-CEM), Universality*$\}$. *A* Σ*-tableau is a tableau built by applying*

$$(E-R)\ \frac{}{x = x}\ (*) \qquad (E-S)\ \frac{x = y}{y = x}$$

$$(E-T)\ \frac{x = y, y = z}{x = z} \qquad (E-<)\ \frac{x = x', y = y', z = z', y <_x z}{y' <_{x'} z'}$$

$$(E-\neg V)\ \frac{x = x', y = y', y : \neg V_x}{y' : \neg V_{x'}} \qquad (E-\phi)\ \frac{x = x', x : \phi}{x' : \phi}\ (**)$$

Fig. 3. Rules for Equality. (*) x is a label in the branch. (**) ϕ is either a formula or a pseudo-formula.

(i) the rules for **PCL** *of Figure 1,*

(ii) the rules in Σ *of Figure 2,*

(iii) the rules for Equality in Figure 3 whenever (R-CS) or (R-CEM) belong to Σ*.*

DEFINITION 4.3 CLOSED BRANCH AND CLOSED TABLEAU. *We say that a branch of a* Σ*-tableau is closed if it contains one of the following formulas (i) or (ii) or (iii):*

(i) $x : \phi$ *and* $x : \neg\phi$*, for any formula or pseudo-formula* ϕ*, or* $x : \bot$*;*

(ii) $y <_x y$*;*

(iii) $y <_x z$ *and* $y : \neg V_x$ *or* $z : \neg V_x$*;*

A Σ*-tableau is closed if every branch is closed.*

DEFINITION 4.4 Σ-PROVABLE. *A formula* ϕ *is* Σ*-provable if there exists a closed* Σ*-tableau for* $x : \neg\phi$*.*

Notice that in the presence of rule (R-UNIV) to deal with Universality, all branches containing $y : \neg V_x$ close. This expresses that every world belongs to every of the subsets W_x of W, which is precisely the semantic condition for Universality. For example, the formula $\neg(\top \Rightarrow \bot)$ is not a theorem of **PCL**. In the absence of rule (R-UNIV), the tableau contains one branch with $x : \neg V_x$, which cannot be closed. This branch closes in presence of the rule (R-UNIV) (the formula is indeed a theorem of **PCL** + Universality). In the presence of (R-UNIV) , the calculus could be simplified by omitting all the branches in which $y : \neg V_x$ is introduced. The fact that in the presence of Universality the calculus can be simplified is related to the remark done before that in the presence of this property, **PCL** -models become simpler.

As an example of how the calculus works, in Figure 4 we show that $\phi \Rightarrow (\psi \wedge \chi) \rightarrow (\phi \wedge \psi \Rightarrow \chi)$ is a theorem of **PCL** .

5. SOUNDNESS AND COMPLETENESS OF THE TABLEAU CALCULUS FOR **PCL** AND ITS EXTENSIONS

5.1 Soundness of the calculus

In order to prove the soundness and completeness of the tableau rules, we have to define the notion of satisfiability of a tableau branch.

Given a branch B, we denote by W_B the set of labels occurring in B.

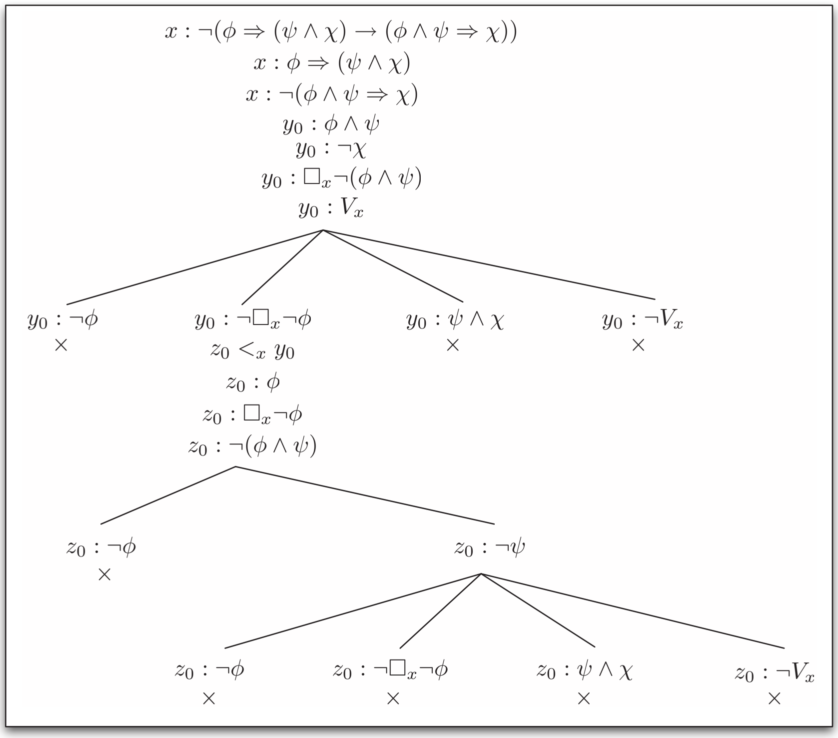

Fig. 4. Example of tableau.

DEFINITION 5.1 **PCL**-MAPPING. *Let $M = (W, \{W_x\}_{x\in W}, \{<_x\}_{x\in W}, I)$ be a* **PCL**-*model. Given a branch B, we say that $f : W_B \to W$ is a* **PCL**-*mapping from B to M, if*

(1) for every $y <_x z \in B$, $f(y), f(z) \in W_{f(x)}$ and $f(y) <_{f(x)} f(z)$ holds in M
(2) for every $y : V_x \in B$, $f(y) \in W_{f(x)}$
(3) for every $x = y \in B$, $f(x) = f(y)$

DEFINITION 5.2 SATISFIABILITY OF A BRANCH. *Given a branch B of a tableau, a* **PCL**-*model M, and a* **PCL**-*mapping f from W_B to W, we say that B is satisfiable under f in M if the following condition holds:*

$$\text{if } x : \phi \in B \text{ then } M, f(x) \models \phi$$

where ϕ is a conditional formula or a pseudo-formula.

A branch B is satisfiable if it is satisfiable in some **PCL**-*model M under some mapping f. A tableau is satisfiable if at least one of its branches is satisfiable.*

In order to show that the tableau rules only prove correct formulas, we first show that they preserve satisfiability.

LEMMA 5.3. *Let T be a tableau satisfiable in a Σ-model. Let T' be obtained from T by applying one of the rules given for* **PCL** *or for Σ in Figures 1, 2, and 3 above. Then*

T' is also satisfiable.

PROOF. If T is a satisfiable tableau, then it contains at least one branch B which is satisfiable, i.e. there is a model $M = (W, \{<_x\}_{x\in W}, \{W_x\}_{x\in W}, I)$ and a $\mathbf{PCL}$-mapping f such that the condition from Definition 5.2 holds for all world formulas in B. We will show that for each tableau rule applied to T, the resulting tableau T' is still satisfiable.

$(class)$ The case of classical connectives is easy and left to the reader.

$(T \Rightarrow)$ Let $x : \phi \Rightarrow \psi \in B$. Let T' be T with B replaced by four new branches: $B_1 = B \cup \{y : \neg\phi\}$, $B_2 = B \cup \{y : \neg V_x\}$, $B_3 = B \cup \{y : \neg\Box_x\neg\phi\}, y : V_x$ and $B_4 = B \cup \{y : \psi\}$. By definition of $(T \Rightarrow)$, $y \in W_B$. We will show that at least one of these branches is satisfiable. Since B is satisfiable, there is a model $M = (W, \{W_x\}_{x\in W}, \{<_x\}_{x\in W}, I)$ and a $\mathbf{PCL}$-mapping f such that $M, f(x) \models \phi \Rightarrow \psi$. Then for all $v \in Min_{<_{f(x)}}(\phi)$, $M, v \models \psi$. This is true iff $\forall v \in W$, either $v \notin Min_{<_{f(x)}}$ or $M, v \models \psi$, i. e. $\forall v \in W$, either $M, v \not\models \phi$ or $M, v \not\models V_{f(x)}$ or $M, v \not\models \Box_{f(x)}\neg\phi$ or $M, v \models \psi$. Since $y \in W_B$, $f(y) \in W$, then one of the following is true:
(1) $M, f(y) \models \neg\phi$
(2) $M, f(y) \models \neg V_x$
(3) $M, f(y) \models \neg\Box_{f(x)}\neg\phi$
(4) $M, f(y) \models \psi$
In the first case, B_1 is satisfiable, in the second case, B_2 is satisfiable, in the third case, B_3 is satisfiable and in the fourth case B_4 is satisfiable.

$(F \Rightarrow)$ Let $x : \neg(\phi \Rightarrow \psi) \in B$. Then $M, f(x) \models \neg(\phi \Rightarrow \psi)$, i. e. there is $v \in Min_{<_{f(x)}}(\phi)$ such that $M, v \models \neg\psi$, i. e. $M, v \models V_{f(x)} \wedge \phi \wedge \Box_{f(x)}\neg\phi$. If the rule $(F \Rightarrow)$ is applied to $x : \neg(\phi \Rightarrow \psi)$ on T, the resulting tableau T' is T with B replaced by $B' = B \cup \{y : \phi, y : V_x, y : \Box_x\neg\phi, y : \neg\psi\}$, where $y \notin W_B$. Since $y \notin W_B$, f is not defined for y. We define a new mapping $f' : W_{B'} \longrightarrow W$ by letting $f'(i) = f(i)$ if $i \neq y$ and $f'(y) = v$. It can be easily verified that f' is a $\mathbf{PCL}$-mapping from $W_{B'}$ to W. Obviously, the new formulas on B' are satisfied by the new model under the mapping f'

$(T\Box)$ Let $y : \Box_x\phi \in B$, and $z <_x y \in B$. After application of rule $(T\Box)$, the resulting tableau T' is T with B replaced by $B' = B \cup \{z : \phi\}$. Since B is satisfiable by M under f, we have that for every v , such that $v <_{f(x)} f(y)$, $M, v \models \phi$. Since $z <_x y \in B$, $f(z) <_{f(x)} f(y)$. Consequently, $M, f(z) \models \phi$, hence B' is also satisfiable.

