PRICING OPTIONS UNDER JUMP-DIFFUSION PROCESSES

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Abstract

This paper derives the appropriate characterization of asset market equilibrium when asset prices follow jump-diffusion processes, and develops the general methodology for pricing options on such assets. Specific restrictions on distributions and preferences are imposed, yielding a tractable option pricing model that is valid even when jump risk is systematic and non-diversifiable. The dynamic hedging strategies justifying the option pricing model are described. Comparisons are made throughout the paper to the analogous problem of pricing options under stochastic volatility.

It is a truism that asset pricing models in general require restrictions on preferences and on technologies or distributions in order to derive characterizations of equilibrium. The option pricing methodology developed by Black and Scholes (1973) is unique in that distributional assumptions alone suffice to generate well-specified option pricing formulas involving mostly observable variables and parameters. The key distributional assumption is that the price of the asset on which the option is written follows a diffusion, the instantaneous variance of which depends at most upon the asset price and time. An additional distributional assumption of perhaps secondary importance for the short option maturities typically considered is that the instantaneous risk-free rate is nonstochastic and constant. Under these plus other "frictionless market" assumptions, the option’s payoff can be replicated by a continuously-adjusted hedge portfolio composed of the underlying asset and short-term bonds. Preclusion of riskless arbitrage opportunities then yields a fairly simple partial differential equation, which when solved gives an option price that does not depend upon attitudes towards risk.

The simple preference-free option pricing formula generated by the Black-Scholes methodology depends critically upon the distributional restriction on the volatility of the underlying asset. The result of that restriction is that the systematic risk of the the option is a function of the systematic risk of the underlying asset only; and the implicit price of that risk embodied in the market price and expected excess return of the underlying asset obviates the need to price that risk by imposing other restrictions.

The plausibility of that distributional restriction has been challenged, in two directions. First, the covariance nonstationarity of stock and foreign currency returns is too pronounced to be explained by the limited covariance nonstationarity of constant elasticity of variance models. The variance itself would appear to be a relevant additional state variable and source of risk affecting option prices. Second, the recent stock market crash is evidence that diffusions can be an inadequate characterization of asset price movements; processes allowing for jumps would appear to be more appropriate.

Unfortunately, stochastic volatility and jump risk option pricing models introduce forms of risk embodied in option prices that are not directly priced by any instrument currently traded in financial markets. The result is that the Black-Scholes arbitrage-based methodology cannot be used; one must again impose preference- and technology-based restrictions in order to price those forms of risk, and consequently to price options. This paper uses a Cox, Ingersoll, and Ross (1985a) general equilibrium framework, generalized to jump-diffusion processes, to examine how options will be priced under jump risk. Comparisons are made throughout the paper to the analogous problem of pricing options under stochastic volatility.

The paper is structured as follows. Section 1 sets up the framework and derives characterizations of general asset market equilibrium under jump-diffusion processes. Section 2 derives the resulting restrictions on options, and specifies an associated system of "risk-neutral" jump-diffusions that can be used to evaluate option prices given any option-specific boundary conditions. Section 3 examines what restrictions on preferences and distributions are useful in simplifying the calculation of option prices. By assuming time-separable power or log utility functions for the representative consumer/investor, and by imposing distributional restrictions on the stochastic evolution of wealth and of the underlying asset price, a Merton (1976)-type option pricing formula is derived for jump-diffusion with random jump amplitudes that is valid even when jump risk is systematic and nondiversifiable. Section 4 discusses arbitrage-based justifications of option pricing models for stochastic volatility and jump risk. The section cautions that while one can construct an arbitrage strategy *justifying* given option pricing formulas (by adding other options to the replicating portfolio for each additional source of risk), one cannot *derive* option pricing models from such arbitrage strategies without implicit restrictions on preferences and on technologies.

1. The Economy

The economy is assumed identical to the one described by Cox, Ingersoll, and Ross (1985a), except that the production processes are augmented by a jump process.[1] There are a large number of infinitely-lived consumers, with identical preferences, endowments, and information sets. Each consumer seeks to maximize a lifetime expected utility function of the form

$$E_0 \int_0^\infty e^{-\rho t}\, U(C_t,\ Y_t)\, dt \tag{1}$$

subject to initial wealth W_0 and initial underlying state of the economy Y_0, where Y is a $K \times 1$ vector. $U(C, \bullet)$ is assumed strictly concave in the consumption flow C, with $U_c(0, \bullet) = \infty$.

There are N investment opportunities available to every investor. The return on each investment follows a state-dependent jump-diffusion with random jumps:

$$dP_i / P_i \;=\; [\alpha_i - \lambda E_Y(k_i)]\, dt + g_i(Y)\, dZ + k_i(Y)\, dq \tag{2}$$

where

$\alpha_i(Y)$ is the state-dependent instantaneous expected return on the process,

Z_t is an $(N+K) \times 1$ vector of independent standard Wiener processes,

$g_i(Y)$ is a $1 \times (N+K)$ state-dependent vector reflecting the sensitivity of returns to the various shocks,

q_t is a Poisson counter with intensity λ (i.e., $Prob(dq = 1) = \lambda dt$), and

[1]For a closely related discussion of discount bond pricing under jump-diffusion processes, see Ahn and Thompson (1987).

$k_i(Y)$ is the random percentage jump amplitude conditional on the Poisson-distributed event occurring.

The state-dependent distribution of k_i is bounded below by -1 because of limited liability, and has mean $E_Y(k_i)$ conditional on state Y. Further restrictions on the distribution will be imposed below as needed.

The instantaneous return dP_i/P_i can be viewed either as the percentage change in an asset price (Merton (1971)), or as the return from putting a unit of a consumption/investment good into a stochastic constant returns to scale production process (Cox et al. (1985a)). Expression (2) gives the stochastic evolution of the asset price (or of a continuously-reinvested unit of the good) as continuous for the most part but subject to discrete, random jumps at random times. The other investment opportunities are analogous, and can be compactly summarized as

$$I_P^{-1} dP = [\alpha(Y) - \lambda E_Y(k)]\, dt + G\, dZ + k\, dq, \qquad \textbf{(3)}$$

where I_P^{-1} is an $N \times N$ diagonal matrix with ith diagonal element P_i^{-1}; k and $E_Y(k)$ are $N \times 1$ vectors; and G is an $N \times (N+K)$ matrix with GG', the covariance matrix per unit time conditional on no jumps, assumed to be positive definite. The underlying variables Y are assumed to evolve similarly:

$$dY = [\mu(Y) - \lambda E_Y(\Delta Y)]\, dt + \sigma_Y(Y)\, dZ + \Delta Y\, dq, \qquad \textbf{(4)}$$

where $\mu(Y)$ is the state-dependent drift in Y, $\sigma_Y(Y)$ is a $K \times (N+K)$ matrix giving the state-dependent sensitivities of the underlying state variables to the various shocks, and ΔY is a $K \times 1$ vector of random increments to the state variables conditional on the Poisson-distributed event occurring. The Poisson shock is assumed to affect all investment opportunities and underlying state variables simultaneously, and is therefore systematic, and nondiversifiable risk.[2]

[2]Independent or semi-independent Poisson shocks (e.g., Merton's (1976) firm-specific jump risk model) could be represented by adding additional independent Poisson counters $q_2, q_3, \ldots$, and zeroing out specific jump amplitudes. This would complicate but not alter fundamentally the analysis below.

