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3 authors, including:

Shamsul Qamar
COMSATS Institute of Inform…

Gerald Warnecke
Otto-von-Guericke-Universit…

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Kinetic schemes for the relativistic gas dynamics*

Matthias Kunik, Shamsul Qamar, Gerald Warnecke

Institute for Analysis and Numerics, Otto-von-Guericke University, 39106 Magdeburg, Germany; e-mail: {matthias.kunik, shamsul.qamar, gerald.warnecke}@mathematik.uni-magdeburg.de

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Summary. A kinetic solution for the relativistic Euler equations is presented. This solution describes the flow of a perfect gas in terms of the particle density n, the spatial part of the four-velocity $\mathbf{u}$ and the inverse temperature β. In this paper we present a general framework for the kinetic scheme of relativistic Euler equations which covers the whole range from the non-relativistic limit to the ultra-relativistic limit. The main components of the kinetic scheme are described now.

(i) There are periods of free flight of duration τ_M, where the gas particles move according to the free kinetic transport equation.
(ii) At the maximization times $t_n = n\tau_M$, the beginning of each of these free-flight periods, the gas particles are in local equilibrium, which is described by Jüttners relativistic generalization of the classical Maxwellian phase density.
(iii) At each new maximization time $t_n > 0$ we evaluate the so called continuity conditions, which guarantee that the kinetic scheme satisfies the conservation laws and the entropy inequality. These continuity conditions determine the new initial data at t_n.
(iv) If in addition adiabatic boundary conditions are prescribed, we can incorporate a natural reflection method into the kinetic scheme in order to solve the initial and boundary value problem.

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Correspondence to: S. Qamar

In the limit $\tau_M \to 0$ we obtain the weak solutions of Euler's equations including arbitrary shock interactions. We also present a numerical shock reflection test which confirms the validity of our kinetic approach.

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1 Introduction

In the kinetic theory of gases the **phase density** $f = f(t, \mathbf{x}, \mathbf{q})$ is a fundamental quantity. It is usually a function of time t, position $\mathbf{x}$ and of the velocity or momentum $\mathbf{q}$ related to the single gas atoms. It usually results from a kinetic equation like the *Boltzmann-* or *BGK-equation*, see Cercignani [2] and Bhatnagar, Gross, Krook [1]. Then the macroscopic, thermodynamic quantities like particle- and energy density, velocity and pressure are tensor-algebraic combinations of some basic integral moments of the phase density f, where the integration is performed with respect to the momentum $\mathbf{q}$ of the gas atoms. Therefore the macroscopic quantities depend only on time and space. They are all completely determined by the phase density f.

Kinetic approaches in order to solve the classical Euler equations of gas dynamics were applied to several initial- and boundary value problems, see for example Dreyer, Kunik and Herrmann [6], [7], [9] and Perthame [23], [24], [25]. In Dreyer & Kunik [8] the reader will also find a kinetic scheme in order to solve an initial and boundary value problem for a phonon Bose gas. Kunik, Qamar and Warnecke [21] also successfully applied the kinetic scheme in order to solve initial value problems for the ultra-relativistic Euler equations. One of the main objectives of this paper is to generalize the kinetic schemes which were previously derived for the classical and ultra-relativistic Euler equations in [6], [7], [9] and [21]. This study includes the kinetic solution of more general Euler equations which cover the whole range from the classical to the ultra-relativistic Euler equations.

The hyperbolic systems that can be treated by the kinetic method are those which may be generated from kinetic transfer equations and from the *Maximum Entropy Principle*. Since these systems lead to a convex entropy function, they enable several rigorous mathematical results, see for example Friedrichs and Lax [14], Godlewski & Raviart [15] as well as Dafermos [5]. In the case of *thermodynamical equilibrium* the Maximum Entropy Principle constitutes a successful method in order to obtain the Maxwellian phase density for the Boltzmann gas as well as the corresponding phase densities for the Fermi- and Bose gas in equilibrium from the corresponding kinetic entropy definitions. But the application of the Maximum Entropy Principle may lead to serious problems in *non-equilibrium*, for example for the moment-systems of the Fokker-Planck equation and of the Boltzmann equation. These difficulties arise due to a singular behaviour of some integral moments of the nonequilibrium phase density. This was shown explicitly by Dreyer, Junk and Kunik [10] for the Fokker-Planck case, but it is not hard to see that the same arguments can also be applied to the Boltzmann case. An exception is the phonon Bose gas considered in [8], where the singular behaviour of the integral moments cannot occure.

After few years of Einstein's famous paper "Zur Elektrodynamik bewegter Körper", Jüttner [18] extended the kinetic theory of gases which was developed by D. Bernoulli, Clausius, Maxwell and Boltzmann, to the domain of relativity. He was succeeded in deriving the relativistic generalization of the Maxwellian equilibrium phase density. Later on this phase density and the whole relativistic kinetic theory was structured in a well organized Lorentz- invariant form, see Chernikov [3], [4], Israel [17], Müller [22] and the textbook of deGroot, van Leeuven and van Weert [16]. Jüttner [20] also established the relativistic form of equilibrium phase densities and the corresponding equations of state for the systems of bosons and fermions.

This paper is an extension of [21], where the kinetic solution of the ultra-relativistic Euler equations was presented. Here we will formulate a kinetic scheme for the solution of the initial value problem for the *relativistic Euler equations* in its more general form. In the last section we will also extend the scheme in order to solve an initial and boundary value problem with an adiabatic wall at rest. We formulate the scheme in such a way that the whole range from the classical Eulerian limit to the ultra-relativistic limit is covered. Euler's equations (relativistic or classic) deal with a *perfect gas*, in which mean free paths and collision free times are so short that perfect isotropy is maintained about any point moving with the gas. In this case the *local equilibrium assumption* is satisfied and the corresponding phase-densities are obtained from the Maximum Entropy Principle in equilibrium. In the textbook of Weinberg [26] one can find a short introduction to special relativity and relativistic hydrodynamics with further literature also for the imperfect fluid (gas), see for example the papers of Eckart [11], [12], [13] for the classical and relativistic thermodynamics.

There are three basic ingredients of the relativistic kinetic schemes. The first one is the relativistic phase density developed by Jüttner. The second one is the solution of a collision free kinetic transport equation, which can be given explicitly in terms of a known initial phase density. For the formulation of the kinetic scheme we prescribe a time step $\tau_M > 0$, define the equidistant times $t_n = n\,\tau_M$ ($n = 0, 1, 2, \ldots$), called maximization times, and solve a collision free kinetic transport equation for each time interval $t_n < t < t_{n+1}$, starting with a relativistic Maxwellian as the initial phase density at each maximization time t_n. The third component consists on the continuity conditions, which guarantee that the conservation laws are also satisfied across the maximization times. They also determine the new initial data for the next free-flight period. Finally, it is also possible to incorporate adiabatic boundary conditions into the kinetic scheme in a quite natural way, which will also be done in this paper. By taking moments of the corresponding phase densities we obtain every macroscopic quantity like particle density, energy density, pressure and velocity four vector. These macroscopic quantities will solve the relativistic Euler equations in the limit $\tau_M \to 0$.

Now we give a short overview of this paper:

In Sect. 2 we will present the basic definitions of the relativistic kinetic theory, namely the macroscopic quantities considered in thermodynamics which are obtained from a kinetic phase density. Moreover the relativistic Maxwellian studied by Jüttner in [18] is introduced and two limiting cases are considered, namely the classical Maxwellian for a cool, non-relativistic gas and the ultra-relativistic Jüttner phase density.

In Sect. 3 we calculate the energy density, pressure and entropy density from Jüttner's phase density. Then we determine the macroscopic moments of this relativistic Maxwellian, which gives the so called constitutive relations. The conservation laws for the particle number, the momentum and energy and these constitutive relations are representing the relativistic Euler equations. The Euler equations are written in differential form as well as in a weak integral form, which takes care for the evolution of shock waves. There holds an entropy inequality in terms of a specific entropy function which satisfies Gibbs equation. In order to prove that the relativistic Maxwellian satisfies the Maximum Entropy Principle we have first proved four lemmas which are needed for this purpose. After this we formulate and prove the Maximum Entropy Principle. In order to get the general formulation of the relativistic Euler equations and the Maximum Entropy Principle, we use some relations concerning the Bessel functions, which can be found in the hand book of Jeffrey [19]. We have also evaluated the limiting cases for the energy density, pressure and entropy density for the non-relativistic limit as well as the ultra-relativic limit. The eigenvalues from the differential form of the Euler equations are presented in this paper. Moreover, the Rankine-Hugoniot shock conditions and the entropy inequality are used in order to derive a simple parameter representation for the admissible single shock fronts.

In Sect. 4 we first formulate the kinetic scheme in order to solve the three-dimensional relativistic Euler equations. This scheme satisfies the conservation laws and the entropy inequality, which is stated in a proposition.

In Sect. 5 we are looking for the spatially one-dimensional solutions which are nevertheless solutions for the three dimensional relativistic Euler equations.

In Sect. 6 we discuss the Eulerian limit $\tau_M \to 0$ of the kinetic scheme where weak solutions are obtained from the initial value problems including arbitrary complicate shock interactions.

In Sect. 7 we compute a spatial one-dimensional numerical test case with the explicitly known single shock solutions from Sect. 3. An adiabatic boundary condition is incorporated into the kinetic scheme in order to study the reflection of the single shock at the adiabatic boundary. Since this reflection can be calculated completely from the shock parametrizations, this constitutes an ideal test case for the kinetic scheme presented in this paper. We will see that the analytical calculations and the numerical results are in very good agreement.

2 Moments of the relativistic kinetic phase density

In this section we describe a relativistic gas consisting of many microscopic structureless particles in terms of the relativistic kinetic phase density. From this fundamental phase density we calculate tensorial moments which give the local macroscopic physical quantities of the gas such as the particle density, the velocity, the pressure, the temperature and so on.

In order to formulate the theory in a Lorentz-invariant form, we make use of the notations for the tensor calculus used in the textbook of Weinberg [26], with only slight modifications:

A) **The space-time coordinates** are $x^\mu,\ \mu = 0, 1, 2, 3,$ with $x^0 := ct$ for the time, x^1, x^2, x^3 for the position.

B) **The metric-tensor** is

$$g_{\mu\nu} = g^{\mu\nu} = \begin{cases} +1, & \mu = \nu = 0\,, \\ -1, & \mu = \nu = 1, 2, 3\,, \\ \ \ 0, & \mu \neq \nu\,. \end{cases} \tag{2.1}$$

C) **The proper Lorentz-transformations** are linear transformations $\Lambda^\alpha{}_\beta$ from one system of space-time with coordinates x^α to another system x'^α. They must satisfy

$$x'^\alpha = \Lambda^\alpha{}_\beta\, x^\beta, \quad g_{\mu\nu} = \Lambda^\alpha{}_\mu\, \Lambda^\beta{}_\nu\, g_{\alpha\beta}, \qquad \Lambda^0{}_0 \geq 1, \ \ \det \Lambda = +1\,.$$

The conditions $\Lambda^0{}_0 \geq 1$ and $\det \Lambda = +1$ are necessary in order to exclude inversion in time and space. Then the following quantity forms a tensor with respect to proper Lorentz-transformations, the so called Levi-Civita tensor:

$$\epsilon_{\alpha\beta\gamma\delta} = \begin{cases} +1, & \alpha\beta\gamma\delta \ \text{ even permutation of } \ 0123, \\ -1, & \alpha\beta\gamma\delta \ \text{ odd permutation of } \ 0123\,, \\ \ \ 0, & \text{otherwise.} \end{cases}$$

Note that in the textbook of Weinberg [26] this tensor as well as the metric tensor both take the sign opposite to the notation used here.