$(F\Box)$ Let $y : \neg\Box_x\phi \in B$. Since B is satisfiable by M under f and $y : \neg\Box_x\phi \in B$ there is a w s. t. $w <_{f(x)} f(y)$ and $M, w \models \neg\phi$. Clearly $w \in W_{f(x)}$, hence $W_{f(x)} \cap [[\neg\phi]] \neq \emptyset$, and therefore by the Limit Assumption there is $v \in Min_{<_{f(x)}}(\neg\phi)$, i. e. $M, v \models \neg\phi \wedge V_x \wedge \Box_x\phi$. After applying $(F\Box)$, we have $B' = B \cup \{z <_x y, z : \neg\phi, z : \Box_x\phi\}$, where $z \notin W_B$. We define a new mapping f' by $f'(u) = f(u)$ for $u \neq z$ and $f'(z) = v$. Since $f'(z) <_{f'(x)} f'(y)$, f' is a $\mathbf{PCL}$-mapping. Moreover, we have $M, f'(z) \models \neg\phi$ and $M, f'(z) \models \Box_x\phi$.

(E - R), (E - T), (E - S) reflexivity, transitivity and symmetry of $=$ hold due to the properties for equality

(E - $<$), (E - $\neg$ V), (E - ϕ) obvious, by predicate substitution

(R-MP) The property trivially holds, since there is no $(MP)-$model satisfying a branch B in which either $x \notin W_x$ or $y <_x x$.

(R-CS) Let B be a branch of T satisfiable by a $(CS)-$model M. Let T' be obtained from T by replacing B by the three branches $B_1 = B \cup \{x <_x y\}$, $B_2 = B \cup \{x = y\}$, and $B_3 = B \cup \{y : \neg V_x\}$. By (S-CS) in M, either $f(x) <_{f(x)} f(y)$ or $f(x) = f(y)$, or $f(y) \notin V_{f(x)}$. In the first case B_1 is satisfiable. In the second case, B_2 is satisfiable. In the third case, B_3 is satisfiable. In both cases, T' is satisfiable.

(R-CV) Let B be branch of T with $y <_x z \in B$, satisfiable in a $(CV)-$model. Let T' be obtained from T by replacing B by the three branches $B_1 = B \cup \{u : \neg V_x\}$, $B_2 = B \cup \{y <_x u\}$, $B_3 = B \cup \{u <_x z\}$ where $u \in W_B$. By (S-CV), either $f(u) \notin W_{f(x)}$ or $f(y) <_{f(x)} f(u)$ or $f(u) <_{f(x)} f(z)$. It follows that either B_1 or B_2 or B_3 are satisfiable, hence T' is satisfiable.

(R-CEM) Let B be branch of T satisfiable by a $(CEM)-$model. Let T' be obtained from T by replacing B by the five branches $B_1 = B \cup \{y : \neg V_x\}$, $B_2 = B \cup \{z : \neg V_x\}$, $B_3 = B \cup \{y = z\}$, $B_4 = B \cup \{y <_x z\}$, $B_5 = \{z <_x y\}$. By (S-CEM), either $f(y) \notin W_{f(x)}$, or $f(z) \notin W_{f(x)}$, or $f(y) = f(z)$, or $f(y) <_{f(x)} f(z)$ or $f(z) <_{f(x)} f(y)$. It follows that one out of $B_1 \dots B_5$ are satisfiable, hence T' is satisfiable.

(R-UNIV) The property trivially holds, since there is no Universal model satisfying a branch B containing $y : \neg V_x$.

□

With the previous lemma at hand, we can prove the following:

THEOREM 5.4 SOUNDNESS. *If ϕ is Σ-provable then it is Σ-valid.*

PROOF. We show the contrapositive. Let ϕ be not Σ-valid, i.e. let $\neg\phi$ be satisfiable by a $\Sigma-$model. We show that the tableau starting with $x : \neg\phi$ does not close, which means that ϕ is not $\Sigma-$provable. Let T be a tableau starting with $x : \neg\phi$. By the previous lemma, each expansion of T is satisfiable, and contains at least one satisfiable branch B (i. e. there is a Σ-model M and a $PCL-$mapping f such that B is satisfiable in M under f). This branch cannot be closed, otherwise by Definition 4.3, it would contain either (i) $x : \phi$ and $x : \neg\phi$, or $x : \bot$ (for some formula ϕ) or (ii) $y <_x y$ or (iii) $y <_x z$ and $y : \neg V_x$ or $z : \neg V_x$. But (i), (ii), and (iii) contradict the fact that B is satisfiable. Indeed, if (i) holds, $M, f(x) \models \phi$ and $M, f(x) \models \neg\phi$, impossible, or $M, f(x) \models \bot$, impossible. If (ii) holds $f(y) <_{f(x)} f(y)$, which contradicts the irreflexivity of $<_{f(x)}$ in M. Last, if (iii) holds, then $f(y) <_{f(x)} f(z)$, and either $f(y) \notin W_{f(x)}$ or $f(z) \notin W_{f(x)}$. Both cases contradict the fact that, by definition, $<_{f(x)}$ ranges over $W_{f(x)}$.

□

5.2 Completeness of the calculus

In order to prove the completeness of the calculus, we restrict our attention to tableaux which can be generated starting from an input formula ϕ.

We first show that no tableau contains infinite descending chains of labels related by the same relation $<_x$, provided it does not contain an infinite number of conditional formulas labelled with the same label. This is of course true if the tableau starts with a finite number of formulas (it is interesting to notice that this holds independently from Blocking Conditions 1 and 2 we will introduce below in order to ensure the termination of the calculus).

LEMMA 5.5. *Let B be a branch of a tableau containing only a finite number of positive conditional formulas $x : \phi_0 \Rightarrow \psi_0$, $x : \phi_1 \Rightarrow \psi_1$, $x : \phi_2 \Rightarrow \psi_2, \ldots, x : \phi_{n-1} \Rightarrow \psi_{n-1}$. Then B does not contain any infinite descending chain of labels $y_1 <_x y_0$, $y_2 <_x y_1, \ldots,$ $y_{i+1} <_x y_i, \ldots$.*

PROOF. Let B contain a descending chain of labels $y_1 <_x y_0$, $y_2 <_x y_1$, ..., $y_{i+1} <_x$ y_i. This chain comes from the successive application of $(T \Rightarrow)$ and $(F\Box)$ to formulas $x : \phi_i \Rightarrow \psi_i$ for $0 \leq i < n$. B then contains the following formulas for $0 \leq i < n$: $y_i : \neg\Box_x\neg\phi_i$, $y_{i+1} <_x y_i$, $y_{i+1} : \phi_i$, $y_{i+1} : \Box_x\neg\phi_i$.

Here $(T \Rightarrow)$ has been applied to every formula $x : \phi_i \Rightarrow \psi_i$ once and with parameter y_i previously (and newly) generated by $(F\Box)$ from $y_{i-1} : \neg\Box_x\neg\phi_{i-1}$. The only way to make the chain longer is by applying $(T \Rightarrow)$ a second time to one of the positive conditional formulas labelled x on B. Let this formula be $x : \phi_k \Rightarrow \psi_k$ where $0 \leq k < n$. Then B contains further $y_{n+1} : \phi_k$ (together with $y_n : \neg\Box_x\neg\phi_k$, $y_{n+1} <_x y_n$, $y_{n+1} : \Box_x\neg\phi_k$).

By the transitivity rule, we get $y_{n+1} <_x y_{k+1}$. Moreover, B contains also y_{k+1} : $\Box_x\neg\phi_k$, from which we obtain by $(T\Box)$ $y_{n+1} : \neg\phi_k$ which closes the branch.

□

We can now show the completeness of the calculus. To this purpose, we need to introduce the following notion of saturated branch. This notion intuitively expresses that all tableau rules which could be applied to the branch have been applied to it.

DEFINITION 5.6 SATURATED BRANCH. *We say that a branch B of a tableau is:*

—saturated *w.r.t.* **PCL** *if:*

1.1 If $x : \phi \wedge \psi \in B$ then $x : \phi \in B$ and $x : \psi \in B$.

1.2 If $x : \neg(\phi \wedge \psi) \in B$ then either $x : \neg\phi \in B$ or $x : \neg\psi \in B$.

1.3 If $x : \neg\neg\phi \in B$ then $x : \phi \in B$.

1.4 If $x : (\phi \Rightarrow \psi) \in B$ then for any label $y \in W_B$, either $y : \neg\phi \in B$ or $y : \neg V_x \in B$ or $y : \neg\Box_x\neg\phi \in B$ or $y : \psi \in B$.

1.5 If $x : \neg(\phi \Rightarrow \psi) \in B$ then there is a label y such that $y : \phi \in B$, $y : V_x \in B$, $y : \neg\psi \in B$ and $y : \Box_x\neg\phi \in B$.

1.6 If $y : \Box_x\phi \in B$ and $z <_x y \in B$ then $z : \phi \in B$.

1.7 If $y : \neg\Box_x\phi \in B$ then there is a label z such that $z <_x y \in B$, $z : \neg\phi \in B$ and $z : \Box_x\phi \in B$.

1.8 If $z <_x u \in B$ and $u <_x y \in B$ then $z <_x y \in B$

—saturated *w.r.t.* $=$ *if:*

2.1 for all $x \in B$, $x = x \in B$;

2.2 if $x = y \in B$, also $y = x \in B$;

2.3 if $x = y \in B$ and $y = z \in B$, also $x = z \in B$;

2.4 if $x = x' \in B$, $y = y' \in B$, $z = z' \in B$, and $y <_x z \in B$, then also $y' <_{x'} z' \in B$;

2.5 if $x = x', y = y', y : \neg V_x \in B$, then $y' : \neg V_{x'} \in B$.

2.6 if $x = x'$ and $x : \phi$ are in B, then also $x' : \phi \in B$.

—saturated *with respect to* **PCL** *+ (MP) if it is saturated w.r.t. to* **PCL** *, and for no x and y, $x : \neg V_x$ or $y <_x x \in B$;*

—saturated *with respect to* **PCL** *+ (CS) if it is saturated w.r.t. to* **PCL** *, saturated w.r.t. to $=$, and for all $x, y \in B$, either $x <_x y \in B$ or $x = y \in B$ or $y : \neg V_x \in B$;*

—saturated *with respect to* **PCL** + *(CV) if it is saturated w.r.t. to* **PCL***, and for all labels* x, y, z, u *in B, if* $y <_x z \in B$*, then also* $y <_x u \in B$ *or* $u <_x z \in B$*;*

—saturated *with respect to* **PCL** + *(CEM) if it is saturated w.r.t. to* **PCL***, saturated w.r.t. to* $=$*, and for all* x, y, z *in B, either* $y = z \in B$ *or* $y <_x z \in B$ *or* $z <_x y \in B$*;*

—saturated *with respect to* **PCL** + *(A0), (A1), (A2) if it is saturated w.r.t. to* **PCL***, and there are no labels* x *and* y*, such that* $y : \neg V_x \in B$*.*

For short, in some of the proofs below we will use the notion of saturated with respect to single axioms to mean that the branch has the single saturation condition associated to the axiom. Hence, for instance, in a branch B saturated w.r.t. to (CV), whenever $y <_x z \in B$, then for all u, $y <_x u \in B$ or $u <_x z \in B$.