In addition to the investment opportunities given by (3), consumers are assumed to have free access to a market for interpersonal, riskless instantaneous borrowing and lending, and free access to markets for contingent claims. The return on the former is given by the endogenously determined spot interest rate r_t; the return on a non-dividend paying contingent claim can be written as

$$dF/F = [\beta - \lambda E_{W,Y}(k_F)]\,dt + h\,dZ + k_F\,dq\,. \tag{5}$$

The state-dependent $1 \times (N + K)$ matrix h and random percentage jump k_F will be determined below using the jump-diffusion version of Ito's lemma. The required expected return β will be determined endogenously. Without loss of generality, it is assumed at present that there is only one contingent claim.

At each instant, each consumer chooses a consumption flow and an investment strategy to maximize expected utility over the consumer's remaining lifetime given state of the economy $[W_t \; Y_t']$. Defining $a(\tau)$ as the $N \times 1$ vector of shares of wealth invested in the N production activities at time τ, $b(\tau)$ as the share of wealth invested in contingent claims, and $(1 - 1'a - b)$ as the share of wealth invested in risk-free lending, the consumer's problem can be stated as

$$\begin{aligned} &\max_{\{C,a,b\}} \; E_t \int_t^\infty e^{-\rho(\tau - t)}\, U(C_\tau, Y_\tau)\, d\tau \quad subject\ to \\ &dW/W = [\alpha_W - \lambda E_Y(k_W)]\,dt + \sigma_W\,dZ + k_W\,dq \\ &dY = [\mu(Y) - \lambda E_Y(\Delta Y)]\,dt + \sigma_Y\,dZ + \Delta Y\,dq \end{aligned} \tag{6}$$

and given initial state $[W_t \; Y_t']$, where

$$\alpha_W = r + a'[\alpha - \lambda E_Y(k) - r1] + b[\beta - \lambda E_{W,Y}(k_F) - r] - C/W \tag{7}$$

$$\sigma_W = a'G + bh, \; and \tag{8}$$

$$k_W = a'k + bk_F\,. \tag{9}$$

Under some regularity conditions that are beyond the scope of this paper,[3] the consumer's optimal consumption and investment strategies are given implicitly by the feedback strategy

$$v(W,Y) = [C(W,Y) \;\; a(W,Y)' \;\; b(W,Y)]$$

and the indirect utility function

$$J(W,Y) = E_t \int_t^{\infty} e^{-\rho(\tau - t)} \, U(C_\tau, \; Y_\tau) \, d\tau \tag{10}$$

that satisfy the Bellman equation

$$\max_v \{U(C,Y) + L^v(J)\} - \rho J = 0 . \tag{11}$$

$L^v(\bullet)$, the differential generator for jump-diffusions associated with control v, is given by

$$\begin{aligned} L^v(J) &= [\alpha_W - \lambda E_Y(k_W)] W J_W + [\mu - \lambda E_Y(\Delta Y)] J_Y \\ &+ \tfrac{1}{2}[W^2 \sigma_W \sigma_W' J_{WW} + 2W\sigma_W \sigma_Y' J_{WY} + tr(\sigma_Y \sigma_Y' J_{YY'})] \\ &+ \lambda E_{W,Y}[J(W + k_W W, \; Y + \Delta Y) - J(W,Y)] \end{aligned} \tag{12}$$

where α_W, k_W and σ_W, given by (7) - (9), are functions of the control v. The differential generator is identical to the one given by Cox *et al* for diffusions, except for the addition of the final, jump-related term in brackets.[4]

Under the assumption that negative investments in the production processes are not possible, necessary first-order conditions for the consumption and investment strategies that maximize

$\psi(v) \equiv U(C,Y) + L^v(J)$ are

$$\begin{aligned} \psi_C &= U_C - J_W = 0 \\ \psi_a &= [\alpha - \lambda E_Y(k) - r1] W J_W + (GG'a + Gh'b) W^2 J_{WW} + WG\sigma_Y' J_{WY} + \lambda E_{W,Y}(J_W^* k) \le 0 \\ a_i \psi_{a_i} &= 0 \;\; for \;\; i = 1, \ldots, N \\ \psi_b &= [\beta - \lambda E_{W,Y}(k_F) - r] W J_W + (hG'a + hh'b) W^2 J_{WW} + WhS' J_{WY} + \lambda E_{W,Y}(J_W^* k_F) = 0 \end{aligned} \tag{13}$$

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[3]See Cox *et al* (1985a) and the references therein for a more formal treatment.

[4]For further discussion of this term, see Merton (1971).

where $J_W^* \equiv J_W[W(1 + a'k + bk_F),\ Y + \Delta Y]$ is the marginal indirect utility of wealth conditional on the Poisson-distributed event occurring. As a function of the random variables k, k_F, and ΔY, J_W^* is itself random.

Since all consumers are identical, the equilibrium in the economy must be characterized by no trades between consumers. Therefore, expected returns on lending and on contingent claims must be *such that*

$$1'a^* = 1 \qquad \textbf{(14)}$$

$$b^* = 0 \qquad \textbf{(15)}$$

where a^* and b^* are the optimal shares of wealth invested in production processes and in contingent claims, respectively. These conditions, along with (13), given the fundamental characterization of equilibrium when jumps are present.

Theorem 1: The equilibrium expected return on any investment satisfies

$$\begin{aligned}\alpha_S - r &= -E_{W,Y}[(dJ_W/J_W)(dS/S)] \\ &= R(W,Y)\,Cov_{NJ}(dW/W,\ dS/S) - J_{WY'}\,Cov_{NJ}(dY, dS/S) \\ &\quad -\lambda E_{W,Y}[(\Delta J_W/J_W)k_S]\end{aligned} \qquad \textbf{(16)}$$

where the equilibrium spot interest rate r satisfies

$$\begin{aligned}r &= (a^*)'\alpha - R(W,Y)\,Var_{NJ}(dW/W) + (J_{WY'}/J_W)\,Cov_{NJ}(dY, dW/W) \\ &\quad + \lambda E[(\Delta J_W/J_W)k_W]\ ,\end{aligned} \qquad \textbf{(17)}$$

and

$R(W,Y) \equiv -WJ_{WW}/J_W$ is the coefficient of relative risk aversion,

$k_S = \Delta S/S$ is the random percentage change in the asset price conditional on a jump,

$\Delta J_W/J_W = [J_W(W + k_W W,\ Y + \Delta Y, t) - J_W(W, Y, t)]/J_W(W, Y, t)$ is the random percentage change in marginal utility of optimally invested wealth conditional on a jump taking place $(k_W = (a^*)'k)$, and

$Cov_{NJ}(dW/W,\ \bullet)$ is the covariance per unit time with optimally invested wealth conditional on no jumps.

Expression (16) holds for investments in production processes $(S = P_i,\ \alpha_S = \alpha_i)$, for investments in contingent claims $(S = F,\ \alpha_S = \beta)$, and for any portfolio of investments (e.g., the market portfolio: $\alpha_S = (a^*)'\alpha = \alpha_W + C/W$).

Theorem 1 gives the appropriate form of the capital asset pricing model for jump-diffusions. Excess returns on any investment are generated by the security's content of the various forms of systematic risk:

1) market risk conditional on no jumps, priced at relative risk aversion R(W, Y);

2) technological risk (i.e., risk of shifts in the investment opportunity set) conditional on no jumps, priced at $-J_{WY}/J_W$ and

3) jump risk, which includes both market and technological risk.

