Relaxations of Parameterized LMIs with Control Applications

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Abstract. A wide variety of problems in control system theory fall within the class of parameterized Linear Matrix Inequalities (LMIs), that is, LMIs whose coefficients are functions of a parameter confined to a compact set. However, in contrast to LMIs, parameterized LMI (PLMIs) feasibility problems involve infinitely many LMIs hence are very hard to solve.

In this paper, we propose several effective relaxation techniques to replace PLMIs by a finite set of LMIs. The resulting relaxed feasibility problems thus become convex and hence can be solved by very efficient interior point methods. Applications of these techniques to different problems such as robustness analysis, or Linear Parameter-Varying (LPV) control are then thoroughly discussed and illustrated by examples.

1 Introduction

Linear matrix inequalities (LMIs) have emerged as a very powerful tool in the analysis and synthesis for robust control problems (see e.g. [6, 8, 12] and references therein). From a computational point of view, LMIs are very attractive because they consist of convex constraints and therefore can be solved by very efficient convex optimization interior points methods with worst-case polynomial complexity [14, 22]. From the viewpoint of robust control analysis and synthesis, LMI formulations offer much more freedom and potential than the usual Riccati/Lyapunov algebraic machinery and thus provide additional flexibility in control applications.

In this paper we are concerned with parameterized LMIs (PLMIs), that is, LMIs depending on a parameter α evolving in a compact set. The parameter α can be given various control theory interpretations and can designate parameter uncertainties or system operating conditions but virtually appears in other branches such as robust filtering, or robust matrix computation problems. Here, a special emphasis is put on PLMIs of the form

$$M_0(z) + \sum_{i=1}^{L} \alpha_i M_i(z) + \sum_{1\le i\le j\le L} \alpha_i\alpha_j M_{ij}(z) < 0 \tag{1}$$

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where z is the decision variable, $M_i(z)$, $M_{ij}(z)$, are affine symmetric matrix-valued functions of z and α is a parameter confined to either the polytope

$$\alpha \in \Gamma := \{\alpha = (\alpha_1, \alpha_2, ..., \alpha_L) : \sum_{i=1}^{L} \alpha_i = 1,\ \alpha_i \geq 0,\ i = 1, 2, .., L\}. \tag{2}$$

or the parameter hyper-rectangle

$$\alpha \in [p, q],\ p \geq 0, q > 0,\ p \in R^L,\ q \in R^L. \tag{3}$$

As will be clarified later, such PLMIs arise quite naturally in important problems of robust control theory such as Lyapunov-based robustness analysis, μ-analysis, robust and LPV (Linear Parameter-Varying) synthesis problems. In the light of this, the importance and generality of PLMIs appear clearly for application purposes. Unfortunately, in contrast to LMIs, PLMIs involve infinitely many LMIs, and therefore are very hard to solve. They are known to be NP-hard [5]. As a consequence, the computational efforts for finding feasible points are expected to be far much higher than those of LMIs. This motivates the need for developing sensible and practical algorithms for solving PLMIs. Possible approaches for solving PLMIs are relaxation techniques where PLMIs are replaced by finitely many LMIs. Such approaches are potentially conservative but often provide practically exploitable solutions of difficult problems with a reasonable computational effort.

From our practical experience, such techniques appear to work pretty well in many practical control problems. In a different paper [2], we have developed a relaxation technique based on directional convexity concepts (DCC) for attacking PLMI problems. The central idea of DCC is to enforce some directional convexity properties of the PLMI (1) so that there is only need to check (1) at the vertices of the polytope (2) or (3). We therefore end up with an easy to solve standard LMI feasibility problem.

In this paper, different relaxation techniques for solving (1) are explored. Regarding problem (1) as the optimization of an indefinite quadratic function of α, we introduce several convexification techniques leading to LMI characterizations. Note that both DCC in [2] and the present work are fundamentally different from that of [1, 23, 24] which exploits polytopic covering techniques to turn PLMI problems into LMIs. The present work is also closely related to the work of Ben-Tal and Nemirovski in [4, 5] which lays the foundations of *robust convex programming* and investigate its theoretical *tractability*, and the work of Oustry, Elghaoui and Lebret in [15] who consider the regularity properties of solutions to PLMIs using the $\mathcal{S}$-procedure and discuss its implications for a variety of topics.

This paper is splitted into six parts. Sections 2 and 3 discuss two different techniques for convexifying problem (1) while Section 4 gives extensions to PLMI problems of the form

$$M_0(z) + \sum_{\nu \in J} \alpha^{[\nu]} M_\nu(z) < 0 \tag{4}$$

where again z is the decision variable and α is constrained by (2) or (3). Here we have used the shorthand

$$\alpha^{[\nu]} := \alpha_1^{\nu_1} \alpha_2^{\nu_2} .... \alpha_L^{\nu_L}$$

where $\nu = (\nu_1, \nu_2, ..., \nu_L)$, $\nu_i \geq 1$, are L-tuples of partial degrees in the finite set J.

Section 5 considers different applications of the proposed techniques ranging from robust analysis to LPV synthesis problems while Section 6 provides illustrative examples.

The notations in this paper are fairly standard. For instance, for a symmetric matrix M, the notation $M < 0$ ($M \leq 0$, resp.) means M is negative definite (negative semidefinite, resp.), the set of all vertices of a polytope Γ is denoted by $\mathrm{vert}\,\Gamma$. In (3), $\alpha \in [p, q]$ must be understood component-wise, that is, $p_i \leq \alpha_i \leq q_i$, $i = 1, 2, ...L$ and similarly, $p \geq 0$ ($q > 0$, resp.) means $p_i \geq 0$ ($q_i > 0$, resp.) for all $1 \leq i \leq L$.

2 Separated convexification method

Before considering the PLMI problem (1), we introduce some auxiliary results on convex overbounding functions for polynomial functions that will be used in the sequel. It is assumed throughout this subsection that $x + y \leq 1$, $x \geq 0$, $y \geq 0$. All proofs for Lemmas 2.1-2.5 are given in the Appendix.

2.1 Convex upper bounds

Lemma 2.1 $-x^2 \leq -x + 0.25$ *and* $-x^\nu \leq -\frac{\nu}{2^{\nu-1}}x + \frac{\nu-1}{2^\nu}$ $\forall \nu \geq 2$.

Lemma 2.2 $-xy \leq \dfrac{x^2+y^2}{2} - \dfrac{x+y}{2} + 0.125$.

Lemma 2.3 $x_1^{\nu_1} x_2^{\nu_2} ... x_L^{\nu_L} \leq \left(\dfrac{x_1^{\nu_1} + x_2^{\nu_2} + ... + x_L^{\nu_L}}{L} \right)^L$

Lemma 2.4 *For every* $\nu_1 \geq 1$, $\nu_2 \geq 1$ *one has*

$$-x^{\nu_1} y^{\nu_2} \leq \frac{1}{2}(x^{2\nu_1} + y^{2\nu_2}) - \frac{1}{2}[2^{-\nu_1}(1-\nu_1) + 2^{-\nu_2}(1-\nu_2) + \nu_1 2^{1-\nu_1} x + \nu_2 2^{1-\nu_2} y] + 0.125$$

Lemma 2.5 *A convex upper bound of function* $-x_1^{\nu_1} x_2^{\nu_2} x_3^{\nu_3}$ *is*

$$\begin{aligned} &\frac{x_1^{2\nu_1}}{2} + \frac{1}{2}\left(\frac{x_2^{2\nu_2} + x_3^{2\nu_3}}{2}\right)^2 - \frac{1}{2}[2^{-\nu_1}(1-\nu_1) + \nu_1 2^{1-\nu_1} x_1] + \frac{1}{4}(x_2^{2\nu_2} + x_3^{2\nu_3}) - \\ &\frac{1}{4}[2^{-\nu_2}(1-\nu_2) + 2^{-\nu_3}(1-\nu_3) + \nu_2 2^{1-\nu_2} x_2 + \nu_3^{1-\nu_3} x_3] + 0.125 \times 1.5 \end{aligned} \tag{5}$$

In virtue of Lemmas 2.4-2.5, it is not difficult to construct a convex upper bound for any function $-\prod_{i=1}^{L} x_i^{\nu_i}$, $\nu_i \geq 1$.

2.2 LMI characterization

Now, refocus of the PLMI problem (1). Obviously,

$$(1) \Leftrightarrow x' M_0(z) x + \sum_{i=1}^{L} \alpha_i x' M_i(z) x + \sum_{1 \leq i \leq j \leq L} \alpha_i \alpha_j x' M_{ij}(z) x < 0, \ \forall ||x|| = 1, \alpha \in \Gamma. \tag{6}$$

For every fixed x and z, the right-hand side (RHS) of (6) is a quadratic function. Since the sign of every $x'M_{ij}(z)x$ is indefinite in general, checking (6) is equivalent to a non-convex quadratic optimization problem.