D) **Einstein's summation convention:**
Any Greek index like α, β, that appears twice, once as a subscript and once as a superscript, is understood to be summed over 0,1,2,3 if not noted otherwise. For spatial indices, which are denoted by Latin indices like i, j, k, we will not apply this summation convention.

First we take a microscopic look at the gas and start with the kinematics of a representative gas atom with particle trajectory $\mathbf{x} = \mathbf{x}(t)$, where the time coordinate t and the space coordinate $\mathbf{x}$ are related to an arbitrary Lorentz-frame. The invariant mass of all structureless particles is assumed to be the same and is denoted by m_0. The microscopic velocity of the gas atom is $\frac{d\mathbf{x}(t)}{dt}$, and its microscopic velocity four-vector is given by $c\,q^{\mu}$, where the *dimensionless microscopic velocity four-vector* q^{μ} is defined by

$$(2.2) \qquad \left(q^0, \mathbf{q}\right)^T , \quad q^0 = q_0 = \sqrt{1+\mathbf{q}^2}\,, \quad \mathbf{q} = \frac{\frac{1}{c}\frac{d\mathbf{x}}{dt}}{\sqrt{1-\left(\frac{1}{c}\frac{d\mathbf{x}}{dt}\right)^2}}\,.$$

The *relativistic phase density* $f(t, \mathbf{x}, \mathbf{q}) \geq 0$ is the basic quantity of the kinetic theory. The following definitions of the macroscopic moments and the entropy four-vector make use of the fact that the so called *proper volume element* d^3q/q_0 is invariant with respect to Lorentz-transformations.

Macroscopic moments and entropy four-vector:

(i) *Particle-density four-vector*

$$(2.3) \qquad N^{\mu} = N^{\mu}(t, \mathbf{x}) = \int_{\mathbb{R}^3} q^{\mu}\, f(t, \mathbf{x}, \mathbf{q})\, \frac{d^3q}{q^0}\,,$$

(ii) *energy-momentum tensor*

$$(2.4) \qquad T^{\mu\nu} = T^{\mu\nu}(t, \mathbf{x}) = m_0 c^2 \int_{\mathbb{R}^3} q^{\mu} q^{\nu}\, f(t, \mathbf{x}, \mathbf{q})\, \frac{d^3q}{q^0}\,,$$

(iii) *entropy four-vector*

$$(2.5) \qquad S^{\mu} = S^{\mu}(t, \mathbf{x}) = -k_B \int_{\mathbb{R}^3} q^{\mu}\, f(t, \mathbf{x}, \mathbf{q})\, \ln\left(\frac{f(t, \mathbf{x}, \mathbf{q})}{\chi}\right) \frac{d^3q}{q^0}\,.$$

Here $k_B = 1.38062 \times 10^{-23} J/K$ is Boltzmann's constant and $\chi = (\frac{m_0 c}{\hbar})^3$ with Planck's constant $\hbar = 1.05459 \times 10^{-34} Jsec$. Note that χ has the same dimension as f, namely 1/volume. We also state here that the entropy formula (2.5) can be generalized easily in such a way, that the well known case of a Fermi- or Bose gas is also included in this kinetic framework. Then formula (2.5) reads in the general case

$$(2.6) \quad S^{\mu} = -k_B \int_{\mathbb{R}^3} q^{\mu} \left[f \ln \frac{f}{\chi} - \eta\chi \left(1 + \eta\frac{f}{\chi}\right) \ln\left(1 + \eta\frac{f}{\chi}\right)\right] \frac{d^3q}{q^0}\,.$$

Here $\eta = 0$ reduces to (2.5), which is valid for the relativistic generalization of Boltzmann's statistic, whereas $\eta = +1$ is required for the Bose-Einstein statistic and $\eta = -1$ for the Fermi statistic.

Note that the spatial part $\mathbf{q} \in \mathbb{R}^3$ of the dimensionless microscopic velocity four-vector is used as an integration variable in the relativistic kinetic theory.

Now we may use the macroscopic moments N^μ, $T^{\mu\nu}$ and S^μ of the relativistic phase density f in order to calculate the other macroscopic quantities of the gas, which are

Tensor algebraic combinations of these moments:

(i) *The proper particle density*

$$n = \sqrt{N^\mu N_\mu}\,, \tag{2.7}$$

(ii) *the dimensionless velocity four-vector*

$$u^\mu = \frac{1}{n} N^\mu\,, \tag{2.8}$$

(iii) *the proper energy density*

$$e = u_\mu u_\nu T^{\mu\nu}\,, \tag{2.9}$$

(iv) *the proper pressure and temperature*

$$p = \frac{1}{3}\,(u_\mu u_\nu - g_{\mu\nu})\, T^{\mu\nu} = k_B\, nT\,, \tag{2.10}$$

(v) *the proper entropy density*

$$\sigma = S^\mu u_\mu\,. \tag{2.11}$$

Remarks

(i) Since $f \geq 0$, it can be shown that N^μ is a time-like vector, $N^\mu N_\mu > 0$, if f does not vanish almost everywhere, which will be assumed in the following. It follows that the particle density n is well defined and positive. In order to see that the energy density is always positive we write it in the form

$$e = m_0 c^2 \int_{\mathbb{R}^3} (u_\mu q^\mu)^2\, f(t, \mathbf{x}, \mathbf{q})\, \frac{d^3 q}{q^0}\,. \tag{2.12}$$

(ii) The *macroscopic velocity* $\mathbf{v}$ of the gas can be obtained easily from the spatial part $\mathbf{u} = (u^1, u^2, u^3)^T$ of the dimensionless velocity four vector by

$$\mathbf{v} = c\,\frac{\mathbf{u}}{\sqrt{1+\mathbf{u}^2}}\,. \tag{2.13}$$

From this formula we can immediately read off that $|\mathbf{v}| < c$, i.e. the absolute value of the velocity is bounded by the speed of light. Note also that $u^0 = \sqrt{1+\mathbf{u}^2}$.

Let be $\mathbf{u} = (u^1, u^2, u^3)^T \in \mathbb{R}^3$ fixed. In order to present a short derivation of the relativistic Euler equations, we will later use the following relations for the so called **Lorentz boost** $\Lambda^{\alpha}_{\ \beta} = \Lambda^{\alpha}_{\ \beta}(\mathbf{u})$,

$$\begin{aligned} \Lambda^0_{\ 0} &= \sqrt{1+\mathbf{u}^2}\,,\quad \Lambda^0_{\ j} = \Lambda^j_{\ 0} = -u^j\,,\\ \Lambda^j_{\ k} &= \delta^j_{\ k} + \frac{u^j u^k}{1+\sqrt{1+\mathbf{u}^2}}\,. \end{aligned} \tag{2.14}$$

where $j,\ k \in \{1, 2, 3\}$ are spatial indices. Let G be the matrix form of (2.1), the above relations (2.14) implies

(a) $G = \Lambda(\mathbf{u})\, G\, \Lambda(\mathbf{u})^T$, $\Lambda^0_{\ 0}(\mathbf{u}) \geq 1$ and $\det(\Lambda(\mathbf{u})) = 1$, i.e. $\Lambda(\mathbf{u})$ is a proper Lorentz-matrix.
(b) $\Lambda^{-1}(\mathbf{u}) = \Lambda(-\mathbf{u})$.

The attribute 'proper' for n, e, p, T and σ denotes quantities, which are invariant with respect to proper Lorentz-transformations. They take their simplest form in the Lorentz rest frame. Since all quantities under consideration are written down in Lorentz-invariant form, we may omit the word 'proper' in the following.
These definitions are valid for any relativistic phase-density $f = f(t, \mathbf{x}, \mathbf{q})$, which has to be determined from a *kinetic equation* of the following form

$$q^{\mu}\frac{\partial f}{\partial x^{\mu}} = Q(f)\,. \tag{2.15}$$

As in the non-relativistic kinetic theory we have a corresponding transport part on the left-hand side and a collision part $Q(f)$ on the right-hand side. In the simplest case $Q(f)$ is determined in such a way that the following five conservation laws hold for the particle number, the energy, and the momentum

$$\frac{\partial N^{\mu}}{\partial x^{\mu}} = 0\,, \qquad \frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = 0\,. \tag{2.16}$$

This simple case holds if the particles interact only during elastic collisions without other forces and radiation. The case $Q(f) = 0$ leads to pure free flight, without any collision of the gas particles. A detailed study of the relativistic Boltzmann-equation can be found in the textbook of de Groot, van Leeuwen and van Weert [16].

Relativistic Jüttner Phase density:

Jüttner extended the classical velocity distribution of Maxwell for a gas in equilibrium to the relativistic case. The resulting *Jüttner distribution* $f_J(n, T, \mathbf{u}, \mathbf{q})$ depends on five constant parameters, which describe the state of the gas in equilibrium, namely the particle density n, the absolute temperature T and the spatial part $\mathbf{u} \in \mathbb{R}^3$ of the dimensionless four-velocity. It is given by

$$\begin{aligned} f_J(n, T, \mathbf{u}, \mathbf{q}) &= \frac{n}{M(\beta)} \exp\left(-\beta\, u_\mu q^\mu\right) \\ &= \frac{n}{M(\beta)} \exp\left(-\beta \left(\sqrt{(1+\mathbf{u}^2)(1+\mathbf{q}^2)} - \mathbf{u}\cdot\mathbf{q}\right)\right), \end{aligned} \tag{2.17}$$

where $\beta = \frac{m_0 c^2}{k_B T}$ and

$$\begin{aligned} M(\beta) &= \int_{\mathbb{R}^3} \exp(-\beta\sqrt{1+\mathbf{q}^2}) d^3q \\ &= 4\pi \int_0^\infty \vartheta^2 \exp(-\beta\sqrt{1+\vartheta^2})\, d\vartheta\,. \end{aligned} \tag{2.18}$$

The function $M(\beta)$ is chosen in such a way that

$$n\, u^\mu = \int_{\mathbb{R}^3} q^\mu\, f_J(n, T, \mathbf{u}, \mathbf{q})\,, \frac{d^3q}{q^0} \tag{2.19}$$

holds for the spatial part $\mathbf{u} = (u^1, u^2, u^3)^T$ of the dimensionless macroscopic velocity four-vector. This is equation (2.8), where $\mathbf{u}$ and n are in addition parameters of Jüttner's relativistic phase density.

