Finite-Dimensional Characterizations of H^∞ Control for Linear Systems with Delays in Input and Output

K. Uchida†, K. Ikeda†, T. Azuma# and A. Kojima‡

†Department of Electrical Electronics and Computer Engineering
Waseda University, Tokyo 169-8555, Japan
#Department of Electrical and Electronic Engineering
Kanazawa University, Ishikawa 920-8667, Japan
‡Department of Electronic Systems Engineering
Tokyo Metropolitan Institute of Technology, Tokyo 191-0065, Japan

Correspondence address:
Kenko Uchida
Dept. of Electrical Electronics and Computer Engineering
Waseda University
Okubo 3-4-1, Shinjuku
Tokyo 169-8555, Japan
fax: +81 3 5273 9507
e-mail: kuchida@uchi.elec.waseda.ac.jp

SUMMARY

We formulate H^∞ control problems for linear systems with delays in input and output, and discuss possibility of finite-dimensional characterizations of solutions. In the case when delay exists in control input and controlled output, first, we derive an output feedback H^∞ control formula of the central solution type, which is given by using solutions of finite- and infinite-dimensional Riccati matrix inequalities. Second, we show that, if the controlled output is chosen such that it satisfies the "prediction condition", the solution to the infinite-dimensional Riccati inequality can be calculated by solving a finite-dimensional Riccati inequality. We provide a system theoretic interpretation for the prediction condition, and show that, if the prediction condition is satisfied, there is an equivalent H^∞ control problem for finite-dimensional linear systems with no delay. Finally, the equivalence result is extended to the case when delay exists also in measurement output.

KEY WORDS: H^∞ control; linear time-delay system; input delay; output delay; prediction condition; finite-dimensional characterization

1. Introduction

This paper is concerned itself with finite-dimensional characterizations of general state space solutions to H^∞ control problems for linear systems with delays in input and output. The fact that the state space of systems with delay is infinite-dimensional leads generally to infinite-dimensional characterizations for analysis and synthesis in systems with delay. Actually, the standard H^∞ control problem for linear systems with delay was solved, as a special case of H^∞ control problems for distributed parameter (infinite-dimensional) systems, in the state space form based on two Riccati operator (infinite-dimensional) equations (see, e.g. [1]). Later, more explicit and feasible solutions, which are based on two algebraic Riccati equations and a differential or transcendential (still, infinite-dimensional in kind) equation, were presented by focusing the cases with delays in control input or measurement output [2-4]. The infinite-dimensional equations are hard obstacles in implementing the solutions. To bypass or overcome this difficulty, a lot of approaches have been proposed: The H^∞ control synthesis via delay-independent linear matrix inequality (LMI) would be one of such approaches [5-7]; further, the delay-dependent LMI approach has been developed for the state feedback case [7-8] and the output feedback case [9]. In [10-11], the infinite-dimensional LMI characterization of the solutions together with a finite-dimensional LMI algorithm has been proposed. The discretization technique of infinite-dimensional Lyapunov functional also should be mentioned [12].

In this paper, we formulate an output feedback H^∞ control problem for linear systems with delays in control input and controlled output, and discuss possibility of finite-dimensional characterizations of the solution along the same line as developed in [13-14] where generalized controlled outputs are introduced so that the spectrum decomposition and/or prediction of the state variable can be exploited for finite-dimensional solutions to LQ control problems in linear systems with delays in state and control input. First, we derive an output feedback H^∞ control formula of the central solution type constructed by using solutions to finite- and infinite-dimensional Riccati matrix inequalities. Second, we show that, if the controlled output is chosen such that it satisfies the "prediction condition", the solution to the infinite-dimensional Riccati inequality can be calculated by solving a finite-dimensional Riccati inequality and the H^∞ control formula is deduced to the finite-dimensional one. We provide a system theoretic interpretation for the prediction condition, and discuss a relation between the prediction condition and the robust stabilization result of [15]. It is also shown that, if the prediction condition is satisfied, there is an equivalent H^∞ control problem for finite-dimensional linear systems with no delay. Finally, the equivalence result is extended to the case when delay exists not only in control input and controlled output but also in measurement output.

