INTRODUCTION TO GEOMETRIC REPRESENTATION THEORY

LECTURES PRESENTED AT THE WORKSHOP ON "GEOMETRY AND INTEGRABILITY", UNIVERSITY OF MELBOURNE, 6-15 FEB, 2008

PETER BOUWKNEGT

Abstract. These notes are a rough write-up of lectures presented at the workshop on "Geometry and Integrability", held at the University of Melbourne, 6-15 Feb, 2008. My interest in Geometric Representation Theory stems from recent developments in the Geometric Langlands program. These lectures are an introduction into material which are a prerequisite to anyone interested in studying the Geometric Langlands program. Most of the results I will discuss can be found in the textbooks by Pressley and Segal [PS], Chriss and Ginzburg [CG], and Hotta, Takeuchi and Tanisaki [HTT], or the lecture notes by Bernstein [Ber] and Gaitsgory [Gai]. Here one can also find references to the original papers. My presentation of the material was very much inspired by, and to some extent based on, lectures given by Ian Grojnowski at the "Introduction to the Geometric Langlands Program" [Gro].

The lectures were aimed at Mathematical Physicists with a similar background as myself. While I have tried to be reasonably precise, I have occasionally compromised and chosen to convey an idea, often by example, rather than spelling out all the mathematical details, in particular those that require more advanced knowledge of algebraic geometry.

Needless to say that in three one-hour lectures one can only scratch the surface of the subject. My hope is that these lectures at least convey the beauty of Geometric Representation Theory and that they may motivate the reader to further study.

Lecture 1

1.1. **Coxeter groups and Weyl groups.**

Definition 1.1. A *Coxeter system* (W,S) is a group W generated by reflections $s\in\mathsf{S}$, such that

$$s_i^2=1\,,\qquad (s_is_j)^{m_{ij}}=1\,,\quad i\neq j\,,\qquad s_i\in\mathsf{S}\,, \tag{1.1}$$

for some integers $m_{ij}=m_{ji}=2,3,4,\ldots,\infty$.

Note that we can write the second relation as

$$\underbrace{s_is_js_i\ldots}_{m_{ij}}=\underbrace{s_js_is_j\ldots}_{m_{ij}} \tag{1.2}$$

An expression $w=s_{i_1}s_{i_2}\ldots s_{i_r}$ is *reduced* if r is minimal. In that case we call $\ell(w)=r$ the *length* of w.

We will be interested in Coxeter groups associated to either Weyl groups associated to finite dimensional Lie algebras $\mathfrak{g}$ or or Weyl groups associated to affine Lie algebras $\widehat{\mathfrak{g}}$ (i.e. untwisted affine Kac-Moody algebras), henceforth referred to as finite and affine Weyl groups, $\mathsf{W}^{\rm fin}$ and $\mathsf{W}^{\rm aff}$, respectively. We recall that the integers m_{ij} are in that case related to the entries of the Cartan matrix a_{ij} by $m_{ij}=2,3,4,6,\infty$, for $a_{ij}a_{ji}=0,1,2,3,\geq 4$, respectively.

If $\{s_1,\ldots,s_\ell\}$, $\ell=\text{rank }\mathfrak{g}$, is a system of generators for $\mathsf{W}^{\rm fin}$, then a system of generators of $\mathsf{W}^{\rm aff}$ is given by adding the reflection s_0 in the additional simple root $\alpha_0=\delta-\theta^\vee$, where θ is the longest root of $\mathfrak{g}$ and δ the imaginary root of $\widehat{\mathfrak{g}}$. One can show that the element s_0s_θ, where s_θ denotes the reflection in the hyperplane orthogonal to θ, acts by translations over $\theta^\vee$, and this leads to the identification

$$\mathsf{W}^{\rm aff}\cong\mathsf{W}^{\rm fin}\ltimes R^\vee \tag{1.3}$$

where $R^\vee$ is the coroot lattice of $\mathfrak{g}$. If we denote by e^λ the element in $\mathsf{W}^{\rm aff}$ corresponding to $\lambda\in R^\vee$, then the semi direct product structure in (1.3) is explicitly given by

$$(we^\lambda)(w'e^{\lambda'})=ww'e^{w'^{-1}(\lambda)+\lambda'}\,,\qquad w,w'\in\mathsf{W}^{\rm fin},\ \lambda,\lambda'\in R^\vee\,, \tag{1.4}$$

or, equivalently,

$$w\,e^\lambda\,w^{-1}=e^{w(\lambda)}\,,\qquad w\in\mathsf{W}^{\rm fin},\ \lambda\in R^\vee\,. \tag{1.5}$$

It is sometimes useful to replace $\mathsf{W}^{\rm aff}$ by a slightly bigger group, referred to as the *extended affine Weyl group* $\widetilde{\mathsf{W}}^{\rm aff}$, by replacing the lattice $R^\vee$ by a slightly finer lattice $Y\supset R^\vee$, invariant under the action of $\mathsf{W}^{\rm fin}$. In our case we will take Y to be the cocharacter lattice a reductive Lie group G with Lie algebra $\mathfrak{g}$ (an even bigger group is obtained by taking the coweight lattice of $\mathfrak{g}$). Note, though, that $\widetilde{\mathsf{W}}^{\rm aff}$ is in general not a Coxeter group.

Example. In the case of $\mathsf{G} = \mathsf{SL}_2$, the Weyl group is generated by a single reflection s_1, in the hyperplane of the (unique) positive root α of $\mathfrak{sl}_2$. The affine Weyl group is generated by two reflections s_0 and s_1, with no further relations. The element $s_0 s_1$ can be identified with e^α, the translation over the root α. The extended Weyl group allows translations over integer multiples of the fundamental weight $\frac{1}{2}\alpha$.