Using the Bessel functions for the non-negative integer numbers j

$$K_j(\beta) = \int_0^\infty \cosh(js)\, \exp(-\beta\, \cosh(s))\, ds\,, \tag{2.20}$$

and applying the integral substitution $\vartheta = \sinh(s)$ we may also write $M(\beta)$ in the form

$$M(\beta) = \frac{4\pi}{\beta} K_2(\beta)\,. \tag{2.21}$$

We have in addition recursion relations for the modified Bessel functions, which can be found in the hand book of Jeffrey [19],

$$K_{j+1}(\beta) = \frac{2j}{\beta} K_j(\beta) + K_{j-1}(\beta)\,, \tag{2.22}$$

where j is the integer order of the modified Bessel functions. Using (2.20), (2.21) and (2.22) we can write

$$M'(\beta) = -4\pi\left(\frac{K_1(\beta)}{\beta} + \frac{3K_2(\beta)}{\beta^2}\right), \tag{2.23}$$

$$\eta(\beta) = 4\pi\int_0^\infty \frac{\vartheta^2}{\sqrt{1+\vartheta^2}}\exp(-\beta\sqrt{1+\theta^2})d\vartheta = \frac{4\pi}{\beta}K_1(\beta). \tag{2.24}$$

Note that

$$\eta'(\beta) = M(\beta). \tag{2.25}$$

The equations (2.21) and (2.23) will be used in Section 3 in order to find the constitutive relations for e, p, σ.

In order to formulate the Euler equations and other relations coming in next sections in a friendly form, we introduce the function

$$\Psi(\beta) = \frac{3}{\beta} + \frac{K_1(\beta)}{K_2(\beta)}. \tag{2.26}$$

In Sect. 3 it turns out that $\Psi(\beta)$ is just the specific energy $\frac{e}{n}$ for the gas in equilibrium.

Limiting cases of relativistic Jüttner Phase density:

Here we discuss two important special cases for this phase density, namely the non-relativistic limit for a cool gas and the ultra-relativistic limit $m_0 \to 0$.

Case 1 The non-relativistic limit (small temperatures, small velocities)

For the first case we rewrite (2.17) in the form

$$f_J(n, T, \mathbf{u}, \mathbf{q}) = \frac{n}{M_1(\beta)}\exp\left(-\beta\frac{(\mathbf{q}-\mathbf{u})^2 + \mathbf{u}^2\mathbf{q}^2 - (\mathbf{u}\cdot\mathbf{q})^2}{1+\sqrt{(1+\mathbf{u}^2)(1+\mathbf{q}^2)} + \mathbf{u}\cdot\mathbf{q}}\right), \tag{2.27}$$

where

$$M_1(\beta) = M(\beta)\,\exp(\beta). \tag{2.28}$$

If we apply for $\vartheta > 0$ the integral substitution

$$\xi = \sqrt{2\beta(\sqrt{1+\vartheta^2} - 1)}, \tag{2.29}$$

then we can rewrite $M_1(\beta)$ in the form

$$M_1(\beta) = \left(\frac{2\pi}{\beta}\right)^{\frac{3}{2}} \cdot 2 \int_0^{\infty} \left(1 + \frac{\xi^2}{2\beta}\right) \sqrt{1 + \frac{\xi^2}{4\beta}} \, \frac{\exp(-\frac{\xi^2}{2})}{\sqrt{2\pi}} \, d\xi \, . \tag{2.30}$$

For β very large compared to 1, i.e. for small temperature, we can conclude from (2.30) that

$$M_1(\beta) = \left(\frac{2\pi}{\beta}\right)^{\frac{3}{2}} + O\big(\beta^{-\frac{5}{2}}\big) = \left(\frac{2\pi K_B T}{m_0 c^2}\right)^{\frac{3}{2}} + O\big(T^{\frac{5}{2}}\big),$$

and the representation (2.27) shows that the Jüttner phase density reduces to the non-relativistic Maxwellian for $|\mathbf{u}|$, $|\mathbf{q}|$ very small, namely

$$f_c(n, T, \mathbf{u}, \mathbf{q}) = n \left(\frac{m_0 c^2}{2\pi K_B T}\right)^{\frac{3}{2}} \exp\left(-\frac{m_0 c^2 (\mathbf{q} - \mathbf{u})^2}{2 K_B T}\right) . \tag{2.31}$$

In order to get the non-relativistic limit (i.e. $\beta \gg 1$) from the formulations coming in next sections, we have the following asymptotic relations for $\beta \to \infty$,

$$\begin{aligned} K_1(\beta) &= \sqrt{\frac{\pi}{2\beta}} \exp(-\beta) \left(1 + \frac{3}{8\beta}\right) + O\left(\frac{\exp(-\beta)}{\beta^{\frac{5}{2}}}\right), \\ K_2(\beta) &= \sqrt{\frac{\pi}{2\beta}} \exp(-\beta) \left(1 + \frac{15}{8\beta}\right) + O\left(\frac{\exp(-\beta)}{\beta^{\frac{5}{2}}}\right), \\ \Psi(\beta) &= 1 + \frac{3}{2\beta} + O\left(\frac{\exp(-\beta)}{\beta^{\frac{5}{2}}}\right). \end{aligned} \tag{2.32}$$

Case 2 The ultra-relativistic limit (zero rest mass of the particles)

The ultra-relativistic limit results for $m_0 \to 0$ and fixed temperature. Then we obtain the ultra-relativistic phase density

$$\begin{aligned} f_J^*(n, T, \mathbf{u}, \mathbf{q}) &= \frac{n\tilde{\beta}^3}{8\pi} \exp\left(-\tilde{\beta}\, u_\mu q^\mu\right) \\ &= \frac{n\tilde{\beta}^3}{8\pi} \exp\left(-\tilde{\beta}|\mathbf{q}| \left(\sqrt{1 + \mathbf{u}^2} - \mathbf{u} \cdot \frac{\mathbf{q}}{|\mathbf{q}|}\right)\right) , \end{aligned} \tag{2.33}$$

where $\tilde{\beta} = \frac{\beta}{m_0} = \frac{c^2}{K_\beta T}$. This was done in details in [21]. Also to get the ultra-relativistic limit (i.e. $\beta \ll 1$) from the formulations coming in next sections, we have the following asymptotic relations for $\beta \to 0$

$$K_1(\beta) = \frac{1}{\beta} + O(\beta \ln \beta)\,, \tag{2.34}$$
$$K_2(\beta) = \frac{2}{\beta^2} + O(1)\,, \quad \Psi(\beta) = \frac{3}{\beta} + O(\beta).$$

Moments in dimensionless form:

Next we will introduce dimensionless quantities by setting $m_0 = c = k_B = \hbar = 1$. Then the relativistic moments and the entropy four-vector given in (2.3), (2.4) and (2.5) simply reduces to

$$N^\mu = N^\mu(t, \mathbf{x}) = \int_{\mathbb{R}^3} q^\mu\, f(t, \mathbf{x}, \mathbf{q})\, \frac{d^3 q}{q^0}\,, \tag{2.35}$$

$$T^{\mu\nu} = T^{\mu\nu}(t, \mathbf{x}) = \int_{\mathbb{R}^3} q^\mu q^\nu\, f(t, \mathbf{x}, \mathbf{q})\, \frac{d^3 q}{q^0}\,, \tag{2.36}$$

$$S^\mu = S^\mu(t, \mathbf{x}) = -\int_{\mathbb{R}^3} q^\mu\, f(t, \mathbf{x}, \mathbf{q}) \ln f(t, \mathbf{x}, \mathbf{q})\, \frac{d^3 q}{q^0}\,, \tag{2.37}$$

where f is first taken as Jüttner phase density as given in (2.17) in its dimensionless form.

3 Relativistic Euler equations

Using the relativistic Jüttner distribution (2.17), the relations (2.21) and (2.23), we calculate the moments (2.35), (2.36) and (2.37). In the global rest frame, where u^μ has components $(1, 0, 0, 0)^T$, we first obtain

$$\begin{aligned} e &= -n\frac{M'(\beta)}{M(\beta)} = n\Psi(\beta)\,, \\ p &= -\frac{n}{3M(\beta)}(M'(\beta) + \eta(\beta)) = nT = \frac{n}{\beta}\,, \\ \sigma &= -n \ln\left(\frac{n\beta}{K_2(\beta)}\right) + \beta n \Psi(\beta) + \gamma n\,. \end{aligned} \tag{3.1}$$

Here the choice of the entropy constant γ is not so important in general. Since e, p and σ are Lorentz invariant, we see that (3.1) is already true in any other Lorentz-frame, without the restriction that $(u^\mu) = (1, 0, 0, 0)^T$.

Starting with the rest frame and then applying the inverse of the Lorentz boost given by (2.14), we get the following relations for the general frame, where $\mathbf{u}$ may or may not be zero,

$$N^\mu = n\, u^\mu\,, \quad T^{\mu\nu} = -p\, g^{\mu\nu} + (e + p) u^\mu u^\nu\,, \quad S^\mu = \sigma u^\mu\,. \tag{3.2}$$

Moreover one can see by simple calculations that σ in $(3.1)_3$ obeys the Gibbs equation

$$T d\left(\frac{\sigma}{n}\right) = p\, d\left(\frac{1}{n}\right) + d\left(\frac{e}{n}\right). \tag{3.3}$$

Since the relativistic moments (3.2) are valid in a special Lorentz frame and since these equations are written in tensor invariant form, they are generally valid in every Lorentz frame. This can also be seen directly without making use of the Lorentz-boosts.

3.1 Maximum Entropy principle

The Jüttner distribution f_J (2.17) has some important properties. First of all it generalizes the classical Maxwellian of a gas in equilibrium to the relativistic case, and secondly f_J satisfies the so called Maximum-Entropy principle in equilibrium, which will be formulated and proved below. For this purpose we need the following lemmas.

Lemma 2.1 *Let be* $\mathbf{u}, \mathbf{q} \in \mathbb{R}^3$. *Also let be* $u^\mu = (\sqrt{1+\mathbf{u}^2}, \mathbf{u})^T$ *and* $q^\mu = (\sqrt{1+\mathbf{q}^2}, \mathbf{q})^T$. *Then the scalar product* $q^\mu u_\mu$ *satisfies the inequality* $q^\mu u_\mu \geq 1$, *where* $q^\mu u_\mu = 1$ *if and only if* $\mathbf{u} = \mathbf{q}$.