2. System Description and Problem Statement

Consider a linear system with delays in control input and controlled output. The system is defined over the interval $[0,\infty)$ and described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_{20} u(t) + B_{21} u(t-h) \\ z(t) &= C_{10} x(t) + \int_{-h}^{0} C_{11}(\beta) B_{21} u(t+\beta) d\beta + D_1 u(t) \\ y(t) &= C_2 x(t) + D_2 w(t) \end{aligned} \tag{1}$$

with the initial condition such that $x(0)=0$ and $u(\beta)=0, -h \le \beta \le 0$. Here, $x(t)$ is the internal variable vector of the system; $w(t)$ is the disturbance vector; $u(t)$ is the control input vector; $z(t)$ is the controlled output vector; $y(t)$ is the measurement output vector. The number h denotes the length of time delay and $h>0$. The parameters $A, B_1, B_{20}, B_{21}, C_{10}, D_1, C_2, D_2$ are constant matrices with appropriate dimensions and the parameter $C_{11}(\beta)$ is a matrix function, whose elements are bounded continuous functions, with appropriate dimension. It is noted that the future trajectory of the internal variable $x(\tau), \tau \ge t$ is uniquely determined by the value $x(t)$ and the function $B_{21}u_t$, where $u_t := (u(t+\beta), -h \le \beta \le 0)$, if the future control input $u(\xi), t \le \xi \le \tau$ and the future disturbance $w(\xi), t \le \xi \le \tau$ are given. From this viewpoint, the pair $(x(t), B_{21}u_t)$ is the state of the system (1), and the state space of the system (1) is infinite-dimensional. Note also that the controlled output $z(t)$ is a general form of linear function of the state $(x(t), B_{21}u_t)$ and the control input $u(t)$.

At each time t, the output $y(t)$ is measured and, in addition, the past history of the input of $u(t)$, that is u_t, is available at each time t, because $u(t)$ is the control input. Thus, we consider the pair $(y(t), u_t)$ as the measurement output, and define each admissible control input $u(t)$ as a bounded linear causal function of the measurement output $(y(t), u_t)$. The H^∞ control problem discussed in this paper is to find an admissible control $u(t)$ which controls the system (1) such that I) the closed loop system is asymptotically stable, and II) the closed loop system satisfies the inequality:

$$\int_0^\infty z^T(t) z(t) dt \le \int_0^\infty w^T(t) w(t) dt, \qquad \forall w \in L_2[0,\infty) \tag{2}$$

where "^{T}" indicates transposition and $L_2[0,\infty)$ denotes the space of vector functions whose elements are square integrable over $[0,\infty)$. Here we note that the inequality (2) is equivalent to

$$\|T_{zw}\|_\infty := \sup_{w \ne 0, w \in L_2} \frac{\|z\|_2}{\|w\|_2} \le 1$$

where T_{zw} denotes the transfer function of the system (1) from the disturbance w to the controlled output z and "$\|\bullet\|_2$" indicates the L_2-norm. An admissible control which satisfies I) and II) is called an H^∞ control.

The H^∞ control problem formulated above is a special case of the standard H^∞ control problem for distributed parameter systems, and the set of solutions can be characterized by two Riccati operator equations (see, e.g. [1]). The operator equations are infinite-dimensional in general and so is this characterization. The infinite-dimensional characterization would give us hard problems in computation and implementation. Our concern is when finite-dimensional characterizations of the solutions are possible.

3. Riccati-Inequality-Based Solutions and Prediction Condition

We start to present an infinite-dimensional solution which is suggested by a general characterization of solutions based on two Riccati operator equations. To simplify presentation, we consider here a special case of the problem formulated in Section 2. Let the system (1) be given in the following form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Gw_1(t) + B_{20}u(t) + B_{21}u(t-h) \\ z(t) &= \begin{bmatrix} F_0 x(t) + \int_{-h}^{0} F_1(\beta) B_{21} u(t+\beta)\, d\beta \\ u(t) \end{bmatrix} \\ y(t) &= C_2 x(t) + w_2(t) \end{aligned} \tag{3}$$

where $w_1(t)$ and $w_2(t)$ are sub-vectors of the disturbance vector $w(t)$ such that $w(t) = (w_1^T(t), w_2^T(t))^T$. This special form assures so-called orthogonality not only between the system disturbance $(w_1^T(t), 0^T)^T$ and the measurement disturbance $(0^T, w_2^T(t))^T$ but also between the term of the state $(x(t), B_{21}u_t)$ and the term of the control input $u(t)$ in the controlled output equation.