1.2. **Hecke algebras.** We now turn to Hecke algebras

Definition 1.2. The *Hecke algebra* associated to the Coxeter group (W, S) is the $\mathbb{Z}[q, q^{-1}]$-algebra $\mathcal{H}_W$ with generators T_s, $s \in S$, subject to the relations

$$\begin{aligned} (T_s + 1)(T_s - q) &= 0\,, \qquad s \in \mathsf{S}\,, \\ \underbrace{T_{s_i} T_{s_j} T_{s_i} \ldots}_{m_{ij}} &= \underbrace{T_{s_j} T_{s_i} T_{s_j} \ldots}_{m_{ij}}\,, \qquad s_i, s_j \in \mathsf{S}\,. \end{aligned} \tag{1.6}$$

Note that the first relation is equivalent to

$$T_s^2 = (q-1)T_s + q\,, \tag{1.7}$$

or

$$T_s^{-1} = q^{-1}T_s - (1 - q^{-1})\,. \tag{1.8}$$

In practice it is difficult to check the second relation in (1.6). An equivalent definition of the Hecke algebra $\mathcal{H}_\mathsf{W}$, is provided by the following

Theorem 1.3. *The Hecke algebra $\mathcal{H}_\mathsf{W}$, associated to W, has a free $\mathbb{Z}[q, q^{-1}]$-basis $\{T_w, w \in \mathsf{W}\}$ such that*

$$\begin{aligned} (T_s + 1)(T_s - q) &= 0\,, && s \in \mathsf{S}\,, \\ T_w T_{w'} &= T_{ww'}\,, && \text{if } \ \ell(w) + \ell(w') = \ell(ww')\,. \end{aligned} \tag{1.9}$$

We will denote the Hecke algebras associated to $\mathsf{W}^{\mathrm{fin}}(\mathsf{G})$ and $\mathsf{W}^{\mathrm{aff}}(\mathsf{G})$ by $\mathcal{H}^{\mathrm{fin}}(\mathsf{G})$, and $\mathcal{H}^{\mathrm{aff}}(\mathsf{G})$, respectively. The extended affine Hecke algebra $\widetilde{\mathcal{H}}^{\mathrm{aff}}(\mathsf{G})$, associated to $\widetilde{\mathsf{W}}^{\mathrm{aff}}$ will be introduced in Section 2.1

1.3. **Convolution algebras.** Let G be a group, acting on a set X. We denote by $\mathcal{F}(X) = \{f : X \to \mathbb{C}\}$ the set of functions on X, and by

$$\mathcal{F}_\mathsf{G}(X) = \{f \in \mathcal{F}(X) \mid f(g \cdot x) = f(x)\}$$

the set of G-invariant functions. We can think of $\mathcal{F}_\mathsf{G}(X)$ as $\mathcal{F}(X/G)$, i.e. functions on the orbit space X/G. The space $\mathcal{F}_\mathsf{G}(X)$ is a vector space over $\mathbb{C}$ with a basis given by the characters $\chi_\mathcal{O}$ corresponding to the orbits $\mathcal{O}$ of G on X. I.e.

$$\chi_\mathcal{O}(x) = \begin{cases} 1 & x \in \mathcal{O}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

If $\phi : X \to Y$ is a G-map, i.e. $g \cdot \phi(x) = \phi(g \cdot x)$ for all $x \in X$, $g \in \mathsf{G}$, then we have maps

$$\phi^* : \mathcal{F}_{\mathsf{G}}(Y) \to \mathcal{F}_{\mathsf{G}}(X)\,, \qquad \phi^*(f)(x) = f(\phi(x))\,,$$

and, provided all the fibers $\phi^{-1}(y)$, $y \in Y$ are finite,

$$\phi_* : \mathcal{F}_{\mathsf{G}}(X) \to \mathcal{F}_{\mathsf{G}}(Y)\,, \qquad \phi_*(f)(y) = \sum_{x\in\phi^{-1}(y)} f(x)\,.$$

Now we apply all of this to the case where X itself is a group. If $X = \mathsf{G}$ is a finite group, then $\mathcal{F}(\mathsf{G})$ is an algebra with unit under convolution, i.e.

$$f_1 \star f_2 = m_*(f_1 \otimes f_2)\,,$$

where $m : \mathsf{G} \times \mathsf{G} \to \mathsf{G}$ denotes group multiplication, and $m_* : \mathcal{F}(\mathsf{G} \times \mathsf{G}) \cong \mathcal{F}(\mathsf{G}) \times \mathcal{F}(\mathsf{G}) \to \mathcal{F}(\mathsf{G})$. Explicitly

$$(f_1 \star f_2)(x) = \sum_{y\in\mathsf{G}} f_1(xy^{-1})f_2(y)\,.$$

The unit, i.e. the orbit of $x = 1$ under the trivial action, is of course given by the function

$$\mathbf{1}(x) = \begin{cases} 1 & x = 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Let K be a subgroup of G. Then the algebra structure on $\mathcal{F}(\mathsf{G})$ descends to an algebra structure on $\mathcal{F}_{\mathsf{K}\times\mathsf{K}}(\mathsf{G})$, where

$$(k_1, k_2) \cdot g = k_1 g k_2^{-1}\,.$$

We define $\mathcal{H} = \mathcal{H}(\mathsf{G},\mathsf{K}) = \mathcal{F}_{\mathsf{K}\times\mathsf{K}}(\mathsf{G})$ as the *Hecke algebra* belonging to the pair (G,K). Obviously, the dimension of $\mathcal{H}$ equals the number of double cosets $\mathsf{K}\backslash\mathsf{G}/\mathsf{K}$. Alternatively, we can think of $\mathcal{H}$ as $\mathcal{F}(\mathsf{K}\backslash\mathsf{G}/\mathsf{K})$, $\mathcal{F}_{\mathsf{K}}(\mathsf{G}/\mathsf{K})$, or $\mathcal{F}_{\mathsf{G}}(\mathsf{G}/\mathsf{K} \times \mathsf{G}/\mathsf{K})$. From the geometric point of view it appears that thinking of $\mathcal{H}$ as $\mathcal{F}_{\mathsf{K}}(\mathsf{G}/\mathsf{K})$ is the most convenient, hence we will use this point of view.

The convolution product on $\mathcal{F}_{\mathsf{K}}(\mathsf{G}/\mathsf{K})$ is explicitly given by

$$(f_1 \star f_2)(x\mathsf{K}) = \sum_{y\in\mathsf{G}/\mathsf{K}} f_1(xy^{-1})f_2(y) = \frac{1}{|\mathsf{K}|}\sum_{y\in\mathsf{G}} f_1(xy^{-1})f_2(y)\,, \tag{1.10}$$

where the last equality holds provided $|\mathsf{K}| < \infty$. [All formulas make sense in the infinite setting as well, with sum replaced by integrals, etc.]

1.4. **The finite Hecke algebra.** Suppose we take $\mathsf{G} = \mathsf{G}(\mathbb{F}_q)$, a reductive group over the field $\mathbb{F}_q$, and $\mathsf{K} = \mathsf{B} = \mathsf{B}(\mathbb{F}_q)$ a Borel subgroup. Then the claim is that $\mathcal{H} = \mathcal{F}_{\mathsf{B}\times\mathsf{B}}(\mathsf{G})$ is the (finite) Hecke algebra based on the group G.