Proof. We consider

$$\begin{aligned} q^\mu u_\mu - 1 &= \sqrt{1+\mathbf{q}^2}\sqrt{1+\mathbf{u}^2} - \mathbf{q}\cdot\mathbf{u} - 1 \\ &= \frac{(\mathbf{q}-\mathbf{u})^2 + \mathbf{q}^2\mathbf{u}^2 - (\mathbf{q}\cdot\mathbf{u})^2}{\sqrt{1+\mathbf{q}^2}\sqrt{1+\mathbf{u}^2} + \mathbf{q}\cdot\mathbf{u} + 1}. \end{aligned} \tag{3.4}$$

Due to Cauchy-Schwarz we know that

$$\mathbf{q}^2\mathbf{u}^2 - (\mathbf{q}\cdot\mathbf{u})^2 \geq 0. \tag{3.5}$$

If $\mathbf{q} \neq \mathbf{u}$ then from (3.4) we have

$$(\mathbf{q}-\mathbf{u})^2 + \mathbf{q}^2\mathbf{u}^2 - (\mathbf{q}\cdot\mathbf{u})^2 > 0, \tag{3.6}$$

and this implies again from (3.4) that $q^\mu u_\mu - 1 > 0$ or $q^\mu u_\mu > 1$. □

Lemma 2.2 *The derivative* $\Psi' : \mathbb{R}^+ \to \mathbb{R}$ *has the representation*

$$\begin{aligned} \Psi'(\beta) &= \frac{d}{d\beta}\left(\frac{3}{\beta} + \frac{K_1(\beta)}{K_2(\beta)}\right) \\ &= -\frac{3}{\beta^2} + \frac{3}{\beta}\cdot\frac{K_1(\beta)}{K_2(\beta)} + \left(\frac{K_1(\beta)}{K_2(\beta)}\right)^2 - 1, \end{aligned} \tag{3.7}$$

and is negative for any $\beta > 0$. Moreover $\Psi(\beta)$ satisfies the inequality

$$\Psi(\beta) > 1\,, \tag{3.8}$$

which indicates that the specific energy is larger than the rest mass energy of a single atom.

Proof. We divide the proof of this lemma in two cases as follow.

Case 1 when $0 < \beta < 1$: From the definition of the modified Bessel functions we have $0 < K_1(\beta) < K_2(\beta)$. So we can write due to (3.7)

$$\begin{aligned} \Psi'(\beta) &= -\frac{3}{\beta^2} + \frac{3}{\beta} \cdot \frac{K_1(\beta)}{K_2(\beta)} + \left(\frac{K_1(\beta)}{K_2(\beta)}\right)^2 - 1 \\ &< -\frac{3}{\beta^2} + \frac{3}{\beta} + 1 - 1 = \frac{3}{\beta^2}(\beta - 1) < 0. \end{aligned} \tag{3.9}$$

Thus we have proved for $0 < \beta < 1$ that $\Psi'(\beta) < 0$.

Case 2 when $\beta \geq 1$: Recall the integral definition (2.20) for the modified Bessel functions $K_j(\beta)$. We also know that $\cosh(s) = 1 + 2\sinh^2(\frac{s}{2})$ and make the substitution $\alpha = \sinh(\frac{s}{2})$ with $ds = \frac{2d\alpha}{\sqrt{1+\alpha^2}}$ in (2.20) in order to get

$$\frac{1}{4}\exp\left(\frac{\beta}{4}\right) K_0\left(\frac{\beta}{4}\right) = \frac{1}{2}\int_0^\infty \frac{\exp\left(\frac{-\beta\alpha^2}{2}\right)}{\sqrt{1+\alpha^2}} d\alpha\,, \tag{3.10}$$

$$\frac{1}{4}\exp\left(\frac{\beta}{4}\right)\left[K_0\left(\frac{\beta}{4}\right) + K_1\left(\frac{\beta}{4}\right)\right] = \int_0^\infty \sqrt{1+\alpha^2}\exp\left(\frac{-\beta\alpha^2}{2}\right) d\alpha\,. \tag{3.11}$$

Also we have the following estimates

$$\frac{1}{\sqrt{1+\alpha^2}} \geq 1 - \frac{\alpha^2}{2}\,, \qquad \sqrt{1+\alpha^2} \leq 1 + \frac{\alpha^2}{2}\,. \tag{3.12}$$

Keeping in view (3.10) and (3.11), we obtain from (3.12)

$$\frac{1}{4}\exp\left(\frac{\beta}{4}\right) K_0\left(\frac{\beta}{4}\right) \geq \frac{1}{2}\sqrt{\frac{\pi}{2\beta}}\left(1 - \frac{1}{2\beta}\right)\,, \tag{3.13}$$

$$\frac{1}{4}\exp\left(\frac{\beta}{4}\right)\left[K_0\left(\frac{\beta}{4}\right) + K_1\left(\frac{\beta}{4}\right)\right] \leq \sqrt{\frac{\pi}{2\beta}}\left(1 + \frac{1}{2\beta}\right)\,. \tag{3.14}$$

Note that also the right hand side of (3.13) is positive due to $\beta \geq 1$. We take the inverse of (3.13) and multiply with (3.14) in order to get

$$1 + \frac{K_1\left(\frac{\beta}{4}\right)}{K_0\left(\frac{\beta}{4}\right)} \leq 2\left(1 + \frac{1}{2\beta}\right) \cdot \left(1 - \frac{1}{2\beta}\right)^{-1}\,. \tag{3.15}$$

Using the recursion relation (2.22) for $j = 1$ and replacing β by $\frac{\beta}{4}$ we get

$$\frac{K_2\left(\frac{\beta}{4}\right)}{K_1\left(\frac{\beta}{4}\right)} - \frac{K_0\left(\frac{\beta}{4}\right)}{K_1\left(\frac{\beta}{4}\right)} = \frac{8}{\beta}\,. \tag{3.16}$$

Now using (3.15) and (3.16) we get the following inequality after some manipulations and replacing $\frac{\beta}{4}$ by β

$$\frac{K_1(\beta)}{K_2(\beta)} \le \frac{1+\frac{3}{8\beta}}{1+\frac{15}{8\beta}+\frac{3}{4\beta^2}}\,. \tag{3.17}$$

Substituting (3.17) in (3.7) for $\frac{K_1(\beta)}{K_2(\beta)}$, we finally get after simplification

$$\Psi'(\beta) \le -\frac{9}{64\beta^6}\cdot\frac{12+60\beta+105\beta^2+69\beta^3+8\beta^4}{\left(1+\frac{15}{8\beta}+\frac{3}{4\beta^2}\right)^2} < 0\,. \tag{3.18}$$

It follows that $\Psi'(\beta) < 0$ for any $\beta > 0$.

Hence we have proved that $\Psi(\beta)$ is a strictly monotonically decreasing function which satisfies due to $(2.32)_3$ the asymptotic relation

$$\lim_{\beta\to\infty}\Psi(\beta) = 1\,. \tag{3.19}$$

Thus we conclude that $\Psi(\beta)$ is strictly bounded below by one. □

Lemma 2.3

(i) Let be $f = f(\mathbf{q}) \ge 0$ for $\mathbf{q} \in \mathbb{R}^3$ any phase density which does not vanish almost everywhere and for which the moments N^μ, $T^{\mu\nu}$ exist. Let be n and e the corresponding particle density and energy density, respectively. Then there holds the inequality chain $0 < n < e$.

(ii) Let be $0 < n < e$ and $\mathbf{u} \in \mathbb{R}^3$ given parameters, corresponding to the particle density, energy density and the spatial part of the macroscopic four-velocity. Then there exists exactly one temperature $T > 0$ such that the Jüttner phase density $f_J(n, T, \mathbf{u}, \mathbf{q})$ gives the prescribed energy density $e > n$.

Proof. In order to prove (i) we use from Lemma 2.1 that $q^\mu u_\mu > 1$ for $\mathbf{q} \neq \mathbf{u}$. Therefore we can write

$$(q^\mu u_\mu)(q^\nu u_\nu) > q^\mu u_\mu\,. \tag{3.20}$$

This implies that

$$\int_{\mathbb{R}^3}(q^\mu u_\mu)(q^\nu u_\nu) f\,\frac{d^3q}{q^0} > \int_{\mathbb{R}^3} q^\mu u_\mu f\,\frac{d^3q}{q^0}\,, \tag{3.21}$$

Now using the definitions (2.35), (2.36) for N^μ and $T^{\mu\nu}$, respectively, we can write (3.21) in the following form

$$u_\mu u_\nu T^{\mu\nu} > u_\mu N^\mu \,. \tag{3.22}$$

Now using the definitions (2.8) and (2.9) for n and e we finally conclude that $e > n$.

Now we prove the part (ii). We know from the part (i) of this lemma that the restriction $e > n > 0$ is necessary. Moreover we know from Lemma 2.2 that $\frac{e}{n} = \Psi(\beta) > 1$ has exactly one solution for $\beta > 0$. Let be $T = \frac{1}{\beta} > 0$ the corresponding temperature. Then we know from $(3.1)_1$ that the Jüttner phase density $f_J(n, T, \mathbf{u}, \mathbf{q})$ leads to the prescribed energy density e.

Remark. The restriction $e > n$ is also natural from the physical point of view since it states that the energy density is always larger then the rest-mass energy.

Lemma 2.4. *For* $u, v > 0$ *we have*

$$v \ln v - u \ln u = [\ln u + 1](v - u) + R(u, v)\,, \tag{2.23}$$

with $R(u, v) \geq 0$.

Proof. Due to Taylors formula there is a $\xi > 0$ between $u, v > 0$ such that

$$v \ln v = u \ln u + (\ln u + 1)(v - u) + \frac{1}{2\xi}(v - u)^2. \tag{3.24}$$

We conclude $R(u, v) = \frac{1}{2\xi}(v - u)^2 \geq 0$. □

Proposition 2.1 (Maximum Entropy Principle): *Let* $f(\mathbf{q}) \geq 0$ *be any phase density which does not vanish everywhere and for which the moments* N^μ *and* $T^{\mu\nu}$ *exist. Let* $n,\ \mathbf{u},\ e$ *be the values resulting from* f *for the particle density, the spatial part of the velocity four vector and the energy density, respectively. Then there is exactly one temperature* $T > 0$ *for which the Jüttner phase density* $f_J(n, T, \mathbf{u}, \mathbf{q})$ *leads to the prescribed energy density e. Let be* σ *and* σ_J *the entropy densities corresponding to* f *and* f_J*, respectively. Then there holds the Maximum Entropy Inequality* $\sigma_J \geq \sigma$.