Now we prepare a notation for defining a quadratic form of the state. Denote by $\{M_0, M_{01}, M_1; R_{02}, R_{12}\}$ a quintet of five matrices M_0, $M_{01}(\beta)$, $M_1(\alpha,\beta)$, R_{02} and $R_{12}(\beta)$ with the same dimensions such that M_0 and R_{02} are constant matrices, $M_{01}(\beta)$ and $R_{12}(\beta)$ are matrix functions whose elements are in $L_2[-h,0]$ and

$M_1(\alpha,\beta)$ is a matrix function whose elements are in $L_2([-h,0]\times[-h,0])$. A quintet $\{M_0,M_{01},M_1;R_{02},R_{12}\}$ is called symmetric if $M_0^T=M_0$ and $M_1^T(\alpha,\beta)=M_1(\beta,\alpha)$. For a given symmetric quintet $\{M_0,M_{01},M_1;R_{02},R_{12}\}$, a quadratic form associated with this quintet is defined as follows:

$$
\begin{aligned}
&(\phi_0,\phi_1;\psi)^T\{M_0,M_{01},M_1;R_{02},R_{12}\}(\phi_0,\phi_1;\psi)\\
&\quad:=\phi_0^T M_0\phi_0+2\phi_0^T\int_{-h}^{0}M_{01}(\beta)\phi_1(\beta)d\beta+\int_{-h}^{0}\int_{-h}^{0}\phi_1^T(\alpha)M_1(\alpha,\beta)\phi_1(\beta)d\alpha\, d\beta\\
&\qquad+2\phi_0^T R_{02}\psi+2\int_{-h}^{0}\phi_1^T(\alpha)R_{12}(\alpha)d\alpha\psi
\end{aligned}
$$

for constant vectors ϕ_0, ψ and a vector function ϕ_1 in $L_2[-h,0]$. A symmetric quintet $\{M_0,M_{01},M_1;R_{02},R_{12}\}$ is called positive (negative) definite, denoted as $\{M_0,M_{01},M_1;R_{02},R_{12}\}>0\,(<0)$, if there exists a positive number ε such that

$$
(\phi_0,\phi_1;\psi)^T\{M_0,M_{01},M_1;R_{02},R_{12}\}(\phi_0,\phi_1;\psi)\ge\varepsilon\phi_0^T\phi_0\ (\le-\varepsilon\phi_0^T\phi_0)
$$

for all (ϕ_0,ϕ_1,ψ). For simplicity, a special quintet with $R_{02}=0$ and $R_{12}=0$ is written as a triplet $\{M_0,M_{01},M_1\}:=\{M_0,M_{01},M_1;0,0\}$, and its quadratic form is defined by $(\phi_0,\phi_1)^T\{M_0,M_{01},M_1\}(\phi_0,\phi_1):=(\phi_0,\phi_1;\psi)^T\{M_0,M_{01},M_1;0,0\}(\phi_0,\phi_1;\psi)$.

3.1 Infinite-dimensional solution. The central solution, which is given by two Riccati operator equations, is a general H^∞ control of infinite-dimension and plays a fundamental role in parameterizing all the solutions [1]. We present a new infinite-dimensional solution that is an extension of the central solution in such a way that the Riccati operator equations, which are equivalent to Riccati type partial differential equations for integral kernels, is relaxed into some Riccati inequalities.