Example. Consider

$$\mathsf{G} = \mathsf{SL}_2(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ a,b,c,d \in \mathbb{F}_q,\ ad-bc=1 \right\},$$

and take

$$\mathsf{K} = \mathsf{B} = \mathsf{B}(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix},\ a,b \in \mathbb{F}_q,\ a \neq 0 \right\},$$

To determine G/B we observe that G acts transitively on the projective line $\mathbb{P}(\mathbb{F}_q^2) = P^1 = \{(z_1,z_2) \in \mathbb{F}_q \mid (z_1,z_2) \sim (az_1,az_2),\ a \neq 0\}$. Since the stabilizer subgroup of the line $[z_1,z_2]=[1,0]$ is precisely B we immediately conclude $\mathsf{G}/\mathsf{B} = P^1$.

Explicitly, each orbit on G/B can be parametrized by $\begin{pmatrix} z_1 & \star \\ z_2 & \star \end{pmatrix}$, and since the action of B on G/B is given by

$$\begin{pmatrix} z_1 & \star \\ z_2 & \star \end{pmatrix} \begin{pmatrix} a & \star \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} az_1 & \star \\ az_2 & \star \end{pmatrix}$$

we have

$$\mathsf{G}/\mathsf{B} = \left\{ \begin{pmatrix} z_1 & \star \\ z_2 & \star \end{pmatrix} \Big| (z_1,z_2) \sim (az_1,az_2) \right\} = P^1 .$$

Note that $|\mathsf{G}/\mathsf{B}| = q+1$.
To determine the orbits of B on G/B we note

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} z_1 & \star \\ z_2 & \star \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 & \star \\ a^{-1}z_2 & \star \end{pmatrix}.$$

Hence there are two orbits of B on G/B. Namely, for $z_2 = 0$ the orbit is given by the coset B, while for $z_2 \neq 0$ the orbit covers all cosets $x\mathsf{B}, x \notin \mathsf{B}$. Denote the characteristic functions of these orbits by χ_1 and χ_s, respectively. We have seen that they form a vector basis of the Hecke algebra $\mathcal{H}(\mathsf{SL}_2(\mathbb{F}_q), \mathsf{B}(\mathbb{F}_q))$. Explicitly,

$$\chi_1(x\mathsf{B}) = \begin{cases} 1 & x \in \mathsf{B}, \\ 0 & x \notin \mathsf{B}, \end{cases} \qquad \chi_s(x\mathsf{B}) = \begin{cases} 1 & x \notin \mathsf{B}, \\ 0 & x \in \mathsf{B}. \end{cases}$$

To determine the algebra structure we need to compute $\chi_1 \star \chi_1$, $\chi_1 \star \chi_s$, and $\chi_s \star \chi_s$. Here we give an explicit computation of $\chi_s \star \chi_s$. By a similar calculation it is easily seen that χ_1

is the identity element.
For $x \in \mathsf{B}$

$$(\chi_s \star \chi_s)(x) = \frac{1}{|\mathsf{B}|} \sum_{y \in \mathsf{G}} \chi_s(xy^{-1})\chi_s(y)$$
$$= \frac{1}{|\mathsf{B}|} \, \#\{y \in \mathsf{G} \,|\, y \notin \mathsf{B},\ xy^{-1} \notin \mathsf{B}\}$$
$$= \frac{1}{|\mathsf{B}|} \, \#\{y \in \mathsf{G} \,|\, y \notin \mathsf{B}\}$$
$$= \frac{1}{|\mathsf{B}|} \, (|\mathsf{G}| - |\mathsf{B}|) = q \,,$$

while for $x \notin \mathsf{B}$

$$(\chi_s \star \chi_s)(x) = \frac{1}{|\mathsf{B}|} \sum_{y \in \mathsf{G}} \chi_s(xy^{-1})\chi_s(y)$$
$$= \frac{1}{|\mathsf{B}|} \, \#\{y \in \mathsf{G} \,|\, y \notin \mathsf{B},\ xy^{-1} \notin \mathsf{B}\}$$
$$= \frac{1}{|\mathsf{B}|} \, \#\{y \in \mathsf{G} \,|\, y \notin \mathsf{B},\ y \notin x\mathsf{B}\}$$
$$= \frac{1}{|\mathsf{B}|} \, (|\mathsf{G}| - |\mathsf{B}| - |\mathsf{B}|) = q - 1 \,.$$

Hence,

$$\chi_s \star \chi_s = (q-1)\chi_s + q\chi_1 \,.$$

Thus we see that $\mathcal{H}(\mathsf{SL}_2(\mathbb{F}_q), \mathsf{B}(\mathbb{F}_q))$ is isomorphic to the finite Hecke algebra of SL_2.

In general we have

Theorem 1.4.

$$\mathcal{H}^{fin}(\mathsf{G}) \ \cong \ \mathcal{H}(\mathsf{G}(\mathbb{F}_q), \mathsf{B}(\mathbb{F}_q)) \,.$$

To see this, we note that due to the Bruhat decomposition

$$\mathsf{G} = \coprod_{w \in \mathsf{W}} \mathsf{B} \, \hat{w} \, \mathsf{B} \,,$$

where W is the Weyl group of G, there exists a basis $\{T_w, w \in \mathsf{W}\}$ of $\mathcal{H}$, where T_w is the character corresponding to the orbit $\mathsf{B}w\mathsf{B}$. We denote $T_1 = \mathbf{1}$. An explicit computation then yields the relations (1.9).

In other words, the orbits of B on G/B are labeled by elements of the Weyl group $\mathsf{W} = N(\mathsf{T})/\mathsf{T}$, and we have a decomposition of G/B into 'Schubert cells' X_w

$$\mathsf{G}/\mathsf{B} = \coprod_{w \in \mathsf{W}} \mathsf{X}_w \,.$$

One can show that $\mathsf{X}_w = (\mathbb{F}_q)^{\ell(w)}$, hence

$$|\mathsf{G}/\mathsf{B}| = \sum_{w\in\mathsf{W}} q^{\ell(w)} = \prod_{i=1}^{\mathrm{rank}\,\mathfrak{g}} \left(\frac{1-q^{d_i+1}}{1-q}\right) \equiv w(q)\,,$$

where the $\{d_i,\ i=1,\ldots,\mathrm{rank}\,\mathfrak{g}\}$ are the so-called *exponents* of $\mathfrak{g}$. E.g. for $\mathsf{G} = \mathsf{SL}_n$ we have $\{d_i\} = \{1,2,3,\ldots,n-1\}$. Note that

$$\lim_{q\to 1} w(q) = \prod_{i=1}^{\mathrm{rank}\,\mathfrak{g}} (d_i+1) = 2|\Delta_+| = \dim_{\mathbb{R}} \left(\mathsf{G}(\mathbb{C})/\mathsf{B}(\mathbb{C})\right)\,.$$

Example. As we have seen, for $\mathsf{G} = \mathsf{SL}_2$ we have two orbits of B on G/B. This corresponds to the Bruhat decomposition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathsf{B}\,, & \text{for } \ c=0\,, \\ \\ \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \in \mathsf{B}\,\hat{s}\,\mathsf{B}\,, & \text{for } \ c\neq 0\,. \end{cases}$$

where

$$\hat{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in N(\mathsf{T})\,,$$

is a representative of the nontrivial element in $\mathsf{W} = N(\mathsf{T})/\mathsf{T} = \{\mathbf{1}, s\}$.