Proof. Due to Lemma 2.3 we have exactly one temperature $T > 0$ for which the Jüttner phase density $f_J(n, T, \mathbf{u}, \mathbf{q})$ gives the prescribed energy density $e > n$ coming from the general phase density $f(\mathbf{q})$. In the following proof we will fix this temperature T and the corresponding phase density $f_J(n, T, \mathbf{u}, \mathbf{q})$. Using the definition (2.37) and (2.11) we have due to the constraint on $\mathbf{u}$

$$\begin{aligned}\sigma_J - \sigma &= S_J^\mu u_\mu - S^\mu u_\mu \\ &= -u_\mu \int_{\mathbb{R}^3} q^\mu \left[(f_J \ \ln f_J)(n, T, \mathbf{u}, \mathbf{q}) - (f \ \ln f)(\mathbf{q})\right] \frac{d^3q}{q^0}\,.\end{aligned} \tag{3.25}$$

In the following we omit the arguments of f and f_J for the sake of simplicity, which will not lead to confusion here. We use Lemma 2.4 for $u = f_J$, $v = f$ and get

$$f \ln f - f_J \ln f_J = (\ln f_J + 1)(f - f_J) + R(f_J, f), \tag{3.26}$$

Using equation (3.26) in (3.25) we have

$$\begin{aligned}\sigma_J - \sigma = {} & u_\mu \int_{\mathbb{R}^3} q^\mu \ln f_J\,(f - f_J)\,\frac{d^3q}{q^0} + u_\mu \int_{\mathbb{R}^3} q^\mu\,(f - f_J)\frac{d^3q}{q^0} \\ & + u_\mu \int_{\mathbb{R}^3} q^\mu\, R(f_J, f)\,\frac{d^3q}{q^0}\,.\end{aligned} \tag{3.27}$$

The second integral in (3.27) is zero due to the constraints on n and $\mathbf{u}$,

$$u_\mu N^\mu - u_\mu N_J^\mu = n - n = 0\,, \tag{3.28}$$

so we are left with

$$\begin{aligned}\sigma_J - \sigma = {} & u_\mu \int_{\mathbb{R}^3} q^\mu \ln f_J\,(f - f_J)\,\frac{d^3q}{q^0} \\ & + u_\mu \int_{\mathbb{R}^3} q^\mu\, R(f_J, f)\,\frac{d^3q}{q^0}\,.\end{aligned} \tag{3.29}$$

From Jüttner's phase density we have

$$\begin{aligned}\ln f_J &= \ln\left(\frac{n}{M(\beta)}\exp\left(-\frac{1}{T}u_\nu q^\nu\right)\right) \\ &= \ln\left(\frac{n}{M(\beta)}\right) - \frac{1}{T}u_\nu q^\nu\,.\end{aligned} \tag{3.30}$$

Using (3.30) in (3.29) and the fact that n and T are independent of the integration variable $\mathbf{q}$ we get

$$\begin{aligned}\sigma_J - \sigma = {} & \ln\left(\frac{n}{M(\beta)}\right) u_\mu \int_{\mathbb{R}^3} q^\mu\,(f - f_J)\,\frac{d^3q}{q^0} \\ & - \frac{1}{T}\,u_\mu u_\nu \int_{\mathbb{R}^3} q^\mu q^\nu\,(f - f_J)\frac{d^3q}{q^0} \\ & + u_\mu \int_{\mathbb{R}^3} q^\mu\, R(f_J, f)\,\frac{d^3q}{q^0}\,.\end{aligned} \tag{3.31}$$

In (3.31) the first integral is zero due to (3.28). Also we know from (2.9) and our constraints on the velocity and energy density that

$$e = u_\mu u_\nu T^{\mu\nu} = e_J = u_\mu u_\nu T_J^{\mu\nu}\,, \tag{3.32}$$

where $T^{\mu\nu}$ and $T_J^{\mu\nu}$ are the energy momentum tensors for f and f_J, respectively. Thus equation (3.31) finally reduces to

$$\sigma_J - \sigma = \int_{\mathbb{R}^3} u_\mu q^\mu \, R(f_J, f) \, \frac{d^3 q}{q^0} \geq 0. \tag{3.33}$$

The integral in (3.33) is positive because $u_\mu q^\mu$ is the scalar product of time-like vectors which is positive due to Lemma 2.1 and $R(f_J, f)$ is positive due to Lemma 2.4. Hence we have proved that Jüttner's phase density satisfies the Maximum Entropy Principle, i.e. $\sigma_J \geq \sigma$. □

3.2 The Constitutive Relations

In order to get the non-relativistic values of the energy density, pressure and entropy density, we substitute the asymptotic relations (2.32) for $K_2(\beta)$ and $\Psi(\beta)$ into the constitutive relations (3.1). We get the following expressions for the non-relativistic limit, where $\beta \to \infty$,

$$e = n + \frac{3}{2} nT \,, \qquad p = nT \,, \qquad \sigma = n \ln \frac{p^{\frac{3}{2}}}{n^{\frac{5}{2}}} + \gamma n \,. \tag{3.34}$$

For the detailed study of the classical case the reader is refered to Dreyer and Kunik [6]. In this paper the particle density n is replaced by the mass density ρ and entropy density σ is denoted by h. The term n in $(3.34)_1$ is the energy density of the rest mass.

In order to get ultra-relativistic limit where $\beta \to 0$, we substitute the asymptotic relations (2.34) into the constitutive relations (3.1) and (3.2), we get

$$e = 3nT \,, \qquad p = nT \,, \qquad \sigma = n \ln \frac{p^3}{n^4} + \gamma n \,, \tag{3.35}$$

$$N^\mu = n\, u^\mu \,, \; T^{\mu\nu} = -p\, g^{\mu\nu} + 4 p u^\mu u^\nu \,,$$

$$S^\mu = N^\mu \, \ln \frac{p^3}{n^4} + \gamma N^\mu \,. \tag{3.36}$$

For the detailed study of the ultra-relativistic case we refer the reader to Kunik, Qamar and Warnecke [21].

Now we use the moments (3.2) and the conservation laws (2.16) in order to get the three dimensional Euler equations at regular points in differential form. For this purpose we introduce the abbreviation

$$\chi(\beta) = \Psi(\beta) + \frac{1}{\beta} \,, \tag{3.37}$$

and obtain for $p = \frac{n}{\beta}$

$$\frac{\partial}{\partial t}(n\sqrt{1+\mathbf{u}^2}) + \sum_{k=1}^{3} \frac{\partial (nu^k)}{\partial x^k} = 0\,, \tag{3.38}$$

$$\frac{\partial}{\partial t}\left(n\chi(\beta)u^i\sqrt{1+\mathbf{u}^2}\right) + \sum_{k=1}^{3} \frac{\partial}{\partial x^k}\left(p\delta^{ik} + n\chi(\beta)u^i u^k\right) = 0\,, \tag{3.39}$$

$$\frac{\partial}{\partial t}\left(-p + n\chi(\beta)(1+\mathbf{u}^2)\right) + \sum_{k=1}^{3} \frac{\partial}{\partial x^k}\left(n\chi(\beta)u^k\sqrt{1+\mathbf{u}^2}\right) = 0\,. \tag{3.40}$$

Remarks.

(i) These equations constitutes a closed system in terms of the unknown fields n, $\mathbf{u}$, and β. Also note that $\beta = \frac{1}{T}$.

(ii) The classical Euler equations result with (2.32) in the following way: From (3.38) we obtain the classical continuity equation by neglecting the second order terms in $\mathbf{u}$, whereas the classical momentum equations are obtained from (3.39) by setting $\chi(\beta)$ equal to one and neglecting the third order terms in $\mathbf{u}$. Finally, the classical energy equation results if we subtract (3.38) from (3.40) and then neglect the fourth order terms in $\mathbf{u}$ and terms which contain $p\mathbf{u}^2$ as a factor.

(iii) The ultra-relativistic Euler equations result directly from (2.34).

Now we are looking for spatial one-dimensional solutions of the three dimensional Euler equations, which will not depend on x^2, x^3 but only on $x = x^1$. Moreover we restrict to a one-dimensional flow field $\mathbf{u} = (u(t,x), 0, 0)^T$

$$\begin{aligned}(n\sqrt{1+u^2})_t + (nu)_x &= 0\,,\\ (n\chi(\beta)u\sqrt{1+u^2})_t + (\tfrac{n}{\beta} + n\chi(\beta)u^2)_x &= 0\,,\\ (-\tfrac{n}{\beta} + n\chi(\beta)(1+u^2))_t + (n\chi(\beta)u\sqrt{1+u^2})_x &= 0\,.\end{aligned} \tag{3.41}$$

Note that these differential equations constitute a strictly hyperbolic system with the sound speed

$$\lambda_* = \left[\frac{1 - 1/(\beta^2\Psi'(\beta))}{1+\beta\Psi(\beta)}\right]^{\frac{1}{2}}\,, \tag{3.42}$$

and the characteristic velocities

$$\begin{aligned}\lambda_1 &= \frac{-\lambda_*\sqrt{1+u^2}+u}{\sqrt{1+u^2}-\lambda_* u}\,,\quad \lambda_2 = \frac{u}{\sqrt{1+u^2}}\,,\\ \lambda_3 &= \frac{\lambda_*\sqrt{1+u^2}+u}{\sqrt{1+u^2}+\lambda_* u}\,.\end{aligned} \tag{3.43}$$

Note that $\Psi'(\beta) < 0$ was proved in Lemma 2.2.

These eigenvalues may first be obtained in the Lorentz zero rest frame where u=0. Then using the additivity law for the velocities in the general Lorentz frame we can easily obtain (3.43).

Note that one can easily get from (3.43), the eigenvalues for the classical and ultra-relativistic cases by using the asympotatic relations (2.32) and (2.34) respectively as given below.

Classical eigenvalues:

$$\lambda_1 = v - \sqrt{\frac{5}{3}T}\,, \quad \lambda_2 = v\,, \quad \lambda_3 = v + \sqrt{\frac{5}{3}T}\,, \tag{3.44}$$

with the velocity $v = \frac{u}{\sqrt{1+u^2}}$, the temperature $T = \frac{1}{\beta}$ and the speed of sound $\lambda_* = \sqrt{\frac{5}{3}T}$.

Ultra-relativistic eigenvalues:

$$\begin{aligned} \lambda_1 &= \frac{2u\sqrt{1+u^2}-\sqrt{3}}{3+2u^2}\,, \ \lambda_2 = \frac{u}{\sqrt{1+u^2}}\,, \\ \lambda_3 &= \frac{2u\sqrt{1+u^2}+\sqrt{3}}{3+2u^2}\,. \end{aligned} \tag{3.45}$$

Here we obtain the speed of sound $\lambda_* = \frac{1}{\sqrt{3}}$.

The differential equations (3.41) are not sufficient if we take shock discontinuities into account. Therefore we need a week integral formulation of the one-dimensional hyperbolic system for the three unknown fields n, u and β, which is given by

$$\begin{aligned} \oint_{\partial\Omega} n\sqrt{1+u^2}dx - nudt &= 0\,, \\ \oint_{\partial\Omega} (n\chi(\beta)u\sqrt{1+u^2})dx - (\tfrac{n}{\beta} + n\chi(\beta)u^2)dt &= 0\,, \\ \oint_{\partial\Omega} (-\tfrac{n}{\beta} + n\chi(\beta)(1+u^2))dx - (n\chi(\beta)u\sqrt{1+u^2})dt &= 0\,. \end{aligned} \tag{3.46}$$

Here $\Omega \subset \mathbb{R} \times \mathbb{R}_0^+$ is a normal region in space-time with piecewise smooth, positive oriented boundary. Note that this weak formulation takes discontinuities into account, since there are no longer derivatives of the fields. If we apply the Gauss Divergence Theorem in regular time-space regions to the weak formulation (3.46) we come back to the differential form of Euler's equation (3.41).