For a triplet $\{S_0,S_{01},S_1\}$ and a constant matrix P, introduce a triplet $\{\Omega_0,\Omega_{01},\Omega_1\}$, a constant matrix Π_{02} and a matrix function Π_{12} such that

$$
\begin{aligned}
\Omega_0(S_0,S_{01},P):=&(A+PF_0^TF_0)^TS_0+S_0(A+PF_0^TF_0)\\
&+F_0^TF_0+S_0PC_2^TC_2PS_0\\
&-\{S_0B_{20}+S_{01}(0)B_{21}\}\{B_{20}^TS_0+B_{21}^TS_{01}^T(0)\}
\end{aligned}
$$

$$
\begin{aligned}
\Omega_{01}(\beta;S_0,S_{01},S_1,P):=&-\frac{\partial}{\partial\beta}S_{01}(\beta)+S_0PF_0^TF_1(\beta)+(A+PF_0^TF_0)^TS_{01}(\beta)\\
&+F_0^TF_1(\beta)+S_0PC_2^TC_2PS_{01}(\beta)\\
&-\{S_0B_{20}+S_{01}(0)B_{21}\}\{B_{20}^TS_{01}(\beta)+B_{21}^TS_1(0,\beta)\}
\end{aligned}
$$

$$\Omega_1(\alpha,\beta;S_0,S_{01},S_1,P) := -(\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\beta})S_1(\alpha,\beta)$$
$$+F_1^T(\alpha)F_0PS_{01}(\beta)+S_{01}^T(\alpha)PF_0^TF_1(\beta)$$
$$+F_1^T(\alpha)F_1(\beta)+S_{01}^T(\alpha)PC_2^TC_2PS_{01}(\beta)$$
$$-\{S_{01}^T(\alpha)B_{20}+S_1(\alpha,0)B_{21}\}\{B_{20}^TS_{01}(\beta)+B_{21}^TS_1(0,\beta)\}$$

$$\Pi_{02}(S_0,S_{01}) := S_0 - S_{01}(-h), \qquad \Pi_{12}(\alpha;S_{01},S_1) = S_{01}^T(\alpha) - S_1(\alpha,-h)$$

Theorem 1. If there exist a positive definite matrix P and a positive definite triplet $\{S_0,S_{01},S_1\}$ which satisfy the finite-dimensional Riccati inequality:

$$AP+PA^T+GG^T-P(C_2^TC_2-F_0^TF_0)P<0 \tag{4}$$

and the infinite-dimensional Riccati inequality:

$$\{\Omega_0,\Omega_{01}B_{21},B_{21}^T\Omega_1B_{21};\Pi_{02}B_{12},B_{12}^T\Pi_{12}B_{12}^T\}<0 \tag{5}$$

then, the output feedback control given by

$$u^*(t) = -\{B_{20}^TS_0+B_{21}^TS_1^T(0)\}\hat{x}(t)$$
$$-\int_{-h}^0\{B_{20}^TS_1(\beta)+B_{21}^TS_2(0,\beta)\}B_{21}u(t+\beta)d\beta \tag{6}$$

$$\dot{\hat{x}}(t) = A\hat{x}(t)+B_{20}u^*(t)+B_{21}u(t-h)+PC_2^T(y(t)-C_2\hat{x}(t))$$
$$+PF_0^T(F_0\hat{x}(t)+\int_{-h}^0F_1(\beta)B_{21}u(t+\beta)d\beta), \qquad \hat{x}(0)=0 \tag{7}$$

is an H^∞ control.

Proof. For an admissible control input u, consider the quadratic functional

$$V(t) := (x(t)-\hat{x}(t))^TP^{-1}(x(t)-\hat{x}(t)) + (\hat{x}(t),B_{21}u_t)^T\{S_0,S_{01},S_1\}(\hat{x}(t),B_{21}u_t) \tag{8}$$

along the trajectory $(x(t),\hat{x}(t)),\ t\geq 0$ defined by the systems (3) and (7) where u^* is replaced with the admissible control u. Differentiating both sides of the quadratic functional $V(t)$ with respect to t, substituting the systems (3) and (7) where u^* is replaced with u into the right hand side, and rearranging terms, we have