We first note that whenever α satisfies (2),

$$\alpha_i\alpha_j x'M_{ij}(z)x \le g_{ijzx}(\alpha_i,\alpha_j) := \max\{-x'M_{ij}(z)x(\frac{\alpha_i^2+\alpha_j^2}{2} - \frac{\alpha_i+\alpha_j}{2} + 0.125), \quad x'M_{ij}(z)x\frac{\alpha_i^2+\alpha_j^2}{2}\} \tag{7}$$

since by Lemma 2.2

$$x'M_{ij}(z)x\alpha_i\alpha_j \le \begin{cases} -x'M_{ij}(z)x(\frac{\alpha_i^2+\alpha_j^2}{2} - \frac{\alpha_i+\alpha_j}{2} + 0.125) & \text{when} \quad x'M_{ij}x \le 0 \\ x'M_{ij}(z)x\frac{\alpha_i^2+\alpha_j^2}{2} & \text{when} \quad x'M_{ij}x \ge 0 \end{cases}$$

Both functions $-x'M_{ij}(z)x(\frac{\alpha_i^2+\alpha_j^2}{2} - \frac{\alpha_i+\alpha_j}{2} + 0.125)$ and $x'M_{ij}(x)x\frac{\alpha_i^2+\alpha_j^2}{2}$ in (7) are either convex or concave in (α_i,α_j) depending on the sign of $x'M_{ij}(z)x$. However, function g_{ijzx} is always convex in (α_i,α_j) since

$$g_{ijzx}(\alpha_i,\alpha_j) = \begin{cases} x'M_{ij}(z)x\frac{\alpha_i^2+\alpha_j^2}{2} & \text{if} \quad x'M_{ij}x \ge 0 \\ -x'M_{ij}(z)x(\frac{\alpha_i^2+\alpha_j^2}{2} - \frac{\alpha_i+\alpha_j}{2} + 0.125) & \text{if} \quad x'M_{ij}x \le 0 \end{cases} \tag{8}$$

and functions $(\frac{\alpha_i^2+\alpha_j^2}{2} - \frac{\alpha_i+\alpha_j}{2} + 0.125)$ and $(\alpha_i^2+\alpha_j^2)/2$ are all convex in the set $\{(\alpha_i,\alpha_j):\ \alpha_i \ge 0,\ \alpha_j \ge 0,\ \alpha_i+\alpha_j \le 1\}$.
Therefore (6) is fulfilled if

$$x'M_0(z)x + \sum_{i=1}^{L}\alpha_i x'M_i(z)x + \sum_{1\le i\le j\le L}\max\{-x'M_{ij}(z)x(\frac{\alpha_i^2+\alpha_j^2}{2} - \frac{\alpha_i+\alpha_j}{2} + 0.125),\ x'M_{ij}(z)x\frac{\alpha_i^2+\alpha_j^2}{2}\} < 0,\ \forall ||x|| = 1, \alpha \in \Gamma. \tag{9}$$

Since the left-hand side (LHS) of (9) is already convex in α, for every fixed (x,z) the maximum of the LHS of (9) over Γ is attained on $\mathrm{vert}\,\Gamma$. The following result formalizes these facts.

Theorem 2.6 *The PLMI problem (1) and (2) has a solution z whenever the following quadratic conditions hold,*

$$x'M_0(z)x + \sum_{i=1}^{L}\alpha_i x'M_i(z)x + \sum_{1\le i\le j\le L}\max\{-x'M_{ij}(z)x(\frac{\alpha_i^2+\alpha_j^2}{2} - \frac{\alpha_i+\alpha_j}{2} + 0.125),\ x'M_{ij}(z)x\frac{\alpha_i^2+\alpha_j^2}{2}\} < 0,\ \forall ||x|| = 1, \alpha \in vert\,\Gamma. \tag{10}$$

The latter conditions are readily rewritten as LMIs.

To see that (10) actually is an LMI characterization, take for instance, $\alpha = e_1 := (1,0,0,...,0) \in \mathrm{vert}\,\Gamma$, then (10) at e_1 reduces to

$$\begin{aligned} & x'M_0(z)x + x'M_1(z)x + \max\{-0.125x'M_{11}(z)x, x'M_{11}x\} \\ & + \sum_{j=2}^{L} \max\{-0.125x'M_{1j}x, 0.5x'M_{1j}x\} + \sum_{2\le i\le j\le L} \max\{-0.125x'M_{ij}(z)x, 0\} \quad < \quad 0 \end{aligned} \tag{11}$$

which in the case $L = 2$ is

$$\begin{aligned} M_0(z) + M_1(z) - 0.125[M_{11}(z) + M_{12}(z)] + aM_{22}(z) &< 0 \\ M_0(z) + M_1(z) - 0.125M_{11}(z) + 0.5M_{12}(z) + aM_{22}(z) &< 0 \\ M_0(z) + M_1(z) + M_{11}(z) - 0.125M_{12}(z) + aM_{22}(z) &< 0 \\ M_0(z) + M_1(z) + M_{11}(z) + M_{12}(z) + aM_{22}(z) &< 0 \\ \forall a \in \{-0.125, 0\} & \end{aligned} \tag{12}$$

Analogously, we can express (10) at any vertex $e_i \in \mathrm{vert}\,\Gamma,\ i = 2,3,...L$ and therefore (10) can be easily expressed as an LMI feasibility problem.

Remark 2.1. It should be noted that $g_{ijzx}(\alpha_i, \alpha_j)$ in (7) is a tight upper bound of $\alpha_i\alpha_j x'M_{ij}(z)x$ with $\alpha_i + \alpha_j \le 1$. Therefore, if the set Γ is alternatively defined as

$$\Gamma := \{\alpha = (\alpha_1, \alpha_2, ..., \alpha_L) : \sum_{i=1}^{L} \alpha_i \le a,\ \alpha_i \ge 0,\ i = 1,2,..,L\}, \tag{13}$$

with $a > 1$, one can use the change of variable $\bar{\alpha}_i = \alpha_i/a$ to recover the case $\bar{\alpha}_i + \bar{\alpha}_j \le 1$. Analogously, applying the change of variable $\bar{\alpha}_i = (p + \alpha_i)/(q - p)$ to the constraint (3) yields the relation

$$\bar{\alpha} \in [0,1]^L. \tag{14}$$

Remark 2.2. When α satisfies constraint (14), there is no need to convexify $x'M_{ij}(z)x\alpha_i\alpha_j$ for $i < j$. In this case, the latter term is multi-convex [2] (see also the proof of Theorem 3.2 below) and (10) is simplified as follows. The quadratic (with respect to α) inequalities (6) hold whenever

$$\begin{aligned} & x'M_0(z)x + \sum_{i=1}^{L} \alpha_i x'M_i(z)x + \sum_{1\le i<j\le L} \alpha_i\alpha_j x'M_{ij}(z)x + \\ & \sum_{i=1}^{L} \max\{-xM_{ii}x \times 0.125, x'M_{ii}(z)x\alpha_i^2\} < 0,\ \forall ||x|| = 1,\ \alpha \in \mathrm{vert}\,[0,1]^L \end{aligned} \tag{15}$$

Here we have used the fact that $\alpha_i \in \{0,1\}$ for $\alpha \in \mathrm{vert}\,\Gamma$, and therefore $\dfrac{\alpha_i^2 + \alpha_j^2}{2} - \dfrac{\alpha_i + \alpha_j}{2} = 0$, for all α in $\mathrm{vert}\,[0,1]^L$. Moreover, if we additionally impose the condition

$$M_{11}(z) \ge 0 \tag{16}$$

then by a straightforward application of the results in [2, 10], the PLMI problem described by (1) and (14) further reduces to

$$M_0(z) + \sum_{i=1}^{L} \alpha_i M_i(z) + \sum_{1\le i\le j\le L} \alpha_i\alpha_j M_{ij}(z) < 0\ \forall \alpha_1 \in \{0,1\},\ \alpha_i \in [0,1],\ i > 1. \tag{17}$$

For instance with $L = 2$, and (16) in force, the PLMI problem (1), (14) becomes

$$\begin{array}{lcl} M_0(z) + \alpha_2 M_2(z) + \alpha_2^2 M_{22}(z) & < & 0 \\ M_0(z) + M_1(z) + \alpha_2 M_2(z) + M_{11}(z) + \alpha_2 M_{12}(z) + \alpha_2^2 M_{22}(z) & < & 0 \end{array} \tag{18}$$

and therefore it remains only to convexify the quadratic term $x' M_{22}(z) x \alpha_2^2$ on $[0, 1]$.

Remark 2.3. Using Lemmas 2.3 and 2.5 the PLMI feasibility problem (4) is also amenable to an LMI feasibility problem. For instance, consider the PLMI feasibility problem

$$M_0(z) + \alpha_1^2 \alpha_2 M_{210}(z) + \alpha_1^2 \alpha_2^2 \alpha_3^2 M_{222}(z) < 0, \ (\alpha_1, \alpha_2, \alpha_3) \in [0, 1]^3. \tag{19}$$

Using Lemmas 2.2 to 2.5, one can check that (19) is satisfied if

$$\begin{array}{l} x' M_0(z) x + \max\{x' M_{210}(z) x {\frac{\alpha_1^2 + \alpha_2}{2}}^2, \ -x' M_{210}(z) x [\frac{1}{2}(\alpha_1^4 + \alpha_2^2) - \frac{1}{2}\alpha_1 \\ \quad -\alpha_2 + 0.5]\} + \max\{x' M_{222}(z) x {\frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}{3}}^3, \ -x' M_{222}(z) x [\frac{\alpha_1^4}{2} \\ \quad\quad + \frac{1}{2} {\frac{\alpha_2^4 + \alpha_3^4}{2}}^2 - \frac{\alpha_1}{2} + \frac{\alpha_2^4 + \alpha_3^4}{4} - \frac{\alpha_2 + \alpha_3}{2} + 0.37\} \ < \ 0 \end{array} \tag{20}$$

for all $||x|| = 1$ and α in vert $[0, 1]^3$. As before, these quadratic constraints are readily recast as an LMI feasibility problem.

