Analytical Modeling of Nonlinear Propagation in a Strongly Dispersive Optical Communication System

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Abstract

Recently an analytical model was presented that treats the nonlinear signal distortion from the Kerr nonlinearity in optical transmission systems as additive white Gaussian noise. This important model predicts the impact of the Kerr nonlinearity in systems operating at a high symbol rate and where the accumulated dispersion at the receiver is large. Starting from the suggested model for the propagating signal, we here give an independent and different calculation of the main result. The analysis is based on the Manakov equation with attenuation included and a complete and detailed derivation is given using a perturbation analysis. As in the case with the published model, in addition to assuming that the input signal can be written on a specific form, two further assumptions are necessary; the nonlinearity is weak and the signal-noise interaction is neglected. The result is then found without any further approximations.

Index Terms

Optical fiber communication, communication system nonlinearities, nonlinear optics, wavelength division multiplexing.

I. Introduction

In an optical transmission system operating at a high symbol rate, the propagating signal is rapidly evolving due to chromatic dispersion (CD). It has been found numerically that after a relatively short distance of propagation, the probability density function of all four quadratures of a polarization-multiplexed signal become Gaussian with zero mean and a variance related to the signal power [1]. This can also be shown analytically by using the central limit theorem and is intuitively understandable since the dispersed signal at every point in time can be viewed as a coherent superposition of many signal pulses. During the propagation, a nonlinear phase shift is induced in proportion to the local power by the Kerr nonlinearity. As the CD is compensated in the receiver, this phase shift will give rise to a residual signal distortion. Numerically, it has been observed that this distortion is very similar to additive white Gaussian noise (AWGN) [1]. This observation is of great importance as it suggests that if no attempt is made to compensate for the nonlinear effects, then modeling the nonlinear signal distortion as AWGN is possible.

In 2011, an analytical model was presented that calculates the noise-like nonlinear distortion for a quite general wavelength division multiplexing (WDM) system [2], [3]. This signal distortion was named *nonlinear interference* (NLI) and recently an extensive paper on the same topic was published [4]. The work reported here is closely related to these publications but it should be pointed out that there are many results in the literature that address the same or similar questions. To name one example, the work for OFDM by Chen and Shieh [5] has many similarities in both the approach and the results. However, the previous work in the area seems to be well described in the introduction of [4] and we will not further elaborate this topic here.

In the publications [2]–[4], a suggestion is given for how to model the signal, the four-wave mixing (FWM) of the different signal spectral components is calculated, and the corresponding power spectral density (PSD) is found. However, the derivation of the main result is quite short. Partly this is because known FWM results are used [6]. In this paper, an independent detailed derivation of the resulting PSD is carried out. The calculation starts from the Manakov equation with power gain and attenuation included and we use the signal model suggested in [2]. The calculation is based on a perturbation approach previously used, e.g., to investigate intrachannel cross-phase modulation (XPM) and intrachannel FWM, which gives rise to "ghost pulses" in systems using on-off keying [7]–[9]. This approach has later been used both to analytically study systems, see for example [10], [11], and to compensate for the NLI, see for example [12]. Using the perturbation approach, we here present a self-contained calculation of the PSD of the NLI directly from the model equation. This work also goes beyond [4] since, as described in Section II-B, we perform the calculation for a more general system.

The organization of this paper is as follows: In Section II, the perturbation analysis is introduced and a formal solution that is valid for all input signals is given. In Section III, we find the solution corresponding to the specific input signal suggested in [2], which is then used in Section IV to calculate the NLI, i.e., the PSD of the perturbation. We then calculate the PSD corresponding to the result in [4, Eq. (18)] by studying a specific system choice in Section V. Finally, we conclude.

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II. Perturbation analysis

The calculation of the NLI in [2] assumes that the FWM is a weak effect. This assumption is stated explicitly as "the pump is undepleted". We proceed in a similar way by introducing the complex envelope of the electric field in the x and y polarizations according to $\mathbf{A} = (A_x, A_y)^{\mathrm{T}}$ and writing this as $\mathbf{A} = \mathbf{A}_l + \mathbf{A}_p$, where $\mathbf{A}_l$ is the linearly propagating signal, i.e., the signal in the absence of any Kerr nonlinearity, and $\mathbf{A}_p$ is a small perturbation. This implies that the nonlinear effects may not become significant and this constitutes the first assumption of the model. The second assumption is that the input signal can be written on the form suggested in [2]. The range of validity for this signal model is a separate question and we do not discuss this question here. A third assumption is that the signal-noise interaction is neglected as the calculation of the NLI does not involve amplifier noise in any way. This is also an assumption of [2] and in order to account for amplifier noise under this assumption, noise corresponding to the total AWGN from the amplification is added just before the receiver. These three assumptions are sufficient and no further approximations are necessary to carry out the analysis.

A. The perturbation equation from the Manakov equation

In order to describe transmission using polarization multiplexing, we start from the Manakov equation [13] and include power gain and attenuation. It should be noted that the Manakov equation is obtained by averaging over the polarization rotations which are assumed to be fast and this equation does not take polarization mode dispersion into account. Denoting the group-velocity dispersion by $\beta_2(z)$, the power gain by $g(z)$, and the power attenuation by $\alpha(z)$ we have

$$i\frac{\partial \mathbf{A}}{\partial z} = \frac{\beta_2(z)}{2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \gamma(z)(\mathbf{A}^{\mathrm{H}}\mathbf{A})\mathbf{A} + i\frac{g(z)-\alpha(z)}{2}\mathbf{A}, \tag{1}$$

where $\mathbf{A}^{\mathrm{H}}\mathbf{A} = |A_x|^2 + |A_y|^2$ is the sum of the power in the x and y polarizations and the nonlinear parameter $\gamma(z) = (8/9)(k_0 n_2/A_{\text{eff}})$. In the expression for γ, k_0 is the wavenumber corresponding to the center frequency, n_2 is the Kerr coefficient, and A_{eff} is the effective area of the optical fiber. The power gain $g(z)$, which is set up using erbium-doped fiber amplifiers (EDFAs) and/or Raman amplification, is assumed to have no frequency dependence, i.e., the gain is flat over the bandwidth of the signal. The perturbation is introduced according to

$$\mathbf{A} = \mathbf{A}_l + \mathbf{A}_p = \begin{pmatrix} A_{x,l} \\ A_{y,l} \end{pmatrix} + \begin{pmatrix} A_{x,p} \\ A_{y,p} \end{pmatrix}, \tag{2}$$

where $\mathbf{A}_l(z,t)$ solves the linear equation obtained by setting $\gamma = 0$ in (1). We find that

$$(\mathbf{A}^{\mathrm{H}}\mathbf{A})\mathbf{A} = (|A_{x,l} + A_{x,p}|^2 + |A_{y,l} + A_{y,p}|^2)\begin{pmatrix} A_{x,l} + A_{x,p} \\ A_{y,l} + A_{y,p} \end{pmatrix}. \tag{3}$$

Our intention is to study A_x and we will consider the two cases that either (i) $|A_{x,l}|$ and $|A_{y,l}|$ are of the same order of magnitude (transmission using polarization multiplexing) or (ii) $A_{y,l} = 0$ (single-polarization transmission). The first assumption above can then be strictly formulated as $|A_{x,p}| \ll |A_{x,l}|$ and $|A_{y,p}| \ll |A_{x,l}|$. This allows us to approximate the nonlinear term to leading order according to

$$(\mathbf{A}^{\mathrm{H}}\mathbf{A})\mathbf{A} \approx (|A_{x,l}|^2 + |A_{y,l}|^2)\begin{pmatrix} A_{x,l} \\ A_{y,l} \end{pmatrix}. \tag{4}$$