$$\begin{aligned}\dot V(t) &= (x(t)-\hat x(t))^T P^{-1}(AP+PA^T+GG^T-P(C_2^TC_2-F_0^TF_0)P)P^{-1}(x(t)-\hat x(t))\\ &+(\hat x(t),u_t;u(t-h))^T\{\Omega_0,\Omega_{01}B_{21},B_{21}^T\Omega_1B_{21};\Pi_{02}B_{21},B_{21}^T\Pi_{12}B_{21}\}(\hat x(t),u_t;u(t-h)) \qquad (9)\\ &-\|z(t)\|^2+\|w(t)\|^2+\|u(t)-u^*(t)\|^2-\|w(t)-w_*(t)\|^2\end{aligned}$$

where "$\|\bullet\|$" indicates the Euclidean norm and $w_*(t)=(w_{*1}^T(t),w_{*2}^T(t))^T$ is given by

$$\begin{aligned}w_{*1}(t) &= G^TP^{-1}(x(t)-\hat x(t))\\ w_{*2}(t) &= -C_2(x(t)-\hat x(t))+C_2P(S_0\hat x(t)+\int_{-h}^{0}S_{01}(\beta)B_{21}u(t+\beta)d\beta)\end{aligned}$$

(Stability) First note that, from the positive definiteness of P and $\{S_0,S_{01},S_1\}$ in (9), there is a positive number ε_1 such that $V(t)\ge\varepsilon_1(\|x(t)-\hat x(t)\|^2+\|\hat x(t)\|^2)$. For the choice of $u=u^*$ and $w=0$, from the negative definiteness of the inequalities (4) and (5) and the identity (9), it follows that there is a positive number ε_2 such that $\dot V(t)\le-\varepsilon_2(\|x(t)-\hat x(t)\|^2+\|\hat x(t)\|^2)$. Thus, we can see that $(x(t),\hat x(t))\to 0$ as $t\to\infty$, and the closed loop system (3) given by (6) and (7) is asymptotically stable.

(L_2-gain inequality) For the choice of $u=u^*$ and for any w in $L_2[0,\infty)$, using the inequalities (4) and (5) in right hand side of (9), we have $\dot V(t)\le-\|z(t)\|^2+\|w(t)\|^2$. Integrating both sides of this inequality with respect to t over $[0,\infty)$ and using $V(0)=0$ which is from zero-initial condition and $V(\infty)=0$ which is from stability in the left hand side, we finally obtain $0\le-\|z\|_2^2+\|w\|_2^2$, which is equivalent to (2).

Q.E.D.

Remark. If a constant matrix P and a triplet $\{S_0,S_{01},S_2\}$ satisfy the infinite-dimensional matrix inequality given by

$$\frac{1}{2}\begin{bmatrix}\Omega_0(S_0,S_{01},P) & \Omega_{01}(\beta;S_0,S_{01},S_1,P)B_{21} & \Pi_{02}(S_0,S_1)B_{21}\\ B_{21}^T\Omega_{01}^T(\alpha;S_0,S_{01},S_1,P) & B_{21}^T\Omega_1(\alpha,\beta;S_0,S_{01},S_1,P)B_{21} & B_{21}^T\Pi_{12}(\alpha;S_1,S_{01})B_{21}\\ B_{21}^T\Pi_{02}^T(S_0,S_{02}) & B_{21}^T\Pi_{12}^T(\beta;S_0,S_{01})B_{21} & 0\end{bmatrix}$$
$$+\frac{1}{2}\begin{bmatrix}\Omega_0(S_0,S_{01},P) & \Omega_{01}(\beta;S_0,S_{01},S_1,P)B_{21} & \Pi_{02}(S_0,S_1)B_{21}\\ B_{21}^T\Omega_{01}^T(\alpha;S_0,S_{01},S_1,P) & B_{21}^T\Omega_1(\alpha,\beta;S_0,S_{01},S_1,P)B_{21} & B_{21}^T\Pi_{12}(\alpha;S_1,S_{01})B_{21}\\ B_{21}^T\Pi_{02}^T(S_0,S_{02}) & B_{21}^T\Pi_{12}^T(\beta;S_0,S_{01})B_{21} & 0\end{bmatrix}^T<0$$
$$\forall\alpha,\beta\in[-h,0]$$

where the elements of the matrix functions S_{01} and S_1 is continuously differentiable, the matrix P and the triplet $\{S_0,S_{01},S_1\}$ satisfy the inequality (5). Such a sufficient condition based on infinite-dimensional matrix inequalities as above are developed for analysis and synthesis problems in linear time-delay systems by Azuma et al. [10] and Azuma et al. [11]. Moreover, they propose a technique reducing infinite-dimensional matrix inequalities to finite-dimensional ones, when the inequalities are rewritten in the linear form.