To a first-order approximation, jump risk can be decomposed into market and technological risk and priced accordingly:

$$\begin{aligned}\alpha_S - r \approx\ & R(W,Y)[Cov_{NJ}(dW/W,\ dS/S) + \lambda E_{W,Y}(k_W\, k_S)] \\ & - (J_{WY'}/J_W)\{Cov_{NJ}(dY, dS/S) + \lambda E_{W,Y}[(\Delta Y)k_S]\} \\ \approx\ & R(W,Y)E_{W,Y}[(dW/W)(dS/S1)] - (J_{WY'}/J_W)E_{W,Y}[(dY)(dS/S)]\end{aligned} \qquad \textbf{(18)}$$

from a first order Taylor expansion of $\Delta J_W/J_W$. The accuracy of this decomposition will of course depend upon the joint distribution of k_W and ΔY and the global curvature of $J_W(\bullet)$.

2. Option Evaluation

Theorem I specifies a relationship between the expected return on any asset and its content of systematic risk. In particular, the theorem restricts the expected return and systematic risk of an option on any underlying asset when the asset's price follows a jump-diffusion with random jump amplitudes:

$$dS/S = [\alpha_S(Y) - \lambda E_Y(k_S)]\,dt + \sigma_S\,dZ + k_S(Y)\,dq \;. \tag{19}$$

Options, as derivative securities, have prices that can be written in the general format $F = F(S, W, Y, t)$. One can use Ito's lemma to rewrite the option's systematic risk in terms of variances and covariances of underlying state variables. A version of Theorem 1 relevant for options (and for derivative securities in general (then follows).

Theorem 1' The equilibrium expected return on an option satisfies[5]

$$(\beta - r)F = [F_S \; F_W \; F_{Y'}]\,[\Phi_S \; \Phi_W \; \Phi_{Y'}]' - \lambda\, E_{W,Y}[(\Delta J_W / J_W)(\Delta F)] \tag{20}$$

where

$$\begin{aligned}
\Phi_S &\equiv -(J_{WW}/J_W)\,Cov_{NJ}(dW, dS) - (J_{WY'}/J_W)\,Cov_{NJ}(dY, dS) = -Cov_{NJ}(dJ_W/J_W, dS) \\
\Phi_W &\equiv -(J_{WW}/J_W)\,Var_{NJ}(dW) - (J_{WY'}/J_W)\,Cov_{NJ}(dY, dW) = -Cov_{NJ}(dJ_W/J_W, dW) \\
\Phi_{Y'} &\equiv -(J_{WW}/J_W)\,Cov_{NJ}(dW, dY') - (J_{WY'}/J_W)\,Cov_{NJ}(dY, dY') = -Cov_{NJ}(dJ_W/J_W, dY') \\
\Delta F &= F(W + k_W W,\; Y + \Delta Y,\; t) - F.
\end{aligned}$$

Ito's lemma also restricts the expected drift in the price of the option:

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[5]The theorem follows directly from the expansion

$$\begin{aligned}
Cov_{NJ}(dW, dF/F) &= F^{-1} Cov_{NJ}(dW,\; F_S\,dS + F_W\,dW + F_{Y'}\,dY \\
&= F^{-1}[F_S Cov_{NJ}(dW, dS) + F_W Var_{NJ}(dW) + F_{Y'} Cov_{NJ}(dY, dW)],
\end{aligned}$$

using a similar expansion for $Cov_{NJ}(dF/F, dY)$, and rearranging terms.

$$
\begin{aligned}
E_{W,Y}(dF) \equiv \beta F &= F_t + L(F) \\
&= F_t + [\alpha_S - \lambda E_{W,Y}(k_S)]SF_S \\
&\quad + [\alpha_W - \lambda E_{W,Y}(k_W)]WF_W + [\mu - \lambda E_Y(\Delta Y)]' F_Y \\
&\quad + \tfrac{1}{2}\Sigma_{S\,W\,Y'}(F) + \lambda E_{W,Y}(\Delta F)\,,
\end{aligned} \tag{21}
$$

where

$$
\Sigma_{V'}(F) \equiv \frac{\partial^2 F}{\partial V \partial V'} \bullet Cov_{NJ}(dV, dV'),\; V' \equiv [S\;\; W\;\; Y']\;. \tag{22}
$$

$\Sigma_{[S\,W\,Y']}(F)$, the dot product of 1) the matrix of second partial derivatives of F with regard to $[S\;\; W\;\; Y']$ and 2) the covariance matrix per unit time conditional on no jumps of $[dS\;\; dW\;\; dY']'$, is the collection of second-order terms from Ito's lemma.

Expression (20), in conjunction with restriction (16) on ß and on $(a^*)'\alpha = \alpha_W + C/W$, yields after some manipulation the fundamental valuation equation for options under jump-diffusions.

Theorem 2: The price of any option $F(S, W, Y, t)$ satisfies the partial differential equation

$$
\begin{aligned}
&F_t + \{r - \lambda E_{W,Y}[(J_W^*/J_W)k_S]\}SF_S \\
&\quad + \{r - \lambda E_{W,Y}[(J_W^*/J_W)k_W] - C^*(W,Y)/W\}WF_W \\
&\quad + [\mu - \lambda E_Y(\Delta Y) - \Phi_{Y'}]' F_Y \\
&\quad + \tfrac{1}{2}\Sigma_{S\,W\,Y'}(F) + \lambda E_{W,Y}[(J_W^*/J_W)(\Delta F)] = rF
\end{aligned} \tag{23}
$$

where $r(W, Y)$ is given by

$$
r(W,Y) = (a^*)'\alpha - \Phi_W(W,Y)/W + \lambda E_{W,Y}[(\Delta J_W/J_W)k_W] \tag{24}
$$

and Φ_Y is the transpose of $\Phi_{Y'}$.

The valuation equation (23) holds for any option, European or American. Evaluating a particular option involves solving (23) subject to terminal boundary value conditions for European options (terminal value plus early-exercise boundary conditions for American options), and conditional on the current state of the economy $[S \;\; W \;\; Y']$.

Cox *et al* (1985a) show that for diffusions, there is an associated system of *risk-neutral diffusions***,** the use of which greatly facilitates evaluation of contingent claims. The same is true for jump-diffusions.

Theorem 3: Options are priced *as if* investors were risk-neutral and $[S \;\; W \;\; Y']$ followed the jump-diffusion process

$$\begin{aligned} dS/S &= [r - \lambda^* E^*_{W,Y}(k^*_S)]\,dt + \sigma_S\, dZ + k^*_S\, dq^* \\ dW/W &= [r - \lambda^* E^*_{W,Y}(k^*_W) - C^*(W,Y)/W]\,dt + \sigma_W\, dZ + k^*_W\, dq^* \\ dY &= [\mu(Y) - \lambda E_Y(\Delta Y) - \Phi_Y(W,Y)]\,dt + \sigma_Y\, dZ + (\Delta Y^*)\, dq^* \end{aligned} \tag{25}$$

where jumps occur with frequency

$$\lambda^*(W,Y) = \lambda\, E_{W,Y}[J_W(W + k_W W,\; Y + \Delta Y)]/J_W \quad (i.e.,\; Prob[dq^* = 1] = \lambda^* dt), \tag{26}$$

$r(W, Y)$ is given by (24) above,

E_Y denotes the expectations operator conditional on state Y relative to the *true* joint probability density function of $[k_S \;\; k_W \;\; \Delta Y']$ (denoted $f(k_S, k_W, \Delta Y \mid Y)$), and

$E^*_{W,Y}$ denotes the expectations operator conditional on state $[W \;\; Y']$ relative to the marginal utility-weighted joint p.d.f. of $[k_S \;\; k_W \;\; \Delta Y']$:

$$f^*(k^*_S,\; k^*_W,\; \Delta Y^* \mid W,\; Y) = \frac{J_W(W + k_W W, Y + \Delta Y)}{E_{W,Y}[J_W(W + k_W W, Y + \Delta Y)]}\, f(k_S,\; k_W,\; \Delta Y \mid Y)\;. \tag{27}$$

Proof of the theorem follows directly from taking $E^*_{W,Y}(dF) = F_t + L^*(F)$ (where L^* is the differential generator relative to system (25)), comparing with Theorem 2, and confirming that $E^*_{W,Y}(dF) = rF$. The instantaneous drifts of S and W using the modified p.d.f. are $E^*(dS) = rS$ and $E^*(dW) = rW - C$, consistent with the interpretation of (25) as "risk-neutral" jump-diffusions. Note that the drift of Y under the modified p.d.f. is $E^*_{W,Y}(dY) = [\mu(Y) - \lambda E_Y(\Delta Y) - \Phi_Y(W,Y)]\,dt$, not $(\mu - \phi_Y)$.