We can now show the following lemma.

LEMMA 5.7. *If* ϕ *is not* Σ*-provable, then there exists a tableau which contains an open branch starting with* $x : \neg\phi$*, and which is saturated with respect to* **PCL** *+ the rules corresponding to the axioms contained in* Σ*.*

PROOF. Since ϕ is not provable, all tableaux starting with $x : \neg\phi$ will contain an open branch. Hence, also the tableau built according to the following systematic procedure contains an open branch: *(step a:)* apply all the static rules as far as possible; in case of branching, make the choice which preserves non-closure (there is always one choice, by hypothesis); *(step b:)* apply the dynamic rules to the new formulas generated in the previous step.

By this strategy, every formula in the branch is eventually considered. It can be easily shown that the branch generated by this systematic procedure is saturated with respect to the rules in **PCL** and with respect to the rules corresponding to the axioms contained in Σ.

□

THEOREM 5.8 COMPLETENESS OF THE CALCULUS. *If* ϕ *is valid in all* Σ*-models, then it is* Σ*-provable.*

PROOF. We show the contrapositive: that if ϕ is not Σ-provable, then $x : \neg\phi$ is satisfiable by a Σ-model. If ϕ is not provable, by Lemma 5.7 the tableau starting with $x : \neg\phi$ contains an open branch B which is saturated with respect to the rules in **PCL** and the rules corresponding to the axioms contained in Σ. Starting from B, we build a canonical model. We have two alternative constructions: one for the case in which neither axiom (CS) nor axiom (CEM) belong to Σ, and another corresponding to the case in which (at least) one of the two axioms belongs to Σ (and therefore the $=$ symbol appears in B).

Case 1: (CS) $\notin \Sigma$ *and (CEM)* $\notin \Sigma$*.*

Let $M_C = \langle W, \{W_x\}_{x \in W}, \{\sqsubset_x\}_{x \in W}, I \rangle$, where:

1) $W = W_B$;
2) For each $x \in W$, $W_x = \{y \mid y : \neg V_x \notin B\}$
3) For each $x \in W$, $y \sqsubset_x z$ if $y <_x z \in B$;
4) For each $x \in W$, $I(x) = \{p \mid p \in ATM$ and $w : p \in B\}$.

If $y <_x z \in B$ then $y : \neg V_x \notin B$ and $z : \neg V_x \notin B$, otherwise B would be closed. It follows that if $y <_x z \in B$, then $y, z \in W_x$ and $y \sqsubset_x z$ is well defined. We show the following facts:

(1) $\sqsubset_x$ *is an irreflexive and transitive relation on* W_x. The transitivity immediately follows from the transitivity of $<_x$ entailed by the definition of saturated branch. The irreflexivity follows from the irreflexivity of $<_x$: by definition of closed branch, 4.3, if for some y $y <_x y \in B$, B would be closed, against the hypothesis.

(2) *Let* $W_x \cap [[\phi]] \neq \emptyset$. *Then* $Min_x(\phi) \neq \emptyset$. Suppose $Min_x(\phi) = \emptyset$; let $y_1 \in W_x \cap [[\phi]]$. Since $y_1 \notin Min_x(\phi)$, there must be an infinite descending chain of elements of $[[\phi]]$ $y_{n+1} \sqsubset_x y_n \sqsubset_x \ldots \sqsubset_x y_1$. By definition of $\sqsubset_x$, this entails that we have an infinite decreasing sequence $y_{n+1} <_x y_n <_x \ldots <_x y_1 \in B$. By Lemma 5.5 we have a contradiction.

We now show that B is satisfiable by M_C. Obviously, the identity $id(x) = x$ is a **PCL**-mapping by the construction of M_C. We show that B is satisfiable by M_C under id. For all formula or pseudo-formula ϕ we show that *(a) if* $x : \phi \in B$, *then* $M_C, x \models \phi$ *and (b) if* $x : \neg\phi \in B$, *then* $M_C, x \models \neg\phi$.

—if $\phi \in ATM$, (a) follows by definition of I. For what concerns (b), $x : \phi \notin B$, otherwise B would be closed. By definition of I, $\phi \notin I(x)$, hence $M_C, x \not\models \phi$, and $M_C, x \models \neg\phi$;

—The case of boolean combination of formulas is easy and left to the reader;

—If ϕ is $\psi \Rightarrow \chi$, to prove (a) we reason as follows: let $y \in W_B$. Then we have four cases:
(i) $y : \neg\psi \in B$. Then, by the induction hypothesis, we have $M_C, y \models \neg\psi$, i.e. $M_C, y \not\models \psi$, thus $y \notin Min_x(\psi)$.
(ii) $y : \neg\Box_x\neg\psi \in B$. Then by saturation of B, there is a label z such that $z <_x y \in B$, $z : \psi \in B$. By construction of the model, we have $z \sqsubset_x y$ and by the induction hypothesis $M_C, z \models \psi$, thus $y \notin Min_x(\psi)$.
(iii) $y : \chi \in B$. Then by the induction hypothesis we get $M_C, y \models \chi$.
(iv) $y : \neg V_x \in B$. Then $y \notin W_x$, hence $y \notin Min_x(\psi)$.
From (i), (ii), (iii) and (iv) we conclude that for all $y \in W_B$, if $y \in Min_x(\psi)$, then $M_C, y \models \chi$, hence $M_C, x \models \psi \Rightarrow \chi$.
To prove (b): since B is saturated, then there is y such that $y : \psi \in B$, $y : \Box_x\neg\psi \in B$, $y : V_x \in B$ and $y : \neg\chi \in B$. By the induction hypothesis, $M_C, y \models \psi$ and $M_C, y \models \neg\chi$. Furthermore, since B is open, $y : \neg V_x \notin B$, hence $y \in W_x$. We show that $y \in Min_x(\psi)$. Suppose that this is not the case; since $M_C, y \models \psi$, there is $z \sqsubset_x y$ and $M, z \models \psi$. By definition of $\sqsubset_x$, we have $z <_x y \in B$ and by saturation $z : \neg\psi \in B$; by induction hypothesis we would have $M_C, z \models \neg\psi$, hence $M_C, z \not\models \psi$, a contradiction. Therefore $y \in Min_x(\psi)$, which proves $M, x \models \neg(\psi \Rightarrow \chi)$.

—If ϕ is $\Box_y\psi$, (a): let $v \sqsubset_y x$. By definition of $\sqsubset_y$, $v <_y x \in B$. From this, by saturation, it follows $v : \psi \in B$, and by the induction hypothesis $M_C, v \models \psi$. We conclude that $M_C, x \models \Box_y\psi$. (b) since B is saturated, there is a label z such that $z <_y x \in B$, $z : \neg\psi \in B$. From this we obtain by construction of the model that $z \sqsubset_y x$, and by the induction hypothesis $M_C, z \models \neg\psi$, from which it follows that $M_C, x \models \neg\Box_y\psi$.

—If (MP) $\in \Sigma$, then M_C satisfies (S - MP). For a contradiction, suppose $y \notin W_x$ or $y \sqsubset_x x$. In the first case, $y : \neg V_x \in B$; in the second case, by definition of $\sqsubset_x$, $y <_x x \in B$. Both cases contradict the saturation condition associated with (MP).

—If (CV) $\in \Sigma$, then M_C satisfies (S-CV). Let $y \sqsubset_x z$. By definition of M_C, $y <_x z \in B$, and by the saturation condition associated with (CV), for all $u \in W_B$, either $u : \neg V_x \in B$ or $y <_x u$ or $u <_x z$. In the first case, $u \notin W_x$, in the second case $y \sqsubset_x u$; in the third case $u \sqsubset_x z$. It follows that (S-CV) is satisfied.

—If (A0), (A1), (A2) $\in \Sigma$, then M_C satisfies Universality. Indeed, by definition of saturated branch, for all $x, y \in W_B$ $y : \neg V_x \notin B$, hence by definition of W_x, $y \in W_x$, i.e. $W_x = W_B = W$.

Case 2: (CS) $\in \Sigma$ *or (CEM)* $\in \Sigma$. Given a branch B saturated with respect to all the rules in Σ, we build the canonical model as follows. For all $x, y \in W_B$, we let $x =_M y$ iff $x = y \in B$. Notice that $=_M$ is an equivalence relation. Indeed, since B is saturated w.r.t. $=$, $=$ (and hence $=_M$) is symmetric, transitive and reflexive according to Definition 5.6, 2.1, 2.2, 2.3.
For all $x \in W_B$, we let $[x] = \{y : x =_M y\}$. We can now define the canonical model $M_C = \langle W, \{W_{[x]}\}_{[x] \in W}, \{\sqsubset_{[x]}\}_{[x] \in W}, I\rangle$, where:

—$W = \{[w]/ =_M\}$, i. e. it is the set of all the equivalence classes of W_B with respect to $=_M$;

—$[y] \sqsubset_{[x]} [z]$ if $y <_x z \in B$. Notice that since B is saturated w.r.t. $=$, this definition is well given, i. e. it does not depend on the choice of the representative of $[x]$, $[y]$, $[z]$, since, according to Definition 5.6, 2.4, for $x' = x$, $y' = y$, and $z' = z$, $y <_x z$ iff $y' <_{x'} z' \in B$;

—$[y] \in W_{[x]}$ if $y : \neg V_x \notin B$. Notice that $W_{[x]}$ is well defined because according to Definition 5.6, 2.5, $[y] \in W_{[x]}$ does not depend on the representatives of the classes $[y]$ and $[x]$ (if $y = y' \in B$ or $x = x' \in B$, then $y : \neg V_x \in B$ iff $y' : \neg V_{x'} \in B$).

—$I([x]) = \{p : p \in ATM$ and $x : p \in B\}$. Notice that also $I([x])$ is well defined, since by 2.6 in Definition 5.6, if $x : p \in B$ and $x = x' \in B$, then also $x' : p \in B$.

Similarly to what done for Case 1 above, we show that:

1 $\sqsubset_{[x]}$ *is irreflexive, transitive, and satisfies the Limit Assumption. Since* $\sqsubset_{[x]}$ *is defined as in Case 1, the proof is the same.*

2 For all formula or pseudo-formula ϕ, we show that *(a) if* $x : \phi \in B$, *then* $M_C, [x] \models \phi$ *and (b) if* $x : \neg\phi \in B$, *then* $M_C, [x] \models \neg\phi$.