3 D.C. convexification method

In section 2, we have presented techniques to convexify all non-convex terms in the left-hand side of (6). Now, we present a method to convexify the left-hand side of (6) in one step, that is, without making a separated treatment of the terms.

The idea of this method is to express the LHS of (6) as a d.c (difference of convex) functions in the form

$$f_{zx}(\alpha) = f_{1zx}(\alpha) - f_{2zx}(\alpha), \tag{21}$$

where f_{1zx}, f_{2zx} are both convex. Then, a convex upperbound of the concave function $-f_{2zx}$ should be introduced. The reader is referred to [13, 21] for a comprehensive theory of d.c functions and their distinguished role in non-convex global optimization.
It is well known (see e.g. [13, Chapter 6]) that as a smooth function, f_{zx} has d.c. representation (21) with

$$f_{1zx}(\alpha) = f_{zx}(\alpha) + r\alpha^2, \ f_{2zx}(\alpha) = r\alpha^2$$

where $r > 0$ is chosen large enough such that the Hessian $\nabla^2 f_{1zx}$ of f_{1zx} which equals $\nabla^2 f_{zx}(\alpha) + 2rI$ is positive semi-definite. It follows that f_{1zx} is convex because of the semi-positiveness of $\nabla^2 f_{1zx}$ while function $r\alpha^2$ is obviously convex.

For our purpose, we rewrite (6) in the form

$$\begin{array}{l} x' M_0(z) x + \sum_{i=1}^{L} x' M_i(z) x \alpha_i + (\sum_{1 \leq i \leq j \leq L} x' M_{ij} x \alpha_i \alpha_j + \sum_{i=1}^{L} r_i x' I x \alpha_i^2) - \sum_{i=1}^{L} r_i x' I x \alpha_i^2 < 0, \\ r_i \geq 0, \ i = 1, 2, ..., L. \end{array} \tag{22}$$

Setting

$$f_{1zx}(\alpha) = x'M_0(z)x + \sum_{i=1}^{L} x'M_i(z)x\alpha_i + (\sum_{1\leq i\leq j\leq L} x'M_{ij}x\alpha_i\alpha_j + \sum_{i=1}^{L} r_i x'Ix\alpha_i^2)$$

one can see that f_{1zx} is convex whenever its Hessian $\nabla^2 f_{1zx}(\alpha)$ is positive semi-definite, i.e. when

$$\nabla^2 f_{1zx}(\alpha) = \begin{bmatrix} 2x'(M_{11}(z)+r_1 I)x & x'M_{12}(z)x & .... & x'M_{1L}(z)x \\ x'M_{12}(z)x & 2x'(M_{22}(z)+r_2 I)x & .... & x'M_{2L}(z)x \\ ... & .... & .... & ... \\ x'M_{1L}(z)x & x'M_{2L}(z)x & .... & 2x'(M_{LL}(z)+r_L I)x \end{bmatrix} \geq 0 \tag{23}$$

For matrices $\mathcal{M}$ of dimension $nL \times nL$ and $\mathcal{A}(x)$ of dimension $nL \times n$ defined as

$$\mathcal{M} = \begin{bmatrix} 2(M_{11}(z)+r_1 I) & M_{12}(z) & .... & M_{1L}(z) \\ M_{12}(z) & 2(M_{22}(z)+r_2 I) & .... & M_{2L}(z) \\ ... & .... & .... & ... \\ M_{1L}(z) & M_{2L}(z) & .... & 2(M_{LL}(z)+r_L I) \end{bmatrix},$$

$$\mathcal{A}(x) := \begin{bmatrix} x & 0 & ... & 0 \\ 0 & x & ... & 0 \\ 0 & 0 & ... & x \end{bmatrix}$$

where n stands for the dimension of the vector x, it is easily seen that

$$\nabla^2 f_{1zx} = \mathcal{A}'(x)\mathcal{M}\mathcal{A}(x) \tag{24}$$

and therefore (23) is enforced by

$$\begin{bmatrix} 2(M_{11}(z)+r_1 I) & M_{12}(z) & .... & M_{1L}(z) \\ M_{12}(z) & 2(M_{22}(z)+r_2 I) & .... & M_{2L}(z) \\ ... & .... & .... & ... \\ M_{1L}(z) & M_{2L}(z) & .... & 2(M_{LL}(z)+r_L I) \end{bmatrix} \geq 0 \tag{25}$$

On the other hand, using Lemma 2.1, we have

$$-\sum_{i=1}^{L} r_i\alpha_i^2 \leq -\sum_{i=1}^{L} r_i(\alpha_i - 0.25), \tag{26}$$

where the RHS is trivially a convex function in α. The following theorem is an immediate consequence of these results.

Theorem 3.1 *The PLMI problem defined by (1) and (2) is feasible whenever the LMIs (in finite number) (25) together with*

$$M_0(z) + \sum_{i=1}^{L}\alpha_i M_i(z) + \sum_{1\leq i\leq j\leq L}\alpha_i\alpha_j M_{ij}(z) + \sum_{i=1}^{L}[r_i\alpha_i^2 I - r_i(\alpha_i - 0.25)I] < 0, \tag{27}$$

where α runs over vertΓ, are feasible.

As mentioned in Remark 2.2, for the more special PLMI problem (1), (14), there is no need for convexifying terms $x'M_{ij}x\alpha_i\alpha_j$, $i < j$. The following result provides solvability conditions for this structurally simpler problem.

Theorem 3.2 *The PLMI problem (1), (14) is feasible whenever the LMIs (27), where α ranges over vert $\Gamma = \{0,1\}^L$, together with the LMIs*

$$M_{ii}(z) + r_i I \geq 0, \;\; i = 1,2,...,L \tag{28}$$

are feasible.

Proof: It suffices to show that for every fixed x and z the maximum of the function

$$\begin{aligned}\bar{f}_{zx}(\alpha) &:= x'[M_0(z) + \sum_{i=1}^{L}\alpha_i M_i(z) + \sum_{1\leq i\leq j\leq L}\alpha_i\alpha_j M_{ij}(z) + \sum_{i=1}^{L}[r_i\alpha_i^2 I - r_i(\alpha_i - 0.25)I]]x \\ &= x'[M_0(z) + \sum_{i=1}^{L}(\alpha_i M_i(z) - r_i(\alpha_i - 0.25)I) + \sum_{1\leq i<j\leq L}\alpha_i\alpha_j M_{ij}]x \\ &\quad + \sum_{i=1}^{L}x'[\alpha_i^2 M_{ii}(z) + r_i\alpha_i^2 I]x \end{aligned} \tag{29}$$

over Γ defined by (14) is achieved on vert Γ.
Let $\hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2, ..., \hat{\alpha}_L) \in \Gamma$ be a maximizer of $\bar{f}_{zx}$. Consider the function $\bar{f}^L$ from $[0,1]$ to $\boldsymbol{R}$ defined as

$$\bar{f}^L_{zx}(\alpha_L) = \bar{f}_{zx}(\hat{\alpha}_1, ..., \hat{\alpha}_{L-1}, \alpha_L).$$

From the representation (29) of $\bar{f}_{zx}$ and also (28), one can see that $\bar{f}^L_{zx}$ is convex on $[0,1]$ and therefore its maximum is attained at some $\alpha^*_L \in \{0,1\}$. But since $\hat{\alpha}$ is a maximizer of $\bar{f}_{zx}$ over Γ we have that $\bar{f}_{zx}(\hat{\alpha}) = \bar{f}^L_{zx}(\alpha^*_L) = \bar{f}_{zx}(\hat{\alpha}_1, \ldots, \hat{\alpha}_{L-1}, \alpha^*_L)$. Analogously, the maximum of the convex function $\bar{f}^{L-1}_{zx}(\alpha_{L-1}) := \bar{f}_{zx}(\hat{\alpha}_1, ..., \hat{\alpha}_{L-2}, \alpha_{L-1}, \alpha^*_L)$ is attained for $\alpha^*_{L-1} \in \{0,1\}$ thus we have $\bar{f}_{zx}(\hat{\alpha}) = \bar{f}_{zx}(\hat{\alpha}_1, ..., \hat{\alpha}_{L-2}, \alpha^*_{L-1}, \alpha^*_L)$. Continuing this process, we can find $\alpha^* = (\alpha^*_1, \alpha^*_2, ..., \alpha^*_L) \in$ vert Γ such that $\bar{f}_{zx}(\hat{\alpha}) = \bar{f}_{zx}(\alpha^*)$. This completes the proof of Theorem 3.2. □

4 Higher-order PLMIs

In this section, the focus is on more complicated PLMIs involving polynomial terms in the parameter. As before, we shall use d. c. concepts to relax the PLMI problem into a finite number of LMIs.