Theorem 3 in principle gives an implementable method of evaluating any option. A European option with strike price X will, for instance, have the value

$$c(S,\ W,\ Y,\ t;\ T,\ X) = E^*_{W,Y}\left[e^{-\int_t^T r(W,Y)d\tau}\ \max(S^*_T - X,\ 0)\right] \quad \textbf{(28)}$$

where the distribution of S^*_T, $f^*_S(S^*_T \mid S_t,\ W_t,\ Y_t,\ t;\ T)$ is given implicitly by system (25). Subject to considerable informational requirements,[6] the option price can be solved numerically via backward induction from the terminal date using risk-neutral probabilities (25). The binomial approach of Cox, Ross, and Rubinstein (1979) or multi-state variable variants thereof[7] can calculate the diffusion part of the recursive option evaluation. The jump-contingent modified expected change in the option price can be evaluated via trapezoidal integration over $f^*(\bullet)$-weighted possible realizations of k^*_S, k^*_W, and ΔY^*. In practice, option evaluation under the full-order system (25) would be exorbitant computationally, and simplifications are necessary.

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[6]I.e., the representative utility function $U(\bullet)$ and the stochastic evolution of $[S\ W\ Y']$ must be known, so that $J_W(W, Y)$, λ^*, k^*_S, Φ_Y etc. can be determined.

[7]E.g., the "hopscotch" method of Gourlay and McKee (1977) used by Wiggins (1987) and Melino and Turnbull (1988) for stochastic volatility models.

3. Reduced-Order Option Pricing Models: Black-Scholes, Stochastic Volatility, and Jump Risk

The greatest simplification possible in (25) is of course the Black-Scholes model, which imposes three distributional restrictions:

1) No jumps in *any* processes: $\lambda = 0$;

2) the underlying asset's price volatility $\sigma_S S$ is not a function of other underlying state variables Y;

3) The riskfree rate is nonstochastic and constant.

The third assumption, while counterfactual, is generally viewed as a plausible approximation. Shifts in the discount rate may not matter much for the value of an option with only three or six months to maturity.

These three assumptions imply that the option price depends only on the state variable S and that, using only the first line of (25), one can write the associated option valuation equation (23) as

$$F_t + rSF_S + \tfrac{1}{2}\sigma_S^2 S^2 F_{SS} = rF \qquad \textbf{(29)}$$

since none of the parameters depends upon omitted state variables. Solving (29) subject to option-specific boundary conditions, one can appeal to the no-arbitrage opportunities condition to validate the solution as the true option price, as is discussed below. The Black-Scholes formula depends neither on the preferences of the representative consumer nor on the distribution of the wealth process. Restricting wealth to following state-dependent geometric Brownian motion is not necessary.

The simplicity of the Black-Scholes model and of the related constant elasticity of variance models reflects the fact that the assumptions ensure that the option contains only one form of systematic risk, the price of which is summarized in the price and expected excess return of a *traded* asset --namely, the underlying asset. Recent challenges to the Black-Scholes model typically involve forms of risk that are not directly priced by the market: volatility risk and jump risk. Consequently, for these models, it is necessary to derive prices for these forms of risk from restrictions on preferences and on distributions. In stochastic volatility models, that "price" is given by the risk premium $\Phi_Y(W, Y)$, where Y proxies volatility. For jump risk models, the price of risk is reflected in the modified jump frequency parameter $\lambda^*(W, Y)$ and the modified jump amplitude distribution $f^*(\bullet)$.

Two different forms of restriction are typically imposed in order to price options under stochastic volatility. The first approach, used *inter alia* by Hull and White (1987), Johnson and Shanno (1987), and Scott (1987), is to assume that volatility on a given asset is uncorrelated with aggregate consumption and volatility risk is therefore nonsystematic and diversifiable. Under this assumption, $\Phi_\sigma = 0$, and the option price can be written $F(S, \sigma, t)$ and solved using (29) and the *true* diffusion for volatility, without need for restrictions on preferences. The evidence of Christie (1982) and others that volatilities of different stocks tend to move together casts doubt, however, on the underlying assumption that volatility risk is nonsystematic.[8]

The second approach, taken by Wiggins (1987), Nelson (1987), and Melino and Turnbull (1988), is to impose restrictions on preferences and on distributions to get a volatility risk premium $\Phi_\sigma(\sigma)$ of a specific form. Log utility is almost invariably assumed; there is also an implicit assumption that wealth follows state-dependent geometric Brownian motion, so that option prices do not depend on wealth. Under these assumptions, $\Phi_\sigma(\sigma) = Cov(d\sigma, dW/W)$, and distributional restrictions are necessary as to how volatility

[8]An analogous approach for pricing options on futures under *interest rate* risk is to be found in Ramaswamy and Sundaresan (1985). They assume no risk premium on interest rate risk, invoking the local expectations hypothesis.

covaries with the market. Different authors make different restrictions,[9] but the end result is an option price $F(S, \sigma, t)$ derived from (29) and the Φ_σ-modified stochastic differential equation for σ.

The same choice of restrictions arises for jump risk, since jump risk is also not priced directly by any financial instrument. Merton (1976) assumed that jump risk was firm-specific and diversifiable. The implication is price zero for jump risk, with the true jump parameters being used in pricing options: $\lambda^*(W, Y) = \lambda,\ f^*(k \mid W, Y) = f(k)$. Given the recent stock market crash, however, viewing jump risk as nonsystematic is clearly untenable. The alternative is again to restrict preferences and distributions. The result of a particularly useful set of restrictions is given below.

Theorem 4: Under either of two sets of assumptions,

A1a) The representative consumer has power utility $U(C) = (C^{1-R} - 1)/(1 - R)$

A1b) There are no jumps in the underlying state variables Y,

or

A2) The representative consumer has log utility (R=1)

and the additional distributional restrictions

A3) The underlying asset price follows a jump-diffusion of type (2) with constant volatility σ_S and random state-independent percentage jump amplitude k_S,

A4) Wealth follows a similar jump-diffusion process, of type (6), with state-independent random percentage jump amplitude k_W but with possibly state-dependent volatility σ_W,

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[9]The three papers cited above all assume that $d\sigma/\sigma = \mu(\sigma)dt + \theta\, dz_\sigma$, with mean reversion in the drift term. Wiggins (1987) suggests either

i) assuming constant market variance, implying that $\Phi_\sigma(\sigma)$ is proportional to σ, or

ii) assuming market volatility is proportional to σ, implying $\Phi_\sigma(\sigma)$ is proportional to σ^2.