  —for $\phi \in ATM$ (a) follows from definition of I. For (b): if $x : \neg\phi \in B$, then $x : \phi \notin B$, otherwise B would be closed. By definition of I, it follows that $\phi \notin I(x)$, hence $M_C, x \not\models \phi$, and $M_C, x \models \neg\phi$;

  —if ϕ is a boolean combination of formulas or a pseudo-formula $\Box_{[y]}\phi$ or a conditional formula $\psi \Rightarrow \chi$, we reason as in Case 1.

3 If (CS) $\in \Sigma$, then M_C satisfies (S-CS). Indeed, by Definition 5.6, either $x = y \in B$ (and $[x] = [y]$) or $x <_x y \in B$, and $[x] \sqsubset_{[x]} [y]$.

4 If (CEM) $\in \Sigma$, then M_C satisfies (S-CEM). Indeed, by Definition 5.6, for all x, y, z, either $y = z \in B$, and by construction of M_C $[y] = [z]$, or $y <_x z \in B$, and by construction of M_C, $[y] \sqsubset_{[x]} [z]$ or $z <_x y \in B$, and by construction of M_C, $[z] \sqsubset_{[x]} [y]$.

5 If (MP) $\in \Sigma$, then M_C satisfies (S-MP). We reason analogously to Case 1.

6 If (CV) $\in \Sigma$, then M_C satisfies (S-CV). We reason as for Case 1.

7 If (A0), (A1), (A2) $\in \Sigma$, then M_C satisfies Universality. We reason as for Case 1.

□

6. TERMINATION OF THE CALCULUS

The calculus presented above for **PCL** and its extensions can lead to non-terminating computations due to the interplay among, on the one side, the rules $(F \Rightarrow)$ and $(F\Box)$ generating new labels, and, on the other side, rule $(T \Rightarrow)$, whose applications to the new labels may generate new formulas to which $(F \Rightarrow)$ or $(F\Box)$ can again be applied. For instance, the tableau construction for the formula $x : \top \Rightarrow \neg(\top \Rightarrow p)$ can produce an infinite branch (containing $x : \neg(\top \Rightarrow p)$, $y : \top$, $y : \Box_x\neg\top$, $y : \neg p$, $y : \neg(\top \Rightarrow p)$, and so on. In this section, we show that the calculus for **PCL** and its extensions can be made terminating. The solution we propose is similar to the one we have adopted to prove the termination of **PCL** in [Giordano et al. 2003]: in order to avoid the generation of infinite branches, we introduce a systematic procedure to build a tableau and we put suitable blocking conditions on the applications of rules $(F \Rightarrow)$ and $(F\Box)$. We show that the systematic procedure terminates, whence it does not lead to generate infinitely many labels. Moreover, we show that the completeness of the calculus is not lost if we adopt the systematic procedure with the mentioned blocking conditions.

First of all, let us assume that the tableau construction does not allow any redundant application of the rules, where a redundant application of a rule is defined as follows:
Redundant application of a rule. Let B be a tableau branch and R a tableau rule applied to B which produces $B_1, \dots B_k$, the extensions of B. The application of R is *redundant* if for some i (with $0 \leq i \leq k$), $B_i = B$. It can be easily seen that a branch B can be extended to a closed branch just in case it can be extended to a closed branch without any redundant application of the rules. In particular, we assume that the $(T \Rightarrow)$ applies exactly once to each formula and label on the branch. The reason is that if we apply the rule twice, the second application is redundant. Additionally, we assume that a branch does not contain repetitions of labelled formulas: labelled formulas which are already on the branch are not added again when applying new rules.

We define a *systematic procedure* for constructing the tableau for a given formula $x : \alpha$.

First, as in [Buchheit et al. 1993], we assume that labels are introduced in a tableau according to the ordering $\prec$, that is to say, given a label y in the tableau, $x \prec y$ for all labels x that are already in the tableau when y is introduced. The relation $\prec$ is a total order.

Furthermore, we say that two labels x and y are *B-equivalent*, written $x \equiv_B y$, if they label the same formulas i.e. if $\{\phi \mid x : \phi \in B\} = \{\phi \mid y : \phi \in B\}$.

Last, we define the following procedure for constructing the tableau for a given $x : \alpha$. This is the same procedure described in the proof of Lemma 5.7. The procedure executes repeatedly two steps: ***(step a)*** applies all the static rules as far as possible; ***(step b)*** applies a dynamic rule $(F\Box)$ or $(F \Rightarrow)$ to a label x such that there is no label y with $y \prec x$ to which a dynamic rule is applicable.

Observe that the procedure respects the following conditions: a dynamic rule is applied to a label x only if no dynamic rule is applicable to a label y such that $y \prec x$; a dynamic rule is applied only if no static rule is applicable.

We observe that (step a) terminates after a finite number of rule applications. Indeed, none of the static rules adds new labels and the number of pseudo-formulas built from a finite set of formulas (subformula property) and from a finite number of labels is finite. As we require that the tableau does not contain redundant applications of rules, each application of a rule in step (a) must at least add a new pseudo-formula to the branch, and thus the number of rule applications must be finite. After (step a) is terminated, the branch is downward saturated except for the formulas of the form $w : \neg(\phi \Rightarrow \psi)$ and $w : \neg\Box_x\phi$. In (step b) the application of a dynamic rule introduces a new world.

To avoid the systematic procedure to repeat forever, by continuing to generate new worlds, we put the following blocking conditions on the rules $(F \Rightarrow)$ and $(F\Box)$. These are essentially loop checking conditions, that can be added as side conditions to the rules. Before we formulate the conditions, we need to introduce the following notation: given a branch B, we let $Box_{B,y,x}$ the set of all pseudo formulas $\Box_y\phi$ such that $x : \Box_y\phi \in B$ or $x' : \Box_y\phi \in B$ and $x <_y x' \in B$ (informally, the positive boxed formulas which hold at x).

The blocking conditions are then listed below. The idea behind Blocking Condition 1 and Blocking Condition 2a is the same: if two labels y and z satisfy the same conditional and propositional formulas, then $<_y$ and $<_z$ can be defined in the same way. For this reason, only one out of $<_y$ and $<_z$ needs to be explicitly built by applying all the rules of the calculus. In particular, if $<_z$ is the relation which is built the first, we can avoid applying rules (F $\Box$) and (F $\Rightarrow$) to $y : \neg(\phi \Rightarrow \psi)$ and $x : \neg\Box_y\phi$ respectively. $<_y$ will then be defined as $<_z$ by step 3 in the construction below. The idea behind Blocking Condition 2b is that applying rule (F $\Box$) to $x : \neg\Box_y\phi$ cannot add anything more on the branch than what has been obtained by applying the same rule to $w : \neg\Box_y\phi$, if all the positive modal formulas that would hold in x' generated by applying (F $\Box$) to $x : \neg\Box_y\phi$, also hold in w', generated by applying (F $\Box$) to $w : \neg\Box_y\phi$.

—*Blocking Condition* 1: do not apply (F $\Rightarrow$) to $y : \neg(\phi \Rightarrow \psi)$ if there is a $z \prec y$ such that $z \equiv_B y$.

—*Blocking Condition* 2: do not apply $(F\Box)$ to $x : \neg\Box_y\phi$ on a branch B if one of the following conditions holds on the branch:

a: there is a $z \prec y$ such that $z \equiv_B y$.;

b: $x : \neg V_y \notin B$, and there is $w : \neg\Box_y\phi$ such that $w \prec x$, and for all y_1 s.t. $y_1 = y \in B$, $Box_{B,y_1,x} \subseteq Box_{B,y_1,w}$.

In case (R-MP) or (R-CS) belong to the calculus, Blocking Condition 2b is applied only if x is distinct from y, and $x = y \notin B$.

The two theorems below show that the calculus with the blocking conditions terminates and is complete.

THEOREM 6.1. *The calculus with Blocking Condition 1 and Blocking Condition 2 terminates.*

PROOF.

Let us suppose that a systematic attempt to prove a formula goes forever. Then, there must be an infinite branch S containing infinitely many different labels because it does not contain repetitions and only a finite number of formulas can appear in the tableau (by the subformula property). Since labels are introduced by rules $(F \Rightarrow)$ and $(F\Box)$, it follows that the branch must either contain infinitely many applications of the rule $(F \Rightarrow)$ to a formula $\neg(\phi \Rightarrow \psi)$ or infinitely many applications of the rule $(F\Box)$ to a formula $\neg\Box_y\phi$ (or both).

For the first case, it is not possible to apply rule $(F \Rightarrow)$ infinitely many times without violating Blocking Condition 1. To see this, each time the rule is applied to a formula

$y : \neg(\phi \Rightarrow \psi)$, by Blocking Condition 1, there is no $z \prec y$ on the branch such that $z \equiv_B y$ (that is, there is some formula $\phi \in \mathcal{L}$ which has a different value in y and in z). This, however, cannot occur infinitely many times as, by the subformula property, the formulas of $\mathcal{L}$ occurring on the branch are finitely many, and their number is linear in the size n of the initial formula. Hence, the number of different subsets of formulas in $\mathcal{L}$ is $O(2^n)$. More precisely, let us consider the sequence of labels $y_1, \ldots, y_k$ to which the dynamic rule $(F \Rightarrow)$ has been applied in the order. By definition of (step b) in the systematic procedure, it must be that $y_1 \prec \ldots \prec y_k$. By Blocking Condition 1, it must be that, for each y_i with $i = 1, \ldots, k$, $y_i \not\equiv_B y_h$ for all $h = 1, \ldots, i-1$, and hence the formulas labelled by each of the labels must be different. Of course, this cannot be true infinitely many times.

For the second case, it is not possible to apply rule $(F\Box)$ infinitely many times without violating Blocking Condition 2. More precisely, it is not possible that rule $(F\Box)$ applies infinitely many times to a formula $x : \neg\Box_y\phi$, for different values of the labels x and y without violating Blocking Condition 2.

Observe that it is not possible to apply $(F\Box)$ infinitely many times to a formula $x : \neg\Box_y\phi$, for different values of the label y, without violating Blocking Condition 2a. The argument is the same as for Blocking Condition 1. The number of possible applications of the rule to different labels y is $O(2^n)$. On the other hand, it is not possible to apply $(F\Box)$ infinitely many times to a formula $x : \neg\Box_y\phi$, for a given y but for different values of the label x, without violating Blocking Condition 2b. Indeed, when $(F\Box)$ is applied to $x : \neg\Box_y\phi$, for each formula $x' : \neg\Box_y\phi$ on the branch there must be at least a positive boxed formula $\Box_y\alpha$ which holds in x but does not hold in x'. Again, this cannot be true infinitely many times. More precisely, the number of possible applications of $(F\Box)$ to a formula $x : \neg\Box_y\phi$, for different values of the label x is $O(2^n)$.