Inserting this into the Manakov equation, we obtain

$$i\frac{\partial}{\partial z}\begin{pmatrix} A_{x,l} + A_{x,p} \\ A_{y,l} + A_{y,p} \end{pmatrix} = \frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}\begin{pmatrix} A_{x,l} + A_{x,p} \\ A_{y,l} + A_{y,p} \end{pmatrix} - \gamma(|A_{x,l}|^2 + |A_{y,l}|^2)\begin{pmatrix} A_{x,l} \\ A_{y,l} \end{pmatrix} + i\frac{g-\alpha}{2}\begin{pmatrix} A_{x,l} + A_{x,p} \\ A_{y,l} + A_{y,p} \end{pmatrix}, \tag{5}$$

where the z-dependence is now implicit for compactness. The fact that $\mathbf{A}_l$ solves (1) when $\gamma = 0$, implies that

$$i\frac{\partial}{\partial z}\begin{pmatrix} A_{x,p} \\ A_{y,p} \end{pmatrix} = \frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}\begin{pmatrix} A_{x,p} \\ A_{y,p} \end{pmatrix} - \gamma(|A_{x,l}|^2 + |A_{y,l}|^2)\begin{pmatrix} A_{x,l} \\ A_{y,l} \end{pmatrix} + i\frac{g-\alpha}{2}\begin{pmatrix} A_{x,p} \\ A_{y,p} \end{pmatrix}. \tag{6}$$

We will study the x-polarized perturbation, which is described by

$$\frac{\partial A_{x,p}}{\partial z} = -i\frac{\beta_2}{2}\frac{\partial^2 A_{x,p}}{\partial t^2} + i\gamma(|A_{x,l}|^2 + |A_{y,l}|^2)A_{x,l} + \frac{g-\alpha}{2}A_{x,p}. \tag{7}$$

For compactness, we temporarily introduce $S(z,t) = i\gamma(|A_{x,l}|^2 + |A_{y,l}|^2)A_{x,l}$ to denote the source term in the partial differential equation for the perturbation, which is then written

$$\frac{\partial A_{x,p}}{\partial z} + i\frac{\beta_2}{2}\frac{\partial^2 A_{x,p}}{\partial t^2} - \frac{g-\alpha}{2}A_{x,p} = S. \tag{8}$$

B. Formal solution

We now derive a general formal solution without any assumptions about the input signal or the system parameters. To describe the power evolution, we introduce $P(z)$ as a function that satisfies the equation

$$\frac{dP}{dz} = [g(z) - \alpha(z)]P. \tag{9}$$

In the absence of any Raman amplification this function varies as $e^{-\alpha z}$ between the amplifiers. The EDFA gain can be modeled by a δ-function in $g(z)$ to obtain the discontinuities in $P(z)$ at each z corresponding to the location of an amplifier. Introducing $A_{x,p}(z,t) = \sqrt{P(z)}\,\psi(z,t)$, we obtain the equation

$$\frac{g-\alpha}{2}\sqrt{P}\psi + \sqrt{P}\frac{\partial \psi}{\partial z} + i\frac{\beta_2}{2}\sqrt{P}\frac{\partial^2 \psi}{\partial t^2} - \frac{g-\alpha}{2}\sqrt{P}\psi = S, \tag{10}$$

or

$$\frac{\partial \psi}{\partial z} + i\frac{\beta_2}{2}\frac{\partial^2 \psi}{\partial t^2} = \frac{S}{\sqrt{P}}. \tag{11}$$

We proceed by describing the accumulated dispersion, $B(z)$, by a function that satisfies

$$\frac{dB}{dz} = \beta_2(z). \tag{12}$$

If there is lumped dispersion compensation (such as a chirped fiber Bragg grating), then B will have discontinuities. This can be modeled by a δ-function in $\beta_2(z)$ to obtain

$$B(z) = \int_0^z \beta_2(\zeta)\,d\zeta. \tag{13}$$

Fourier transforming[1] the equation, and denoting this operation by tilde, we obtain

$$\frac{\partial \tilde{\psi}}{\partial z} - i(2\pi f)^2\frac{\beta_2}{2}\tilde{\psi} = \frac{\tilde{S}}{\sqrt{P}}, \tag{14}$$

which can be written

$$\frac{\partial}{\partial z}\left(e^{-i(2\pi f)^2 B/2}\tilde{\psi}\right) = e^{-i(2\pi f)^2 B/2}\frac{\tilde{S}}{\sqrt{P}}. \tag{15}$$

Integrating $\int_0^z \cdot\, d\zeta$, we use

$$\left[e^{-i(2\pi f)^2 B/2}\tilde{\psi}\right]_{\zeta=0}^{\zeta=z} = \{\tilde{\psi}(0,f)=0\} = e^{-i(2\pi f)^2 B(z)/2}\tilde{\psi}(z,f) \tag{16}$$

to obtain

$$\tilde{\psi}(z,f) = e^{i(2\pi f)^2 B(z)/2}\int_0^z e^{-i(2\pi f)^2 B(\zeta)/2}\frac{\tilde{S}(\zeta,f)}{\sqrt{P(\zeta)}}\,d\zeta. \tag{17}$$

The final perturbation at the receiver is described by $\tilde{\psi}(L,f)$, where L is the total system length (possibly containing many spans with fibers and amplifiers). Assuming that perfect EDC is performed, the exponential before the integration is canceled and we obtain

$$\tilde{\psi}(L,f) = \int_0^L e^{-i(2\pi f)^2 B(z)/2}\frac{\tilde{S}(z,f)}{\sqrt{P(z)}}\,dz. \tag{18}$$

Assuming that the power at $z = L$ is equal to the power at the transmitter, P_0, we have $A_{x,p}(L,t) = \sqrt{P(L)}\,\psi(L,t) = \sqrt{P_0}\,\psi(L,t)$ and we can write the general expression for the perturbation as

$$\tilde{A}_{x,p}(L,f) = \int_0^L e^{-i(2\pi f)^2 B(z)/2}\frac{\tilde{S}(z,f)}{\sqrt{P(z)}}\sqrt{P_0}\,dz = \int_0^L e^{-i(2\pi f)^2 B(z)/2}\frac{\tilde{S}(z,f)}{\sqrt{p(z)}}\,dz, \tag{19}$$

where $p(z) = P(z)/P_0$ has been introduced. This function describes the normalized power evolution through the system. Using the definition of S, we find

$$\tilde{A}_{x,p}(L,f) = \int_0^L \frac{e^{-i(2\pi f)^2 B(z)/2}}{\sqrt{p(z)}}\mathcal{F}[i\gamma(|A_{x,l}|^2 + |A_{y,l}|^2)A_{x,l}]\,dz, \tag{20}$$

where $\mathcal{F}$ denotes Fourier transformation.

[1] We use the same definition for the Fourier transform as Proakis [14], i.e., $\tilde{u}(f) = \int_{-\infty}^{\infty} u(t)e^{-i2\pi ft}dt$ and $u(t) = \int_{-\infty}^{\infty} \tilde{u}(f)e^{i2\pi ft}df$.