Wiggins claims the results are not sensitive to the choice of specification. Nelson (1987), examining the S&P 500 as a proxy for the market portfolio, uses the latter specification. Melino and Turnbull (1988), looking at foreign exchange options, use the former specification.

A5) (One plus) the percentage jump amplitudes in the underlying asset price and wealth have a joint log-normal distribution:

$$\begin{bmatrix} \ln(1+k_S) \\ \ln(1+k_W) \end{bmatrix} \sim N\left[\begin{pmatrix} \gamma_S - \tfrac{1}{2}\delta_S^2 \\ \gamma_W - \tfrac{1}{2}\delta_W^2 \end{pmatrix}, \begin{pmatrix} \delta_S^2 & \delta_{SW} \\ \delta_{SW} & \delta_W^2 \end{pmatrix}\right]$$

A6) The instantaneous riskfree rate is (approximately) nonstochastic and constant

then

option prices depend only on the underlying asset price and time ($F = F(S, t)$), and are evaluated using a risk-neutral jump-diffusion with log-normal random jumps:

$$dS/S = [r - \lambda^* E^*(k^*)]\,dt + \sigma_S\,dZ + k^*\,dq^* \tag{31}$$

where

$$\begin{aligned} Prob(dq^* = 1) &= \lambda^* dt \\ &= \lambda E[(1+k_W)^{-R}]\,dt \\ &= \lambda \exp[-R\gamma_W + \tfrac{1}{2}(1+R)\delta_W^2]\,dt \end{aligned} \tag{32}$$

and

$$\ln(1+k_W^*) \sim N[(\gamma_S - R\delta_{SW} - \tfrac{1}{2}\delta_S^2),\ \delta_S^2] \equiv N[\gamma^* - \tfrac{1}{2}\delta_S^2,\ \delta_S^2]\,. \tag{33}$$

For example, the value of a European call with strike price X and time T to maturity is

$$c(S,\ T,\ X) = \sum_{n=0}^{\infty} w_n\, c(S e^{n\gamma^* - \lambda^* E^*(k_S) T},\ T,\ X,\ \sigma_S^2 + n\delta_S^2) \tag{34}$$

where $w_n = e^{-\lambda^* T}(\lambda^* T)^n / n!$ is the probability of n jumps in the lifetime of the option, and $c(\bullet)$, the discounted expected value (using the modified p.d.f.) of the option conditional on n jumps, uses the Black-Scholes formula:

$$\begin{aligned} c(S,\ T;\ X,\ v^2) &= e^{-rT}[S e^{rT} N(d_1) - X N(d_2)]\,, \\ d_1 \equiv [\ln(S e^{rT}/X) + \tfrac{1}{2}v^2 T] / v\sqrt{T}, &\quad d_2 \equiv d_1 - v\sqrt{T}\,. \end{aligned} \tag{35}$$

The proof is straightforward. The restriction on preferences A1a&b) or A2), along with the process A4) for wealth, imply that the separable indirect utility function given by Cox *et al* (1985b) for diffusions also holds for jump-diffusions:

$$J(W, Y) \;=\; \begin{cases} f(Y)U(W) \;+\; g(Y) \;\; \textit{for power utility} \\ \ln(W)/\rho \;+\; g(Y) \;\; \textit{for} \; \log \; \textit{utility} \end{cases} \tag{36}$$

with the familiar result from Cox *et al* that contingent claims prices and the riskfree interest rate may depend on the underlying state variables Y but will not depend on wealth. Assumption A3), combined with A1b) or A2), ensures that the option price will not depend on Y, since no parameter of the modified dS/S process depends on Y (or on W). Stochastic volatility is ruled out by assumption; the marginal utility-modified jump parameters are the true, state-independent parameters adjusted by $J_W^* / J_W \;=\; (1 + k_W)^{-R}$, which does not depend on Y or W; and the effect of Y on discounting risk is ruled out by A6). Assumption A5) is a convenient approximation[10] that implies J_W^*/J_W is log-normally distributed. Using (27) and the convenient properties of products of log-normal variables, the result is that one plus the "risk-neutral" jump amplitude is also log-normally distributed, with mean $E^*(1 + k_S) \;=\; e^{\gamma_S - R\delta_{SW}} \;\equiv\; e^{\gamma^*}$. Merton's (1976) formula for European call options on jump-diffusions with log-normal jumps can then be applied directly, using appropriately modified parameters.

Expressions (32) and (33) clarify the issue of "diversifiable" jump risk discussed in Merton (1976). Even if the jumps in the underlying asset price are independent of k_W (i.e., $\delta_{SW} = 0$), the fact that the jumps occur at the same time implies that jump risk on the asset is nondiversifiable and systematic, with a price reflected in the modified jump frequency $\lambda^* \;=\; \lambda E(1 + k_W)^{-R}$. Only if the jumps are truly firm-specific and

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[10]If the jumps $1 + k$ in underlying production processes are log-normally distributed, then $1 + k_W \;=\; 1 + (a*)'k$ is approximately but not exactly log-normally distributed.

do not affect aggregate wealth $(k_W \equiv 0)$ as is assumed in Merton (1976), can one use the true distribution of dS/S in pricing options.

A second result from (32) and (33) is that, insofar as jumps in the underlying asset price are positively correlated with jumps in the market (as one would expect for S&P 500 futures, for instance), the effect is to bias downwards the mean expected jump $E^*(k_S)$ implicit in option prices. If the true mean jump $E(k_S)$ were zero, the implicit mean jump $E^*(k_S)$ would be negative, leading to negative asymmetries relative to Black-Scholes option prices (overvalued in-the-money, undervalued out-of-the money calls relative to at-the-money calls) of the sort found in recent years in stock options. It is unlikely, however, that this effect alone suffices to explain the strong moneyness bias in S&P 500 futures options prices in 1987.[11] It seems more likely that a crash was expected: $E(k_S) < 0$.

[11]For example, suppose the S&P 500 is a good proxy for the market portfolio, and an infrequent jump with mean 0 ($\gamma_S = 0$) and standard deviation $\delta_S = 0.10$ is expected. The jump distribution implicit in option prices for risk aversion $R = 2$ has a mean of $\exp(-R\delta_{SW}) - 1 = \exp(2\delta_S^2) - 1 \approx .02$. A negative mean of this size does not induce much bias in option prices.

4. Arbitrage-Based Justifications of Option Pricing Models

The no-arbitrage justification for the Black-Scholes option pricing formula is based upon synthesizing an option by means of a dynamic trading strategy in the underlying asset and riskfree bonds.[12] The self-financing trading strategy involves holding at each instant of time $F_S(S, t)$ shares of the underlying asset and $[F(S, t) - S_t F_S(S, t)]$ in riskfree bonds,[13] where F(S, t) is the theoretical option price that solves the partial differential equation

$$F_t \quad + \quad rSF_S \quad + \quad ½\sigma^2 S^2 F_{SS} \quad = \quad rF \qquad \textbf{(37)}$$

subject to the terminal value conditions of the option (and subject to additional early-exercise conditions if the option is American). By construction, the trading strategy pays off identically to the actual option (if the Black-Scholes assumptions are correct), so any deviation between the initial cost $F(S, 0)$ of the trading strategy and the price F_0 of the actual option would allow a riskfree arbitrage opportunity.[14] Hence, the theoretical option price F(S, 0) must equal the market price for the option.