□

THEOREM 6.2. *The calculus with Blocking Condition 1 and Blocking Condition 2 is complete.*

PROOF.We show that given an open branch B (obtained in the calculus with the blocking conditions), we can build an open saturated branch w.r.t. Definition 5.6. We know from the proof of Theorem 5.8 that this branch is satisfiable, from which we can conclude, similarly to what done for Theorem 5.8.

For the moment, we only consider the basic calculus for **PCL** . We saturate the branch by the following steps. We first consider Blocking Condition 2b and then Blocking Conditions 1 and 2a.

Step 1: For each $x : \neg\Box_y\phi$ to which Blocking Condition 2b has been applied, we proceed as follows: if $x = x_1 \in B$, and Blocking Condition 2b has not been applied to $x_1 : \neg\Box_y\phi$, nothing needs to be done (since the 'right' relations have been put by rule $(E-<)$ applied to B). If this is not the case, let $w : \neg\Box_y\phi$ be the formula (minimal with respect to $\prec$) that caused the application of Blocking Condition 2b. Since $(F\Box)$ has been applied to $w : \neg\Box_y\phi$, there is $x' <_y w$ in B such that $x' : \Box_y\phi$, $x' : \neg\phi$. We let $x' <_y x$. Furthermore, if $y = y_1 \in B$, $x' = x'_1 \in B$, we also introduce $x'_1 <_{y_1} x$ in the branch.

Step 2: For all $<_y$ if $x'' <_y x'$ and $x' <_y x$ are in the branch, we add to the branch $x'' <_y x$.

We call B' the branch obtained from Step 1 and Step 2 above.

We prove the following fact for B':

Fact 1 *If* $x' <_y x \in B'$*, then* $Box_{B,y,x} \subseteq Box_{B,y,x'}$*, and if* $\Box_y \phi \in Box_{B,y,x}$*, then* $x' : \phi \in B'$*.* We reason by induction on the number n of applications of transitivity needed to introduce the relation $x' <_y x$ in B' at Step 2. If $n = 0$, then either $x' <_y x \in B$ or $x' <_y x$ has been introduced by Step 1. In the first case, in B it holds that $Box_{B,y,x} \subseteq Box_{B,y,x'}$ by definition of $Box_{B,y,x}$. Furthermore, if $\Box_y \phi \in Box_{B,y,x}$, then either $x : \Box_y \phi \in B$ or $z : \Box_y \phi \in B$ and $x <_y z$ in B. In the first case, we immediately conclude that $x' : \phi \in B$ since rule $(T\Box)$ has been applied in B. Hence also $x' : \phi \in B'$. In the second case, by (Trans) (that has been applied to B) also $x' <_y z \in B$; we therefore conclude $x' : \phi \in B$ since rule $(T\Box)$ has been applied in B. Also in this case, $x' : \phi \in B'$.

If $x' <_y x$ has been introduced at Step 1, then there are w, x_1, y_1, x'_1 such that $x'_1 <_{y_1} w \in B$, $y = y_1 \in B$, $x' = x'_1 \in B$, and $Box_{B,y,x} \subseteq Box_{B,y,w}$. Since rule (E - $<$) has been applied to B, also $x' <_y w \in B$, hence by what said just above, $Box_{B,y,w} \subseteq Box_{B,y,x'}$, hence we infer that $Box_{B,y,x} \subseteq Box_{B,y,x'}$. Furthermore, if $\Box_y \phi \in Box_{B,y,x}$, then also $\Box_y \phi \in Box_{B,y,w}$, and we conclude that $x' : \phi \in B$ by reasoning as done above.

If $n > 0$, then $x' <_y x$ has been introduced by transitive closure at Step 2 from (i) $x' <_y x''$ and (ii) $x'' <_y x$. By inductive hypothesis, the property holds for (i) and (ii), hence $Box_{B,y,x} \subseteq Box_{B,y,x''} \subseteq Box_{B,y,x'}$. If $\Box_y \phi \in Box_{B,y,x}$, we infer that $\Box_y \phi \in Box_{B,y,x''}$; from this, by inductive hypothesis on (i), we conclude that $x' : \phi \in B'$.

By using Fact 1, we can prove that the following saturation conditions hold in B' (Fact 2):

Fact 2

(i) If $x : \Box_y \phi \in B'$ and $x' <_y x \in B'$ then also $x' : \phi \in B'$. Indeed, if $x : \Box_y \phi \in B'$, then also $x : \Box_y \phi \in B$ i.e. $\Box_y \phi \in Box_{B,y,x}$; by Fact 1, we derive that $x' : \phi \in B'$.

(ii) If $x : \neg\Box_y \phi \in B'$, *and Blocking Condition 2a* has not been applied to the formula, then there is in B' $x' <_y x$, $x' : \Box_y \phi$, $x' : \neg\phi$. We distinguish two cases: if Blocking Condition 2b has been applied to the formula, this holds by construction of B' (Step 1); if Blocking Condition 2b has not been applied to the formula, the property holds since rule $(F\Box)$ has been applied to the formula in B.

(iii) $<_y$ is transitive. This follows by definition of Step 2.

(iv) B' is open, since it cannot happen that one of the closure conditions below is satisfied. Indeed it cannot happen that:

[-] $x : \phi \in B'$ and $x : \neg\phi \in B'$, since this does not happen in B, which is open, and no labelled formula has been modified in B'.

[-] *For no label* x *and* y, $x <_y x \in B'$. For a contradiction, suppose $x <_y x \in B'$. Then clearly $x <_y x \notin B$ (otherwise B would be closed, against the hypothesis). This means that $x <_y x$ has been inserted in B' either by Step 1 or by Step 2. Suppose it has been inserted at Step 1. Then there are x_1, y_1, such that $x_1 = x$, $y_1 = y$, $x_1 : \neg\Box_{y_1} \phi \in B$ and there is w s.t. $w : \neg\Box_{y_1} \phi$, $x_1 <_{y_1} w$ and $x_1 : \Box_y \phi$ are in B, hence B would be closed, against the hypothesis.

If $x <_y x$ has been inserted at Step 2 by transitivity from relations $<_y$ previously introduced, then there must be a chain $x <_y x_1 \ldots <_y x_n <_y x$ where all these relations $<_y$ either belong to B or are introduced by Step 1. Notice that there must be at least a $x_i <_y x_j$ that was not already in B and that has been introduced by Step 1 (otherwise by (Trans) $x <_y x \in B$, and B would be closed, against the hypothesis). This means that for some x_{i_1}, y_1 such that $x_{i_1} = x_i, y_1 = y \in B$, $x_{i_1} <_{y_1} x_j$, and for some ϕ, $x_j : \neg\Box_{y_1} \phi$, $x_{i_1} : \Box_{y_1} \phi \in B$, $x_{i_1} : \neg\phi$ are in B. By rule (E- ϕ), also $x_i : \Box_{y_1} \phi \in B$

and $x_i : \neg\phi \in B$. By definition of Box_{B,y,x_i}, $\Box_{y_1}\phi \in Box_{B,y,x_i}$. Notice that if $x <_y x_1 \ldots <_y x_n <_y x$, by construction also $x <_{y_1} x_1 \ldots <_{y_1} x_n <_{y_1} x$, and by transitive closure of Step 2, also $x_i <_{y_1} x_i$, and by Fact 1 above, $x_i : \phi \in B$. However, this would entail that B is closed, against the hypothesis.
We conclude that for no x, y it can happen that $x <_y x \in B'$, i.e. $<_y$ is irreflexive.

[-] $x <_y z, x : \neg V_y$ or $z : \neg V_y$. Indeed, if $x <_y z \in B$ this cannot hold, since it does not hold in B (otherwise B would be closed) and no new formula $x : \neg V_x$ has been introduced. If $x <_y z$ has been introduced by Step 1 or Step 2, we can reason as follows. First, it cannot be that $z : \neg V_y \in B'$, otherwise $z : \neg V_y \in B$, and Blocking Condition 2b would not have been applied to any $z : \neg\Box_y\phi \in B$, hence the relation would not have been added. For what concerns $x : \neg V_y$, we can easily verify that if a relation $x <_y z$ has been inserted by Step 1 or Step 2, there should already be in B a w such that $x <_y w$, and if $x : \neg V_y \in B$, B would be closed, contradiction. We conclude, since $x : \neg V_y$ cannot be introduced by step 1 or 2.

Until now, we have dealt with Blocking Condition 2b. In order to deal with the other blocking conditions, we continue our saturation process by Step 3.
Step 3: Let $Blocked_{1,2a}$ be the set of labels to which Blocking Condition 1 or Blocking Condition 2a have been applied. For all $y_i \in Blocked_{1,2a}$, let z_i be the label minimal w.r.t. $\prec$ such that $z_i \equiv_B y_i$. Notice that $Blocked_{1,2a}$ is disjoint from the set of all corresponding z_i (indeed, to these labels neither Blocking Condition 1 nor Blocking Condition 2a have been applied); we can hence perform all the substitutions described below in one single step.

We proceed as follows in order to build the new branch B'' from B'.

1 - We remove from branch B' all the relations $<_{y_i}$, and all the formulas $x : \neg\Box_{y_i}\phi, x : \Box_{y_i}\phi, x : V_{y_i}$ and $x : \neg V_{y_i}$;

2 - We reinsert on the branch $<_{y_i}$ each time there is $<_{z_i}$;

3 - We insert $x : \Box_{y_i}\phi$, $x : \neg\Box_{y_i}\phi$, $x : V_{y_i}$ and $x : \neg V_{y_i}$ each time there is $x : \Box_{z_i}\phi$, $x : \neg\Box_{z_i}\phi, x : V_{z_i}$ and $x : \neg V_{z_i}$ respectively;

Fact 3: *The branch B'' so obtained is open and saturated w.r.t* **PCL** :
[saturated:]

(i) for the saturation conditions that concern the propositional operators, this is obvious (indeed, for any non-boxed formula ϕ, if $x : \phi \in B$, then $x : \phi \in B'$, and B' is saturated since B is saturated w.r.t. propositional rules)

(ii) if $x : \phi \Rightarrow \psi \in B''$, we distinguish two cases.
1: $x \notin Blocked_{1,2a}$. In this case, the property holds since $x : \phi \Rightarrow \psi \in B$, hence $(T \Rightarrow)$ has been applied to it, and it can be easily shown that the formulas introduced by $(T \Rightarrow)$ in B still hold in B''.
2: if x is $y_i \in Blocked_{1,2a}$, then also $z_i : \phi \Rightarrow \psi$ is in B and in B''. Furthermore, B'' is saturated w.r.t. $z_i : \phi \Rightarrow \psi$, i.e. for all x' either $x' : \neg\phi$ or $x' : \neg\Box_{z_i}\phi$ or $x' : \psi$ or $x' : \neg V_{z_i}$ is in B''. By construction, if $x' : \neg\Box_{z_i}\phi \in B''$ also $x' : \neg\Box_{y_i}\phi \in B''$, and if $x' : \neg V_{z_i} \in B''$, also $x' : \neg V_{y_i} \in B''$. It follows that B'' is also saturated w.r.t. $y_i : \phi \Rightarrow \psi$.

