Throughput-optimal number of relays in delay-bounded multi-hop ALOHA networks

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This letter addresses the throughput of an ALOHA-based Poisson-distributed multi-hop wireless network which is subject to a delay constraint in packet delivery. We derive an exact expression and also a tight approximation, based on the Central Limit Theorem, for the end-to-end multi-hop throughput. The result also determines the throughput-maximizing number of relays to be used between each source-destination pair.

Introduction: Stochastic geometry has recently gained considerable interest to study the throughput of distributed ad-hoc networks [1, 2]. Effects of randomness in the network due to the Poisson node distribution, ALOHA channel access, and fading are captured by means of a packet reception model that requires the signal-to-interference-plus-noise ratio (SINR) to be larger than a decoding threshold for successful reception. However a shortcoming of this framework is that it accounts for single-hop transmissions, considering only a snapshot of the network. Lately an extension of this approach was proposed to account for multi-hop transmissions and possible retransmissions on each hop [3]. In this letter we derive the exact, and also a closed-form tight approximation, of the end-to-end throughput of a multi-hop setting. The result also renders the optimal number of relays between each source-destination pair.

System model: We consider a network in which sources are distributed as a 2-D homogenous Poisson Point Process (P.P.P) of intensity ζ. At each slot, sources transmit with probability p independent of each other, hence the process of transmitting sources is also a P.P.P with intensity $\lambda = \zeta p$. Each source node (S) has a corresponding destination node (D) that is a distance R away in a random direction. As in [3], a packet is allowed to traverse M hops (via $M-1$ relays placed on a straight line) from the source to the destination, hence $R = \sum_{m=1}^{M} r_m$ where r_m is the length of the m^{th} hop (see Fig. 1). For each $S-D$ pair we allow for only a single transmission at a time along the entire multi-hop route. This assumption does not affect the achievable throughput as other $S-D$ pairs can exploit the unused space instead. With this assumption the set of interferers, Ψ, in each time slot is a P.P.P. of the same intensity λ. All transmitters have a fixed transmit power ρ and the noise power is denoted by η. Wave propagation effects are characterized by path loss and fading. Hence, for a single hop transmission the received power at the destination is $\rho h R^{-\alpha}$, where $\alpha > 2$ is the path loss exponent and $h \sim \exp(1)$ is the Rayleigh fading coefficient. For a typical $S-D$ pair, the SINR on the m^{th} hop is:

$$SINR = \frac{\rho h_m r_m^{-\alpha}}{\sum_{i \in \Psi} \rho h_i |X_i|^{-\alpha} + \eta} \tag{1}$$

where $|X_i|$ is the distance of the i^{th} interferer to the receiving relay node. Here, due to the stationarity of Ψ, we have assumed that the receiver of each hop is at the origin. Further, if β is the decoding threshold, the success probability of each hop is:

$$p_s(r_m) = \mathbb{P}(SINR > \beta) = \exp\left\{-\frac{\beta\eta}{\rho r_m^{-\alpha}} - \lambda \beta^{\frac{2}{\alpha}} C_\alpha r_m^2\right\} \tag{2}$$

Where $C_\alpha = 2\pi^2/(\alpha \sin(2\pi/\alpha))$. The reader is referred to [1] for a proof of (2) .

End-to-end multi-hop throughput: In the multi-hop network, source packets are routed through the relays to reach the destination. On each hop Automatic Repeat-reQuest is used and packets are retransmitted until they are successfully delivered. With no delay constraints, it would take a total of $T(M) = \sum_{m=1}^{M} T_m(M)$ transmissions for a packet to reach the destination with $T_m(M)$ being the number of transmissions on hop m. When a delay constraint is to be satisfied, a packet is said to be successfully delivered to the destination when $T(M) \leq A$, where A is the total allowable number of transmissions for each packet from the source to the destination [3]. With M hops between each $S - D$ pair, the end-to-end multi-hop throughput is:

$$C(M) = \lambda \mathbb{P}(T(M) \leq A) \frac{\log(1+\beta)}{\mathbb{E}(\min(T(M),\ A))} R \tag{3}$$

which quantifies the aggregate bit-meters per second that can be supported for a density λ of sources, given a transmission range R, decoding threshold β, and delay constraint A. It is assumed that the destination node informs the source node upon a successful reception so that it can send the next packet. Note that $\min(T(M),\ A)$ represents the actual number of per-packet transmissions considering that a packet would be dropped midway and the next packet sent by the source in case the delay constraint is violated. Hence the denominator accounts for the loss in rate due to the amount of time the source would have to wait before transmitting the next packet.