III. Signal model

Provided that the first assumption is fulfilled, (20) is the solution to (1) without further approximations. However, we need to choose the system parameters, select the boundary condition at $z = 0$, i.e., the input signal, and perform the integration. When trying to do this with an accurate modeling of the signal pulses, the calculations become cumbersome and the final expression needs to be averaged over the data. We here instead use the second assumption, i.e., we write the initial field in the way suggested in [2]. This model is

$$\tilde{A}_{x,l}(0,f) = \sqrt{f_0} \sum_{k=-\infty}^{\infty} \xi_k \sqrt{G_x(kf_0)}\, \delta(f - kf_0), \tag{21}$$

$$\tilde{A}_{y,l}(0,f) = \sqrt{f_0} \sum_{k=-\infty}^{\infty} \zeta_k \sqrt{G_y(kf_0)}\, \delta(f - kf_0), \tag{22}$$

where ξ_k and ζ_k are complex independent Gaussian random variables of unit variance and the input signal PSD of the x and y polarizations are denoted by $G_x(f)$ and $G_y(f)$, respectively. In the following, we will suppress the infinite summation limits for notational convenience. We account for dispersion and power variations during the propagation by modifying this to

$$\tilde{A}_{x,l}(z,f) = \sqrt{f_0 p(z)} \sum_k \xi_k \sqrt{G_x(kf_0)}\, \delta(f - kf_0) e^{i(2\pi k f_0)^2 B(z)/2}, \tag{23}$$

$$\tilde{A}_{y,l}(z,f) = \sqrt{f_0 p(z)} \sum_k \zeta_k \sqrt{G_y(kf_0)}\, \delta(f - kf_0) e^{i(2\pi k f_0)^2 B(z)/2}. \tag{24}$$

Using this linear solution, we can now calculate the perturbation using (20). In order to do this we need the expression

$$\mathcal{F}[i\gamma(|A_{x,l}|^2 + |A_{y,l}|^2)A_{x,l}] = i\gamma\mathcal{F}[A_{x,l}^2 A_{x,l}^*] + i\gamma\mathcal{F}[A_{x,l}A_{y,l}A_{y,l}^*]. \tag{25}$$

The solution (20) can be written

$$\tilde{A}_{x,p}(L,f) = i\int_0^L \gamma e^{-i(2\pi f)^2 B/2} \frac{\mathcal{F}[A_{x,l}^2 A_{x,l}^*]}{\sqrt{p}}\, dz + i\int_0^L \gamma e^{-i(2\pi f)^2 B/2} \frac{\mathcal{F}[A_{x,l}A_{y,l}A_{y,l}^*]}{\sqrt{p}}\, dz, \tag{26}$$

where we again omit the z dependence for notational compactness. We have

$$A_{x,l}(z,t) = \sqrt{f_0 p} \sum_k \xi_k \sqrt{G_x(kf_0)}\, e^{i(2\pi k f_0)t} e^{i(2\pi k f_0)^2 B/2}, \tag{27}$$

$$A_{y,l}(z,t) = \sqrt{f_0 p} \sum_k \zeta_k \sqrt{G_y(kf_0)}\, e^{i(2\pi k f_0)t} e^{i(2\pi k f_0)^2 B/2}, \tag{28}$$

and get

$$\begin{aligned} A_{x,l}^2 A_{x,l}^* = &\left(\sqrt{f_0 p} \sum_k \xi_k \sqrt{G_x(kf_0)}\, e^{i(2\pi k f_0)t} e^{i(2\pi k f_0)^2 B/2}\right)\left(\sqrt{f_0 p} \sum_l \xi_l \sqrt{G_x(lf_0)}\, e^{i(2\pi l f_0)t} e^{i(2\pi l f_0)^2 B/2}\right) \\ &\times \left(\sqrt{f_0 p} \sum_m \xi_m^* \sqrt{G_x(mf_0)}\, e^{-i(2\pi m f_0)t} e^{-i(2\pi m f_0)^2 B/2}\right), \end{aligned} \tag{29}$$

$$\begin{aligned} A_{x,l}A_{y,l}A_{y,l}^* = &\left(\sqrt{f_0 p} \sum_k \xi_k \sqrt{G_x(kf_0)}\, e^{i(2\pi k f_0)t} e^{i(2\pi k f_0)^2 B/2}\right)\left(\sqrt{f_0 p} \sum_l \zeta_l \sqrt{G_y(lf_0)}\, e^{i(2\pi l f_0)t} e^{i(2\pi l f_0)^2 B/2}\right) \\ &\times \left(\sqrt{f_0 p} \sum_m \zeta_m^* \sqrt{G_y(mf_0)}\, e^{-i(2\pi m f_0)t} e^{-i(2\pi m f_0)^2 B/2}\right). \end{aligned} \tag{30}$$

These expressions can be simplified to

$$A_{x,l}^2 A_{x,l}^* = (f_0 p)^{3/2} \sum_{k,l,m} \xi_k \xi_l \xi_m^* \sqrt{G_x(kf_0)G_x(lf_0)G_x(mf_0)} e^{i2\pi(k+l-m)f_0 t} e^{i(2\pi)^2(k^2+l^2-m^2)f_0^2 B/2}, \tag{31}$$

$$A_{x,l}A_{y,l}A_{y,l}^* = (f_0 p)^{3/2} \sum_{k,l,m} \xi_k \zeta_l \zeta_m^* \sqrt{G_x(kf_0)G_y(lf_0)G_y(mf_0)} e^{i2\pi(k+l-m)f_0 t} e^{i(2\pi)^2(k^2+l^2-m^2)f_0^2 B/2}. \tag{32}$$

The Fourier transforms are

$$\mathcal{F}[A_{x,l}^2 A_{x,l}^*] = (f_0 p)^{3/2} \sum_{k,l,m} \xi_k \xi_l \xi_m^* \sqrt{G_x(kf_0)G_x(lf_0)G_x(mf_0)}\delta(f-(k+l-m)f_0)e^{i(2\pi)^2(k^2+l^2-m^2)f_0^2 B/2}, \tag{33}$$

$$\mathcal{F}[A_{x,l} A_{y,l} A_{y,l}^*] = (f_0 p)^{3/2} \sum_{k,l,m} \xi_k \zeta_l \zeta_m^* \sqrt{G_x(kf_0)G_y(lf_0)G_y(mf_0)}\delta(f-(k+l-m)f_0)e^{i(2\pi)^2(k^2+l^2-m^2)f_0^2 B/2}. \tag{34}$$

The integrands of (26) can then be written

$$\begin{aligned} \gamma e^{-i(2\pi f)^2 B/2} \frac{\mathcal{F}[A_{x,l}^2 A_{x,l}^*]}{\sqrt{p}} &= \gamma e^{-i(2\pi f)^2 B/2} f_0^{3/2} p \sum_{k,l,m} \xi_k \xi_l \xi_m^* \sqrt{G_x(kf_0)G_x(lf_0)G_x(mf_0)} \\ &\quad \times \delta(f-(k+l-m)f_0)e^{i(2\pi)^2(k^2+l^2-m^2)f_0^2 B/2}, \end{aligned} \tag{35}$$

$$\begin{aligned} \gamma e^{-i(2\pi f)^2 B/2} \frac{\mathcal{F}[A_{x,l} A_{y,l} A_{y,l}^*]}{\sqrt{p}} &= \gamma e^{-i(2\pi f)^2 B/2} f_0^{3/2} p \sum_{k,l,m} \xi_k \zeta_l \zeta_m^* \sqrt{G_x(kf_0)G_y(lf_0)G_y(mf_0)} \\ &\quad \times \delta(f-(k+l-m)f_0)e^{i(2\pi)^2(k^2+l^2-m^2)f_0^2 B/2}. \end{aligned} \tag{36}$$