Lecture 2

2.1. **The affine Hecke algebra.** The affine Hecke algebra is defined as in (1.9), but now where $w \in \mathsf{W}^{\mathrm{aff}}$. Recall that $\mathsf{W}^{\mathrm{aff}} = \mathsf{W}^{\mathrm{fin}} \ltimes R^\vee$, where $R^\vee$ is the co-root lattice of G.

Obtaining the affine Hecke algebra as a convolution algebra is completely analogous to Theorem 1.4, using an 'affinization' of G. Concretely, we let $\mathcal{F} = \mathbb{F}_q((z))$ be the field of Laurent series (with finite negative part), and $\mathcal{O} = \mathbb{F}_q[[z]]$ be the subfield of Taylor series. We consider the algebraic groups $\mathsf{G}(\mathcal{F})$ and $\mathsf{G}(\mathcal{O})$.

We have an evaluation homomorphism $\mathrm{ev} : \mathsf{G}(\mathcal{O}) \to \mathsf{G}(\mathbb{F}_q)$, defined by $\mathrm{ev}(\gamma(z)) = \gamma(0)$ (i.e. evaluation at $z = 0$). The Iwahori subgroup $\mathsf{I} \subset \mathsf{G}(\mathcal{O})$ is defined by

$$\mathsf{I} = \mathrm{ev}^{-1}(\mathsf{B}) = \{\gamma \in \mathsf{G}(\mathcal{O}) \mid \gamma(0) \in \mathsf{B}\}\,.$$

Using the affine analogue of the Bruhat decomposition

$$\mathsf{G}(\mathcal{F}) \;=\; \coprod_{w \in \mathsf{W}^{\mathrm{aff}}} \mathsf{I}\,\hat{w}\,\mathsf{I}\,, \tag{2.1}$$

one can show, in complete analogy with the finite dimensional case, that

Theorem 2.1 (Iwahori-Matsumoto)**.**

$$\mathcal{H}^{\mathit{aff}}(\mathsf{G}) \;\cong\; \mathcal{H}(\mathsf{G}(\mathcal{F}), \mathsf{I})\,.$$

2.2. **The center of $\mathcal{H}^{\mathbf{aff}}$.** Recall that $\mathsf{W}^{\mathrm{aff}} \cong \mathsf{W}^{\mathrm{fin}} \ltimes R^\vee$. A similar statement holds for the associated Hecke algebra

Theorem 2.2.

$$\mathcal{H}^{\mathit{aff}} \cong \mathcal{H}^{\mathit{fin}} \otimes \mathbb{Z}[R^\vee]$$

Note, in particular, that only the finite Weyl group part of $\mathsf{W}^{\mathrm{aff}}$ gets deformed.

The affine Hecke algebra has a vector space basis $\{T_{we^\lambda}\}$ indexed by elements of $\mathsf{W} \ltimes R^\vee$. Naively, given the statement of the theorem above, one might think that the elements $\{T_{e^\lambda}\ \ \lambda \in R^\vee\}$ span the commutative subalgebra $\mathbb{Z}[R^\vee]$, but in fact the elements T_{e^λ} neither form a subalgebra, nor commute.

Example. Consider SL_2. We have $e^\alpha = s_0 s_1$, while $e^{-\alpha} = s_1 s_0$. Consequently

$$\begin{aligned} T_{s_0s_1}T_{s_1s_0} &= T_{s_0}\left(T_{s_1}T_{s_1}\right)T_{s_0} = T_{s_0}\left((q-1)T_{s_1} + q\right)T_{s_0} \\ &= (q-1)T_{s_0s_1s_0} + qT_{s_0}T_{s_0} = (q-1)T_{s_0s_1s_0} + q(q-1)T_{s_0} + q^2\,, \end{aligned}$$

while similarly

$$T_{s_1s_0}T_{s_0s_1} = (q-1)T_{s_1s_0s_1} + q(q-1)T_{s_1} + q^2\,.$$

This shows both that $T_{e^\alpha}T_{e^{-\alpha}} \neq T_{e^{-\alpha}}T_{e^\alpha}$, and $T_{e^\alpha}T_{e^{-\alpha}} \notin \mathrm{span}\{T_{e^\lambda}\,,\ \lambda \in R^\vee\}$.

However, from the observation that

$$\ell(e^\lambda) = \langle 2\rho, \lambda \rangle \,, \qquad \lambda \in R^\vee_+$$

where $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ is the Weyl vector, and $R^\vee_+ = R^\vee/W$ the set dominant coroots, it follows immediately that $\{T_{e^\lambda}\,,\ \lambda \in R^\vee_+\}$ is a commutative subalgebra of $\mathsf{W}^{\rm aff}$, i.e.

Corollary 2.3. *For $\lambda, \lambda' \in R^\vee_+$ we have*

$$\ell(e^\lambda) + \ell(e^{\lambda'}) = \ell(e^{\lambda+\lambda'}) \,,$$

and thus

$$T_{e^\lambda} T_{e^{\lambda'}} = T_{e^{\lambda+\lambda'}} = T_{e^{\lambda'}} T_{e^\lambda}$$

Now, for $\lambda \in R^\vee$ we choose $\lambda_1, \lambda_2 \in R^\vee_+$ such that $\lambda = \lambda_1 - \lambda_2$. Then we define

$$\Theta_\lambda \equiv T_{e^{\lambda_1}} {T_{e^{\lambda_2}}}^{-1} q^{-(\rho,\lambda)} \,. \tag{2.2}$$

It can be shown that this definition does not depend on the choice of λ_1, λ_2. Also note that $T_{e^{\lambda_1}}$ and $T_{e^{\lambda_2}}$, and hence $T_{e^{\lambda_1}}$ and ${T_{e^{\lambda_2}}}^{-1}$ commute. This also immediately implies that $\Theta_\lambda \Theta_{\lambda'} = \Theta_{\lambda'} \Theta_\lambda$, for all $\lambda, \lambda' \in R^\vee$.