To find an exact expression for $C(M)$ we assume that the per-hop success probabilities are independent across different time slots. Based on [4] this is a fair assumption whose accuracy increases as λ decreases. Let us first find an expression for $\mathbb{P}(T(M) \leq A)$. Note that the event $\{T(M) \leq A\}$ is equivalent to the event $\{T^{'}(M) \leq A - M\}$ where $T^{'}(M)$ is the total number of failures before the packet is successfully delivered to the final destination on the M^{th} success (hop). Now, assuming equidistant hops between the typical $S - D$ pair (i.e. $r_m = R/M$), it is evident that $T^{'}(M)$ has a negative binomial distribution. That is: $T^{'}(M) \sim \mathcal{NB}(M, p_s)$, where p_s, with the dropped argument, denotes $p_s(R/M)$ hereafter for ease of representation. Hence:

$$\mathbb{P}(T(M) \leq A) = \mathbb{P}(T^{'}(M) \leq A - M) = I_{p_s}(M, A - M + 1) \tag{4}$$

Where $I(.,.)$, being the regularized incomplete beta function, is the cumulative distribution function of the negative binomial distribution. Further:

$$\mathbb{E}(\min(T(M),\ A)) = \mathbb{E}(\min(T^{'}(M),\ A - M)) + M \tag{5}$$

$$= \sum_{k=0}^{A-M-1} k \binom{k+M-1}{k} p_s^M (1-p_s)^k + (A - M)\left(1 - I_{p_s}(M, A - M)\right) + M$$

Equations (4) and (5) are then substituted into (3) to obtain the exact end-to-end multi-hop throughput.

We now derive a tight approximation for the end-to-end multi-hop throughput using the Central Limit Theorem (C.L.T.) that uses a Gaussian representation for $T(M)$ as a sum of M independent geometric random variables $T_m(M)$. First we check the necessary conditions for the C.L.T. (namely finite means and variances). Note that due to (2), $p_s(r_m)$ is always non-zero. Hence, the mean and variance of each $T_m(M)$ which are $\frac{1}{p_s(r_m)}$ and $\frac{1-p_s(r_m)}{p_s^2(r_m)}$, respectively, are always bounded. This way and initially assuming $r_m = R/M$ (and hence identically distributed $T_m(M)$'s), we have:

$$T(M) \sim \mathcal{N}(\mu_M, \sigma_M^2) \tag{6}$$

Where:

$$\mu_M = \sum_{m=1}^{M} \mathbb{E}\left(T_m(M)\right) = \frac{M}{p_s} \tag{7}$$

$$\sigma_M^2 = \sum_{m=1}^{M} Var\left(T_m(M)\right) = \frac{M(1-p_s)}{p_s^2} \tag{8}$$

Using (6) along with (7) and (8) to manipulate (3), we have:

$$C(M) = \frac{\left(\Phi\left(\frac{A-\mu_M}{\sigma_M}\right)-\Phi\left(-\frac{\mu_M}{\sigma_M}\right)\right)\lambda R log\,(1+\beta)}{\frac{\sigma_M}{\sqrt{2\pi}}\left(\exp\left(\frac{-\mu_M^2}{2\sigma_M^2}\right)-\exp\left(\frac{-(A-\mu_M)^2}{2\sigma_M^2}\right)\right)+\mu_m\left(\Phi\left(\frac{A-\mu_M}{\sigma_M}\right)-\Phi\left(\frac{-\mu_M}{\sigma_M}\right)\right)+A\left(1-\Phi\left(\frac{A-\mu_M}{\sigma_M}\right)\right)} \tag{9}$$

Where Φ denotes the cumulative distribution function of the standard normal distribution. Note that due to the geometric distribution of the $T_m(M)$'s, the convergence to a Gaussian random variable is fast.

It can be also shown that when the hops are not equidistant (i.e. the $T_m(M)$'s are not identically distributed), a mild Lindeberg's condition [5] can be applied to obtain $C(M)$. Note that the equidistant hop assumption is optimal in terms of maximizing the throughput. This is because due to the fast decaying exponential expression for (2), the reduction of success probabilities on long hops overshadows its increase on shorter hops. Hence the non-equidistant hop assumption is only considered when there is an extrinsic limitation imposed on the network (e.g. when one or more of the relays have lower transmission power than others).