We can move the exponential functions containing B after the summation signs to obtain

$$\begin{aligned} \gamma e^{-i(2\pi f)^2 B/2} \frac{\mathcal{F}[A_{x,l}^2 A_{x,l}^*]}{\sqrt{p}} &= f_0^{3/2} \gamma p \sum_{k,l,m} \xi_k \xi_l \xi_m^* \sqrt{G_x(kf_0)G_x(lf_0)G_x(mf_0)} \\ &\quad \times \delta(f-(k+l-m)f_0)e^{i(2\pi)^2(k^2+l^2-m^2)f_0^2 B/2} e^{-i(2\pi)^2(k+l-m)^2 f_0^2 B/2}, \end{aligned} \tag{37}$$

$$\begin{aligned} \gamma e^{-i(2\pi f)^2 B/2} \frac{\mathcal{F}[A_{x,l} A_{y,l} A_{y,l}^*]}{\sqrt{p}} &= f_0^{3/2} \gamma p \sum_{k,l,m} \xi_k \zeta_l \zeta_m^* \sqrt{G_x(kf_0)G_y(lf_0)G_y(mf_0)} \\ &\quad \times \delta(f-(k+l-m)f_0)e^{i(2\pi)^2(k^2+l^2-m^2)f_0^2 B/2} e^{-i(2\pi)^2(k+l-m)^2 f_0^2 B/2}. \end{aligned} \tag{38}$$

We use that

$$(k^2+l^2-m^2)-(k+l-m)^2 = -2(k-m)(l-m) \tag{39}$$

to obtain

$$\begin{aligned} \gamma e^{-i(2\pi f)^2 B/2} \frac{\mathcal{F}[A_{x,l}^2 A_{x,l}^*]}{\sqrt{p}} &= f_0^{3/2} \gamma p \sum_{k,l,m} \xi_k \xi_l \xi_m^* \sqrt{G_x(kf_0)G_x(lf_0)G_x(mf_0)} \\ &\quad \times \delta(f-(k+l-m)f_0)e^{-i4\pi^2(k-m)(l-m)f_0^2 B} \end{aligned} \tag{40}$$

$$\begin{aligned} \gamma e^{-i(2\pi f)^2 B/2} \frac{\mathcal{F}[A_{x,l} A_{y,l} A_{y,l}^*]}{\sqrt{p}} &= f_0^{3/2} \gamma p \sum_{k,l,m} \xi_k \zeta_l \zeta_m^* \sqrt{G_x(kf_0)G_y(lf_0)G_y(mf_0)} \\ &\quad \times \delta(f-(k+l-m)f_0)e^{-i4\pi^2(k-m)(l-m)f_0^2 B}. \end{aligned} \tag{41}$$

The solution can therefore be written

$$\begin{aligned} \tilde{A}_{x,p}(L,f) &= i\int_0^L f_0^{3/2} \gamma p \sum_{k,l,m} \xi_k \xi_l \xi_m^* \sqrt{G_x(kf_0)G_x(lf_0)G_x(mf_0)}\, \delta(f-(k+l-m)f_0)e^{-i4\pi^2(k-m)(l-m)f_0^2 B}\, dz \\ &\quad + i\int_0^L f_0^{3/2} \gamma p \sum_{k,l,m} \xi_k \zeta_l \zeta_m^* \sqrt{G_x(kf_0)G_y(lf_0)G_y(mf_0)}\, \delta(f-(k+l-m)f_0)e^{-i4\pi^2(k-m)(l-m)f_0^2 B}\, dz. \end{aligned} \tag{42}$$

Inverse Fourier transformation gives

$$\begin{aligned} A_{x,p}(L,t) &= if_0^{3/2} \int_0^L \gamma p \sum_{k,l,m} \xi_k \xi_l \xi_m^* \sqrt{G_x(kf_0)G_x(lf_0)G_x(mf_0)} e^{i2\pi(k+l-m)f_0 t} e^{-i4\pi^2(k-m)(l-m)f_0^2 B}\, dz \\ &\quad + if_0^{3/2} \int_0^L \gamma p \sum_{k,l,m} \xi_k \zeta_l \zeta_m^* \sqrt{G_x(kf_0)G_y(lf_0)G_y(mf_0)} e^{i2\pi(k+l-m)f_0 t} e^{-i4\pi^2(k-m)(l-m)f_0^2 B}\, dz. \end{aligned} \tag{43}$$

We can also rearrange integration and summation to obtain

$$A_{x,p}(L,t) = if_0^{3/2} \sum_{k,l,m} \xi_k \xi_l \xi_m^* \sqrt{G_x(kf_0)G_x(lf_0)G_x(mf_0)} e^{i2\pi(k+l-m)f_0 t} \int_0^L \gamma p e^{-i4\pi^2(k-m)(l-m)f_0^2 B}\, dz$$
$$+ if_0^{3/2} \sum_{k,l,m} \xi_k \zeta_l \zeta_m^* \sqrt{G_x(kf_0)G_y(lf_0)G_y(mf_0)} e^{i2\pi(k+l-m)f_0 t} \int_0^L \gamma p e^{-i4\pi^2(k-m)(l-m)f_0^2 B}\, dz. \tag{44}$$

Considering $\gamma(z)$, $p(z)$, $B(z)$, and L to be given system parameters, we introduce

$$\mathcal{C}_{klm} \equiv \mathcal{C}(kf_0, lf_0, mf_0) \equiv \int_0^L \gamma p e^{-i4\pi^2(k-m)(l-m)f_0^2 B}\, dz \tag{45}$$

to write this as

$$\begin{aligned} A_{x,p}(L,t) &= if_0^{3/2} \sum_{k,l,m} \mathcal{C}_{klm} \xi_k \xi_l \xi_m^* \sqrt{G_x(kf_0)G_x(lf_0)G_x(mf_0)} e^{i2\pi(k+l-m)f_0 t} \\ &\quad + if_0^{3/2} \sum_{k,l,m} \mathcal{C}_{klm} \xi_k \zeta_l \zeta_m^* \sqrt{G_x(kf_0)G_y(lf_0)G_y(mf_0)} e^{i2\pi(k+l-m)f_0 t} \\ &= if_0^{3/2} \sum_{k,l,m} \mathcal{C}_{klm} e^{i2\pi(k+l-m)f_0 t} \xi_k \sqrt{G_x(kf_0)} \left(\xi_l \xi_m^* \sqrt{G_x(lf_0)G_x(mf_0)} + \zeta_l \zeta_m^* \sqrt{G_y(lf_0)G_y(mf_0)} \right). \end{aligned} \tag{46}$$

For later use we notice that

$$|\mathcal{C}_{klm}| \leq \int_0^L \left| \gamma p e^{-i4\pi^2(k-m)(l-m)f_0^2 B} \right| dz = \int_0^L \gamma p\, dz \leq \hat{\gamma}\hat{p}L, \tag{47}$$

where $\hat{p}$ and $\hat{\gamma}$ are the maximum values of $p(z)$ and $\gamma(z)$ in the interval $z \in [0, L]$. This means that $|\mathcal{C}_{klm}|$ is upper bounded by a constant as k, l, m, and f_0 are changed.