A similar no-arbitrage justification holds for stochastic volatility or interest risk models. Assuming negligible transactions costs in options markets and relatively few relevant state variables $[W\ (Y^*)']$ relative to the number of actively traded options,[15] the theoretical option prices given by the no-jump

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[12]See, e.g., Ingersoll (1987), ch. 17.

[13]When the underlyin987) for a careful examination of the conditions under which the no-arbitrage justification for option pricingg asset is a futures contract, the strategy is to hold $F_S(S, t)$ futures contract (at no cost) and F(S,t) bonds.

[14]See Dybvig and Huang (1987) for a careful examination of the conditions under which the no-arbitrage justification for option pricing models holds.

[15]$[W\ (Y^*)']$, a $1 \times (1 + K^*)$ vector, is a subset of $[W\ Y']$. As noted above, stochastic volatility models typically assume only one relevant state variable: $Y^* = \sigma$ and W irrelevant because of log utility

version of Theorem 3 ($\lambda^* = 0, dq^* \equiv 0$) must be the actual option prices. For if not, one can again synthesize an option via a self-financing dynamic trading strategy based on the theoretical prices, using the underlying asset, $K^* + 1$ *other* options, and riskfree bonds. Taking $[F(S, W, Y^*, t) \;\; G(S, W, Y^*, t)']$ as the $K^* + 2$ theoretical option prices, the replicating strategy for the option with price F_0 is to hold at each instant

$N_S(S, W, Y^*, t)$ shares of the underlying asset,

$N_G(S, W, Y^*, t)$ units of the $K^* + 1$ other options, and

$[F - SN_S - G' N_G]$ in riskfree bonds,[16]

where N_S and N_G are given implicitly by

$$\begin{pmatrix} F_S \\ F_W \\ F_{Y^*} \end{pmatrix} = \begin{pmatrix} 1 & G_S' \\ 0 & G_W \\ 0 & G_{Y^*}' \end{pmatrix} \begin{pmatrix} N_S \\ N_G \end{pmatrix} \tag{38}$$

By construction, the trading strategy again replicates the terminal payoffs of the actual option with initial price F_0. Therefore, a deviation of F_0 from $F(S, W, Y^*, 0)$ would allow a riskless arbitrage opportunity, which cannot exist in equilibrium.

There are some important differences between the stochastic volatility synthetic option and that of Black and Scholes. The informational requirements are much greater for the former: one must know the preferences of the representative investor, and the nature of the stochastic evolution of *W* and *Y** in order to calculate the theoretical option prices used in constructing the hedge. Correspondingly, one cannot use

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and state-dependent geometric Brownian motion. Interest risk models (e.g., Ramaswamy and Sundaresan (1985), Ahn and Thompson (1987)) typically assume $Y^* = r$, and *W* irrelevant.

[16]For replicating options on futures contracts, N_S and N_G are the same but one holds $F - G' N_G$ in bonds.

the no-arbitrage condition to *derive* option prices under stochastic volatility, but can only use the condition to justify formulas obtained from preference- and technology-based restrictions. The two problems are related. In the Black-Scholes model, the only relevant risk is impounded in the price and expected return of the underlying asset, which is traded. The problem for stochastic volatility models, as noted by Wiggins (1987), is that there is no corresponding traded asset with value based solely on volatility -- no σ-based asset" such as one that pays off σ_T at time T. Extra options can hedge σ-risk, and can therefore replicate another option, as shown above. But since option prices also reflect other sources of risk, one cannot extract the value of a σ-based asset and the price of options without other restrictions.

The case of options on jump-diffusions with stochastic jump amplitudes adds additional complexities to the above considerations. A perfectly replicating synthetic option cannot be constructed in general, since a finite number of other assets will not typically span the space of jump-contingent possible outcomes. To illustrate this, consider replicating an option as above using an arbitrarily large number of other options, where theoretical option prices are $[F(S, W, Y^*, t) \;\; G(S, W, Y^*, t)']$. (In principle, Y^* could equal Y, since all underlying state variables that affect jump-contingent marginal utility of wealth are relevant.) Taking any self-financing portfolio $[N_S \;\; N_G' \;\; (P - SN_S - G'N_G)]$ in stocks, options and bonds that satisfies (38) above for N_S and N_G, the stochastic evolution of the value P of this portfolio relative to the theoretical price F is described by

$$dP - dF = r_t(P - F)\,dt - \lambda E_{W,Y}[(J_W^*/J_W)(\Delta P - \Delta F)]\,dt + (\Delta P - \Delta F)\,dq \qquad \textbf{(39)}$$

where

$$\Delta P = N_S(k_S S) + N_G'[G(S + k_S S, W + k_W W, Y + \Delta Y, t) - G] \text{ and}$$

$$\Delta F = F(S + k_S S, W + k_W W, Y + \Delta Y, t) - F$$

are the random changes conditional on a jump occurring. Since $P(\bullet)$ and $F(\bullet)$ are nonlinear functions of the random jump sizes, it will not in general be possible to choose N_G such that $\Delta P \equiv \Delta F$. Therefore, even choosing $P(S, W, Y^*, 0) = F(S, W, Y^*, 0)$, $P(S, W, Y^*, T)$ will differ from $F(S, W, Y^*, T)$ in some realizations -- i.e., the portfolio will not replicate the terminal boundary conditions of the option in all states of nature.

One approach, used by Jones (1984) and Bates (1988), is to restrict the jump-contingent outcomes to a finite number: e.g., to 1, in which case $K^* + 2$ options, the underlying asset, and short-term bonds are needed in the portfolio to replicate perfectly another option ($K^* + 1$ options for the $[W\ (Y^*)']$ diffusion risks, and 1 for the jump risk). The further restrictions on preferences and distributions of Theorem 4, which imply option prices of the form $F(S, t)$, considerably simplify the replicating portfolio to only the underlying asset, *one* other option, and bonds. Only under these restrictions, however, is it precisely correct (as asserted in Jones (1984)) that deterministic jump risk in the underlying asset price can be hedged by adding one option per jump. Furthermore, as noted in the stochastic volatility case, the existence of a perfectly replicating strategy can be used to justify option pricing formulas but cannot be used to derive them.[17] The problem is similar to but more pronounced than the case of stochastic volatility: there are no jump-contingent Arrow-Debreu securities traded that could be used to evaluate options.

Restricting jump amplitudes to a finite number of possible realizations is somewhat unsatisfactory and, fortunately, not necessary. While there is not a perfectly replicating portfolio when jump amplitudes are

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[17]The Jones (1984) 'derivation" of a Merton-type formula involved setting $\lambda^*(W, Y) = \lambda E_{W,Y}[J_W(W + k_W W, Y + \Delta Y) / J_W(W, Y)]$ constant, implicitly imposing the restrictions of Theorem 4 on preferences and on the stochastic evolution of *W* and *Y*.