Furthermore we require that the weak solution (3.46) must also satisfy the one-dimensional *entropy-inequality*

$$\oint_{\partial\Omega} S^0 dx - S^1 dt \geq 0\,, \tag{3.47}$$

where

$$(3.48)\qquad \begin{aligned} S^0 &= -n\sqrt{1+u^2}\left(\ln \tfrac{n\beta}{K_2(\beta)} + \beta\Psi(\beta)\right), \\ S^1 &= -nu\left(\ln \tfrac{n\beta}{K_2(\beta)} + \beta\Psi(\beta)\right). \end{aligned}$$

Now we consider bounded and integrable *initial data* for a positive particle density n, transformed velocity u and absolute temperature T, which may have jumps

$$(3.49)\quad n(0,x) = n_0(x) > 0, \quad u(0,x) = u_0(x), \quad T(0,x) = T_0(x) > 0.$$

If $x = x(t)$ is a shock-discontinuity of the weak solution (3.46) with speed $v_s = \dot{x}(t)$, $W_- = (n_-, u_-, p_-)$ the state left to the shock and $W_+ = (n_+, u_+, p_+)$ the state to the right, then (3.46) leads to the RANKINE-HUGONIOT *jump conditions*

$$(3.50)\qquad \begin{aligned} v_s(N^0_+ - N^0_-) &= N^1_+ - N^1_-, \\ v_s(T^{01}_+ - T^{01}_-) &= T^{11}_+ - T^{11}_-, \\ v_s(T^{00}_+ - T^{00}_-) &= T^{01}_+ - T^{01}_-, \end{aligned}$$

where

$$\begin{aligned} &N^0_\pm = n_\pm\sqrt{1+u^2_\pm}, \quad N^1_\pm = n_\pm u_\pm, \quad T^{01}_\pm = n_\pm\chi(\beta_\pm)u_\pm\sqrt{1+u^2_\pm}, \\ &T^{11}_\pm = p_\pm + n_\pm\chi(\beta_\pm)u^2_\pm, \quad T^{00}_\pm = -p_\pm + n_\pm\chi(\beta_\pm)(1+u^2_\pm). \end{aligned}$$

Also in singular points the local form of (3.47) reads

$$(3.51)\qquad -v_s(S^0_+ - S^0_-) + (S^1_+ - S^1_-) \geq 0,$$

which must be satisfied at each shock curve of (3.46). The shock that satisfies (3.50) and (3.51) is called *entropy shock*.

Now we give parameter representations for the single entropy shocks. For this purpose we choose the initial data as follows:

Let be $(n_*, u_*, \beta_*) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ and define $p_* = n_* T_* = \frac{n_*}{\beta_*}$.

We use the inverse temperature β as a shock-parameter and impose the restriction $\beta < \beta_*$ in order to obtain from (3.50) and (3.51) the following parametrization of the particle density and the pressure

$$(3.52)\qquad \begin{aligned} \gamma(\beta) &= \Psi(\beta_*)\,\chi(\beta_*) - \Psi(\beta)\,\chi(\beta), \\ n(\beta) &= \frac{n_*\beta}{2\chi(\beta_*)}\left[\sqrt{\gamma(\beta)^2 + 4\frac{\chi(\beta_*)\chi(\beta)}{\beta_*\beta}} - \gamma(\beta)\right], \\ p(\beta) &= \frac{n(\beta)}{\beta}. \end{aligned}$$

For the parametrization of the velocities we have

$$
\begin{aligned}
\hat{u}(\beta) &= \left(\frac{(p(\beta) - p_*) \cdot (n(\beta)\,\Psi(\beta) - n_*\Psi(\beta_*))}{n(\beta)\, n_* \chi(\beta) \chi(\beta_*)} \right)^{\frac{1}{2}} , \\
u(\beta) &= u_* \sqrt{1 + \hat{u}(\beta)^2} \pm \hat{u}(\beta) \sqrt{1 + u_*^2} ,
\end{aligned}
$$

$$
\hat{u}_s(\beta) = \left[\frac{(p(\beta) - p_*)\,(p_* + n(\beta)\Psi(\beta))}{n_*\,\chi(\beta_*) \left[n(\beta)\,(\Psi(\beta) - \frac{1}{\beta}) - n_*(\Psi(\beta_*) - \frac{1}{\beta_*}) \right]} \right]^{\frac{1}{2}} , \tag{3.53}
$$

$$
\begin{aligned}
u_s(\beta) &= u_* \sqrt{1 + \hat{u}_s(\beta)^2} \pm \hat{u}_s(\beta) \sqrt{1 + {u_*}^2} , \\
v_s &= \frac{u_s}{\sqrt{1 + u_s^2}} , \; v = \frac{u}{\sqrt{1 + u^2}} , \; v_* = \frac{u_*}{\sqrt{1 + u_*^2}} .
\end{aligned}
$$

Remarks. The restriction $\beta < \beta_*$ guarantees that all the expressions under the square roots in (3.52) and (3.53) are positive because

(i) $\gamma(\beta)$ is negative since $\Psi(\beta) > 0$ and $\chi(\beta) > 0$ are strictly monotonically decreasing functions due to Lemma 2.2,

(ii) $n(\beta)$ is a strictly monotonically decreasing and positive function,

(iii) $\Psi(\beta) - \frac{1}{\beta}$ is also a strictly monotonically decreasing and positive function, the same with $p(\beta) > p_*$.

Now there results the following parametrization for the different kind of shock waves:

- The "+" sign in (3.53) and $\beta < \beta_*$ gives the so called *3-shocks* with the constant state (n_*, u_*, β_*) on the right

  $$(n_-, u_-, \beta_-) = (n(\beta), u(\beta), \beta), \quad (n_+, u_+, \beta_+) = (n_*, u_*, \beta_*).$$

  These 3-shocks satisfy both the Rankine-Hugoniot conditions (3.50) as well as the entropy condition (3.51).
- The "-" sign in (3.53) and $\beta < \beta_*$ gives the so called *1-shocks* with the constant state (n_*, u_*, β_*) on the left:

  $$(n_-, u_-, \beta_-) = (n_*, u_*, \beta_*), \quad (n_+, u_+, \beta_+) = (n(\beta), u(\beta), \beta).$$

  These 1-shocks satisfy both the Rankine-Hugoniot conditions (3.50) as well as the entropy condition (3.51).

Now we define the *2-shocks*, that turn out to be *contact-discontinuities without entropy-production*:

Only for these we choose $n > 0$ instead of β as a shock parameter and set

$$(n_-, u_-, \beta_-) = (n_*, u_*, \beta_*), \quad (n_+, u_+, \beta_+) = \left(n, u_*, \frac{n\beta_*}{n_*} \right).$$

These shocks satisfy the RANKINE-HUGONIOT- and entropy conditions. Note that velocity and pressure are constant across a 2-shock. Here the shock-speed is $v_s = v_* = \frac{u_*}{\sqrt{1+u_*^2}}$.

Remark. One can prove that the only shocks satisfying (3.50) and (3.51) are 1-, 2- and 3-shocks.

4 The kinetic scheme for the Relativistic Euler equations

We first formulate the scheme for the three dimensional Euler equations. After that we solve the one dimensional Euler equations, using a special integration technique. Recalling the relativistic Jüttner phase density (2.17), we start with the given initial data $n_I(\mathbf{x}) = n(0, \mathbf{x})$, $T_I(\mathbf{x}) = T(0, \mathbf{x})$, $\mathbf{u}_I(\mathbf{x}) = \mathbf{u}(0, \mathbf{x})$. We prescribe a time step $\tau_M > 0$ and let $t_n = n\,\tau_M$, $n = 0, 1, 2, 3....$ Then we define the moments and the entropy four-vector in the free flight for $0 < \tau < \tau_M$ as

$$(4.1)\qquad \begin{aligned} N^\mu(t_n+\tau, \mathbf{x}) &= \int_{\mathbb{R}^3} q^\mu\, f_n(\mathbf{x} - \tau\frac{\mathbf{q}}{q^0}, \mathbf{q})\frac{d^3q}{q^0}\,, \\ T^{\mu\nu}(t_n+\tau, \mathbf{x}) &= \int_{\mathbb{R}^3} q^\mu q^\nu\, f_n(\mathbf{x} - \tau\frac{\mathbf{q}}{q^0}, \mathbf{q})\frac{d^3q}{q^0}\,, \end{aligned}$$

$$(4.2)\qquad S^\mu(t_n+\tau, \mathbf{x}) = -\int_{\mathbb{R}^3} q^\mu\,(f_n \ln f_n)(\mathbf{x} - \tau\frac{\mathbf{q}}{q^0}, \mathbf{q})\frac{d^3q}{q^0}\,,$$

with the relativistic initial phase density at the maximization time t_n

$$(4.3)\qquad f_n(\mathbf{y}, \mathbf{q}) = f_J(n(t_n, \mathbf{y}), T(t_n, \mathbf{y}), \mathbf{u}(t_n, \mathbf{y}), \mathbf{q})\,.$$

Moreover n, T, u^μ are calculated from N^μ and $T^{\mu\nu}$ for the next time step from the following generally valid definitions

$$(4.4)\qquad n = \sqrt{N^\mu N_\mu}, \quad u^\mu = \frac{1}{n}N^\mu\,, \quad T = \frac{1}{3n}(u_\mu u_\nu - g_{\mu\nu})\,T^{\mu\nu}\,.$$

In order to initialize the kinetic scheme for the next time step, we first require the following continuity conditions for the zero-components of the moments across the maximization time t_n, $n \geq 1$

$$(4.5)\qquad \begin{aligned} N^0(t_n^+, \mathbf{x}) &= N^0(t_n^-, \mathbf{x})\,, \\ T^{0k}(t_n^+, \mathbf{x}) &= T^{0k}(t_n^-, \mathbf{x}),\ k = 1, 2, 3\,, \\ T^{00}(t_n^+, \mathbf{x}) &= T^{00}(t_n^-, \mathbf{x})\,. \end{aligned}$$

Here we have used the following abbreviations for the one-sided limits across the maximization time t_n, $n \geq 1$, where ε is a positive real number

$$N^{\mu}(t_n^{\pm}, \mathbf{x}) = \lim_{\varepsilon \to 0} N^{\mu}(t_n \pm \varepsilon, \mathbf{x}),$$
$$T^{\mu\nu}(t_n^{\pm}, \mathbf{x}) = \lim_{\varepsilon \to 0} T^{\mu\nu}(t_n \pm \varepsilon, \mathbf{x}).$$

These conditions are necessary in order to guarantee the conservation laws for mass, momentum and energy across the maximization time t_n. Moreover we start again with a relativistic Jüttner distribution for the next time step. Then we obtain, using the constitutive relations (3.2) for the three dimensional Euler equations which are valid for the t_n^+ side of the maximization time

$$\begin{aligned} N^0(t_n^+, \mathbf{x}) &= n(t_n^+, \mathbf{x})\sqrt{1 + \mathbf{u}^2(t_n^+, \mathbf{x})}, \\ T^{0k}(t_n^+, \mathbf{x}) &= n(t_n^+, \mathbf{x})\chi(\beta(t_n^+, \mathbf{x}))u^k(t_n^+, \mathbf{x})\sqrt{1 + \mathbf{u}^2(t_n^+, \mathbf{x})}, \\ T^{00}(t_n^+, \mathbf{x}) &= -p(t_n^+, \mathbf{x}) + n(t_n^+, \mathbf{x})\chi(\beta(t_n^+, \mathbf{x}))(1 + \mathbf{u}^2(t_n^+, \mathbf{x})). \end{aligned} \tag{4.6}$$

Here $k = 1, 2, 3$ again denote a spatial index. Since these components of the moments are continuous across the maximization time t_n, we can replace them by the free-flight moments for t_n^- and solve the equations (4.6) for $p, \mathbf{u}, n$ in order to initialize the kinetic scheme for the next time step.
Using for positive γ the definitions

$$\gamma^2 = \frac{\sum_{k=1}^{3}(T^{0k})^2}{(N^0)^2}, \quad F(\beta) = \frac{T^{00}}{N^0}\sqrt{1 + \frac{\gamma^2}{\chi(\beta)^2}} - \frac{\gamma^2}{\chi(\beta)} - \psi(\beta), \tag{4.7}$$

we will find the following relations for the fields at the next time step

$$F(\beta(t_n^+, \mathbf{x})) = 0, \tag{4.8}$$

$$u^k(t_n^+, \mathbf{x}) = \frac{T^{0k}}{N^0\,\chi(\beta(t_n^+, \mathbf{x}))}, \tag{4.9}$$

$$n(t_n^+, \mathbf{x}) = \frac{N^0}{\sqrt{1 + \frac{\gamma^2}{\chi(\beta(t_n^+, \mathbf{x}))^2}}}. \tag{4.10}$$

In the formulas (4.7)-(4.10), N^0, T^{00} and T^{0k} are abbreviations for the free flight moments $N^0(t_n^-, \mathbf{x})$, $T^{00}(t_n^-, \mathbf{x})$ and $T^{0k}(t_n^-, \mathbf{x})$, respectively.