Numerical results: We used Monte Carlo simulations to validate our results. Fig. 2 shows the end-to-end multi-hop throughput for the selection of parameters $\alpha = 4$, $\beta = 5$, $R = 1$, $\lambda = 1$, and $SNR = \rho R^{-\alpha}/\eta = 5$. The figure shows the results for delay constraints $A = 16$ and $A = 25$. We compare our results against the upper bound obtained in [3]. There, the authors use their upper bound to derive the optimal number of relays which maximizes $C(M)$ and claim that it is at most one away from the true value. We observed that this is not necessarily true for tight delay constraints (e.g. $A = 16$). In Fig. 2, the upper bound of [3], our proposed exact expression and the approximation by (9) accompany Monte Carlo simulations. As can be seen, when $A = 16$ the upper bound of [3] is maximized at $M = 15$, whereas the simulations and our results for $C(M)$ are all maximized at $M = 11$. However, when $A = 25$ they all predict the optimal number of hops to be $M = 15$. Note that since the upper bound assumes $A \to \infty$, it becomes tighter as A increases and eventually all four curves coincide for large enough A. Furthermore, it is evident from the figure that our C.L.T. approximation (9) is almost as good as the exact result. As expected, the approximation gets better as A increases.

Conclusions: We addressed the end-to-end throughput in a delay-bounded multi-hop fading network with Poisson node distribution and distributed channel access. The analysis is used to determine the throughput-optimal number of relays between source-destination pairs.

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References

1 Baccelli, F., Blaszczyszyn, B., and Muhlethaler, P.: ‘An ALOHA protocol for multihop mobile wireless networks’, *IEEE Trans. Inf. Theory*, vol. 52, No. 2, pp. 421-436, 2006.

2 Nardelli, P.H.J., Cardieri, P., Latva-aho, M.: ‘Maximizing transmission capacity of ad hoc networks via transmission system design’, *IET Electron. Lett.*, vol. 47, No. 5, pp. 348-349, 2011.

3 Andrews, J., Weber, S., Kountouris, M., and Haenggi, M.: ‘Random Access Transport Capacity’, *IEEE Trans. Wireless Commun.*, vol. 9, No. 6, pp. 2101-2111, 2010.

4 Ganti, R., Haenggi, M.: ‘Spatial and temporal correlation of interference in ALOHA ad hoc networks’, *IEEE Commun. Lett.*, vol. 13, No. 9, pp. 631-633, 2009.

5 Billingsley, P.: ‘Probability and measure’, third edition, John Wiley and Sons, 1995.

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Figure captions:

Fig. 1 Transmitting sources distributed as a P.P.P. of intensity λ each have a destination distance R away in a random direction, which they reach via $M-1$ relays.

Fig. 2 End-to-end multi-hop throughput as a function of the number of hops for $A=16$ (left), and $A=25$ (right).