3.2. Prediction condition and finite-dimensional solution. The most hard problem in implementing the H^{∞} control (6) is to find the solution of the infinite-dimensional Riccati inequality (5). Here, we consider a particular case when this infinite-dimensional inequality is reduced to a finite-dimensional one. To the controlled output equation in the system (3), we introduce a structural assumption described as

(C1) $F_1(\beta)B_{21} = F_0 e^{-A(\beta+h)} B_{21}, \quad -h \le \beta \le 0$

Theorem 2. Suppose that the condition (C1) is satisfied. If there exists positive definite matrices P and S which satisfy the finite-dimensional Riccati inequalities (4) and

$$(A+PF_0^T F_0)^T S + S(A+PF_0^T F_0) + F_0^T F_0 - S(B_h B_h^T - PC_2^T C_2 P)S < 0 \tag{10}$$

where $B_h := B_{20} + e^{-Ah} B_{21}$, then, the positive definite matrix P and the positive definite triplet $\{S_0, S_{01}, S_1\}$ defined by

$$\begin{aligned} &S_0 := S \\ &S_{01}(\beta) := Se^{-A(\beta+h)} \\ &S_1(\alpha,\beta) := e^{-A^T(\alpha+h)} S e^{-A(\beta+h)} \end{aligned} \tag{11}$$

satisfy the infinite-dimensional Riccati inequality (5). Moreover, in this case, the H^{∞} control (6) is rewritten as

$$u^*(t) = -B_h^T S \hat{p}(t) \tag{12}$$

$$\begin{aligned} \dot{\hat{p}}(t) = A\hat{p}(t) + B_h u^*(t) + PC_2^T (y(t) + C_2 \int_{-h}^{0} e^{-A(\beta+h)} B_{21} u(t+\beta) d\beta - C_2 \hat{p}(t)) \\ + PF_0^T F_0 \hat{p}(t), \qquad \hat{p}(0) = 0 \end{aligned} \tag{13}$$

Proof. The former half of this theorem is shown by direct substitution of the formula (11) with (10) into the inequality (5) on the assumption of (C1). The form of (12) and (13) is derived from the original form (6) and (7) by substituting the formula (11) into (6) and (7) on the assumption of (C1) and setting

$$\hat{p}(t) := \hat{x}(t) + \int_{-h}^{0} e^{-A(\beta+h)} B_{12} u(t+\beta) d\beta \tag{14}$$

Q.E.D.

Here we present a system theoretic interpretation of the condition. When the condition (C1) is satisfied, the controlled output in the system (3) is reduced to the form:

$$z(t) = \begin{pmatrix} F_0 p(t) \\ u(t) \end{pmatrix} \tag{15}$$

$$p(t) := x(t) + \int_{-h}^{0} e^{-A(\beta+h)} B_{21} u(t+\beta) d\beta \tag{16}$$

From the formula (16), we can see that $p(t)$ (more precisely, $e^{Ah} p(t)$) corresponds to the predictive value $x(t+h)$ of the internal variable $x(t)$. Thus, if the condition (C1) is satisfied, the problem becomes to control the predictive value of the internal variable, and we call (C1) the "prediction condition". It is also noted that $p(t)$ defined by (11) satisfies a finite-dimensional linear system given as

$$\dot{p}(t) = Ap(t) + Gw_1(t) + B_h u(t) \tag{17}$$

Then, comparing our solution given by (12) and (13) with the central solution for lumped parameter systems, we see that the corresponding measurement output should be given in the following form

$$q(t) := y(t) + C_2 \int_{-h}^{0} e^{-A(\beta+h)} B_{21} u(t+\beta) d\beta = C_2 p(t) + w_2(t) \tag{18}$$

As discussed in the next section, the finite-dimensional characterizations presented in Theorem 2 come from the resultant descriptions of (15), (17) and (18).