First, for simplicity of description let us consider the following particular case of (4), (14):

$$\begin{aligned} M_0(z) + \alpha_1^2 M_{20} + \alpha_2^2 M_{02} + \alpha_1^3 M_{30} + \alpha_1^2\alpha_2 M_{21} + \alpha_1\alpha_2^2 M_{12} + \alpha_2^3 M_{03} < 0 \\ (\alpha_1, \alpha_2) \in [0,1]^2, \end{aligned} \tag{30}$$

i.e. (30) is (4), (14) with $L = 2$ and

$$J \subset \{\nu \in \boldsymbol{N}^2 : \sum_{i=1}^{2}\nu_i \leq 3, \;\; \nu_i \geq 0, \;\; i = 1,2\} \tag{31}$$

Again, setting

$$f_{zx}(\alpha) = x'[M_0(z) + \alpha_1^2 M_{20} + \alpha_2^2 M_{02} + \alpha_1^3 M_{30} + \alpha_1^2\alpha_2 M_{21} + \alpha_1\alpha_2^2 M_{12} + \alpha_2^3 M_{03}]x$$
$$f_{1zx}(\alpha) = f_{zx}(\alpha) + \sum_{i=1}^{2}(r_i x' I x \alpha_i^2 + s_i x' I x \alpha_i^3)$$

then

$$f_{zx}(\alpha) = f_{1zx}(\alpha) - \sum_{i=1}^{2}(r_i x' I x \alpha_i^2 + s_i x' I x \alpha_i^3). \quad (32)$$

The Hessian $\nabla^2 f_{1zx}(\alpha)$ of $f_{1zx}(\alpha)$ is

$$\begin{bmatrix} \begin{array}{l} x'[2M_{20}(z) + 6\alpha_1 M_{30}(z) \\ +2\alpha_2 M_{21}(z) + (2r_1 + 6s_1\alpha_1)I]x \end{array} & x'[2\alpha_1 M_{21}(z) + 2\alpha_2 M_{12}(z)]x \\ x'[2\alpha_1 M_{21}(z) + 2\alpha_2 M_{12}(z)]x & \begin{array}{l} x'[M_{02}(z) + 6\alpha_2 M_{03}(z) \\ +2\alpha_1 M_{12}(z) + (2r_2 + 6s_2\alpha_2)I]x \end{array} \end{bmatrix} \quad (33)$$

Therefore, using a relation similar to (24), it can be shown that f_{1zx} is convex on Γ if

$$\begin{bmatrix} \begin{array}{l} 2M_{20}(z) + 6\alpha_1 M_{30}(z) \\ +2\alpha_2 M_{21}(z) + (2r_1 + 6s_1\alpha_1)I \end{array} & 2\alpha_1 M_{21}(z) + 2\alpha_2 M_{12}(z) \\ 2\alpha_1 M_{21}(z) + 2\alpha_2 M_{12}(z) & \begin{array}{l} M_{02}(z) + 6\alpha_2 M_{03}(z) \\ +2\alpha_1 M_{12}(z) + (2r_2 + 6s_2\alpha_2)I \end{array} \end{bmatrix} \geq 0 \quad \forall \alpha \in [0,1]^2. \quad (34)$$

Now, it is already seen that as the LHS of (34) is affine in α, (34) is equivalent to the following LMI

$$\begin{bmatrix} \begin{array}{l} 2M_{20}(z) + 6\alpha_1 M_{30}(z) \\ +2\alpha_2 M_{21}(z) + (2r_1 + 6s_1\alpha_1)I \end{array} & 2\alpha_1 M_{21}(z) + 2\alpha_2 M_{12}(z) \\ 2\alpha_1 M_{21}(z) + 2\alpha_2 M_{12}(z) & \begin{array}{l} M_{02}(z) + 6\alpha_2 M_{03}(z) \\ +2\alpha_1 M_{12}(z) + (2r_2 + 6s_2\alpha_2)I \end{array} \end{bmatrix} \geq 0 \quad \forall \alpha \in \text{vert}[0,1]^2. \quad (35)$$

with the decision variables z, r, s.
By Lemma 2.1

$$-\sum_{i=1}^{2} s_i\alpha_i^3 \leq -\sum_{i=1}^{2} s_i(0.75\alpha_i - 0.25) \quad (36)$$

Using (26), (36) we have

Proposition 4.1 *PLMI (30) is feasible if LMIs (35) and*

$$\begin{array}{l} M_0(z) + \alpha_1^2 M_{20} + \alpha_2^2 M_{02} + \alpha_1^3 M_{30} + \alpha_1^2\alpha_2 M_{21} + \alpha_1\alpha_2^2 M_{12} \\ +\alpha_2^3 M_{03} + \sum_{i=1}^{2}[(r_i\alpha_i^2 + s_i^3\alpha_i^3)I - (r_i + 0.75s_i)\alpha_i I + 0.5I] \quad < \quad 0, \quad \forall \alpha \in vert[0,1]^2, \\ r_i \geq 0, \;\; s_i \geq 0, \quad i = 1,2, \end{array} \quad (37)$$

are feasible.

Clearly, PLMI (30) with α belonging to (2) instead of (14) can be treated in a similar manner and moreover, the above method can be generalized for the general case of PLMI (4), (14) as follows.

Obviously, for every J in (4) there is $k > 0$ integer such that

$$J \subset \{\nu \in R^L : \sum_{i=1}^{L} \nu_i \leq k,\ \nu_i \geq 0,\ i = 1, 2, ..., L\}. \tag{38}$$

The minimum of such k will be called *the order* of J and J itself will be called the index set of (4). Then the d.c. convexification process for solving (4), (2) is as follows.

Step 1. *Using a d.c representation like (32) to relax the feasibility problem for PLMI (4), (2) to the feasibility problem for PLMI (37) and (34). PLMI (34) belongs to class (4) but the order of its index set decreases to* $k-2$.
Step 2. *Do the d.c. relaxation process for PLMI (34) until get a PLMI like (4) with the order of the index set less than or equal* 1. *This last PLMI is already equivalent to an LMIs system.*

To illustrate the above steps let us consider more the following PLMI

$$\sum_{i=0}^{6} \theta^i M_i(z) < 0\ \forall \theta \in [0, 1]. \tag{39}$$

First, expressing the LHS of (39) in the form

$$[\sum_{i=0}^{6} \theta^i M_i(z) + \sum_{i=2}^{6} r_i \theta^i I] - \sum_{i=2}^{6} r_i \theta^i I$$

and using

$$-\sum_{i=2}^{6} r_i \theta^i I \leq -\sum_{i=2}^{6} r_i (\frac{i}{2^{i-1}}\theta - \frac{i-1}{2^i})I$$

by Lemma 2.1 we see by d.c. convexification process that (39) holds true whenever there are $r_i \geq 0,\ i = 2, 3, 4, 5, 6$ such that

$$[\sum_{i=0}^{6} \theta^i M_i(z) + \sum_{i=2}^{6} r_i \theta^i I] - \sum_{i=2}^{6} r_i (\frac{i}{2^{i-1}}\theta - \frac{i-1}{2^i}) < 0\ \forall \theta \in \{0, 1\}, \tag{40}$$

$$\begin{aligned} &\sum_{i=2}^{6} i(i-1)\theta^{i-2}(M_i(z) + r_i I) \geq 0\ \forall \theta \in [0, 1] \quad \Leftrightarrow \\ &-\sum_{i=2}^{6} i(i-1)\theta^{i-2}(M_i(z) + r_i I) \leq 0\ \forall \theta \in [0, 1]. \end{aligned} \tag{41}$$

Rewrite (40) in the form of LMIs as

$$M_0(z) + \sum_{i=2}^{6} r_i \frac{i-1}{2^i} I \quad < \quad 0 \tag{42}$$

$$\sum_{i=0}^{6} M_i(z) + \sum_{i=2}^{6} r_i (1 - \frac{i}{2^{i-1}} + \frac{i-1}{2^i}) I \quad < \quad 0 \tag{43}$$

It remains to do the d.c. convexification for (41). Again expressing LHS of (41) in the form

$$[-\sum_{i=2}^{6} i(i-1)\theta^{i-2}(M_i(z)+r_iI)+(s_2\theta^2+s_3\theta^3+s_4\theta^4)I]-(s_2\theta^2+s_3\theta^3+s_4\theta^4)I$$

and using

$$-(s_2\theta^2+s_3\theta^3+s_4\theta^4)\leq -s_2(\theta-0.25)I-s_3(0.75\theta-0.25)I-s_4(0.5\theta-0.1875)I$$

one can see that (41) is implied by the following

$$\begin{array}{rcl} [-\sum_{i=2}^{6} i(i-1)\theta^{i-2}(M_i(z)+r_iI)+(s_2\theta^2+s_3\theta^3+s_4\theta^4)I] & & \\ -s_2(\theta-0.25)I-s_3(0.75\theta-0.25)I-s_4(0.5\theta-0.1875)I & \leq & 0\ \forall\theta\in\{0,1\} \quad (44)\\ -\sum_{i=4}^{6}\frac{i!}{(i-4)!}\theta^{i-4}(M_i(z)+r_iI)+(2s_2+6s_3\theta+12s_4\theta^2)I & \geq & 0\ \forall\theta\in[0,1]. \quad (45)\end{array}$$