IV. Power spectral density

When the input signal is modeled by (21) and (22), then the perturbation is given by (46). The next step is to calculate the corresponding PSD. Using the Wiener-Khinchin theorem [14, p. 67], we have that the PSD is the Fourier transform of the autocorrelation according to

$$G_{x,p}(f) = \mathcal{F}[R(\tau)] = \int_{-\infty}^{\infty} R(\tau) e^{-i2\pi f\tau}\, d\tau, \tag{48}$$

where

$$R = \mathbb{E}\{A_{x,p}(L,t_1)A_{x,p}^*(L,t_2)\} \tag{49}$$

and $\mathbb{E}\{\cdot\}$ denotes the expectation operator. As will be shown, the latter expression can be written as a function of the time difference $\tau = t_1 - t_2$. We will suppress the infinite limits for notational convenience. We have

$$\begin{aligned} &A_{x,p}(L,t_1)A_{x,p}^*(L,t_2) = \\ &\left[if_0^{3/2} \sum_{k,l,m} \mathcal{C}_{klm} e^{i2\pi(k+l-m)f_0 t_1} \xi_k \sqrt{G_x(kf_0)} \left(\xi_l \xi_m^* \sqrt{G_x(lf_0)G_x(mf_0)} + \zeta_l \zeta_m^* \sqrt{G_y(lf_0)G_y(mf_0)} \right) \right] \times \\ &\left[-if_0^{3/2} \sum_{k',l',m'} \mathcal{C}_{k'l'm'}^* e^{-i2\pi(k'+l'-m')f_0 t_2} \xi_{k'}^* \sqrt{G_x(k'f_0)} \left(\xi_{l'}^* \xi_{m'} \sqrt{G_x(l'f_0)G_x(m'f_0)} + \zeta_{l'}^* \zeta_{m'} \sqrt{G_y(l'f_0)G_y(m'f_0)} \right) \right] = \\ &f_0^3 \sum_{\substack{k,l,m \\ k',l',m'}} \mathcal{C}_{klm} \mathcal{C}_{k'l'm'}^* e^{i2\pi(k+l-m)f_0 t_1} e^{-i2\pi(k'+l'-m')f_0 t_2} \xi_k \xi_{k'}^* \sqrt{G_x(kf_0)} \sqrt{G_x(k'f_0)} \times \\ &\left(\xi_l \xi_m^* \sqrt{G_x(lf_0)G_x(mf_0)} + \zeta_l \zeta_m^* \sqrt{G_y(lf_0)G_y(mf_0)} \right) \left(\xi_{l'}^* \xi_{m'} \sqrt{G_x(l'f_0)G_x(m'f_0)} + \zeta_{l'}^* \zeta_{m'} \sqrt{G_y(l'f_0)G_y(m'f_0)} \right). \end{aligned} \tag{50}$$

We see that the complete autocorrelation function will consist of four terms

$$R = R_1 + R_2 + R_3 + R_4 \tag{51}$$

where

$$R_1 = \mathbb{E}\Bigg\{ f_0^3 \sum_{\substack{k,l,m\\k',l',m'}} \mathcal{C}_{klm}\mathcal{C}^*_{k'l'm'} e^{i2\pi(k+l-m)f_0t_1} e^{-i2\pi(k'+l'-m')f_0t_2}$$
$$\times \xi_k\xi_l\xi_m^*\xi_{k'}^*\xi_{l'}^*\xi_{m'}\sqrt{G_x(kf_0)G_x(lf_0)G_x(mf_0)G_x(k'f_0)G_x(l'f_0)G_x(m'f_0)}\Bigg\}, \quad (52)$$

$$R_2 = \mathbb{E}\Bigg\{ f_0^3 \sum_{\substack{k,l,m\\k',l',m'}} \mathcal{C}_{klm}\mathcal{C}^*_{k'l'm'} e^{i2\pi(k+l-m)f_0t_1} e^{-i2\pi(k'+l'-m')f_0t_2}$$
$$\times \xi_k\xi_l\xi_m^*\xi_{k'}^*\zeta_{l'}^*\zeta_{m'}\sqrt{G_x(kf_0)G_x(lf_0)G_x(mf_0)G_x(k'f_0)G_y(l'f_0)G_y(m'f_0)}\Bigg\}, \quad (53)$$

$$R_3 = \mathbb{E}\Bigg\{ f_0^3 \sum_{\substack{k,l,m\\k',l',m'}} \mathcal{C}_{klm}\mathcal{C}^*_{k'l'm'} e^{i2\pi(k+l-m)f_0t_1} e^{-i2\pi(k'+l'-m')f_0t_2}$$
$$\times \xi_k\zeta_l\zeta_m^*\xi_{k'}^*\xi_{l'}^*\xi_{m'}\sqrt{G_x(kf_0)G_y(lf_0)G_y(mf_0)G_x(k'f_0)G_x(l'f_0)G_x(m'f_0)}\Bigg\}, \quad (54)$$

$$R_4 = \mathbb{E}\Bigg\{ f_0^3 \sum_{\substack{k,l,m\\k',l',m'}} \mathcal{C}_{klm}\mathcal{C}^*_{k'l'm'} e^{i2\pi(k+l-m)f_0t_1} e^{-i2\pi(k'+l'-m')f_0t_2}$$
$$\times \xi_k\zeta_l\zeta_m^*\xi_{k'}^*\zeta_{l'}^*\zeta_{m'}\sqrt{G_x(kf_0)G_y(lf_0)G_y(mf_0)G_x(k'f_0)G_y(l'f_0)G_y(m'f_0)}\Bigg\}. \quad (55)$$

These four terms in the autocorrelation will give rise to four different terms in the total PSD, which we denote by $G_{x,p,1}$, $G_{x,p,2}$, $G_{x,p,3}$, and $G_{x,p,4}$, respectively. We need to find these four terms individually.

A. The random variables

It is seen that in (52)–(55), all terms except the ξ and ζ are deterministic. In order to carry out the expectation operation, in this section we therefore need to investigate these random variables.

The ξ and ζ are complex independent Gaussian random variables of unit variance, i.e., $\mathbb{E}\{\xi_k\} = 0$, $\mathbb{E}\{\xi_k^2\} = 0$, $\mathbb{E}\{|\xi_k|^2\} = 1$, $\forall k$. Analogous expressions hold for ζ_k. Using this, we can simplify the six-dimensional sums (52)–(55) in the following way. First assume that one of the summation variables, say k, has a value different from all other summation variables, i.e., k is unique. Due to the independence, we then have

$$\mathbb{E}\{\xi_k\xi_l\xi_m^*\xi_{k'}^*\xi_{l'}^*\xi_{m'}\} = \mathbb{E}\{\xi_k\}\,\mathbb{E}\{\xi_l\xi_m^*\xi_{k'}^*\xi_{l'}^*\xi_{m'}\} = 0. \quad (56)$$