4. Prediction Condition and Finite-Dimensional Problem

We come back to the general H^∞ control problem formulated in Section 2. In the general case, the condition corresponding to the condition (C1) in the special case is

(C2) $\quad C_{11}(\beta) B_{21} = C_{10} e^{-A(\beta+h)} B_{21}, \quad -h \le \beta \le 0$

The implication of the condition (C2) is the same to that of the condition (C1). So we call (C2) also the prediction condition. On the condition (C2), we show a stronger result in finite-dimensional characterizations of solutions without using Riccati equations or inequalities.

Theorem 3. Suppose that the condition (C2) is satisfied. Then, the H^∞ control problem described by the criterion (2) and the system (1) is equivalent to the H^∞ control problem described by the criterion (2) and the following finite-dimensional system:

$$\begin{aligned} \dot{p}(t) &= Ap(t) + B_1 w(t) + B_h u(t) \\ z(t) &= C_{10} p(t) + D_1 u(t) \\ q(t) &= C_2 p(t) + D_2 w(t) \end{aligned} \tag{19}$$

where $p(t)$ is the internal variable with $p(0) = 0$, $q(t)$ is the measurement output and $B_h = B_{20} + e^{-Ah} B_{21}$.

Proof. Let $u(t) = \Gamma(t, q(\cdot), u_\cdot)$ be a solution to the finite-dimensional H^∞ control problem described by (2) and (19). Defining

$$\begin{aligned} x(t) &:= p(t) - \int_{-h}^{0} e^{-A(\beta+h)} B_{21} u(t+\beta) d\beta \\ y(t) &:= q(t) - C_2 \int_{-h}^{0} e^{-A(\beta+h)} B_{21} u(t+\beta) d\beta \end{aligned} \tag{20}$$

and using (C2), we can show from (19) that $x(t)$ and $y(t)$ defined above obey the system (1) and the control $u(t) = \Gamma(t, y(\cdot) + C_2 \int_{-h}^{0} e^{-A(\beta+h)} B_{21} u(\cdot+\beta) d\beta, u_\cdot)$ is an solution to the H^∞ control problem described by (2) and (1). Conversely, let $u(t) = \Delta(t, y(\cdot), u_\cdot)$ be a solution to the H^∞ control problem described by (2) and (1). Defining

$$\begin{aligned} p(t) &:= x(t) + \int_{-h}^{0} e^{-A(\beta+h)} B_{21} u(t+\beta) d\beta \\ q(t) &:= y(t) + C_2 \int_{-h}^{0} e^{-A(\beta+h)} B_{21} u(t+\beta) d\beta \end{aligned} \tag{21}$$

and using (C2), we can show from (1) that $p(t)$ and $q(t)$ defined above obey the system (19) and the control $u(t) = \Delta(t, q(\cdot) - C_2 \int_{-h}^{0} e^{-A(\beta+h)} B_{21} u(\cdot+\beta)\beta, u_\cdot)$ is an solution to the finite-dimensional H^∞ control problem described by (2) and (19). Thus, we obtain the conclusion of Theorem 3. Q.E.D.

Controlled outputs are chosen correspondingly to purposes of control design. We comment briefly on meanings of the prediction condition (C2) (or (C1)) from the viewpoint of control design. If it is required to control the predictive value of the internal variable, the prediction condition will be satisfied automatically. In the previous work [15], the authors proved the same equivalence as in Theorem 3 under the assumption that the condition

(C3) $C_{11}(\beta)B_{21} = 0, \quad -h \le \beta \le 0, \qquad C_{10}A^i B_{21} = 0, \quad i = 0, 1, 2, \cdots$

and showed that the framework of H^∞ control problems for input delayed systems satisfying (C3) includes the robust stabilization problems against additive or multiplicative perturbations (including uncertain delay case). We can verify immediately that the condition (C3) is a sufficient condition for the prediction condition (C2) to hold, and therefore see that such robust stabilization problems can be handled also in the framework of this paper. To clarify more general meanings of the prediction condition from the viewpoint of control design is still an open problem of interest.