It can be easily shown that (45) is fulfilled if there is $q\geq 0$ such that

$$\begin{array}{r}[\sum_{i=4}^{6}\frac{i!}{(i-4)!}\theta^{i-4}(M_i(z)+r_iI)-(2s_2+6s_3\theta+12s_4\theta^2)I+q\theta^2I] \\ -q(\theta-0.25)I\leq 0\ \forall\theta\in\{0,1\},\end{array} \quad (46)$$

$$6!(M_6+r_6I)-24s_4I+2qI\geq 0 \quad (47)$$

Rewriting (44), (46) in the LMIs form

$$-2M_2(z)-2r_2I+(0.25s_2+0.25s_3+0.1875s_4)I \leq 0 \quad (48)$$

$$-\sum_{i=2}^{6} i(i-1)(M_i(z)+r_iI)+(0.25s_2+0.5s_3+0.6875s_4)I \leq 0 \quad (49)$$

$$24(M_4+r_4)I-2s_2I+0.25qI \leq 0 \quad (50)$$

$$\sum_{i=4}^{6}\frac{i!}{(i-4)!}(M_i(z)+r_iI)-(2s_2+6s_3+12s_4+0.25q)I \leq 0 \quad (51)$$

we finally obtain the following

Proposition 4.2 *PLMI (39) is feasible if the LMIs*

$$r_i\geq 0,\ i=2,3,4,5,6;\ s_2\geq 0,\ s_3\geq 0,\ s_4\geq 0,\ q\geq 0$$

and (42), (43), (48), (47)-(51) with the decision variables $z,r_2,r_3,r_4,r_5,r_6,s_2,s_3,s_4,q$ *are feasible*

Remark 4.1. One can modify (22) in the form

$$x'M_0(z)x+\sum_{i=1}^{L}x'M_i(z)x\alpha_i+(\sum_{1\leq i\leq j\leq L}x'M_{ij}x\alpha_i\alpha_j+\sum_{i=1}^{M}r_i\alpha_i^2x'Ix)-\sum_{i=1}^{M}r_ix'Ix\alpha_i^2<0 \quad (52)$$

for some $M < L$. Then, imposing the convexity for

$$x'M_0(z)x + \sum_{i=1}^{L} x'M_i(z)x\alpha_i + (\sum_{1\le i\le j\le L} x'M_{ij}x\alpha_i\alpha_j + \sum_{i=1}^{M} r_i\alpha_i^2 x'Ix)$$

it remains to convexify $-\sum_{i=1}^{M}\alpha_i^2$ instead of $-\sum_{i=1}^{L}\alpha_i^2$.

Remark 4.2. One can use a hybrid convexification for (4), (2). For instance at Step 2, one can use the separated convexification method to relax (34) or DCC [2].

5 Applications

The techniques and tools presented in Sections 2-4 are applicable to

- Lyapunov-based stability and performance robustness analysis,
- μ analysis, see [7, 2] for a detailed discussion,
- LPV synthesis.

For brevity, we only report applications to the first and third issue. The reader is referred to reference [2] for other applications and a detailed discussion.

5.1 Robust stability

We consider the linear uncertain system

$$\dot{x} = A(\alpha)x, \quad A(\alpha) := \alpha_1 A_1 + \ldots + \alpha_L A_L, \tag{53}$$

where α is a fixed uncertain parameter evolving in the unit simplex (2). It follows that the uncertain matrix $A(\alpha)$ ranges over a matrix polytope

$$A(\alpha) \in \mathrm{conv}\,\{A_1, \ldots, A_L\}$$

We are seeking a quadratic parameter-dependent Lyapunov function with similar structure

$$V(x,\alpha) := x'\,(\,\alpha_1 X_1 + \ldots + \alpha_L X_L\,)\,x$$

establishing stability of the uncertain system for all admissible dynamics.

Working out the condition $\frac{d}{dt}V(x,\alpha) < 0$, and exploiting Theorems 2.6 and 3.1 to convert the problem into a finite number of LMI feasibility conditions, the following sufficient tests for robust stability is obtained.

Proposition 5.1 *Assume one of the A_i's is stable. Then, the uncertain system (53) is stable whenever there exist symmetric matrices X_1, ... , X_L such that either*

a) *(10) holds for $z := (X_1, X_2, ..., X_L)$ and the definitions*

$$\begin{array}{l} M_i(z) = 0,\ i = 0, 1, ..., L, \\ M_{ij}(z) = A_i'X_j + X_jA_i' + A_j'X_i + X_iA_j,\ i < j, \\ M_{ii}(z) = A_i'X_i + X_iA_i,\ i = 1, 2, ..., L, \end{array} \tag{54}$$

or

b) *(25) and (27) hold for M_i, M_{ij} defined by (54).*

In such case the Lyapunov function $V(x,\alpha)$ establishes stability of the uncertain system 53.

Proof: For $V(x,\alpha)$ as a Lyapunov function candidate, the uncertain system (53) is stable if

$$(\sum_{i=1}^{L}\alpha_i A_i)'(\sum_{i=1}^{L}\alpha_i P_i) + (\sum_{i=1}^{L}\alpha_i P_i)(\sum_{i=1}^{L}\alpha_i A_i) < 0 \ \forall \alpha \in \Gamma \tag{55}$$

$$\sum_{i=1}^{L}\alpha_i P_i > 0 \ \forall \alpha \in \Gamma. \tag{56}$$

The proof is obtained by identifying the terms in (55) with those of (1) and by direct application of Theorems 2.6 or 3.1. Note for completeness that, (56) follows by continuity from inequality (55) and the fact that one of the A_i's is stable.

□

5.2 Linear Parameter-Varying Control

In this section, we more thoroughly investigate how the concepts and tools introduced can be utilized in the context of LPV control. For clarity, we recall the general statement of the problem.

We are considering an LPV plant with state-space realization

$$\begin{aligned} \dot{x} &= A(\theta)x + B_1(\theta)w + B_2(\theta)u \\ z &= C_1(\theta)x + D_{11}(\theta)w + D_{12}(\theta)u \\ y &= C_2(\theta)x + D_{21}(\theta)w, \end{aligned} \tag{57}$$

where

$$A \in \boldsymbol{R}^{n\times n}, \quad D_{12} \in \boldsymbol{R}^{p_1\times m_2}, \text{and} \quad D_{21} \in \boldsymbol{R}^{p_2\times m_1}$$

define the problem dimension. It is assumed that

(A1) the state-space data $A(\theta)$, $B_1(\theta)$, ... are bounded continuous functions of θ,

(A2) the time-varying parameter $\theta(t) := [\theta_1(t), \cdots, \theta_N(t)]'$ and its rate of variation $\dot{\theta}(t)$, defined at all times and continuous, evolve in hyper-rectangles H and H_d, that is,

$$H: \quad \theta_i(t) \in [\underline{\theta}_i, \bar{\theta}_i], \quad \forall t \geq 0, \tag{58}$$

$$H_d: \quad \dot{\theta}_i(t) \in [\underline{\nu}_i, \bar{\nu}_i], \quad \forall t \geq 0\,. \tag{59}$$

The assumptions **(A1)** and **(A2)** are general. They secure existence and uniqueness of the solutions to (57) for given initial conditions and also specify the parameter trajectories under consideration.

With these assumptions in place, the general LPV control problem with guaranteed L_2-gain performance consists of finding a dynamic LPV controller with state-space equations

$$\begin{array}{rcl} \dot{x}_K & = & A_K(\theta,\dot{\theta})x_K + B_K(\theta,\dot{\theta})y \\ u & = & C_K(\theta,\dot{\theta})x_K + D_K(\theta,\dot{\theta})y \end{array} \tag{60}$$

which ensures internal stability and a guaranteed L_2-gain bound γ for the closed-loop operator (57)-(60) from the disturbance signal w to the error signal z, that is,

$$\int_0^T z'z\,d\tau \le \gamma^2 \int_0^T w'w\,d\tau, \quad \forall T \ge 0,$$

for all admissible parameter trajectories $\theta(t)$.

Sufficient solvability conditions for this problem can be derived using a suitable extension of the Bounded Real Lemma [25], and by confining the search of (Lyapunov) variables to some finite-dimensional subspace of functions of θ. The next theorem provides such a set of conditions for the general LPV control problem. An alternative approach, based on polytopic covering techniques, is proposed in [23]. We also assume, see [2], that

(A3) the matrices $[\,B_2'(\theta) \quad D_{12}'(\theta)\,]$, $\;[\,C_2(\theta) \quad D_{21}(\theta)\,]$ have full row-rank over H.

Note that the dependence of data and variables on θ, or $\dot{\theta}$ is generally dropped for simplicity in the sequel. A proof of the following theorem can be found in [2].