(iii) if $x : \neg(\phi \Rightarrow \psi) \in B''$, and $x \notin Blocked_{1,2a}$, then for some x', $x' : V_x$, $x' : \Box_x\neg\phi$, $x' : \phi$, $x' : \neg\psi \in B$. All these formulas are in B''. If x is $y_i \in Blocked_{1,2a}$, then also

$z_i : \neg(\phi \Rightarrow \psi \in B'')$, and for some x', $x' : V_{z_i}, x' : \Box_{z_i}\neg\phi$, $x' : \phi, x' : \neg\psi \in B''$. By construction, also $x' : V_{y_i}$, $x' : \Box_{y_i}\neg\phi$, and B'' is saturated w.r.t. $x : \neg(\phi \Rightarrow \psi)$.

(iv) if $x : \neg\Box_{x'}\phi \in B''$ and $x' \notin Blocked_{1,2a}$, then by Fact 2 (ii) above, there is in B' $x'' <_{x'} x$ s.t. $x'' : \Box_{x'}\phi$, $x'' : \neg\phi$. All these formulas still hold in B'', which is hence saturated w.r.t. $x : \neg\Box_{x'}\phi$. If x' is $y_i \in Blocked_{1,2a}$, then by construction also $x : \neg\Box_{z_i}\phi$ for z_i corresponding to y_i. Furthermore, there is $x'' <_{z_i} x$, $x'' : \Box_{z_i}\phi$, $x'' : \neg\phi$ in B', hence in B''. By construction of B'', also $x'' <_{y_i} x$, $x'' : \Box_{y_i}\phi$, and B'' is saturated w.r.t. negative boxed formulas.

(v) if $x : \Box_{x'}\phi \in B''$ and $x' \notin Blocked_{1,2a}$, then by Fact 2 (i), B'' is saturated w.r.t. the formula. If x' is $y_i \in Blocked_{1,2a}$, then the formula has been inserted only if $x : \Box_{z_i} \in B''$ for z_i corresponding to y_i. The property follows from the fact that B'' is saturated w.r.t. $x : \Box_{z_i} \in B'$, and $<_{y_i}$ has been defined as $<_{z_i}$.

(vi) It can be easily shown that all the obtained $<_x$ are transitive, by construction and from Fact 2 (iii).

[open:]
B'' is open, since it cannot happen that one of the closure conditions below is satisfied. Indeed it cannot happen that:

- $x : \phi$ and $x : \neg\phi$, since by Fact 2 (iv) this does not happen in B', and the only formulas that have been modified in B'' are the boxed formulas with index y_i, which behave well, since the corresponding boxed formulas with index z_i behave well.
- $x <_y x$. By Fact 2 (iv) above this does not happen in B', and it does not happen for the $<_y$ modified at step 3, since it does not hold for the corresponding $<_{z_i}$.
- $x <_y z, x : \neg V_y$ or $z : \neg V_y$. By Fact 2 (iv) above, this does not hold in B'. If $y \in Blocked_{1,2a}$, this does not hold, since it does not hold for the corresponding z_i.

We still have to show that if some of the rules for the extensions of **PCL** are included in the calculus, the branch B'' obtained by the construction above is saturated with respect to the corresponding conditions.

Fact 4: If any of the following rules belong to the calculus, then B'' is saturated with respect to the corresponding conditions.

(Rules for =) *If the rules for the equality belong to the calculus, branch B'' is saturated w.r.t. $=$.*
For what concerns conditions 2.1, 2.2, 2.3 of Definition 5.6, this is obvious, since it holds for B, and the equalities are not concerned by the construction. For what concerns 2.4, let $x = x'$, $y = y'$, $z = z'$ and $y <_x z$ in B''. If $y <_x z \in B''$, it can be easily shown that $y' <_{x'} z'' \in B''$. Indeed, $<_x$ is not modified by step 3, this holds by rule $(E- <)$ if $y <_x z \in B$; if $y <_x z \in B'$ this holds by construction: for y' and x' this is specified at step 1. For z', by (E- ϕ) also $z' : \neg\Box_z\phi \in B$. If $z' : \neg\Box_z\phi \in B$ has not been blocked, then the relation has not been introduced at step 1. If $z' : \neg\Box_z\phi \in B$ has been blocked, the chosen w must be the same as for $z : \neg\Box_z\phi \in B$, hence $y <_x z'$ has been introduced in B', and also $y' <_{x'} z'$. If $<_x$ has been modified by step 3, and set equal to some $<_z$, this holds since it holds for $<_z$.

(R-CEM) Since (R-CEM) has been applied to all labels in B, for all x, y, z, it holds that $y = z \in B$ or $y <_x z \in B$ or $z <_x y \in B$. If $<_x$ is not modified by step 3, then $y = z \in B''$ or $y <_x z \in B''$ or $z <_x y \in B''$. If $<_x$ is modified by step 3, then it is defined as $<_z$ for which the property holds. In all cases, B'' is saturated w.r.t. **PCL** + (CEM).

(R-UNIV) In the presence of rule (R-UNIV), there is no formula $y : \neg V_x$ in B, hence there is no such formula in B'', and B'' is saturated w.r.t. **PCL** + (A0),(A1),(A2).

(R-CV) Even if (R-CV) has been applied to B, the saturation condition associated to (CV) might not hold for B'', since new relations might have been inserted by steps 1, 2 or 3. In this case, we proceed as follows: we saturate w.r.t. (CV) the branch B' obtained from step 1 and step 2. It can be easily seen that if B' is saturated w.r.t. (CV) (i.e. all $<_x$ are modular), the same property holds for B''. In order to saturate B', we introduce the following step 2' just after step 2 in the construction above:
Step 2': We define the saturated branch $B^{'*}$ as follows. We first let $B^{'*} = B'$. We then consider each $y <_x z \in B$: if $Box_{B,x,z} \subset Box_{B,x,y}$ (strictly included), then for each z_0 s.t. $Box_{B,x,z} = Box_{B,x,z_0}$, we let $y <_x z_0 \in B^{'*}$.
We show that $B^{'*}$ is saturated w.r.t. the conditions of Fact 2, w.r.t. (CV) , and open.
[*saturated:*]
(i) *If* $y : \Box_x \phi \in B^{'*}$, *and* $z <_x y \in B^{'*}$, *then* $z : \phi \in B^{'*}$. First of all, notice that $y : \Box_x \phi \in B$, since the boxed formulas have not been changed by steps 1, 2, 2'. If $z <_x y \in B'$, this follows from Fact 2 (i). If $z <_x y$ has been added at step 2', then there is y' s.t. $z <_x y' \in B$, and $Box_{B,x,y} = Box_{B,x,y'}$. The property follows by the application of rules (Trans) and (T $\Box$) in B.
(ii) holds since if $y' <_x y \in B'$, then also $y' <_x y \in B^{'*}$.
Before dealing with transitivity (property (iii)) we show that all the relations $<_x$ in $B^{'*}$ are asymmetric and modular. Transitivity will be a consequence of these two facts
$<_x$ is asymmetric: suppose for a contradiction that $y <_x z \in B^{'*}$, and $z <_x y \in B^{'*}$. Then clearly at least one of the two relations must have been added by step 2' above (the relation is asymmetric in B', by irreflexivity and transitivity): if $z <_x y$ has been introduced at step 2', $Box_{B,x,y} \subset Box_{B,x,z}$. This means that $y <_x z$ cannot be introduced by step 2', and that it cannot belong to B' (otherwise by Fact 1 $Box_{B,x,z} \subseteq Box_{B,x,y}$, contradiction). We can reason in the same way if $y <_x z$ is introduced at step 2'. We therefore conclude that either $y <_x z \in B^{'*}$ or $z <_x y \in B^{'*}$ but not both, i.e. $<_x$ is asymmetric.
The relation $<_x$ is modular, i.e. for all x, y, z, u if $y <_x z \in B^{'*}$, then either $y <_x u \in B^{'*}$ or $u <_x z \in B^{'*}$. If $y <_x z \in B$, this is obvious by the application of (R-CV) in B. If $y <_x z \notin B$ but $y <_x z \in B'$, then there is a chain $y <_x z_0 <_x \ldots <_x z_n <_x z$ (where $z_0 <_x \ldots <_x z_n$ might be empty) such that each single relation either belongs to B or is inserted at step 1. Consider now $y <_x z_0$ (if $z_0 <_x \ldots <_x z_n$ is empty, consider $y <_x z$). We distinguish the following cases. 1) If $y <_x z_0 \in B$, then by (R- CV) applied to B it can be easily shown that either $y <_x u \in B$ or $u <_x z_0 \in B$, hence $y <_x u \in B^{'*}$ or $u <_x z_0 \in B^{'*}$, i.e. by transitivity of $<_x$, $u <_x z \in B^{'*}$, and the property holds. 2) If $y <_x z_0$ has been inserted by step 1 from $y <_x w \in B$, we know by step 1 that $Box_{B,x,w} \subset Box_{B,x,y}$ (indeed, by step 1 for some ϕ, $w : \neg\Box_x \phi \in B$ whereas $y : \Box_x \phi \in B$). By definition of step 1, we know that $Box_{B,x,z_0} \subseteq Box_{B,x,w}$, and by Fact 1 that $Box_{B,x,z} \subseteq Box_{B,x,z_0}$, from which we conclude that $Box_{B,x,z} \subseteq Box_{B,x,w}$. We distinguish two cases: by (R- CV) applied to B, either $y <_x z \in B$ or $z <_x w \in B$. The first case is not possible (since we are working under the hypothesis that $y <_x z \notin B$). From the second case, and Fact

1, we conclude that $Box_{B,x,w} \subseteq Box_{B,x,z}$, hence $Box_{B,x,z} = Box_{B,x,w}$, and by step 2', $y <_x z \in B^{'*}$.

We are left with the case in which $y <_x z$ has been inserted at step 2'. In this case, there is z' such that $Box_{B,x,z} = Box_{B,x,z'}$, $y <_x z' \in B$, and $Box_{B,x,z'} \subset Box_{B,x,y}$. By (R- CV) applied to B, either $y <_x u \in B$ or $u <_x z' \in B$. In the first case, we are done. In the second case, by Fact 1 we know that $Box_{B,x,z'} \subseteq Box_{B,x,u}$. We distinguish two cases: if $Box_{B,x,z'} = Box_{B,x,u}$, then $y <_x u$ has been inserted by step 2' in $B^{'*}$. If $Box_{B,x,z'} \subset Box_{B,x,u}$, then $u <_x z$ has been inserted by step 2' in $B^{'*}$. The property therefore holds.