5. Delay in Measurement Output and Prediction Condition

We finally discuss a possibility of applying the idea of prediction condition to the case with delay in measurement output. Consider a linear system with delays not only in control input and controlled output but also in measurement output, which is described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_{20}u(t) + B_{21}u(t-h) \\ z(t) &= C_{10}x(t) + \int_{-h}^{0} C_{11}(\beta)B_{21}u(t+\beta)d\beta + D_1 u(t) \\ \bar{y}(t) &= C_{20}x(t) + C_{21}x(t-r) + \int_{-r}^{0} C_{22}(\beta)B_1 w(t+\beta)d\beta + D_2 w(t) \end{aligned} \tag{22}$$

Here $\bar{y}(t)$ is the measurement output and, except for the measurement equations, the system (22) and the system (1) is the same. In addition to the assumption for the system (1), we assume that $x(\beta) = 0$ and $w(\beta) = 0, \ -r \le \beta \le 0$, where $r > 0$ is the measurement delay, C_{20}, C_{21} are constant matrices with appropriate dimensions and $C_{22}(\beta)$ is a matrix function, whose elements are bounded continuous functions, with appropriate dimension. For the measurement output in the system (22), consider the following condition:

(C4) $C_{22}(\beta)B_1 = C_{21}e^{-A(\beta+r)}B_1, \quad -r \le \beta \le 0$

From the role of this condition in predicting the internal variable, as we will see in the proof of the next theorem, we call also this condition (C4) the prediction condition.

Theorem 4. Suppose that the condition (C4) is satisfied. Then, the H^∞ control problem described by the criterion (2) and the system (22) is equivalent to the H^∞ control problem described by the criterion (2) and the system (1) where C_2 in the measurement output equation is replaced with $C_r := C_{20} + C_{21}e^{-Ar}$ such that

$$y(t) = C_r x(t) + D_2 w(t)$$

Proof. This theorem is proved along the same line as in the proof of Theorem 3, if we can show that the measurement data $(\bar{y}(t), u_t)$ in (22) is equivalent to the measurement data $(y(t), u_t)$ in (1). First note the formula:

$$e^{-Ar}x(t) = x(t-r) + \int_{-r}^{0} e^{-A(\beta+r)} B_1 w(t+\beta)\,d\beta + \int_{-r}^{0} e^{-A(\beta+r)} (B_{20}u(t+\beta) + B_{21}u(t+\beta-r))\,d\beta$$

Using this formula and the condition (C4), we can see that

$$\begin{aligned}\bar{y}(t) &= C_r x(t) + D_2 w(t) - C_{21}\int_{-r}^{0} e^{-A(\beta+r)} (B_{20}u(t+\beta) + B_{21}u(t+\beta-r))\,d\beta \\ &= y(t) - C_{21}\int_{-r}^{0} e^{-A(\beta+r)} (B_{20}u(t+\beta) + B_{21}u(t+\beta-r))\,d\beta\end{aligned}$$

Thus, the equivalence follows. Q.E.D.

Differently from the case of controlled output, the form of the measurement output equation in system (22) satisfying the prediction condition (C4) is rather restrictive, because measurement outputs are generally determined a priori, e.g. by physical constraints and not chosen purposely by designers. On the other hand, if the disturbance $B_1 w(t)$ can be measured, so that the past history of the disturbance $\{B_1 w(s), 0 \le s \le t\}$ is available at each time t, we can always obtain the form of measurement output equation in (22) satisfying the prediction condition (C4) from the measurement output equation of the form

$$y(t) = C_{20}x(t) + C_{21}x(t-r) + D_2 w(t)$$

6. Conclusion

We discussed the output feedback H^{∞} control problem for linear systems with delays in control input, controlled output and measurement output, and showed that the finite-dimensional characterization of the solution is possible if the prediction condition on the controlled output and/or the measurement output is satisfied. We also discussed the meaning of the prediction condition from the viewpoint of control design.

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