We first solve the implicit equation $F(\beta(t_n^+, \mathbf{x})) = 0$ by using Newton's method. We have checked that for sufficiently small value of β the function $F(\beta)$ is negative, while $F(\beta)$ has a positive limit for $\beta \to \infty$. So we conclude from the mean-value theorem that $F(\beta)$ has at least one real root. Unfortunately it is very difficult to prove that $F(\beta)$ is a monotonically increasing function, which could only be checked numerically in many cases. Once we get the value of $\beta(t_n^+, \mathbf{x})$, the other quantities $u^k(t_n^+, \mathbf{x})$ and $n(t_n^+, \mathbf{x})$ can be easily calculated in the prescribed order from the free flight moments N^0, T^{00} and T^{0k}. Since the continuity conditions initialize the scheme for the next time step they conclude the formulation of the kinetic scheme.

Proposition 4.2 *Let $\Omega \subset \mathbb{R}_0^+ \times \mathbb{R}^3$ be any bounded convex region in space and time. By do_ν we denote the positive oriented boundary element of $\partial\Omega$. Let $\tau_M > 0$ be a fixed time step. The moment representations* (4.1) *and* (4.2) *calculated by the iterated scheme have the following properties:*

(i) There hold the conservation laws for the particle number, the momentum and energy:

$$\oint_{\partial\Omega} N^\nu do_\nu = 0, \quad \oint_{\partial\Omega} T^{\mu\nu} do_\nu = 0. \tag{4.11}$$

(ii) The following entropy inequality is satisfied:

$$\oint_{\partial\Omega} S^\nu do_\nu \geq 0. \tag{4.12}$$

Here the covariant vector do_ν is a positive oriented surface element to the boundary $\partial\Omega$. It can be written in covariant form as

$$do_\kappa = \varepsilon_{\kappa\lambda\mu\nu} \sum_{i,j,m=1}^{3} \frac{\partial x^\lambda}{\partial u^i} \frac{\partial x^\mu}{\partial u^j} \frac{\partial x^\nu}{\partial u^m} du^i\, du^j\, du^m \,,$$

where $x^\alpha = x^\alpha(u^1, u^2, u^3)$ is a positive oriented parameterization of the boundary $\partial\Omega$.

Remark. Note that these weak formulations are the weak representations of the differential forms

$$\begin{aligned} &\frac{\partial N^\nu}{\partial x^\nu}(t_n + \tau, \mathbf{x}) = 0, \qquad \frac{\partial T^{\mu\nu}}{\partial x^\nu}(t_n + \tau, \mathbf{x}) = 0, \\ &\frac{\partial S^\nu}{\partial x^\nu}(t_n + \tau, \mathbf{x}) = 0. \end{aligned} \tag{4.13}$$

The proof of this proposition runs along the same lines as was carried out in [6], [21] for the classical and ultra-relativistic Euler equations, and may therefore be omitted.

5 The one-dimensional kinetic scheme

In the following we are looking for the spatially one dimensional solutions, which are nevertheless solutions to the full three dimensional equations. In order to initialize the scheme, we consider the initial data

$$n_I(\mathbf{x}) = n(0, x^1),\ \mathbf{u}_I(\mathbf{x}) = (u(0, x^1), 0, 0)^T,\ p_I(\mathbf{x}) = p(0, x^1). \tag{5.1}$$

This special form of the initial data will lead to a kinetic solution of the same form for all later times $t > 0$ and $x = x^1$, namely

$$n = n(t, x),\ \mathbf{u} = (u(t, x), 0, 0),\ p = p(t, x). \tag{5.2}$$

Thus we have got a one dimensional solution from the three dimensional kinetic scheme. We will use the (generally valid) equation $p = nT = \frac{n}{\beta}$ and start with the full three dimensional scheme.

In order to formulate the scheme we first consider the rest frame where $u^0 = 1$ and $u = 0$. Then we will apply a Lorentz transformation in order to come back to a general frame where u may or may not be zero. In the rest frame the spatial part of the dimensionless microscopic four vector is denoted by $\mathbf{q}' = (q'^1, q'^2, q'^3) \in \mathbb{R}^3$. For the integration variable $\mathbf{q}'$ we will apply the substitution

$$q'^1 = \vartheta\xi\,, \quad q'^2 = \vartheta\sqrt{1-\xi^2}\sin\varphi\,, \quad q'^3 = \vartheta\sqrt{1-\xi^2}\cos\varphi\,, \tag{5.3}$$

with $|\mathbf{q}'| = \vartheta$, where the domains of the new variables are given by $0 \le \vartheta < \infty$, $-1 \le \xi \le 1$ and $0 \leqq \varphi \leqq 2\pi$. In order to avoid the unnecessary integration over the whole range of ϑ, we wish to find a finite domain for ϑ which only contributes to the solution, and remove the rest range which has only a very small contribution to the integral. The Jüttner phase density (2.17) in the rest frame can be written as

$$f_J(n, T, \mathbf{0}, \mathbf{q}') = \frac{n}{M(\beta)\exp(\beta)} \exp\left(-\beta(\sqrt{1+\mathbf{q}'^2} - 1)\right). \tag{5.4}$$

So we want to have $\exp\left(-\beta(\sqrt{1+\underline{\mathrm{q}}'^2} - 1)\right) \le \varepsilon$ for sufficiently small value of ε. This gives

$$|\mathbf{q}'| \le R(T, \varepsilon) := \sqrt{\left(1 + T\,\ln\frac{1}{\varepsilon}\right)^2 - 1}\,. \tag{5.5}$$

From the above equation it is clear that we have a sphere of center zero and radius $R(T, \varepsilon)$ in the rest frame. Also it is important to note that this value of $R(T, \varepsilon)$ covers the whole range from non-relativistic to ultra-relativistic limit. Here ε is a constant value, for example $\varepsilon = 10^{-4}$. Now we transform to a general frame where the initial speed u may or may not be zero, using the inverse of the Lorentz boost given in (2.14)

$$q^\mu = \Lambda^\mu_{\ \nu}(-\hat{u}, 0, 0)q'^\nu, \qquad \mu = 0, 1, 2, 3\,. \tag{5.6}$$

Here $\hat{u} \in \mathbb{R}$ is a fixed one-dimensional velocity parameter, which will be given later for the formulation of the kinetic scheme. Using the equations (5.3) and (5.6), we get the values for q^μ, namely

$$q^1 = \hat{u}\sqrt{1+\vartheta^2} + \vartheta\xi\sqrt{1+\hat{u}^2}\,, \tag{5.7}$$

$$q^2 = \vartheta\sqrt{1-\xi^2}\sin\varphi\,, \tag{5.8}$$

$$q^3 = \vartheta\sqrt{1-\xi^2}\cos\varphi\,, \tag{5.9}$$

$$q^0 = \sqrt{1+(q^1)^2+(q^2)^2+(q^3)^2}\,. \tag{5.10}$$

Also the Jacobian of integration is given by

$$J(\hat{u}, \xi, \vartheta) = \frac{\hat{u}\vartheta^3\xi}{\sqrt{1+\vartheta^2}} + \vartheta^2\sqrt{1+\hat{u}^2}\,. \tag{5.11}$$

Here the parameter $\hat{u}$ is not depending on ϑ, ξ and φ. Recall that the spatial initial data (5.1) lead to the restrictions (5.2) for the fields at all later times. Due to (5.2) the fields $n,\ T$ and u in the free flight phase density f_n are not depending on the variable φ. This fact enables us to do the integration with respect to φ directly. Thus we have the following reduced integrals in this case

$$\begin{aligned} N^0(t_n+\tau, \mathbf{x}) &= 2\pi \int_0^R \int_{-1}^1 J\, \hat{f}_n(x - \tau\tfrac{q^1}{q^0}, \mathbf{q}) d\xi\, d\vartheta\,, \\ N^1(t_n+\tau, \mathbf{x}) &= 2\pi \int_0^R \int_{-1}^1 J\, \tfrac{q^1}{q^0} \hat{f}_n(x - \tau\tfrac{q^1}{q^0}, \mathbf{q}) d\xi\, d\vartheta\,, \end{aligned} \tag{5.12}$$

$$\begin{aligned} T^{00}(t_n+\tau, x) &= 2\pi \int_0^R \int_{-1}^1 J\, q^0 \hat{f}_n(x - \tau\tfrac{q^1}{q^0}, \mathbf{q}) d\xi\, d\vartheta\,, \\ T^{01}(t_n+\tau, x) &= 2\pi \int_0^R \int_{-1}^1 J\, q^1 \hat{f}_n(x - \tau\tfrac{q^1}{q^0}, \mathbf{q}) d\xi\, d\vartheta\,, \\ T^{11}(t_n+\tau, x) &= 2\pi \int_0^R \int_{-1}^1 J\, \tfrac{(q^1)^2}{q^0} \hat{f}_n(x - \tau\tfrac{q^1}{q^0}, \mathbf{q}) d\xi\, d\vartheta\,, \\ T^{22}(t_n+\tau, x) &= 2\pi \int_0^R \int_{-1}^1 J\, \tfrac{(q^2)^2}{q^0} \hat{f}_n(x - \tau\tfrac{q^1}{q^0}, \mathbf{q}) d\xi\, d\vartheta\,, \end{aligned} \tag{5.13}$$

where

$$\hat{f}_n(y, \mathbf{q}) := f_n(y, 0, 0, \mathbf{q})\,, \quad y \in \mathbb{R}\,,\ \mathbf{q} = (q^1, q^2, q^3)^T \in \mathbb{R}^3\,, \tag{5.14}$$

with the function $f_n(\mathbf{y}, \mathbf{q})$ which is given in (4.3), the Jacobian $J := J(u(t_n, x), \xi, \vartheta)$ for $\hat{u} = u(t_n, x)$ and the radius $R := R(T, \varepsilon)$ for the temperature $T := \max\{T(t_n, y) \,|\, x - \tau \le y \le x + \tau\}$. Moreover we obtain

$$
\begin{aligned}
N^2(t_n + \tau, x) &= N^3(t_n + \tau, x) = 0\,,\\
T^{10}(t_n + \tau, x) &= T^{01}(t_n + \tau, x)\,,\\
T^{22}(t_n + \tau, x) &= T^{33}(t_n + \tau, x)\,,
\end{aligned}
$$

where all the other components of $T^{\mu\nu}$ are zero. So in the one dimensional case n, u and T can be find from the generally valid relations in equation (4.4) as follows:

$$
n(t,x) = \sqrt{(N^0(t,x))^2 - (N^1(t,x))^2}\,, \tag{5.15}
$$

$$
u(t,x) = \frac{1}{n} N^1(t,x)\,, \tag{5.16}
$$

$$
\begin{aligned}
p(t,x) =& \frac{1}{3}[u^2(t,x)T^{00}(t,x) - 2u\sqrt{1+u^2(t,x)}\,T^{01}(t,x)\\
&+ (1+u^2(t,x))T^{11}(t,x) + 2T^{22}(t,x)]\,.
\end{aligned} \tag{5.17}
$$

Note that we still need the resolved continuity conditions (4.7)-(4.10) in order to initialize the scheme for the next time step. Since only the zero-components N^0, $T^{0\mu}$ are continuous across the maximization times t_n, we have to distinguish between the left- and right-handed limits $n(t_n^{\pm}, x)$, $u(t_n^{\pm}, x)$ and $p(t_n^{\pm}, x)$ at the maximization times t_n in (5.15)- (5.17).