Asymmetry and modularity of $<_x$ entail transitivity (vi): let $u <_x y$ and $y <_x z$ in $B^{'*}$. By modularity, either $y <_x u \in B^{'*}$ or $u <_x z \in B^{'*}$. The first case is not possible by asymmetry of $<_x$, hence we conclude that $u <_x z$.

[*open:*] The only possible closure condition that might have changed with saturation step 2' is that $y <_x y$ might have been introduced in $B^{'*}$. However, it can be easily verified that this cannot happen.

Fact 3 above shows that step 3 applied to $B^{'*}$ (that satisfies Fact 2) is open and saturated. It can be easily seen that B'' obtained by step 3 from $B^{'*}$ is saturated with respect to (CV). Furthermore, it can be easily verified that if (R-CEM) and (R-UNIV) belong to the calculus, then B'' is also saturated w.r.t. (CEM) and (UNIV). Last, if the rules for equality belong to the calculus B'' is saturated w.r.t. $=$. We can conclude that if (R - CV) belongs the calculus, the calculus with Blocking Condition 1 and Blocking Condition 2 is complete w.r.t. to modular models.

(R- CS) Since (R- CS) has been applied to B, for all x, y, $x <_x y \in B$. However, $<_x$ might have been modified at step 3, hence the property might not hold in B''. If $<_x$ has been modified at step 3, and it has been put equal to $<_z$, we adjust B'' by the following step 4:

Step 4: We define $B^{''*}$ from B'' as follows: for all $<_x$ modified at step 3, for all x_1 equal to x or such that $x_1 = x \in B''$:

—we remove all formulas $z <_{x_1} x$;

—we introduce $x <_{x_1} y$ for all y s.t. $x = y \notin B''$ and $y : \neg V_x \notin B''$

—we remove $x : \neg\Box_{x_1}\phi$;

—if $x : \phi \in B''$, then we add $x : \Box_{x_1}\neg\phi$ to B'', and remove $y : \Box_{x_1}\neg\phi$ from B''.

The obtained branch is obviously saturated w.r.t. (CS). Furthermore, we show that the obtained branch satisfies all the properties of Fact 3. For (i), (v) this is obvious.

As far as (ii) is concerned: if $x : \phi \Rightarrow \psi \in B^{''*}$, (i.e. $x : \phi \Rightarrow \psi$ in B'' and in B), then for all y s.t. $y = x \notin B$ it can be easily shown that the saturation condition corresponding to the conditional is satisfied (by Fact 3 (ii)). If $x = y \in B$, then $y : \neg\Box_x\neg\phi$ might have been removed from B''. However, it can be shown that if $y : \neg\Box_x\neg\phi \in B''$, this can only be because $<_x$ has been modified by step 3 (otherwise in the presence of (R-CS) a branch containing $y : \neg\Box_x\neg\phi$ and $y = x$ would be closed, indeed Blocking Condition 2b is not applied in this case). At step 3, $<_x$ has been defined as $<_z$ for some z s.t. for all $\phi \in \mathcal{L}$, $z : \phi \in B$ iff $x : \phi \in B$. In particular, $z : \phi \Rightarrow \psi \in B$. Then, either $z : \neg\phi \in B$ or $z : \psi \in B$ (all the other branches being closed, in the presence of (R-CS)). Hence also $x : \neg\phi \in B$ or $x : \psi \in B$, i.e. $x : \neg\phi \in B''$ or $x : \psi \in B''$, and the branch is saturated w.r.t. $x : \phi \Rightarrow \psi$.

For (iii), let $x : \neg(\phi \Rightarrow \psi) \in B^{''*}$. Also $x : \neg(\phi \Rightarrow \psi) \in B''$, and by Fact 3 (ii), there is y s.t. $y : \phi$, $y : \Box_x \neg\phi$, $y : \neg\psi$ are in B''. If $<_x$ has not been modified at step 4, this still holds. Otherwise, if $<_x$ has been modified at step 4, then it had been defined equal to some $<_z$ at step 3. We distinguish two cases. In the first case, $x : \neg\phi \in B''$, hence no formula $y : \Box_x \neg\phi$ is removed from the branch, and the property still holds. In the second case, $x : \phi \in B''$, (hence $x : \phi \in B$) and by the construction above $y : \Box_x \neg\phi$ has been removed from the branch in all cases in which $y = x \notin B''$. Notice that in this case, by step 3, also $z : \phi \in B$ and $y : \Box_z \neg\phi \in B$. By (R-CS), either $z <_z y \in B$ or $z = y \in B$. The first case is not possible (otherwise, $z : \neg\phi \in B$, i.e. $z : \neg\phi \in B$, contradiction). It must hence be $z = y \in B$, i.e. also $z : \neg\psi \in B$, and therefore also $y : \neg\psi \in B$. From this we conclude that $x : \neg\psi \in B^{''*}$, furthermore by step 4 also $x : \Box_x \neg\phi$ has been added to $B^{''*}$. It follows that the saturation condition associated to $x : \neg(\phi \Rightarrow \psi)$ is satisfied.

(iv) continues to hold, since the only relations that have been changed are the relations of kind $y <_x x$, and these can only affect the property for formulas $x : \neg\Box_x \phi$; however, these formulas have been removed from the branch;

(v) Consider $x : \Box_{x'} \phi$. If $<_{x'}$ has not been changed by step 3, this holds by Fact 3 (v). Otherwise, $x' : \phi \in B''$ (otherwise the boxed formula would have been removed from the branch). The only added relation is $x' <_{x'} x$, and the property continues to hold.

(vi) It can be easily verified that the relation $<_x$ so obtained is still transitive.

It can be easily shown that if (R-CEM), (R-UNIV) belong to the calculus, then $B^{''*}$ is saturated w.r.t (R-CEM) and (R-UNIV) respectively. If the rules for equality belong to the calculus $B^{''*}$ is saturated w.r.t. $=$. If (R-CV) belongs to the calculus and step 2' has been performed between step 2 and step 3, $B^{''*}$ is saturated w.r.t. (R- CV): let $y <_x z \in B^{''*}$. We distinguish two cases. First case: $y <_x z \notin B''$, hence the relation has been added in $B^{''*}$ by step 4, i.e. $y = x \in B''$. Notice that in this case for all u, either $y = u \in B''$, and $u <_x z$ has been introduced in $B^{''*}$ or $y = u \notin B''$, and $y <_x u$ has been introduced in $B^{''*}$, hence the property holds. Second case: $y <_x z \in B''$. By what shown above (in the case of (R-CV)), we know that in B'' for all u either $y <_x u \in B''$ or $u <_x z \in B''$. If the relation $<_x$ has not been modified by step 4, this holds in $B^{''*}$. If $<_x$ has been modified, consider two cases for all u. If $y <_x u \in B''$, this might not hold in $B^{''*}$ because $x = u \in B$, and the relation has been erased by step 4. However, in this case $u <_x z$ has been inserted by step 4. If $u <_x z \in B''$ but $u <_x z \notin B^{''*}$, then the relation might have been erased by step 4 because $x = z \in B''$. However this case is not possible since also $y <_x z$ would have been erased by step 4 (whereas by hypothesis $y <_x z \in B^{''*}$).

Furthermore, the modified B'' is still open. Indeed, it can be easily verified that none of the closure conditions have been inserted in $B^{''*}$.

(R-MP) First, notice that if (R- CS) belongs to the calculus, we do not need any extra construction, since it can be easily shown that $B^{''*}$ obtained by step 4 above (case of (R-CS)) is also saturated w.r.t. (MP). We consider the case in which (R- CS) does not belong to the calculus, and we describe the following construction which is alternative to the construction for (R- CS). In order to build a branch saturated w.r.t. (MP), we need to perform the following step 4.

Step 4: for all x, for all x_1 equal to x or such that $x_1 = x \in B''$, for all y, remove from B'' $y <_{x_1} x$ and $x : \neg\Box_{x_1} \phi$. If (R-CV) belongs to the calculus, and $y <_{x_1} r \in B''$,

and $y <_{x_1} r$ has not been removed from B'', we introduce $x <_{x_1} r$. We let $B^{''*}$ be the obtained branch.
We show that $B^{''*}$ is saturated w.r.t. **PCL** + (MP). All the properties of Fact 3 still hold in $B^{''*}$. As far as (i), (iii) are concerned, this is obvious. For (ii), let $x : \phi \Rightarrow \psi \in B^{''*}$. Also $x : \phi \Rightarrow \psi \in B$, and for all z, rule $(T \Rightarrow)$ has introduced in B either $z : \neg\phi$ or $z : \psi$ or $z : \neg\Box_x\neg\phi$ or $z : \neg V_x$. If $z = x \notin B''$, the same formulas hold in $B^{''*}$, hence the property holds. If $z = x \in B''$, then also $z = x \in B$, and by (R- MP) the branches containing $z : \neg\Box_x\neg\phi$ or $z : \neg V_x$ would be closed (recall that in this case Blocking Condition 2b would not be applied to $z : \neg\Box_x\neg\phi$). In this case, we hence know that either $z : \neg\phi$ or $z : \psi$ are in B and hence in $B^{''*}$. We conclude that also in this case the property holds. As far as (iv) is concerned, for $x : \neg\Box_z\phi$ with $z = x \notin B''$, the property still holds; on the other hand, if $z = x \in B''$, then $x : \neg\Box_z\phi$ has been removed. For what concerns (v), the sensitive case is if $x <_{x_1} r$ has been introduced at step 4 and $r : \Box_{x_1}\phi \in B^{''*}$. We distinguish two cases on the relation $y <_{x_1} r$ that originated the insertion of $x <_{x_1} r$ in $B^{''*}$: if $y <_{x_1} r \in B$, then, given that (R-CV) and (R-MP) have been applied to B, $x <_{x_1} r \in B$, hence $x : \phi \in B^{''*}$. If $y <_{x_1} r \in B$ has been introduced at step 1, then $y <_{x_1} r_0 \in B$, with $Box_{B,x_1,r} \subseteq Box_{B,x_1,r_0}$. By (R-CV) and (R - MP) applied to B, we conclude that $x <_{x_1} r_0 \in B$, hence $x : \phi \in B$, and $x : \phi \in B^{''*}$. We can repeat the same reasoning to conclude that the same holds if $y <_{x_1} r$ has been inserted at step 2. If it has been inserted at step 2', then for some r_0 s.t. $Box_{B,x_1,r} = Box_{B,x_1,r_0}$, $y <_{x_1} r_0 \in B$. By (R-CV) and (R-MP) applied to B, we conclude that $x <_{x_1} r_0 \in B$, hence $x : \phi \in B$, and $x : \phi \in B^{''*}$. Last, if $y <_{x_1} r$ has been inserted at step 3, then $<_{x_1}$ has been defined equal to some $<_z$ such that for all $\psi \in \mathcal{L}$, $x_1 : \psi \in B$ iff $z : \psi \in B$. If the relation has been introduced, then $y <_z r \in B^{'*}$ obtained from step 2'. By repeating the same reasoning done for the previous cases (applied to z instead of x_1), we derive that $z : \phi \in B$, hence also $x_1 : \phi \in B$, and by $(E - \phi)$ applied to B also $x : \phi \in B$, i.e. $x : \phi \in B^{''*}$, from which we conclude. As far as (vi) is concerned, it can be easily shown that transitivity still holds: the sensitive case is the one in which $x <_{x_1} r$ has been inserted at step 4, and $r <_{x_1} r' \in B^{''*}$. In this case by step 4 also $x <_{x_1} r'$ has been inserted, hence transitivity holds. Furthermore, $B^{''*}$ is open: the only closure condition that could be inserted by step 4 is $x <_{x_1} x$, however it can be easily verified that this cannot happen. It can be easily shown that if (R-CEM), (R-UNIV) belong to the calculus, then $B^{''*}$ is saturated w.r.t (R-CEM) and (R-UNIV) respectively. If the rules for equality belong to the calculus $B^{''*}$ is saturated w.r.t. $=$. If (R-CV) belongs to the calculus: let $y <_x z \in B^{''*}$. If $y <_x z \in B''$, by what said for (R- CV), for all u $y <_x u \in B''$ or $u <_x z \in B''$. In the first case, it could be $y <_x u \notin B^{''*}$ in case $x = u$. However, in this case, $u <_x z$ has been introduced at step 4. In the second case, it might be $u <_x z \notin B^{''*}$ because $x = z \in B''$. However this case is not possible since also $y <_x z$ would have been removed from B''. If $y <_x z$ has been inserted at step 4, then $y = x \in B$ and for some r, $r <_x z \in B^{''*}$ (without being inserted at step 4). By what said above, either $r <_x u \in B^{''*}$ or $u <_x z \in B^{''*}$. In the second case, the property holds. In the first case, $y <_x u$ has been inserted at step 4, and we can conclude.