The other crucial relation is

$$T_s \Theta_\lambda - \Theta_{s(\lambda)} T_s = (q-1) \frac{\Theta_\lambda - \Theta_{s(\lambda)}}{1 - \Theta_{-\alpha^\vee}} \,, \qquad s \in \mathsf{S},\ \lambda \in R^\vee \,. \tag{2.3}$$

In fact, using (2.3), we can extend $\mathcal{H}^{\rm aff}$ to a slightly bigger algebra by allowing $\lambda \in Y$, where Y is the cocharacter lattice of G. So, we define (cf. Theorem 2.2) $\widetilde{\mathcal{H}}^{\rm aff} = \mathcal{H}^{\rm fin} \otimes \mathbb{Z}[Y]$. I.e. $\widetilde{\mathcal{H}}^{\rm aff}$ is the Hecke algebra associated to the extended affine Weyl group $\widetilde{\mathsf{W}}^{\rm aff}$.

Note that it immediately follows that, for $s \in \mathsf{S}, \lambda \in Y$,

$$T_s(\Theta_\lambda + \Theta_{s(\lambda)}) = (\Theta_\lambda + \Theta_{s(\lambda)}) T_s \,. \tag{2.4}$$

It follows

Corollary 2.4. *For $\lambda \in Y$ we have*

$$c_\lambda \equiv \sum_{w \in \mathsf{W}^{\mathit{fin}}} \Theta_{w(\lambda)} \in \mathsf{Z}(\widetilde{\mathcal{H}}^{\mathit{aff}})$$

In fact, the center is generated by the elements c_λ

Theorem 2.5.

$$\mathsf{Z}(\widetilde{\mathcal{H}}^{\mathit{aff}}) \;\cong\; \langle c_\lambda \rangle_{\lambda \in Y_+} \,.$$

2.3. **Classical Satake correspondence.** Define $\mathcal{H}^{\rm sph} = \mathcal{H}(\mathsf{G}(\mathcal{K}), \mathsf{G}(\mathcal{O}))$.

Theorem 2.6 (Classical Satake correspondence)**.** *We have*

$$\mathcal{H}^{sph} \cong \mathbb{Z}[Y]^{\mathsf{W}} = \mathsf{Z}(\widetilde{\mathcal{H}}^{aff})\,.$$

$$\mathsf{G}(\mathcal{F}) \cong \coprod_{\lambda\in Y} \mathsf{G}(\mathcal{O})\ z^\lambda\ \mathsf{G}(\mathcal{O})$$

where we have denoted by z^λ the image of $z \in \mathbb{C}^\times$ in T under $\lambda \in Y$.

We will now show that $\mathcal{H}^{\rm sph}$ is commutative. This is based on a technique known as the Gelfan'd trick. The trick consists of constructing an anti-automorphism of $\mathsf{G}(\mathcal{F})$, which descends to $\mathcal{H}^{\rm sph}$, and acts as the identity on $\mathcal{H}^{\rm sph}$. First we observe that we have an involution ϑ of G, uniquely defined by the requirements

$$\vartheta\mathsf{B} \cap \mathsf{B} = \mathsf{T}\,, \qquad \vartheta(t) = t^{-1}\,, \quad t \in \mathsf{T}\,.$$

For instance, for $\mathsf{G} = \mathsf{SL}_n$, we take $\vartheta(x) = (x^{-1})^T$. We then use ϑ to define a map $\widetilde{\vartheta} : \mathsf{G}(\mathcal{F}) \to \mathsf{G}(\mathcal{F})$ by

$$\widetilde{\vartheta}(x) = \vartheta(x)^{-1}\,.$$

The map $\widetilde{\vartheta}$ has the following properties

(i) $\widetilde{\vartheta}$ is an anti-automorphism, i.e. $\widetilde{\vartheta}(xy) = \widetilde{\vartheta}(y)\widetilde{\vartheta}(x)$
(ii) $\widetilde{\vartheta}$ preserves $\mathsf{G}(\mathcal{O})$, i.e. $\widetilde{\vartheta}(\mathsf{G}(\mathcal{O})) \subset \mathsf{G}(\mathcal{O})$
(iii) $\widetilde{\vartheta}(z^\lambda) = z^\lambda$

Now, (ii) implies that $\widetilde{\vartheta}$ descends to an anti-automorphism $\widetilde{\vartheta}^*$ on $\mathcal{H}^{\rm sph}$, while (iii) implies that $\widetilde{\vartheta}^*(\chi_\lambda) = \chi_\lambda$ for all characters in $\mathcal{H}^{\rm sph}$, i.e. $\widetilde{\vartheta}^* = 1$ on $\mathcal{H}^{\rm sph}$. But then

$$fg = \widetilde{\vartheta}^*(fg) = \widetilde{\vartheta}^*(g)\widetilde{\vartheta}^*(f) = gf$$

for all $f, g \in \mathcal{H}^{\rm sph}$, hence $\mathcal{H}^{\rm sph}$ is commutative.

Lecture 3

3.1. **The Langlands dual group.** Let G be a reductive algebraic group over $\mathbb{C}$, B a Borel subgroup, and $\mathsf{T} \subset \mathsf{B}$ a maximal torus. We denote the corresponding Lie algebras by $\mathfrak{g}$, $\mathfrak{b}$ and $\mathfrak{t}$, respectively. We denote by

$$X = \mathrm{Hom}(\mathsf{T}, \mathbb{C}^\times) = \mathrm{char}_{\mathsf{G}}\,,$$
$$Y = \mathrm{Hom}(\mathbb{C}^\times, \mathsf{T}) = \mathrm{cochar}_{\mathsf{G}}\,,$$

the character and cocharacter lattices of G, respectively, and by

$$\Delta = \{\alpha \in X\backslash\{0\} \,|\, \mathfrak{g}_\alpha \neq 0\}\,,$$

the set of roots of G. For each root α we have a coroot $\alpha^\vee \in Y$ defined as follows: The root α defines an embedding $(\mathsf{SL}_2)_\alpha \to \mathsf{G}$, such that the maximal torus $\mathbb{C}^\times \subset (\mathsf{SL}_2)_\alpha$ maps to T, hence it defines a map $\alpha^\vee : \mathbb{C}^\times \to \mathsf{T}$. The set of coroots is defined as

$$\Delta^\vee = \{\alpha^\vee : \mathbb{C}^\times \to (\mathsf{SL}_2)_\alpha \to \mathsf{G} \,|\, \alpha \in \Delta\} \subset Y\,.$$

Let R and $R^\vee$ be the lattices generated by Δ and $\Delta^\vee$, respectively. The *root datum* of G is defined to be the quadruple $(X, Y, \Delta, \Delta^\vee)$. A classical theorem states that the group G can be reconstructed from its root datum. If, in addition, we denote the lattices dual to $R^\vee$ and R by P and $P^\vee$, respectively (the so-called *weight* and *coweight* lattices), then we have the following inclusions

$$\begin{array}{ccccc} R & \subset & X & \subset & P\,, \\ R^\vee & \subset & Y & \subset & P^\vee\,. \end{array}$$

Moreover, we have, $R = X$, $P^\vee = Y$ iff $Z(\mathsf{G}) = 0$, and $R^\vee = Y$, $P = X$ iff $\pi_1(\mathsf{G}) = 0$.