random, it is possible to come close. Imposing the restrictions of Theorem 4, the problem is to closely replicate $\Delta F \equiv F(S + k_S S, t) - F$ for all realizations of k_S, using a portfolio that jumps $\Delta P = N_S(S k_S) + N_G'[G(S + k_S S, t) - G]$. Since option prices are smooth, convex functions of S, it is not difficult to closely mimic the distribution of ΔF using a finite and, in fact, rather small number of options in the replicating portfolio. Possible criteria for choosing N_S and N_G include

$$\min_{N_S, N_G} E[(\Delta P - \Delta F)^2] \tag{40}$$

or

$$\min_{N_S, N_G} E[(J_W^*/J_W)(\Delta P - \Delta F)^2] \Leftrightarrow \min_{N_S, N_G} E^*[(\Delta P - \Delta F)^2] \tag{41}$$

subject to the constraint (on N_S and N_G) that $P_S = F_S$, and the constraint (on bond holdings) that $P = F$. Under the latter criterion, J_W can reflect the preferences of the representative investor or of an investor actually considering the arbitrage. Either specification gives N_S and N_G as the solution to

$$\begin{pmatrix} E[(\Delta F)(\Delta S)] \\ E[(\Delta F)(\Delta G)] \\ F_S \end{pmatrix} = \begin{pmatrix} E[(\Delta S)^2] & E[(\Delta S)(\Delta G')] & 1 \\ E[(\Delta S)(\Delta G)] & E[(\Delta G)(\Delta G')] & G_S \\ 1 & G_S' & 0 \end{pmatrix} \begin{pmatrix} N_S \\ N_G \\ m \end{pmatrix} \tag{42}$$

where m is the Lagrangian multiplier on the constraint $P_S = F_S$. E^* replaces E in (42) when the second criterion is used, and the expected products can be calculated for a given option pricing model using trapezoidal integration over the domain of k_S.

Figures 1 and 2 illustrate the result of replicating deep in-the-money and out-of-the-money European options on S&P 500 futures, using futures contracts and near-the-money options of identical maturity. One plus the true percentage jump size is assumed distributed log-normally, with $E(k) = 0$. It is assumed that the S&P 500 is the market portfolio,[18] so that the "risk-neutral" jump size used in determining $[N_S \; N_G{}']$ under the second criterion has mean $E^*(k) = e^{-R\delta^2} - 1 \approx -R\delta^2$. Using one option in the replicating portfolio results essentially in a "delta-gamma-neutral" strategy that does well for small jumps, less well for large ones.[19] Using three near-the-money options plus the underlying asset, the option is almost perfectly replicated under either criterion.

5. Conclusion

To recapitulate, this paper has

1) Derived the appropriate characterization of asset market equilibrium when asset prices follow jump-diffusion processes,

2) Derived the resulting restrictions on option prices, and developed a system of "risk-neutral" diffusions that can be used to evaluate options for given restrictions of preferences and technologies,

3) Developed a tractable option pricing model valid even when jump risk is systematic, and

4) Described the appropriate hedging strategy justifying the option pricing formula, while cautioning that the existence of such strategies cannot be used to derive the formulas without imposing additional, implicit restrictions.

Other option pricing models, such as for volatility risk when volatility follows a jump-diffusion, could also be handled within this framework -- at the cost of additional complexity.

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[18]Or, equivalently, that $\delta_{SW} = \delta_S^2 \equiv \delta^2$.

[19]The "delta-gamma-neutral" portfolio insurance approach chooses N_S and N_G to replicate both the slope ("delta:" $P_S = F_S$) and the curvature ("gamma:" $P_{SS} = F_{SS}$) of the desired option, in order to reduce portfolio adjustments over small- to medium-sized movements in S. Given that the jump distribution has greatest mass near zero for the parameters chosen, the strategy given by (40) or (41) is closely related to such a strategy. A minor caveat is that deltas and gammas calculated by portfolio insurers using Black-Scholes will be only approximately correct when the true process is a jump-diffusion.

References

Ahn, Chang Mo and Howard E. Thompson (1987). "Jump-Diffusion Processes and the Term Structure of Interest Rates," U. of Wisconsin-Madison working paper, May.

Ball, Clifford A. and Walter N. Torous (1983). "A Simplified Jump Process for Common Stock Returns," *Journal of Financial and Quantitative Analysis* 18:1, March, 53-65.

---------- (1985). "On Jumps in Common Stock Prices and Their Impact on Call Option Pricing," *Journal of Finance* 40:1, March, 155-173.

Black, Fischer and Myron Scholes (1973). "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81:3, May/June, 637-654.

Christie, Andrew A. (1982). The Stochastic Behavior of Common Stock Variances: Value, Leverage and Interest Rate Effects," *Journal of Financial Economics* 10, 407-432.

Cox, John C. Jonathan E. Ingersoll, and Stephen A. Ross (1985a). "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica* 53:2, March, 363-384.

---------- (1985b). "A Theory of the Term Structure of Interest Rates," *Econometrica* 53:2, March, 385-407.

Cox, John C., Stephen A. Ross, and Marc Rubinstein (1979). "Option Pricing: A Simplified Approach," *Journal of Financial Economics* 7:3, Sept. 229-263.

Gourlay, A. R. and S. McKee (1977). "The Construction of Hopscotch Methods for Parabolic and Elliptic Equations in Two Space Dimensions with a Mixed Derivative," *Journal of Computational and Applied Mathematics* 3, 201-206.

Hull, John and Alan White (1987). "The Pricing of Options on Assets with Stochastic Volatility," *Journal of Finance* 42:2, June, 281-300.

Ingersoll, Jonathan E. (1987). *Theory of Financial Decision Making*. Tetowa, NJ: Rowman & Littlefield.

Johnson, Herb and David Shanno (1987). "Option Pricing when the Variance is Changing," *Journal of Financial and Quantitative Analysis* 22:2, June, 143-151.

Jones, E. Phillip (1984). "Option Arbitrage and Strategy with Large Price Changes," *Journal of Financial Economics* 13:1, March, 91-113.

Melino, Angelo and Stuart Turnbull (1988). "Pricing Foreign Currency Options with Stochastic Volatility," University of Toronto manuscript, June.

Merton, Robert C. (1971). "Optimal Consumption and Portfolio Rules in a Continuous-Time Model," *Journal of Economic Theory* 3:4, Dec., 373-413.

---------- (1976). "Option Pricing when Underlying Stock Returns are Discontinuous," *Journal of Financial Economics* 3:1/2, Jan./March, 125-144.

Nelson Daniel B. (1987). "Conditional Heteroskedasticity in Asset Returns," M.I.T. manuscript, Dec.

Ramaswamy, Krishna and Suresh M. Sundaresan (1985). "The Valuation of Options on Futures Contracts," *Journal of Finance* 40:5, Dec., 1319-1340.

Scott, Louis O. (1987). "Option Pricing When the Variance Changes Randomly: Theory, Estimation, and an Application," *Journal of Financial and Quantitative Analysis* 22:4, Dec.

Wiggins, James B. (1987). "Option Values Under Stochastic Volatility: Theory and Empirical Estimates," *Journal of Financial Economics* 19, 351-372.

Figure 1

Replication of jump-contingent values of a European call option on an S&P 500 futures contract with strike price 6% out-of-the-money, using futures contracts, near-the-money options, and T-bills.