6 From the kinetic scheme to the Eulerian limit ($\tau_M \to 0$)

In the previous sections we have calculated the solution of the kinetic scheme in the three- and one-dimensional case, respectively. This was done for the prescribed initial data of n, u and p for a given free-flight time step $\tau_M > 0$. If we calculate these solutions for $\tau_M \to 0$ then we get the Eulerian limit

$$
N^\mu \to n\, u^\mu\,, \quad T^{\mu\nu} \to -pg^{\mu\nu} + (e+p)u^\mu u^\nu\,, \quad S^\mu \to \sigma\, u^\mu\,, \tag{6.1}
$$

where p, e and σ are given by (3.1). First we pass to the Eulerian limit (6.1) at the points of smoothness in the following way using (2.15) with $Q(f) = 0$

$$
\begin{aligned}
\lim_{\tau\to 0} \frac{\partial}{\partial\tau} N^0(t_n + \tau, \mathbf{x}) &= \lim_{\tau\to 0} \frac{\partial}{\partial\tau} (n(t_n+\tau,\mathbf{x})\sqrt{1+u^2(t_n+\tau,\mathbf{x})}\,)\\
&= \lim_{\tau\to 0} \frac{\partial}{\partial\tau} \int_{\mathbb{R}^3} q^0 f_n(\mathbf{x} - \tau\frac{\mathbf{q}}{q^0}, \mathbf{q}) \frac{d^3q}{q^0}
\end{aligned}
$$

$$
\begin{aligned}
&= -\lim_{\tau\to 0}\int_{\mathbb{R}^3} q^0 \sum_{k=1}^{3} \frac{q^k}{q^0}\frac{\partial}{\partial x^k} f_n(\mathbf{x}-\tau\frac{\mathbf{q}}{q^0},\mathbf{q})\frac{d^3q}{q^0}\\
&= -\int_{\mathbb{R}^3} \sum_{k=1}^{3} q^k \frac{\partial}{\partial x^k} f_n(\mathbf{x},\mathbf{q})\frac{d^3q}{q^0}\\
&= -\sum_{k=1}^{3}\frac{\partial}{\partial x^k}(u^k(t_n^+,\mathbf{x})n(t_n^+,\mathbf{x}))\\
&= -\nabla\cdot(\mathbf{u}(t_n^+,\mathbf{x})n(t_n^+,\mathbf{x}))\,,
\end{aligned}
$$

$$
\Rightarrow \frac{\partial}{\partial t}(n\sqrt{1+u^2(t_n^+,\mathbf{x})}\,) + \nabla\cdot(\mathbf{u}(t_n^+,\mathbf{x})n(t_n^+,\mathbf{x})) = 0\,, \tag{6.2}
$$

which is a first Euler equation (3.38). Similarly we get the other two Euler equations (2.5) and (2.6) if we differentiate $T^{00}(t_n+\tau,\mathbf{x})$ and $T^{0k}(t_n+\tau,\mathbf{x})$ with respect to τ and pass to the limit $\tau\to 0$.

Secondly on the left hand side of (6.1) there are the moments N^μ, $T^{\mu\nu}$ and S^μ as calculated by the kinetic scheme see (4.1) and (4.2). Since the solution of the kinetic scheme satisfies the conservation laws and the entropy inequality as stated in Proposition 4.1, we conclude from (6.1) that these also result for the weak entropy solution in the Eulerian limit $\tau_M\to 0$. The weak entropy solution in the Eulerian limit in one space dimension is given by (3.46), (3.47) and (3.48).

7 A single shock reflection problem

In this section we test our spatially one-dimensional kinetic scheme for a single shock problem. Moreover we introduce an adiabatic wall at $x=0$ and the corresponding boundary condition in order to study the single shock reflection which can also be calculated using the shock parametrization given in Section 3 for the relativistic Euler equations. We start with a 1-shock running to the lower adiabatic wall. After the reflection there results an outcoming 3-shock. Figure 1 shows a sketch of this single shock reflection.

The initial and boundary value problem for the relativistic Euler equations is defined for $t\geq 0$ and $x\geq 0$, where $x=0$ is the boundary for the lower adiabatic wall. The initial data for n, u and β are prescribed for $x>0$, whereas for $t\geq 0$ we prescribe the boundary data $u(t,0)=0$. Following the idea presented by Dreyer, Kunik and Herrmann in [7], [8] in order to solve the initial and boundary value problem for the classical Euler equations, we conclude that we can use the one dimensional scheme presented in Section 5 with the following two modifications:

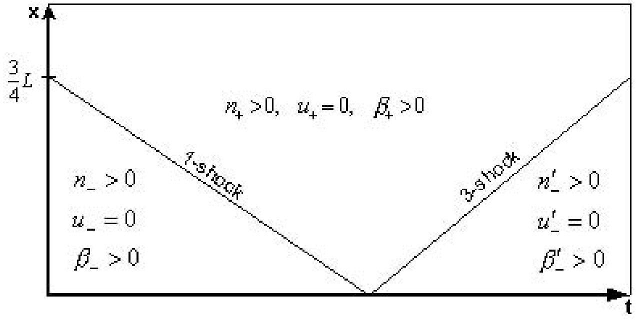

Fig. 1. Sketch of a single shock reflection

(i) In (5.12) and (5.13) we have to replace the phase density $\hat{f}_n(x - \tau \frac{q^1}{q^0}, \mathbf{q})$ by the new one $\hat{f}_n(|x - \tau \frac{q^1}{q^0}|, \mathbf{q}^*)$, where $\mathbf{q}^*$ is given by

(7.1) $$q^{*1} = \operatorname{sgn}(x - \tau \frac{q^1}{q^0})\,(\hat{u}\sqrt{1+\vartheta^2} + \vartheta\xi\sqrt{1+\hat{u}^2}),$$

(7.2) $$q^{*2} = \vartheta\sqrt{1-\xi^2}\sin\varphi,$$

(7.3) $$q^{*3} = \vartheta\sqrt{1-\xi^2}\cos\varphi,$$

(7.4) $$q^{*0} = \sqrt{1 + (q^{*1})^2 + (q^{*2})^2 + (q^{*3})^2}.$$

Recall that the function $\hat{f}_n(y, \mathbf{q})$ is defined in (5.14).

(ii) In (5.12) and (5.13) we have to replace the temperature T for the integration radius $R = R(T, \varepsilon)$, see (5.5), by the new one

(7.5) $$T := \max\{T(t_n, y) \mid\ y \geq 0 \text{ and } x - \tau \leq y \leq x + \tau\}.$$

Further details on the implementation of our kinetic scheme may be found in Section 7 of [21]. Here we have proceeded analogously as in the ultra-relativistic case presented there.

Figure 2, which is computed from the kinetic scheme, represents a plot for the particle density n, the velocity u and the temperature T in the time range $0 \leq t \leq 2.63125$ and in the space range $0 \leq x \leq L = 1$. We have calculated the final time in such a way that the outcoming 3-shock finally arrives at the same position $x = \frac{3}{4}L$ where the original 1-shock has initially started. The Riemannian initial data for the 1-shock with the jump at $x = \frac{3}{4}L$ are chosen according to the shock parametrization given in Section 3. The initial data are

$$n_- = 1.0 \quad u_- = 0.0 \quad \beta_- = 0.5 \quad p_- = 2.0,$$
$$n_+ = 1.35396 \quad u_+ = -0.175227 \quad \beta_+ = 0.45 \quad p_+ = 3.0088,$$

where 200 maximization times are considered here, so that $\tau_M = 0.0131563$.

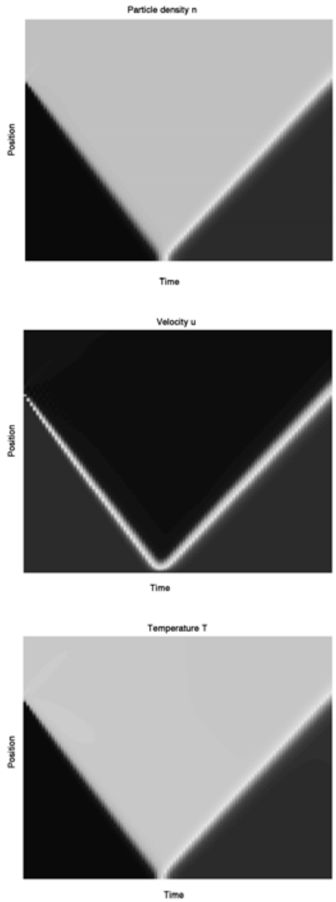

Fig. 2. A single shock reflection

Moreover we obtained the constant states left and right to the reflected 3-shock analytically from the Rankine-Hugoniot shock parametrizations (3.52), (3.53) as well as numerically from the kinetic scheme. Both are compared below, where we use the notations from Fig. 1.

The 3-shock data from the Rankine-Hugoniot jump conditions are
$n'_- = 1.83235 \quad u'_- = 0.0 \quad \beta'_- = 0.405264 \quad p'_- = 4.52136\,,$
$n_+ = 1.35396 \quad u_+ = -0.175227 \quad \beta_+ = 0.45 \quad p_+ = 3.0088\,.$

The 3-shock data from the kinetic scheme are
$n'_- = 1.83200 \quad u'_- = 0.0 \quad \beta'_- = 0.40852 \quad p'_- = 4.48448\,,$
$n_+ = 1.35394 \quad u_+ = -0.175236 \quad \beta_+ = 0.450024 \quad p_+ = 3.0086\,.$

We see a very good agreement between the analytical and numerical results.

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