The crucial observation is that $(X, Y, \Delta, \Delta^\vee)$ is a root datum iff $(Y, X, \Delta^\vee, \Delta)$ is a root datum. This leads to

Definition 3.1. If G is a reductive group with root datum $(X, Y, \Delta, \Delta^\vee)$, then the *Langlands dual group* ${}^L\mathsf{G}$ is defined to be the reductive group corresponding to the root datum $(Y, X, \Delta^\vee, \Delta)$.

Example. For $\mathsf{G} = \mathsf{SL}_2(\mathbb{C})$, the elements in $\mathrm{Hom}(\mathsf{T}, \mathbb{C}^\times)$ are given by

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^n\,, \qquad n \in \mathbb{Z} \tag{3.1}$$

hence $X = \mathbb{Z}$, while the nonzero (positive) root α follows from the observation

$$\left(\mathrm{Ad}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right)\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & t^2 x \\ 0 & 0 \end{pmatrix} = t^2 \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\,, \tag{3.2}$$

hence

$$\alpha\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) = t^2 \,. \tag{3.3}$$

I.e. $\Delta = \{2\} \in \mathbb{Z}$. Similarly, the elements in $\mathrm{Hom}(\mathbb{C}^\times, \mathsf{T})$ are given by

$$t \mapsto \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix} \,, \qquad n \in \mathbb{Z} \,, \tag{3.4}$$

while the coroot $\alpha^\vee : \mathbb{C}^\times \to \mathsf{T}$ is clearly given by

$$\alpha^\vee(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \,, \tag{3.5}$$

hence $\Delta^\vee = \{1\} \in \mathbb{Z} = Y$. To summarize, the root datum for $\mathsf{SL}_2(\mathbb{C})$ is given by $(X, Y, \Delta, \Delta^\vee) = (\mathbb{Z}, \mathbb{Z}, \{2\}, \{1\})$.
On the other hand, for $\mathsf{G} = \mathsf{PSL}_2(\mathbb{C}) = \mathsf{SL}_2(\mathbb{C})/\mathbb{Z}_2$, i.e. matrices

$$\left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2 \;\middle|\; A \sim -A \right\}$$

the elements in $\mathrm{Hom}(\mathsf{T}, \mathbb{C}^\times)$ are given by

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^{2n} \,, \qquad n \in \mathbb{Z} \,, \tag{3.6}$$

such that A and $-A$ in T have the same image, while elements in $\mathrm{Hom}(\mathbb{C}^\times, \mathsf{T})$ are given by

$$t \mapsto \begin{pmatrix} t^{n/2} & 0 \\ 0 & t^{-n/2} \end{pmatrix} \,, \qquad n \in \mathbb{Z} \,. \tag{3.7}$$

Here we are allowed to take a square root since $A \sim -A$ in PSL_2. The expressions for the positive root and coroot are the same as before. Hence the root datum for $\mathsf{PSL}_2(\mathbb{C})$ is given by $(X, Y, \Delta, \Delta^\vee) = (\mathbb{Z}, \mathbb{Z}, \{1\}, \{2\})$. This shows that for $\mathsf{G} = \mathsf{SL}_2(\mathbb{C})$ we have ${}^L\mathsf{G} = \mathsf{PSL}_2(\mathbb{C})$.

The following table lists a few more examples of Langlands dual groups

G	${}^L\mathsf{G}$
GL_n	GL_n
SL_n	PSL_n
Spin_{2n}	$\mathsf{SO}_{2n}/\mathbb{Z}_2$
SO_{2n+1}	Sp_{2n}
E_8	E_8

3.2. **Geometric Satake correspondence.** If we denote by Rep G the abelian semigroup of finite dimensional representations of G, and by $K(\mathrm{Rep}\,\mathsf{G})$ the corresponding Grothendieck goup, the *representation ring of* G, then from the previous section it is clear that $\mathbb{Z}[Y]^{\mathsf{W}} \cong K(\mathrm{Rep}\,{}^L\mathsf{G})$. Hence we can reinterpret the classical Satake correspondence as the statement that

$$\mathcal{H}^{\mathrm{sph}} \;\cong\; K(\mathrm{Rep}\,{}^L\mathsf{G})\,. \tag{3.8}$$

This correspondence underlies many of the recent developments in the geometric Langlands program, as it states that somehow the Langlands dual group ${}^L\mathsf{G}$ acts on objects having to do with G (e.g. the moduli space of G-bundles, etc).

The *geometric Satake correspondence* is a 'categorification' of (3.8), i.e. a refinement of the statement at the level of objects and morphisms. It is clear how to categorify the right hand side of (3.8), we simply replace it by the category of (finite dimensional) representations of ${}^L\mathsf{G}$, i.e. $\underline{\mathrm{Rep}}{}^L\mathsf{G}$. We note that $\underline{\mathrm{Rep}}{}^L\mathsf{G}$ is a so-called *Tannakian category* (i.e. an abelian tensor category with additional structure).