Parameters: S = 100, T = 0.25, r = 0.10, σ = 0.10
Jump parameters: E(k) = 0.00, Var[ln(1+k)] = $(0.10)^2$
Modified jump parameters: λ* = 0.25 E*(k) = -0.02 Var*[ln(1+k)] = $(0.10)^2$

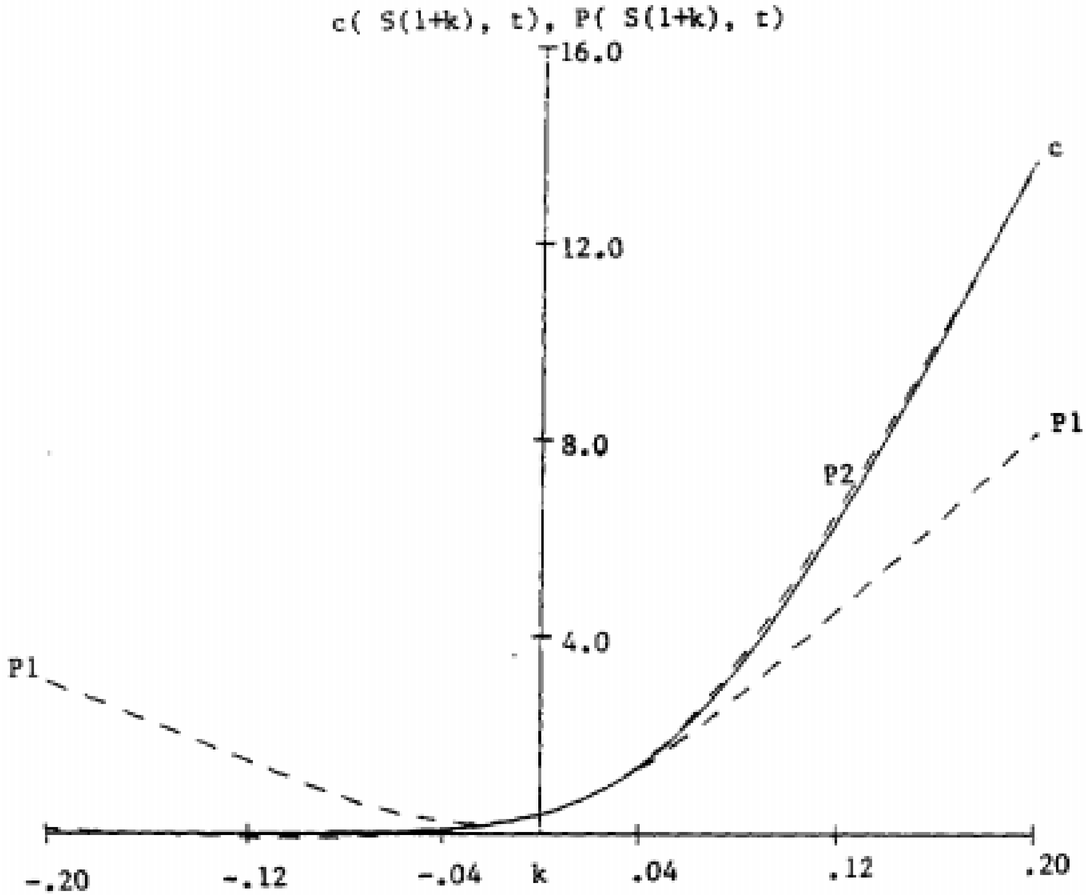

Table 1

Replicating portfolios: Futures contracts, options, and T-bills

Port-folio	No. of Futures	Number of options with strike price 96	98	100	102	104	Value of T-bills
P1)	-0.24			0.88			-1.45
P2)	-0.03		2.42	-6.07	4.64		-0.58
P3)	0.00	-0.71	2.17	-1.78	-1.07	2.37	0.04
Option prices:		4.60	3.21	2.10	1.28	0.73	

Figure 2

Replication of jump-contingent values of a European call option on an S&P 500 futures contract with strike price 6% in-the-money, using futures contracts, near-the-money options, and T-bills.

Parameters: S = 100, T = 0.25, r = 0.10, σ = 0.10
Jump parameters: E(k) = 0.00, Var[ln(1+k)] = $(0.10)^2$
Modified jump parameters: λ^* = 0.25 E*(k) = -0.02 Var*[ln(1+k)] = $(0.10)^2$

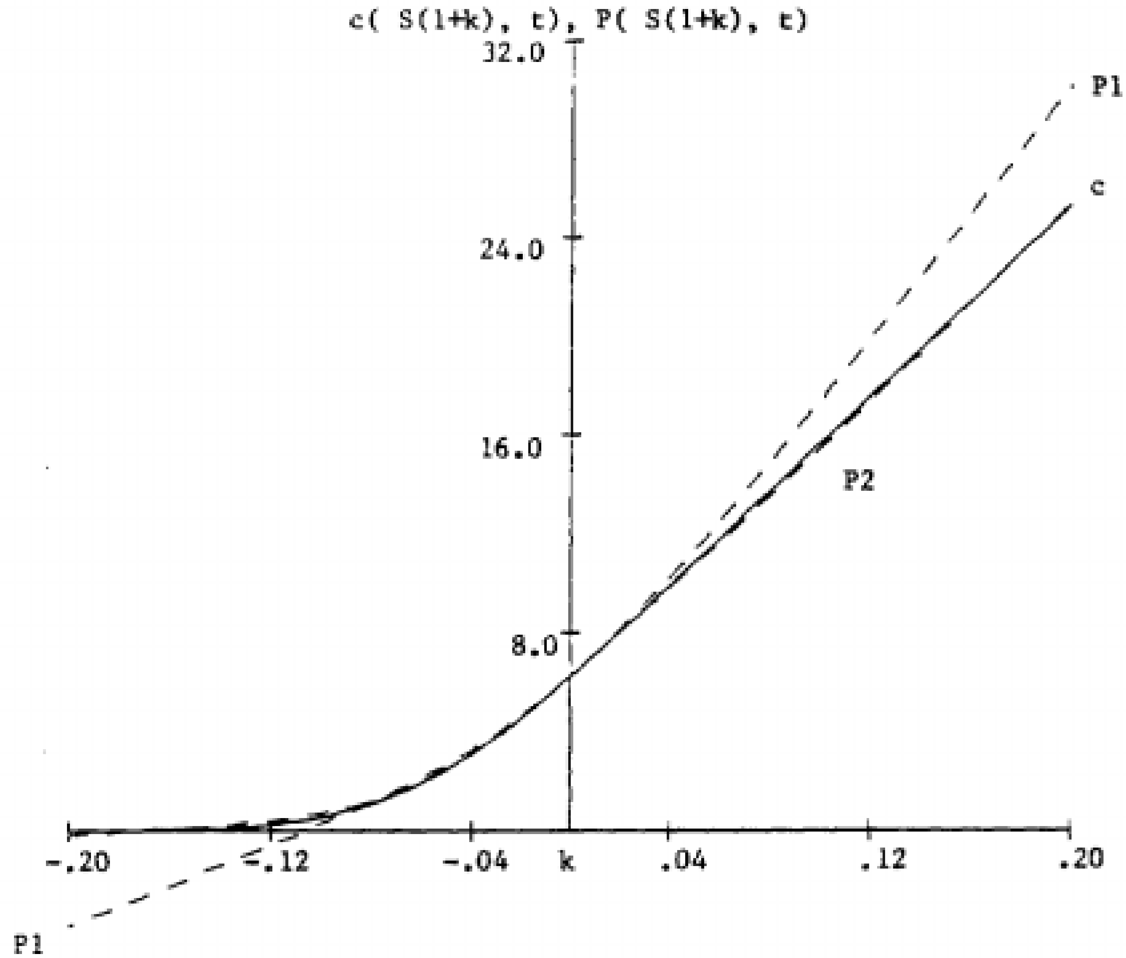

Table 2

Replicating portfolios: Futures contracts, options, and T-bills

Port-folio	No. of Futures	Number of options with strike price 96	98	100	102	104	Value of T-bills
P1)	0.41			0.91			4.32
P2)	0.06		4.85	-6.57	2.70		0.97
P3)	0.01	2.86	-2.06	-0.02	1.33	-0.58	0.16
Option prices:		4.60	3.21	2.10	1.28	0.73	