An identical argument holds if some of the ξ are replaced by ζ. This implies that the expected value is always zero when one of the summation variables is unique. Thus, we have the condition that no summation variable can be unique. Second we assume that no summation variable is unique, but there are three pairwise equal values, where each pair has a unique value. Assume for example that $k = l$, $k' = l'$, $m = m'$. Then

$$\mathbb{E}\{\xi_k\xi_l\xi_m^*\xi_{k'}^*\xi_{l'}^*\xi_{m'}\} = \mathbb{E}\{\xi_k\xi_k\xi_m^*\xi_{k'}^*\xi_{k'}^*\xi_m\} = \mathbb{E}\{\xi_k\xi_k\}\,\mathbb{E}\{\xi_{k'}^*\xi_{k'}^*\}\,\mathbb{E}\{\xi_m^*\xi_m\} = \mathbb{E}\{\xi_k^2\}\,\mathbb{E}\{(\xi_{k'}^*)^2\}\,\mathbb{E}\{|\xi_m|^2\} = 0. \quad (57)$$

We conclude that each random variable must be paired up with the complex conjugated version of the same random variable. This conclusion reduces the dimensionality of the summation to three or less. However, the dimensionality cannot be less than three, because then the expression $G_{x,p} \to 0$ as $f_0 \to 0$. To see this we first notice that

$$|G_{x,p}| \le |G_{x,p,1}| + |G_{x,p,2}| + |G_{x,p,3}| + |G_{x,p,4}|, \quad (58)$$

and since all terms behave similarly in this respect, we can study, say, $G_{x,p,1}$. Let us select the one-dimensional case $k = l = m = k' = l' = m'$. We then have

$$R_1 = \mathbb{E}\left\{ f_0^3 \sum_k |\mathcal{C}_{kkk}|^2 e^{i2\pi kf_0\tau}|\xi_k|^6 G_x^3(kf_0)\right\} = f_0^3 \sum_k |\mathcal{C}_{kkk}|^2 e^{i2\pi kf_0\tau}\,\mathbb{E}\{|\xi_k|^6\}G_x^3(kf_0). \quad (59)$$

We get

$$G_{x,p,1} = f_0^3 \sum_k |\mathcal{C}_{kkk}|^2 \, \mathbb{E}\{|\xi_k|^6\} G_x^3(kf_0)\delta(f - kf_0). \tag{60}$$

We see that $G_{x,p,1} \to 0$ as $f_0 \to 0$ by identifying the Riemann sum and writing the expression as

$$G_{x,p,1} = f_0^2 \int |\mathcal{C}(f_1, f_1, f_1)|^2 \, \mathbb{E}\{|\xi_k|^6\} G_x^3(f_1)\delta(f - f_1)\, df_1 = f_0^2 |\mathcal{C}(f, f, f)|^2 \, \mathbb{E}\{|\xi_k|^6\} G_x^3(f). \tag{61}$$

Remembering that the value of $|\mathcal{C}|^2$ is upper bounded by a constant, the fact that $G_{x,p,1} \to 0$ is now obvious. An analogous argument can be made for the case of a two-dimensional sum. We conclude that the summation must have dimension exactly three, i.e., each random variable must be paired up with the complex conjugated version of the same random variable and all three pairs must have unique values.

B. The first term in the PSD

We now study the first expression, i.e., $G_{x,p,1}(f) = \mathcal{F}[R_1(\tau)]$. Following the above rules for how the indices can be chosen, we have six possible combinations:

$$\begin{array}{ccc} k = k' & l = l' & m = m', \\ k = k' & l = m & m' = l', \\ k = l' & l = k' & m = m', \\ k = l' & l = m & m' = k', \\ k = m & l = k' & m' = l', \\ k = m & l = l' & m' = k'. \end{array} \tag{62}$$

We need to study these different possibilities individually.

1) The case $k = k'$, $l = l'$, $m = m'$: We then have

$$\begin{aligned} R_1 &= \mathbb{E}\left\{ f_0^3 \sum_{k,l,m} |\mathcal{C}_{klm}|^2 e^{i2\pi(k+l-m)f_0 t_1} e^{-i2\pi(k+l-m)f_0 t_2} |\xi_k|^2 |\xi_l|^2 |\xi_m|^2 G_x(kf_0)G_x(lf_0)G_x(mf_0) \right\} \\ &= f_0^3 \sum_{k,l,m} |\mathcal{C}_{klm}|^2 e^{i2\pi(k+l-m)f_0\tau} G_x(kf_0)G_x(lf_0)G_x(mf_0), \end{aligned} \tag{63}$$

$$G_{x,p,1}(f) = f_0^3 \sum_{k,l,m} |\mathcal{C}_{klm}|^2 G_x(kf_0)G_x(lf_0)G_x(mf_0)\delta(f - (k + l - m)f_0). \tag{64}$$

Letting $f_0 \to 0$, we get

$$\begin{aligned} G_{x,p,1}(f) &= \iiint |\mathcal{C}(f_1, f_2, f_3)|^2 G_x(f_1)G_x(f_2)G_x(f_3)\delta(f - f_1 - f_2 + f_3)\, df_1 df_2 df_3 \\ &= \iint |\mathcal{C}(f_1, f_2, f_1 + f_2 - f)|^2 G_x(f_1)G_x(f_2)G_x(f_1 + f_2 - f)\, df_1 df_2. \end{aligned} \tag{65}$$

2) The other cases: Compared to the case above, the case $k = l'$, $l = k'$, $m = m'$ is obtained by swapping k' and l'. Since $\mathcal{C}_{klm} = \mathcal{C}_{lkm}$ and the rest of the expression is clearly invariant under this change, this yields an identical result as the case above. The other four cases are all equivalent to the case $k = k'$, $l = m$, $m' = l'$. We get

$$\begin{aligned} R_1 &= \mathbb{E}\left\{ f_0^3 \sum_{kll'} \mathcal{C}_{kll}\mathcal{C}^*_{kl'l'} e^{i2\pi(k+l-l)f_0 t_1} e^{-i2\pi(k+l'-l')f_0 t_2} \xi_k \xi_l \xi_l^* \xi_k^* \xi_{l'}^* \xi_{l'} \sqrt{G_x(kf_0)G_x(lf_0)G_x(lf_0)G_x(kf_0)G_x(l'f_0)G_x(l'f_0)} \right\} \\ &= \mathbb{E}\left\{ f_0^3 \sum_{kll'} \mathcal{C}_{kll}\mathcal{C}^*_{kl'l'} e^{i2\pi kf_0 t_1} e^{-i2\pi kf_0 t_2} |\xi_k|^2 |\xi_l|^2 |\xi_{l'}|^2 G_x(kf_0)G_x(lf_0)G_x(l'f_0) \right\} \\ &= f_0^3 \sum_{kll'} \mathcal{C}_{kll}\mathcal{C}^*_{kl'l'} e^{i2\pi kf_0\tau} G_x(kf_0)G_x(lf_0)G_x(l'f_0), \end{aligned} \tag{66}$$

$$G_{x,p,1}(f) = f_0^3 \sum_{kll'} \mathcal{C}_{kll}\mathcal{C}^*_{kl'l'} G_x(kf_0)G_x(lf_0)G_x(l'f_0)\delta(f - kf_0). \tag{67}$$

Letting $f_0 \to 0$, we get

$$\begin{aligned} G_{x,p,1}(f) &= \iiint \mathcal{C}(f_1, f_2, f_2)\mathcal{C}^*(f_1, f_3, f_3)G_x(f_1)G_x(f_2)G_x(f_3)\delta(f - f_1)\, df_1 df_2 df_3 \\ &= \iint \mathcal{C}(f, f_2, f_2)\mathcal{C}^*(f, f_3, f_3)G_x(f)G_x(f_2)G_x(f_3)\, df_2 df_3. \end{aligned} \tag{68}$$