The categorification of the left hand side is more subtle. One may ask what the geometric significance is of the function $\chi_\lambda \in \mathcal{H}^{\mathrm{sph}}$, $\lambda \in X$, which maps under (3.8) to ch_{L_λ}, the character of the irreducible representation with highest weight λ. It turns out, as discovered by Lusztig, that $\chi_\lambda = \chi_{K_\lambda}$, the characteristic function of the irreducible D-module (or, equivalently, perverse sheaf) attached to the orbit $K_\lambda = \mathsf{G}(\mathcal{O})z^\lambda\mathsf{G}(\mathcal{O})$. More generally, if we define $\mathcal{P}_{\mathsf{G}}$ to be the abelian category of $\mathsf{G}(\mathcal{O})$-equivariant D-modules (perverse sheaves) on the loop Grassmannian $\mathrm{Gr}_{\mathsf{G}} = \mathsf{G}(\mathcal{F})/\mathsf{G}(\mathcal{O})$, then the geometric Satake correspondence is the statement

Theorem 3.2 (Geometric Satake correspondence)**.** *We have an equivalence of categories*

$$\mathcal{P}_{\mathsf{G}} \;\cong\; \underline{\mathrm{Rep}}{}^L\mathsf{G}\,.$$

The proof of this theorem is based on the fact that $\mathcal{P}_{\mathsf{G}}$ is a semi-simple Tannakian category, which admits fibre functors. This implies that $\mathcal{P}_{\mathsf{G}}$ is equivalent to the category of representations of a reductive group H. Finally one shows that $\mathsf{H} = {}^L\mathsf{G}$. Note, in fact, that this gives us an alternative, intrinsic, definition of the Langlands dual group, i.e. not using character lattices and root/weight systems.

To prove, or even explain all ingredients in, Theorem 3.2 is beyond the scope of these lectures. Instead we will just give a brief introduction into D-modules and their relevance to representation theory (i.e. the Beilinson-Bernstein theorem).

3.3. **D-modules.** let X be an algebraic variety (or, in the analytic context, a manifold) and let $\mathcal{D}_X$ be the sheaf of rings of differential operators on X. $\mathcal{D}_X$ is an associative algebra generated by (algebraic) functions $\mathcal{O}_X$ and vector fields Θ_X, subject to the (local) relations

$$\begin{aligned} \xi f - f\xi &= \mathcal{L}_\xi f\,, \qquad f\in\mathcal{O}_X,\ \xi\in\Theta_X\,,\\ \xi_1\xi_2-\xi_2\xi_1 &= [\xi_1,\xi_2]\,, \qquad \xi_1,\xi_2\in\Theta_X\,, \end{aligned} \tag{3.9}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative and $[\ ,\ ]$ the Lie bracket of vector fields. We can think of $\mathcal{D}_X$ as the universal enveloping algebra of Θ_X.

As an example, for the affine line $\mathbb{A}^1$, we have $\mathcal{D}_{\mathbb{A}^1} = \mathbb{C}\langle x,\partial_x\rangle/(x\partial_x-\partial_x x=1)$, i.e. the Weyl algebra.

Definition 3.3. A *D-module* is a sheaf of modules over the sheaf of rings $\mathcal{D}_X$.

Note that there exists an equivalence between (finite rank) algebraic vector bundles over X and (locally free) $\mathcal{O}_X$-modules. In that sense a D-module is just a generalization of a vector bundle.

3.4. **Beilinson-Bernstein localization.** Suppose we are given a variety X with an action of G. By considering the infinitesimal action of G on $U\subset X$ we obtain a map $\mathfrak{g}\to\Theta_X$. Consequently, we have a map

$$\Gamma\ :\ U(\mathfrak{g})\to\Gamma(X,\mathcal{D}_X)\,,$$

where $\Gamma(X,\mathcal{D}_X)$ denotes the sheaf of sections of $\mathcal{D}_X$. Thus, by pull-back

$$\Gamma^*\ :\ \mathrm{Mod}(\mathcal{D}_X)\to\mathrm{Mod}(U(\mathfrak{g}))\,.$$

Beilinson-Bernstein localization is the existence of a map in the opposite direction

$$\Delta\ :\ \mathrm{Mod}(U(\mathfrak{g}))\to\mathrm{Mod}(\mathcal{D}_X)\,.$$

defined by

$$\Delta M = \mathcal{D}_X\otimes_{U(\mathfrak{g})} M\,.$$

[Cf. the associated vector bundle construction, i.e. for a principal G-bundle P, and a G-module V, one can construct the associated vector bundle $P\times_{\mathsf{G}} V$.]
Interesting results are obtained by taking suitable spaces X, e.g. by taking $X=\mathsf{G}/\mathsf{B}$. In this case we find a 1–1 correspondence (or, more precisely, an equivalence of categories) between certain $U(\mathfrak{g})$-modules and D-modules. In order to get a general class of $U(\mathfrak{g})$-modules we need to slightly generalize the construction above, i.e. we need to twist the differential operators by a G-equivariant line bundle.

An algebraic vector bundle $E\to X$ is an *equivariant vector bundle* if we are given an action of the algebraic group G on the algebraic variety E, satisfying $g(E_x)=E_{gx}$ for $x\in X$, $g\in\mathsf{G}$, and the map $g: E_x\to E_{gx}$ is a linear isomorphism. Recall that G-equivariant vector bundles E over $X=\mathsf{G}/\mathsf{B}$ are in 1–1 correspondence with (finite dimensional) modules V of B. It is clear that the fiber E_B at $B\in\mathsf{G}/\mathsf{B}$ is a B-module. [Recall that the set X of

Borel subgroups of G is in bijection with G/B, namely, G acts on X by conjugation and the stabilizer of an element $B \in X$ in G is B itself.] Conversely, for any B-module V we can construct a G-invariant vector bundle E on X such that $E_B = V$ through the associated vector bundle construction $E = \mathsf{G} \times_{\mathsf{B}} V$, i.e. by quotienting $\mathsf{G} \times V$ by the action of B given by $b \cdot (g, v) = (gb^{-1}, b \cdot v)$. Clearly, E carries an action of G by $g' \cdot (g, v) = (g'g, v)$.

In particular, G-equivariant line bundles are in 1–1 correspondence to 1-dimensional B-modules. Since the action of the unipotent radical $\mathsf{N} = R_u(\mathsf{B})$ of B is necessarily trivial, 1-dimensional B-modules correspond to characters $\lambda \in X = \mathrm{Hom}(\mathsf{T}, \mathbb{C}^\times)$ of $\mathsf{T} = \mathsf{B}/\mathsf{N}$. Denote the line-bundle over G/B, corresponding to $\lambda \in X$, by $\mathcal{L}(\lambda)$, and let $\mathrm{Mod}(\mathcal{D}^\lambda_{\mathsf{G}/\mathsf{B}}) \equiv \mathrm{Mod}(\mathcal{D}_{\mathcal{L}(\lambda)})$ denote the category of *twisted D-modules.*

On the other hand we let $\mathrm{Mod}(U(\mathfrak{g}))_\chi$ denote the abelian category of $U(\mathfrak{g})$-modules with central character χ, i.e. $U(\mathfrak{g})$-modules modules M such that

$$z \cdot m = \chi(z) m\,, \qquad \forall z \in \mathsf{Z}(U(\mathfrak{g})), m \in M\,. \tag{3.10}$$

Because of the Harish-Chandra homomorphism $\mathsf{Z}(U(\mathfrak{g})) \to U(\mathfrak{h})$, central characters are in 1–1 correspondence with $\lambda \in \mathfrak{h}^*$. We denote by χ_λ the character corresponding to $\lambda \in \mathfrak{h}^*$.