Figure 1

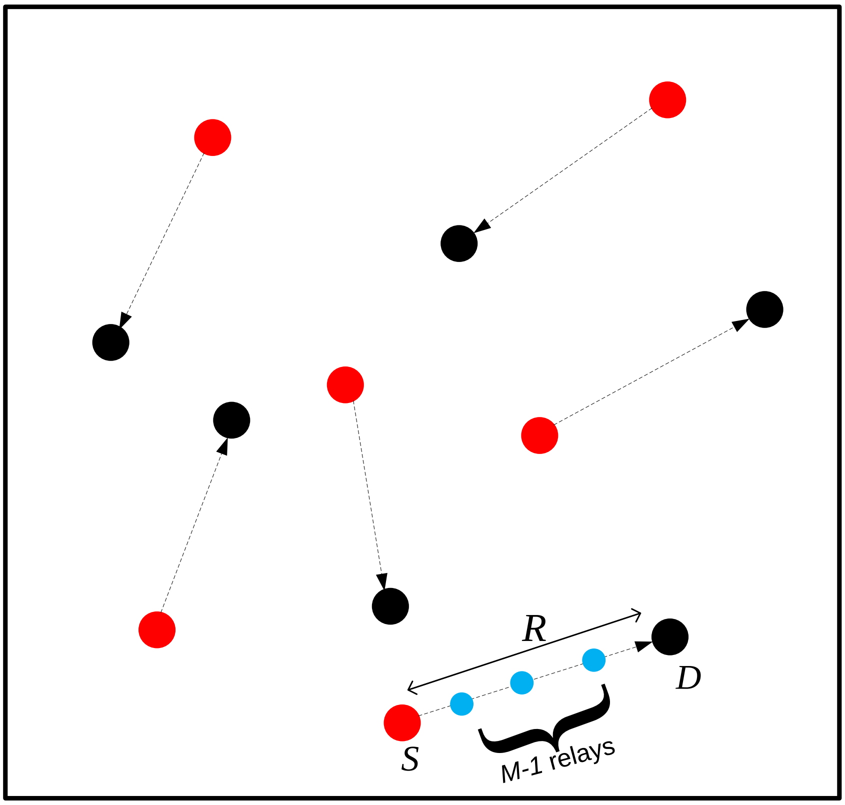
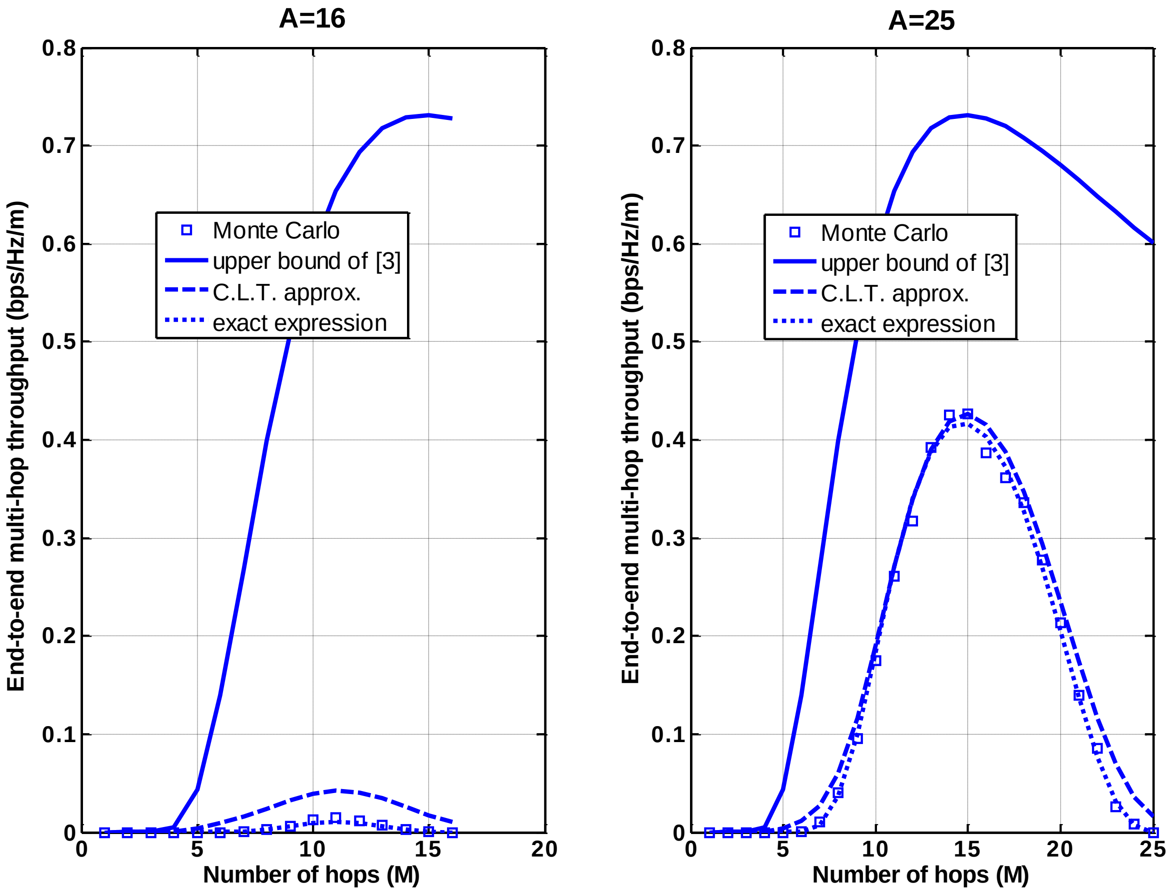


Figure 2


A=16
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
20
Number of hops (M)
End-to-end multi-hop throughput (bps/Hz/m)
Monte Carlo
upper bound of [3]
C.L.T. approx.
exact expression
A=25
0
5
10
15
20
25