However

$$\mathcal{C}(f_1, f_2, f_2) = \mathcal{C}(f_1, f_3, f_3) = \mathcal{C}(f_2, f_1, f_2) = \mathcal{C}(f_3, f_1, f_3) = \int_0^L \gamma(z)p(z)\, dz \equiv \mathcal{C}_0, \tag{69}$$

which is a real number. This gives

$$G_{x,p,1}(f) = \mathcal{C}_0^2 G_x(f) \iint G_x(f_2)G_x(f_3)\, df_2 df_3 = \mathcal{C}_0^2 G_x(f) \int G_x(f_2)\, df_2 \int G_x(f_3)\, df_3 = \mathcal{C}_0^2 P_x^2 G_x(f). \tag{70}$$

3) The total PSD for the first term: The six possible index selection cases split into two groups of degeneracy two and four, respectively. We get the total expression

$$G_{x,p,1}(f) = 2 \iint |\mathcal{C}(f_1, f_2, f_1 + f_2 - f)|^2 G_x(f_1)G_x(f_2)G_x(f_1 + f_2 - f)\, df_1 df_2 + 4\mathcal{C}_0^2 P_x^2 G_x(f). \tag{71}$$

C. The second term in the PSD

We now study the second expression, i.e., $G_{x,p,2}(f) = \mathcal{F}[R_2(\tau)]$. Following the above rules for how the indices can be chosen, we have two combinations

$$\begin{array}{lll} k = k' & l = m & m' = l' \\ k = m & l = k' & m' = l' \end{array} \tag{72}$$

The reason that we have fewer possibilities is that the expression contains both ξ and ζ, which are independent.

1) The case $k = k'$, $l = m$, $m' = l'$: We then have

$$\begin{aligned} R_2 &= \mathbb{E}\left\{ f_0^3 \sum_{k,l,l'} \mathcal{C}_{kll}\mathcal{C}^*_{kl'l'} e^{i2\pi k f_0 t_1} e^{-i2\pi k' f_0 t_2} |\xi_k|^2 |\xi_l|^2 |\zeta_{l'}|^2 G_x(kf_0)G_x(lf_0)G_y(l'f_0) \right\} \\ &= f_0^3 \sum_{k,l,l'} \mathcal{C}_{kll}\mathcal{C}^*_{kl'l'} e^{i2\pi k f_0 \tau} G_x(kf_0)G_x(lf_0)G_y(l'f_0), \end{aligned} \tag{73}$$

$$G_{x,p,2}(f) = f_0^3 \sum_{k,l,l'} \mathcal{C}_{kll}\mathcal{C}^*_{kl'l'} G_x(kf_0)G_x(lf_0)G_y(l'f_0)\delta(f - kf_0). \tag{74}$$

Letting $f_0 \to 0$, we get

$$\begin{aligned} G_{x,p,2}(f) &= \iiint \mathcal{C}(f_1, f_2, f_2)\mathcal{C}^*(f_1, f_3, f_3)G_x(f_1)G_x(f_2)G_y(f_3)\delta(f - f_1)\, df_1 df_2 df_3 \\ &= \mathcal{C}_0^2 \iint G_x(f)G_x(f_2)G_y(f_3)\, df_2 df_3 \\ &= \mathcal{C}_0^2 P_x P_y G_x(f). \end{aligned} \tag{75}$$

2) The case $k = m$, $l = k'$, $m' = l'$: We notice that this case is obtained from the above case by swapping k and l and will therefore give the same result.

3) The total PSD for the second term: We get the final expression

$$G_{x,p,2}(f) = 2\mathcal{C}_0^2 P_x P_y G_x(f). \tag{76}$$

D. The third term in the PSD

We now study the third expression, i.e., $G_{x,p,3}(f) = \mathcal{F}[R_3(\tau)]$. Following the above rules for how the indices can be chosen, we now have two combinations

$$\begin{array}{lll} k = k' & l = m & m' = l' \\ k = l' & l = m & m' = k' \end{array} \tag{77}$$

The first case is identical to the first case for the second term in the PSD. The second case is obtained from the first case by swapping k' and l' and will therefore give the same result. We get

$$G_{x,p,3}(f) = 2\mathcal{C}_0^2 P_x P_y G_x(f). \tag{78}$$

E. The fourth term in the PSD

We now study the fourth expression, i.e., $G_{x,p,4}(f) = \mathcal{F}[R_4(\tau)]$. Following the above rules for how the indices can be chosen, we have two combinations

$$\begin{aligned} k = k' \quad l = l' \quad m = m' \\ k = k' \quad l = m \quad m' = l' \end{aligned} \tag{79}$$

1) The case $k = k'$, $l = l'$, $m = m'$: We then have

$$\begin{aligned} R_4 &= \mathbb{E}\left\{ f_0^3 \sum_{k,l,m} |\mathcal{C}_{klm}|^2 e^{i2\pi(k+l-m)f_0\tau} |\xi_k|^2 |\zeta_l|^2 |\zeta_m|^2 G_x(kf_0) G_y(lf_0) G_y(mf_0) \right\} \\ &= f_0^3 \sum_{k,l,m} |\mathcal{C}_{klm}|^2 e^{i2\pi(k+l-m)f_0\tau} G_x(kf_0) G_y(lf_0) G_y(mf_0), \end{aligned} \tag{80}$$

$$G_{x,p,4}(f) = f_0^3 \sum_{k,l,m} |\mathcal{C}_{klm}|^2 G_x(kf_0) G_y(lf_0) G_y(mf_0) \delta(f - (k+l-m)f_0). \tag{81}$$

Letting $f_0 \to 0$, we get

$$\begin{aligned} G_{x,p,4}(f) &= \iiint |\mathcal{C}(f_1, f_2, f_3)|^2 G_x(f_1) G_y(f_2) G_y(f_3) \delta(f - f_1 - f_2 + f_3)\, df_1 df_2 df_3 \\ &= \iint |\mathcal{C}(f_1, f_2, f_1 + f_2 - f)|^2 G_x(f_1) G_y(f_2) G_y(f_1 + f_2 - f)\, df_1 df_2. \end{aligned} \tag{82}$$

2) The case $k = k'$, $l = m$, $m' = l'$: We get

$$\begin{aligned} R_4 &= \mathbb{E}\left\{ f_0^3 \sum_{k,l,l'} \mathcal{C}_{kll} \mathcal{C}^*_{kl'l'} e^{i2\pi k f_0 t_1} e^{-i2\pi k f_0 t_2} |\xi_k|^2 |\zeta_l|^2 |\zeta_{l'}|^2 G_x(kf_0) G_y(lf_0) G_y(l'f_0) \right\} \\ &= f_0^3 \sum_{k,l,l'} \mathcal{C}_{kll} \mathcal{C}^*_{kl'l'} e^{i2\pi k f_0 \tau} G_x(kf_0) G_y(lf_0) G_y(l'f_0), \end{aligned} \tag{83}$$

$$G_{x,p,4}(f) = f_0^3 \sum_{k,l,l'} \mathcal{C}_{kll} \mathcal{C}^*_{kl'l'} G_x(kf_0) G_y(lf_0) G_y(l'f_0) \delta(f - kf_0). \tag{84}$$