Without elaborating on exactly what kind of sheaves or $U(\mathfrak{g})$-modules we allow, the main theorem in this context is, loosely speaking,

Theorem 3.4 (Beilinson-Bernstein)**.** *The map Δ provides an equivalence of categories*

$$\mathrm{Mod}(U(\mathfrak{g}))_{\chi_\lambda} \;\cong\; \mathrm{Mod}(\mathcal{D}^\lambda_{\mathsf{G}/\mathsf{B}})\,.$$

We illustrate the above in an example.

Example. $\mathsf{SL}_2(\mathbb{C})$:

As we have seen before, for $\mathsf{G} = \mathsf{SL}_2(\mathbb{C})$, we can identify $\mathsf{G}/\mathsf{B} = \mathbb{C}P^1$. Explicitly

$$\mathsf{G}/\mathsf{B} = \left\{ \begin{pmatrix} z_1 & \star \\ z_2 & \star \end{pmatrix} \mid (z_1, z_2) \sim (az_1, az_2),\ a \in \mathbb{C}^\times \right\}.$$

We can cover G/B by two open sets U_1 and U_2, with local coordinates z and x, respectively, i.e.

$$U_1 = \{[z_1, z_2] \in \mathbb{C}P^1 \,|\, z_2 \neq 0\}\,, \qquad z = \frac{z_1}{z_2}\,,$$
$$U_2 = \{[z_1, z_2] \in \mathbb{C}P^1 \,|\, z_1 \neq 0\}\,, \qquad z = \frac{z_2}{z_1}$$

such that on the overlap $U_1 \cap U_2$, we have

$$z = \frac{1}{x}\,.$$

The generating vectorfield on U_1 and U_2 are obviously given by d/d and d/dx, respectively. I.e. we have

$$\mathcal{D}_{\mathsf{G/B}}(U_1) = \langle z, \frac{d}{dz}\rangle / (z\frac{d}{dz} - \frac{d}{dz}z = -1)\,,$$

and similarly for U_2, such that on the overlap $U_1 \cap U_2$

$$\frac{d}{dz} = -x^2 \frac{d}{dx}\,.$$

The action of G on $\mathsf{G/B}$ follows from

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 & \star \\ z_2 & \star \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 & \star \\ cz_1 + dz_2 & \star \end{pmatrix}\,. \tag{3.11}$$

I.e. on U_1 we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}\,, \tag{3.12}$$

while on U_2

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{cx+d}{ax+b}\,. \tag{3.13}$$

The infinitesimal action on functions on $\mathsf{G/B}$ is given by

$$x \cdot f(z) = \frac{d}{dt} f(\exp(-tx)\cdot z)\Big|_{t=0}\,, \qquad \text{for } x \in \mathfrak{g}\,, \tag{3.14}$$

such that, for the generators of $\mathfrak{g}$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,, \tag{3.15}$$

the map $\Gamma : U(\mathfrak{g}) \to \mathcal{D}_{\mathsf{G/B}}(U_i)$, on U_1 and U_2, respectively, is given by

$$\begin{array}{lll} e \mapsto & -\dfrac{d}{dz} & x^2\dfrac{d}{dx}\,, \\ h \mapsto & -2z\dfrac{d}{dz} & 2x\dfrac{d}{dx}\,, \\ f \mapsto & z^2\dfrac{d}{dz} & -\dfrac{d}{dx}\,. \end{array} \tag{3.16}$$

To twist this construction by an equivariant line bundle, recall that equivariant line bundles $\mathcal{L}(n)$ over $\mathsf{G/B} = \mathbb{C}P^1$ are parametrized by $n \in \mathbb{Z} = \mathrm{Hom}(\mathsf{T}, \mathbb{C}^\times)$. Explicitly, the representation $\rho_n : \mathsf{B} \to \mathbb{C}^\times$ is given by

$$\rho_n\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = a^n\,.$$

Accordingly, in order to construct $\mathcal{L}(n)$, we glue $U_1 \times \mathbb{C}$ and $U_2 \times \mathbb{C}$, by identifying

$$(z,u) \sim (x,v)\,, \qquad x = \frac{1}{z},\ v = z^n u\,, \qquad \text{on } (U_1 \cup U_2) \times \mathbb{C}\,,$$

while the action of G on $U_1 \times \mathbb{C} \subset \mathcal{L}(n)$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z,u) = (\frac{az+b}{cz+d}, (cz+d)^n u)\,, \tag{3.17}$$

and similarly on $U_2 \times \mathbb{C}$. This results in the following map $\Gamma : U(\mathfrak{g}) \to \mathcal{D}_{\mathsf{G}/\mathsf{B}}(U_i)$, for U_1 and U_2, respectively, defined on generators by

$$\begin{aligned} e &\mapsto \quad -\frac{d}{dz} && x^2\frac{d}{dx} + nx\,, \\ h &\mapsto \quad -2z\frac{d}{dz} - n && 2x\frac{d}{dx} + n\,, \\ f &\mapsto \quad z^2\frac{d}{dz} + nz && -\frac{d}{dx}\,. \end{aligned} \tag{3.18}$$

References

[Ber] J. Bernstein, *Algebraic theory of D-modules*, Lecture Notes. Available from http://www.math.uchicago.edu/~mitya/langlands.html

[CG] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, (Birkhäuser, New York, 1997).

[Gai] D. Gaitsgory, *Geometric representation theory*, Lecture Notes, Fall 2005, Available from http://www.math.harvard.edu/~gaitsgde/267y/cat0.pdf

[Gro] I. Grojnowski, Lectures at "Introduction to the Geometric Langlands Program", Australian National University, July 16 - August 3, 2007. Unpublished.

[HTT] R. Hotta, K. Takeuchi and T. Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics **236** (Birkhäuser, Boston, 2008).

[PS] A. Pressley and G. Segal, *Loop groups*, (Oxford University Press, 1987)

Department of Theoretical Physics, Research School of Physical Sciences and Engineering, and Department of Mathematics, Mathematical Sciences Institute, The Australian National University, Canberra, ACT 0200, Australia

E-mail address: peter.bouwknegt@anu.edu.au