Theorem 5.2 *With the assumptions* **(A1)-(A3)** *in force, the following conditions are equivalent:*

(i) The Bounded Real Lemma conditions with L_2-gain performance level γ hold for some quadratic Lyapunov function

$$V(x,x_K,\theta) := \begin{bmatrix} x \\ x_K \end{bmatrix}' P(\theta) \begin{bmatrix} x \\ x_K \end{bmatrix},$$

where $P(\theta)$ is continuously differentiable, and for some LPV controller (60).

(ii) There exist continuously differentiable parameter-dependent symmetric matrices $X(\theta)$ and $Y(\theta)$ and a scalar σ solving the PLMI problem:

$$\begin{bmatrix} \dot{X} + XA + A'X & XB_1 & C_1' \\ B_1'X & -\gamma I & D_{11}' \\ C_1 & D_{11} & -\gamma I \end{bmatrix} - \sigma \begin{bmatrix} C_2' \\ D_{21}' \\ 0 \end{bmatrix} [\,C_2 \quad D_{21} \quad 0\,] < 0 \tag{61}$$

$$\begin{bmatrix} -\dot{Y} + YA' + AY & YC_1' & B_1 \\ C_1Y & -\gamma I & D_{11} \\ B_1' & D_{11}' & -\gamma I \end{bmatrix} - \sigma \begin{bmatrix} B_2 \\ D_{12} \\ 0 \end{bmatrix} [\,B_2' \quad D_{12}' \quad 0\,] < 0 \tag{62}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \tag{63}$$

for all $(\theta,\dot{\theta})$ on $H \times H_d$.

For simplicity of the presentation, it is first assumed the the state-space data in (57) and the Lyapunov variables are affine functions of the parameter θ, that is,

(A4) $$A(\theta) := A_0 + \sum_{i=1}^{N} \theta_i A_i, \quad B_1(\theta) := B_{10} + \sum_{i=1}^{N} \theta_i B_{1i} ....$$

The following Theorem is a direct consequence of Theorem 5.2 and Theorems 2.6, 3.2 with

$$X(\theta) = \sum_{i=0}^{N} \theta_i X_i, \ Y(\theta) = \sum_{i=0}^{M} \theta_i Y_i$$

in (61)-(63).

Theorem 5.3 *With the assumptions* **(A1)-(A4)** *above, there exists an LPV controller (60) solution to the LPV control problem with guaranteed* L_2*-gain performance with level* γ *whenever there exist symmetric matrices* X_0, X_1, ... ,X_N *and* Y_0, Y_1, ...,Y_N *and scalars* γ *and* σ *such that*

$$\begin{bmatrix} X_0 + \sum_{i=1}^{N} \theta_i X_i & I \\ I & Y_0 + \sum_{i=1}^{N} \theta_i Y_i \end{bmatrix} > 0, \qquad \forall \ \theta \in vert\, H \tag{64}$$

and either

a) *(10) holds true simultaneously for* z, $M_i(z)$, *and* $M_{ij}(z)$ *defined in the items 1) and 2) below:*

1) $z = (X_0, X_1, ..., X_N, \sigma, \gamma)$, $\alpha = (\theta, \dot{\theta})$, $\Gamma = H \times H_d$, $L = 2N$,

$$M_0(z) = \begin{bmatrix} X_0 A_0 + A_0' X_0 & X_0 B_{10} & C_{10} \\ B_{10}' X_0 & -\gamma I & D_{110}' \\ C_{10}' & D_{110} & -\gamma I \end{bmatrix} \tag{65}$$

$$-\sigma \begin{bmatrix} C_{20}' \\ D_{210}' \\ 0 \end{bmatrix} [\, C_{20} \quad D_{210} \quad 0\,], \tag{66}$$

$$M_i(z) = \begin{bmatrix} X_i A_0 + A_0' X_i + X_0 A_i + A_i' X_0 & X_i B_{10} + X_0 B_{1i} & C_{1i}' \\ B_{1i}' X_0 + B_{10}' X_i & 0 & D_{11i}' \\ C_{1i} & D_{11i} & 0 \end{bmatrix}$$

$$-\sigma \left(\begin{bmatrix} C_{2i}' \\ D_{21i}' \\ 0 \end{bmatrix} [\, C_{20} \quad D_{210} \quad 0\,] + \begin{bmatrix} C_{20}' \\ D_{210}' \\ 0 \end{bmatrix} [\, C_{2i} \quad D_{21i} \quad 0\,] \right),$$

$$1 \le i \le N, \tag{67}$$

$$M_i(z) = \begin{bmatrix} X_{i-N} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad N+1 \le i \le 2N, \tag{68}$$

$$M_{ij}(z) = \begin{bmatrix} X_i A_j + A_j' X_i + X_j A_i + A_i' X_j & X_i B_{1j} + X_i B_{1j} & 0 \\ B_{1i}' X_j + B_{1j}' X_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-\sigma\left(\begin{bmatrix} C'_{2i} \\ D'_{21i} \\ 0 \end{bmatrix}[\,C_{2j} \quad D_{21j} \quad 0\,] + \begin{bmatrix} C'_{2j} \\ D'_{21j} \\ 0 \end{bmatrix}[\,C_{2i} \quad D_{21i} \quad 0\,]\right),$$

$$1 \le i < j \le N, \tag{69}$$

$$M_{ii}(z) = \begin{bmatrix} X_iA_i + A'_iX_i & X_iB_{1i} & 0 \\ B'_{1i}X_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \sigma \begin{bmatrix} C'_{2i} \\ D'_{21i} \\ 0 \end{bmatrix}[\,C_{2i} \quad D_{21i} \quad 0\,],$$

$$1 \le i \le N, \tag{70}$$

$$M_{ij}(z) = 0, \quad \textit{for } i > N \textit{ or } j > N \tag{71}$$

2) $z = (Y_0, Y_1, ..., Y_N, \sigma, \gamma)$, $\alpha = (\theta, \dot{\theta})$, $\Gamma = H \times H_d$, $L = 2N$,

$$M_0(z) = \begin{bmatrix} Y_0A'_0 + A_0X_0 & Y_0C'_{10} & B_{10} \\ C_{10}Y_0 & -\gamma I & D_{110} \\ B_{10} & D_{110} & -\gamma I \end{bmatrix} \tag{72}$$

$$-\sigma \begin{bmatrix} B_{20} \\ D_{120} \\ 0 \end{bmatrix}[\,B'_{20} \quad D'_{120} \quad 0\,], \tag{73}$$

$$M_i(z) = \begin{bmatrix} Y_iA'_0 + A_0Y_i + Y_0A'_i + A_iY_0 & Y_iC'_{10} + Y_0C'_{1i} & B_{1i} \\ C_{1i}Y_0 + C_{10}Y_i & 0 & D_{11i} \\ B'_{1i} & D'_{11i} & 0 \end{bmatrix}$$

$$-\sigma\left(\begin{bmatrix} B_{2i} \\ D_{12i} \\ 0 \end{bmatrix}[\,B'_{20} \quad D'_{120} \quad 0\,] + \begin{bmatrix} B_{20} \\ D_{120} \\ 0 \end{bmatrix}[\,B'_{2i} \quad D'_{12i} \quad 0\,]\right),$$

$$1 \le i \le N, \tag{74}$$

$$M_i(z) = \begin{bmatrix} Y_{i-N} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad N+1 \le i \le 2N, \tag{75}$$

$$M_{ij}(z) = \begin{bmatrix} Y_iA'_j + A_jY_i + Y_jA'_i + A_iY_j & Y_iC'_{1j} + Y_iC'_{1j} & 0 \\ C_{1i}Y_j + C_{1j}Y_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-\sigma\left(\begin{bmatrix} B_{2i} \\ D_{12i} \\ 0 \end{bmatrix}[\,B'_{2j} \quad D'_{12j} \quad 0\,] + \begin{bmatrix} B_{2j} \\ D_{12j} \\ 0 \end{bmatrix}[\,B'_{2i} \quad D'_{11i} \quad 0\,]\right),$$

$$1 \le i < j \le N, \tag{76}$$

$$M_{ii}(z) = \begin{bmatrix} Y_iA'_i + A_iX_i & X_iC'_{1i} & 0 \\ C_{1i}Y_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \sigma \begin{bmatrix} B_{2i} \\ D_{12i} \\ 0 \end{bmatrix}[\,B'_{2i} \quad D'_{12i} \quad 0\,],$$

$$1 \le i \le N, \tag{77}$$

$$M_{ij}(z) = 0, \quad \textit{for } i > N \textit{ or } j > N \tag{78}$$

or alternatively,

b) *The LMIs (25) and (27) are jointly feasible with* M_i, M_{ij} *defined by (66)-(71) and (73)-(78).*

The conditions in Theorem 5.3 constitute a standard semi-definite programming problem. The linear objective γ should be minimized subject to a finite number of LMI constraints, and a number of efficient softwares are available for this purpose (see e.g. [9]). Given a solution z to the LMI problems in Theorem 5.3, the LPV controller state-space data are easily constructed for any value of $(\theta, \dot{\theta})$ in $H \times H_d$ using a simple scheme detailed in [2, Subsection 5.4.1].