□

In the proof of Theorem 6.1, we have already observed that the number different labels to which rule $(F \Rightarrow)$ can be applied on a branch is $O(2^n)$, where n is the size of the initial input formula. As there are $O(n)$ distinct negative conditionals $\neg(\phi \Rightarrow \psi)$ to which $(F \Rightarrow)$ can be applied in a label, the overall number of $(F \Rightarrow)$ applications on a branch is $O(2^n)$.

Concerning rule $(F\Box)$, we have seen that it can be applied to a formula $x : \neg\Box_y\phi$ on a branch: $O(2^n)$ times, for the different choices of the label y, and $O(2^n)$ times, for the different choices of label x. The number of labels that can be introduced on a branch is therefore $O(2^n)$.

In order to establish an upper bound on the complexity of the systematic procedure above, we need to estimate the overall number of rule applications, including those ones in step (a). To this purpose, we can observe that the number of pseudo-formulas which can be built from $O(n)$ formulas and $O(2^n)$ labels is $O(2^n)$. As we do not admit redundant applications of rules, on each branch the overall number of tableau rule applications determined by the systematic procedure is $O(2^n)$. Thus, our systematic procedure allows a non-deterministic algorithm for testing satisfiability in **PCL** to be defined: at each step the algorithm first saturates with respect to the static rules (step a), then guesses a branch and applies a dynamic rule (step b). By what said just above, the algorithm eventually terminates, and returns ”satisfiable” if it finds an open branch and ”unsatisfiable” if it does not. Since the number of rules applied on the branch is $O(2^n)$, we can conclude that:

THEOREM 6.3. *For all extensions* **PCL**+Σ *of* **PCL**, *the systematic procedure decides the satisfiability of a formula in non-deterministic exponential time with respect to the size of the input.*

The above result seems to be in partial accordance with [Friedman and Halpern 1994], where the authors show that a formula is satisfiable iff it is satisfiable in an exponentially sized structure. Moreover, the authors argue that deciding satisfiability can be done in EXPTIME (that we understand as Deterministic Exponential Time), even if the algorithm they suggest is not described in full details (Theorem 0.13). In future investigation we will study whether we can obtain a *deterministic* EXPTIME decision procedure based on our tableau calculus, thus matching the known upper bound for these logics.

7. RELATED WORK

To put our tableau method in a context, we briefly recall and compare related works on proof methods for conditional logics.

[Giordano et al. 2003] and [Giordano et al. 2005] contain preliminary versions of some of the results contained in this paper.

De Swart [de Swart 1983] and Gent [Gent 1992] give sequent/tableau calculi for conditional logics **VC** (= **PCL** +CV+MP+CS) and **VCS**. The kind of systems they propose are based on the entrenchment connective $\leq$, from which the conditional operator can be defined. Their systems are analytic and comprise an infinite set of rules $\leq F(n, m)$, with a uniform pattern, to decompose each sequent with m negative and n positive entrenchment formulas.

Crocco and Fariñas [Crocco and Fariñas del Cerro 1995] present sequent calculi for some conditional logics including minimal **CK**, **C2**, **CO** (= **PCL** without CA) and others. Their calculi comprise two levels of sequents: principal sequents with $\vdash_P$ correspond to the basic deduction relation, whereas auxiliary sequents with $\vdash_a$ correspond to the conditional operator: thus the constituents of $\Gamma \vdash_P \Delta$ are sequents of the form $X \vdash_a Y$, where X, Y

are sets of formulas. However, their calculi are not analytic, and do not give a decision procedure.

Artosi, Governatori, and Rotolo [Artosi et al. 2002] develop labelled tableau for the flat fragment of conditional logic **CO** (they call it **CU** and it corresponds to cumulative KLM logic **C**). Formulas are labelled by path of worlds containing also variable worlds. Since they adopt a selection function semantics, they have to cope with the *normality condition*: if $[[A]]^M = [[A']]^M$ then $f(A, w) = f(A', w)$. To this purpose, they use an efficient unification procedure to propagate positive conditionals, and the unification procedure takes care of checking the equivalence of antecedents. Their tableau contains a cut-rule, called PB, which is not eliminable. They also discuss the extension of their method to **PCL** and stronger systems on the one hand, and to nested conditionals, on the other; however the feasibility of these extensions is rather problematic.

Broda, Gabbay, Lamb, and Russo develop an elegant natural deduction system for Boutilier's conditional logic of normality and some variants of it [Boutilier 1994], [Broda et al. 2002]. Their proof system uses labels following the methodology of *Labelled Deductive Systems* [Broda et al. 2002], where the objects involved in the proofs are structured configurations of formulas, worlds, and relations thereof. In this respect, their approach is rather similar to ours. However, as we already observed, Boutilier's conditional logic has a simpler semantics defined in terms of standard modal logic without world-indexed relations or modalities (and thus it cannot handle nested and iterated conditionals). Moreover, it is not evident if one can extract a decision procedure for Boutilier's logic from their natural deduction system.

Lamarre [Lamarre 1993] presents tableau systems for conditional logics **V**(= **PCL** +CV), **VN**, **VC** and **VW**(= **PCL** +CV+MP). Lamarre's method is a consistency-checking procedure which tries to build a system of spheres falsifying the input formulas. The method makes use of a subroutine to compute the *core*, that is defined as the set of formulas characterizing the minimal sphere. The computation of the core needs in turn the consistency checking procedure. Thus, there is a mutual recursive definition between the procedure for checking consistency and the one to compute the core.

Groeneboer and Delgrande [Groeneboer and Delgrande 1988; Delgrande and Groeneboer 1990] have developed a tableau method for conditional logic **VN** which is based on the translation of this logic into modal logic **S4.3**.

In [Olivetti et al. 2007], sequent calculi for the minimal conditional logic **CK** and some extensions of it are presented (namely almost all extensions of **CK** combining (ID), (CS), (CEM), and (MP)). The calculi are based on the selection function semantics for these logics and make use of labels. It is shown that, by means of these calculi, one can obtain complexity bounds, sometimes tighter than what was previously known, for the respective logics.

Finally, in [Giordano et al. 2005], the authors present tableau calculi for the logics for default reasoning proposed by Kraus, Lehmann and Magidor in [Kraus et al. 1990]. In particular, [Giordano et al. 2005] provides calculi for the systems **P**, **C**, **CL** and **R** by Kraus Lehmann and Magidor. The systems **P** and **R** can be seen as the flat fragment of the logics **PCL** and **PCL** $+$**CV** studied here, whereas the other two are weaker and do not fit directly into the preferential semantics. The two mentioned logics **P** and **R** of the KLM framework are hence simpler than the logics studied in the present work, whose language is not restricted to the flat fragment. Therefore it is not surprising that the calculus for the latter systems (presented in this work) is more complex than the calculi for the former, although they are based on similar ideas: namely expressing by modal rules the Limit Assumption or, in the context of KLM systems, the so-called *smoothness condition*. For the same reason, the complexity of the calculi proposed in [Giordano et al. 2005] is lower than the complexity we argue for the logics studied here.

8. CONCLUSION

In this work we have studied the conditional logics of preferential structures, namely **PCL** and its most important extensions. First, we have given a general and direct completeness result for a wide range of logics (with and without the universality condition) with respect to preferential structures under the Limit Assumption. This result extends the seminal result presented by Burgess [Burgess 1981] for a semantics without the Limit Assumption. Then we have presented a tableau proof procedure for these logics. The proposed tableau system is uniform and modular in the sense that each specific semantic condition is captured by a tableau rule. Moreover, our proof method makes use of labels and is based on the use of hybrid modal formulas, where the modalities are indexed by worlds. We have been able to obtain a terminating procedure by a loop-checking mechanism similar to standard *blocking*. Our method gives a practical decision procedure for the class of conditional logics investigated in this work. Moreover, by the proof method we constructively obtain a complexity upper bound for these logics.

In future work, we intend to carry on a finer grained complexity analysis of **PCL** extensions by means of our tableau method. Moreover, we intend to investigate several improvements on the tableau system, including loop-checking mechanisms, a better treatment of equality via label-substitution and specific proof-search strategies. We also plan to implement this method along the lines of [Giordano et al. 2007]. Finally, we think that our methods can be adapted to provide reasoning methods for *concept comparative similarity* in description logics presented in [Sheremet et al. 2005], as this notion is strongly related to the systems of conditional logics studied in this work.

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Received October 2007; revised April 2008; accepted May 2008