6 Illustrative Examples

6.1 Stability analysis

We consider the following example from [20]. The A matrix of the uncertain system is given in the form:

$$A(\theta_1, \theta_2) = \begin{bmatrix} -2+\theta_1 & 0 & -1+\theta_1 \\ 0 & -3+\theta_2 & 0 \\ -1+\theta_1 & -1+\theta_2 & -4+\theta_1 \end{bmatrix}$$

We are seeking the maximum rectangle in the (θ_1, θ_2)- space for which stability is guaranteed. In this context, Proposition 5.1 is directly applicable to the polytope of extreme values of the parameters θ_1 and θ_2. By both the separated and d.c. convexification methods of Sections 2 and 3, the uncertain system is found stable for all values of θ_1 and θ_2 in the rectangle

$$-1e6 \le \theta_1 \le 1.7499, \qquad -1e6 \le \theta_2 \le 2.99\,.$$

This result is consistent with the true domain of stability ($\theta_1 < 1.75$, $\theta_2 < 3$), and outperforms existing results [20].

6.2 LPV synthesis example

The following example provides an illustration of the LPV control synthesis techniques proposed in Theorem 5.3. The discussion emphasizes the complexity and cost associated with various LPV synthesis strategies. The problem setup comes from [18]. It has been slightly complicated to incorporate two time-varying parameters for illustration purpose, while retaining the same design specifications.

The LPV model of the longitudinal dynamics of the missile are given as:

$$\begin{aligned}
\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} &= \begin{bmatrix} -0.89 & 1 \\ -142.6 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} 0 & -0.89 \\ 178.25 & 0 \end{bmatrix} \begin{bmatrix} w_{\theta_1} \\ w_{\theta_2} \end{bmatrix} + \begin{bmatrix} -0.119 \\ -130.8 \end{bmatrix} \delta_{\text{fin}} \\
\begin{bmatrix} w_{\theta_1} \\ w_{\theta_2} \end{bmatrix} &= \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} \\
\begin{bmatrix} \eta_z \\ q \end{bmatrix} &= \begin{bmatrix} -1.52 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix}
\end{aligned}$$

where α, q, η_z and δ_{fin} denote the angle of attack, the pitch rate, the vertical accelerometer measurement, the fin deflection, respectively; and θ_1, θ_2 are two time-varying parameters, measured in real time, resulting from changes in missile aerodynamic conditions (angle of attack from 0 up to 20 degrees). The synthesis structure used in this problem is depicted in Figure 1

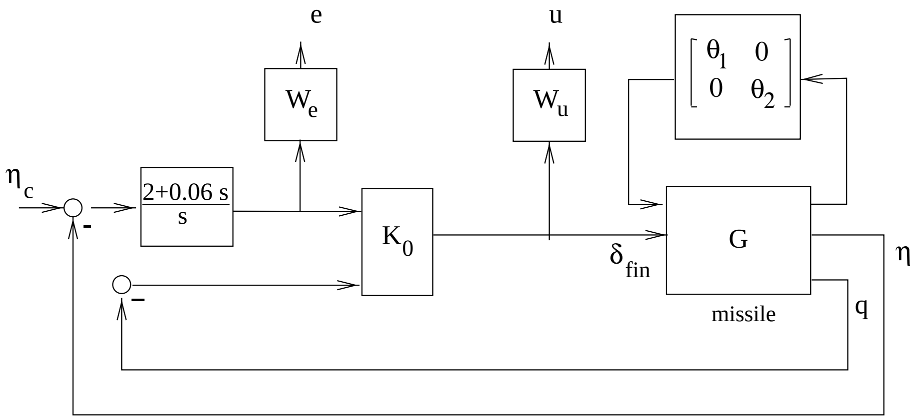

Figure 1: Synthesis interconnection

The problem specifications are as follows:

- A settling time of 0.2 second with minimal overshoot and zero steady-state error for the vertical acceleration η_z in response to a step command η_c.
- The controller must achieve an adequate high-frequency roll-off for noise attenuation and withstand neglected dynamics and flexible modes. Magnitude constraints of 2 are also imposed to the control signal δ_{fin}.

Moreover, those specifications must be met for all parameter values:

$$|\theta_1| \leq 1, \qquad |\theta_2| \leq 1.$$

An integrator has been introduced on the acceleration channel to ensure zero steady-state error. It turns out that the resulting LPV controller K is obtained as the composition of the operators K_0 and

$$\begin{bmatrix} \frac{2+0.06\,s}{s} & 0 \\ 0 & 1 \end{bmatrix}.$$

The weighting functions W_e and W_u were chosen to be

$$W_e = 0.8, \qquad W_u = \frac{0.001s^3 + 0.03s^2 + 0.3s + 1}{1e - 5s^3 + 3e - 2s^2 + 30s + 10000}.$$

It is not difficult to rewrite this example in the form (57) with $A(\theta_1, \theta_2) = A_0 + \theta_1 A_1 + \theta_2 A_2$, where the other state-space data do not depend on (θ_1, θ_2).

The design synthesis consists in the computation of a parameter-dependent controller, $K_0(\theta_1, \theta_2)$ such that all specifications above are met. For simplicity of the discussion, we

method	achieved perf. level γ	cputime
DCC in [2]	0.1290	170.71 sec.
separated convexification	0.1284	256.00 sec.
d.c. convexification	0.1290	273.3600 sec.

Table 1: Numerical results of LPV synthesis techniques

assume that the LPV model can be considered as a parameterized family of linear time-invariant systems. Similar conclusions can be drawn with time-varying parameters with bounded rates of variation. Using the techniques in Theorem 5.3, the following results are obtained on a PC with with CPU Pentium II 330 Mhz.

Table 1 displays the achieved performance level γ for three different techniques. It appears that all our methods provide almost the same performance level. Also important is the fact that the number of LMIs associated with the separated convexification method is significantly larger than that of DCC and d.c. convexification methods in this example. However, cputime for the separated convexification method in this example is less than that for d.c. convexification method because LMI solver [9] for the former requires less iterations than that for the later.

Figures 6.2 and 6.2 show the time-domain simulations, corresponding for each derived LPV controller, to parameter values at the vertices of the (θ_1, θ_2)-range. The roll-off properties of the controllers obtained at "frozen" values of the parameters θ_1 and θ_2 have been checked to be satisfactory for each technique.

As the DCC-based LPV synthesis in [2], the d. c. LPV synthesis technique considered in this section turns out to be less conservative than LFT (Linear Fractional Transformation) gain-scheduling techniques [16, 3, 11, 19] which disregard the parameter variation rates. About 10 percents degradation of the performance level has been observed in this application. This result is encouraging and gives motivations for the practical use of the proposed techniques.

6.3 Larger problems and complexity issues

To illustrate the proposed techniques for higher-order polynomial dependences, we consider the system (57) where $A(\theta)$ now reads

$$A(\theta) = A_0 + \theta A_1 + \theta^2 A_2 + \theta^3 A_3$$

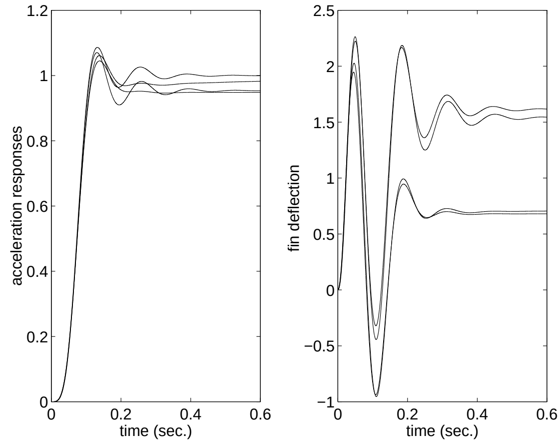

Figure 2: Time domain responses - LPV controller designed with separated convexification

with the data

$$A_1 = \begin{bmatrix} 2.0 & 1.0 & 3.0 & 0.0 & 5.0 & -1.0 \\ 0.0 & 3.0 & 0.0 & 5.0 & 2.0 & 1.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 1.0 & -1.0 \\ 0.0 & 0.0 & 7.0 & 0.0 & 0.54 & 0.0 \\ 1.0 & 1.0 & 0.0 & 0.0 & 1.0 & -1.0 \\ -1.0 & 2.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2.0 & 0.5 & 3.0 & 0.0 & 5.0 & -2.0 \\ 0.0 & 3.0 & 0.0 & 5.0 & 2.0 & 1.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 1.0 & -1.0 \\ 0.0 & 0.0 & 7.0 & 0.0 & -0.42 & 0.0 \\ 1.0 & 1.0 & 0.0 & 0.0 & 1.0 & -1.0 \\ -0.5 & 2.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -19.80 & -3.30 & -29.70 & 0.0 & -49.50 & 29.70 \\ 0.0 & -29.70 & 0.0 & -49.50 & -19.80 & -9.90 \\ 0.0 & 0.0 & -9.90 & 0.0 & -9.90 & 9.90 \\ 0.0 & 0.0 & -69.30 & 0.0 & 9.80 & 0.0 \\ -99.0 & -9.90 & 0.0 & 0.0 & -9.90 & 9.90 \\ 3.30 & -19.80 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

and other data are the same as in the LPV missile example.

The matrices $X(\theta), Y(\theta)$ in (61)–(63) are sought using one of the following more or less complex parametrizations

$$X(\theta) = X_0, \; Y(\theta) = Y_0 \tag{79}$$

$$X(\theta) = X_0 + \theta X_1, \; Y(\theta) = Y_0 + \theta Y_1 \tag{80}$$

$$X(\theta) = X_0 + \theta X_1 + \theta^2 X_2, \; Y(\theta) = Y_0 + \theta Y_1 + \theta^2 Y_2, \tag{81}$$

$$X(\theta) = X_0 + \theta X_1 + \theta^2 X_2 + \theta^3 X_3, \; Y(\theta) = Y_0 + \theta Y_1 + \theta^2 Y_2 + \theta^3 Y_3 \,. \tag{82}$$

With these functions, one can see from Theorem 5.3 that we shall obtain PLMIs of order from 3 to 6 with respect to variable θ. The methods of Sections 2 and 4 are immediately applicable. The numerical results are given in Table 2. The benefits of utilizing higher-order parameter-dependent Lyapunov functions appear clearly. The achieved performance is significantly improved for the parametrizations (79) to (82). This also implies that traditional fixed quadratic Lyapunov functions (79) are very conservative.

Let us consider one more complicated example where the system (57) is given as

$$A(\theta_1, \theta_2, \theta_3) = A_0 + \theta_1 A_1 + \theta_2 A_2 + \theta_3 A_3$$

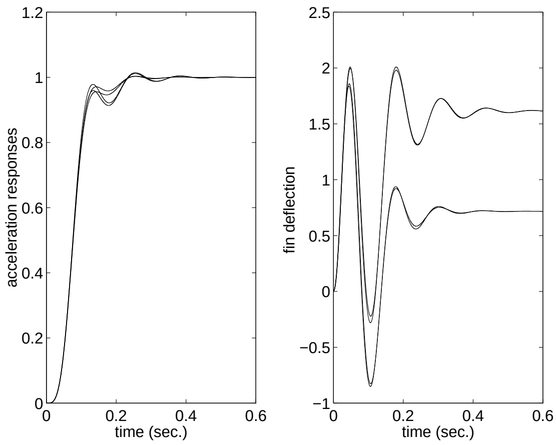

Figure 3: Time domain responses - LPV controller designed with D. C. convexification

Lyapunov function	achieved perf. level γ	cputime in Pentium II 330Mhz
(79)	19.7147	80.20 sec.
(80)	16.2512	89.25 sec.
(81)	6.5355	239.75 sec.
(82)	1.5705	484.999 sec.

Table 2: Numerical results of high order PLMIs

and

$$A_0 = [A_{01}\ A_{02}], \qquad A_1(5,4) = -356.5,\ A_2(4,4) = -1.78,\ A_3(6,6) = -0.76.$$

with all other entries of A_1, A_2 and A_3 being zero and

$$A_{01} = \begin{bmatrix} -100.001 & -99.9 & 0 & 0 & 0 & 0 & 0 \\ 0 & -100.0010 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -100.001 & -99.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & -99.111 & 0 & 0 & 0 \\ 0 & 0 & 0 & 178.2500 & -0.0010 & 0 & \\ 0 & 0 & 0 & 0 & 0 & -0.0010 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.0010 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 427.0980 \end{bmatrix},$$

method	achieved perf. level γ	cputime in Pentium II 330Mhz
separated convexification	0.8022	1.623e+004 sec.
d.c. convexification	0.8229	9.273e+003 sec.

Table 3: Numerical results of a large dimensional PLMI

$$A_{02} = \begin{bmatrix} 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 1.0 & 0 & \\ 0 & 0 & 1.0 & \\ 0 & 0 & 0 & \\ -0.0010 & 0 & 0 & \\ -46.8341 & -0.0010 & 0 & \\ 232.0679 & 120.4649 & 0 & -0.001 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 & -0.001 \\ 0 & -0.001 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.1787 & 0.0003 \\ -0.8364 & 0.0001 \end{bmatrix},\ B_2 = \begin{bmatrix} -9.9950 \\ -9.9950 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2.6773 \\ 1.4274 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.4743 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ D_{11} = 0,\ D_{12} = 0,$$
$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -0.3162 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.3162 & 0 & -0.1000 \end{bmatrix},\ D_{21} = 0,$$

Note that we choose data $B_1, B_2, \ldots$ independent on $(\theta_1, \theta_2, \theta_3)$ and also simple A_1, A_2, A_3 to save a space. Obviously, the performance of our methods does not change with additional dependences. The numerical results for this example are given in Table 3. The achieved performance level by the separated convexification is less conservative than that of the d.c. convexification. Note that cputime for this examples is much higher since the LMI solver [9] requires about 400 iterations for each methods.

From Tables 2 and 3, we observe that the cputime is more sensitive to the system sizes (number of states and inputs/outputs) rather than the number of the parameters. Practical experience shows that both relaxation methods can work well up to 4 parameters in (2) or (3). The d.c. convexification method can work well for larger numbers L since the polytope (2) has L vertices whereas the hyper-rectangle (3) has 2^L vertices. Moreover, since LMI solvers are improving very quickly, our methods are expected to be practical for significantly more complex system representations in the nearest future. Certainly, by virtue of the infinite-dimensional nature of PLMI problems the proposed techniques introduce some degree of conservatism that can only be reduced at the price of heavy computations. The tradeoff accuracy/cost thus appears to be essential for the tractability of these techniques. There is however some flexibility to monitor this tradeoff by selecting more or less complicated design variables. The designer, however, should keep in mind that perfect accuracy is generally out of reach whatever technique is used.

7 Conclusion

Two general techniques for relaxing PLMIs into a finite number of LMIs have been introduced. These techniques are applicable to PLMIs with polynomial parameter-dependence where the parameter ranges in a polytope. In the light of the proposed examples, these techniques reveal of practical interest in Lyapunov-based stability analysis, μ analysis and LPV synthesis where they provide less conservative refinements of earlier techniques. This list is however non-exhaustive and other relevant applications such as robust filtering or robust least square problems can be handled similarly.

APPENDIX

Proof of lemma 2.1. It is obvious since $-\frac{i}{2^{i-1}}x+\frac{i-1}{2^i}$ is the tangent to the curve $y=-x^\nu$ at point $x=1/2$.

□

Proof of Lemma 2.2. From Lemma 2.1

$$\begin{aligned} -xy &= \frac{x^2+y^2}{2}-\frac{1}{2}(x+y)^2 \\ &\leq \frac{x^2+y^2}{2}+\frac{1}{2}(-x-y+0.25) \end{aligned}$$

and the Lemma follows.

□

Proof of Lemma 2.4. By Lemma 2.2 we have

$$-x^{\nu_1}y^{\nu_2} \leq \frac{1}{2}(x^{2\nu_1}+y^{2\nu_2})-\frac{1}{2}(x^{\nu_1}+y^{\nu_2})+0.125.$$

Since $2^{-\nu_1}(1-\nu_1)+\nu_1 2^{1-\nu_1}x$ ($2^{-\nu_2}(1-\nu_2)+\nu_1 2^{1-\nu_2}y$, resp.) is the tangent to the curve x^{ν_1} (y^{ν_2}, resp.) at $x=2^{-1}$ ($y=2^{-1}$, resp.) we have

$$\begin{aligned} -x^{\nu_1} &\leq -[2^{-\nu_1}(1-\nu_1)+\nu_1 2^{1-\nu_1}x] \\ -y^{\nu_2} &\leq -[2^{-\nu_2}(1-\nu_2)+\nu_2 2^{1-\nu_2}y] \end{aligned} \tag{83}$$

and then the lemma follows.

□

Proof Lemma 2.5. Applying Lemma 2.2 we have

$$-x_1^{\nu_1}x_2^{\nu_2}x_3^{\nu_3} \leq \frac{x_1^{2\nu_1}+x_2^{2\nu_2}x_3^{2\nu_3}}{2}-\frac{x_1^{\nu_1}+x_2^{\nu_2}x_3^{\nu_3}}{2}+0.125$$

Now, by Lemmas 2.3 and (83) we have

$$\begin{aligned} & x_2^{2\nu_2}x_3^{2\nu_3} \leq \left(\frac{x_2^{2\nu_2}+x_3^{2\nu_3}}{2}\right)^2 \\ & -x_1^{\nu_1} \leq -[2^{-\nu_1}(1-\nu_1)+\nu_1 2^{1-\nu_1}x_1] \\ & -x_2^{\nu_2}x_3^{\nu_3} \leq \frac{1}{2}(x_2^{2\nu_2}+x_3^{2\nu_3})-\frac{1}{2}[2^{-\nu_2}(1-\nu_2)+2^{-\nu_3}(1-\nu_3)+\nu_2 2^{1-\nu_2}x_2+\nu_3 2^{1-\nu_3}x_3]+0.125 \end{aligned}$$

and (5) follows.

□

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