Letting $f_0 \to 0$, we get

$$\begin{aligned} G_{x,p,4}(f) &= \iiint \mathcal{C}(f_1, f_2, f_2) \mathcal{C}^*(f_1, f_3, f_3) G_x(f_1) G_y(f_2) G_y(f_3) \delta(f - f_1)\, df_1 df_2 df_3 \\ &= \iint \mathcal{C}(f, f_2, f_2) \mathcal{C}^*(f, f_3, f_3) G_x(f) G_y(f_2) G_y(f_3)\, df_2 df_3 \\ &= \mathcal{C}_0^2 P_y^2 G_x(f). \end{aligned} \tag{85}$$

3) The total PSD for the fourth term: We get the final expression

$$G_{x,p,2}(f) = \iint |\mathcal{C}(f_1, f_2, f_1 + f_2 - f)|^2 G_x(f_1) G_y(f_2) G_y(f_1 + f_2 - f)\, df_1 df_2 + \mathcal{C}_0^2 P_y^2 G_x(f). \tag{86}$$

F. The total PSD

We can now find the total PSD by summing up all the terms according to

$$\begin{aligned} G_{x,p} &= G_{x,p,1} + G_{x,p,2} + G_{x,p,3} + G_{x,p,4} \\ &= 2 \iint |\mathcal{C}(f_1, f_2, f_1 + f_2 - f)|^2 G_x(f_1) G_x(f_2) G_x(f_1 + f_2 - f)\, df_1 df_2 \\ &\quad + \iint |\mathcal{C}(f_1, f_2, f_1 + f_2 - f)|^2 G_x(f_1) G_y(f_2) G_y(f_1 + f_2 - f)\, df_1 df_2 \\ &\quad + \mathcal{C}_0^2 (4P_x^2 + 4P_x P_y + P_y^2) G_x(f). \end{aligned} \tag{87}$$

We notice that the x polarization acting on itself is twice as effective as the y polarization acting on the x polarization. As is clear from the derivation, the reason for this is the level of degeneracy. This is similar to the case of the double influence of XPM as compared to SPM, when the nonlinear Schrödinger equation is approximated by a coupled system with one equation for each WDM channel [15, p. 264]. The final term of (87) is of no consequence for the transmission. The origin of this term is the phase modulation from the entire propagating field acting on itself. In practice, we expect this to just cause a rotation of the received signal constellations. However, in the perturbation analysis, this type of phase modulation gives rise to these extra terms in the PSD. As (87) can be considered the main result of this calculation, we repeat the expression for $\mathcal{C}$ for ease of reference

$$\mathcal{C}(f_1, f_2, f_1+f_2-f) \equiv \int_0^L \gamma(z)p(z)e^{-i4\pi^2(f_1-f)(f_2-f)B(z)}\, dz. \tag{88}$$

It should be noticed that $\mathcal{C}$ is determined when the physical parameters of the channel are selected and $|\mathcal{C}|^2$ is a measure of the FWM efficiency. Then, by choosing the input signal PSD we obtain the PSD of the NLI.

V. Transmission system example

We now calculate the resulting expression for a system of the type considered in [2]. The system consists of N_{SMF} spans, each containing a standard single-mode fiber (SMF) followed by an EDFA. There are no dispersion-compensating fibers (DCF) or any other optical dispersion compensation. Instead, the CD will be compensated for by using DSP in the receiver. The first span starts at $z = z_0 = 0$ and the last span ends at $z = z_{N_{\text{SMF}}}$. The fiber parameters α, β_2, and γ are assumed to have no z-dependence. Furthermore, we assume that identical signals are launched in the two polarizations, i.e., we set $G_y = G_x$ and we remove the terms that are not due to FWM to rewrite (87) as

$$G_{x,p} = 3\iint |\mathcal{C}(f_1, f_2, f_1+f_2-f)|^2 G_x(f_1)G_x(f_2)G_x(f_1+f_2-f)\, df_1 df_2. \tag{89}$$

We have $B(z) = \beta_2 z$ and temporarily introducing $\kappa = 4\pi^2(f_1-f)(f_2-f)$ we find

$$\begin{aligned}
\mathcal{C}(f_1, f_2, f_1+f_2-f) &= \int_0^L \gamma p e^{-i\kappa B(z)}\, dz \\
&= \gamma \sum_{n=1}^{N_{\text{SMF}}} \int_{z_{n-1}}^{z_n} e^{-\alpha(z-z_{n-1})}e^{-i\kappa\beta_2 z}\, dz \\
&= -\frac{\gamma}{\alpha+i\kappa\beta_2}\sum_{n=1}^{N_{\text{SMF}}}\left(e^{-\alpha(z_n-z_{n-1})}e^{-i\kappa\beta_2 z_n} - e^{-i\kappa\beta_2 z_{n-1}}\right) \\
&= -\frac{\gamma}{\alpha+i\kappa\beta_2}\sum_{n=1}^{N_{\text{SMF}}}\left(e^{-\alpha L_{\text{SMF}}}e^{-i\kappa\beta_2 n L_{\text{SMF}}} - e^{-i\kappa\beta_2 (n-1) L_{\text{SMF}}}\right) \\
&= \frac{\gamma}{\alpha+i\kappa\beta_2}(1-e^{-\alpha L_{\text{SMF}}}e^{-i\kappa\beta_2 L_{\text{SMF}}})\frac{1-e^{-i\kappa\beta_2 L_{\text{SMF}} N_{\text{SMF}}}}{1-e^{-i\kappa\beta_2 L_{\text{SMF}}}},
\end{aligned} \tag{90}$$

where we also used the assumption that all SMFs have the same length, denoted by $L_{\text{SMF}} = z_n - z_{n-1}$. We find

$$\begin{aligned}
|\mathcal{C}(f_1, f_2, f_1+f_2-f)|^2 &= \gamma^2\left|\frac{1-e^{-\alpha L_{\text{SMF}}}e^{-i\kappa\beta_2 L_{\text{SMF}}}}{\alpha+i\kappa\beta_2}\right|^2\left|\frac{1-e^{-i\kappa\beta_2 L_{\text{SMF}} N_{\text{SMF}}}}{1-e^{-i\kappa\beta_2 L_{\text{SMF}}}}\right|^2 \\
&= \gamma^2\left|\frac{1-e^{-\alpha L_{\text{SMF}}}e^{-i\kappa\beta_2 L_{\text{SMF}}}}{\alpha+i\kappa\beta_2}\right|^2\frac{\sin^2(\kappa\beta_2 L_{\text{SMF}} N_{\text{SMF}}/2)}{\sin^2(\kappa\beta_2 L_{\text{SMF}}/2)}.
\end{aligned} \tag{91}$$

We then find

$$\begin{aligned}
G_{x,p} = 3\gamma^2 \iint &\left|\frac{1-e^{-\alpha L_{\text{SMF}}-i4\pi^2(f_1-f)(f_2-f)\beta_2 L_{\text{SMF}}}}{\alpha+i4\pi^2(f_1-f)(f_2-f)\beta_2}\right|^2\frac{\sin^2[2\pi^2(f_1-f)(f_2-f)\beta_2 L_{\text{SMF}} N_{\text{SMF}}]}{\sin^2[2\pi^2(f_1-f)(f_2-f)\beta_2 L_{\text{SMF}}]} \\
&\times G_x(f_1)G_x(f_2)G_x(f_1+f_2-f)\, df_1 df_2.
\end{aligned} \tag{92}$$

In order to compare this expression with the coherent expression in [4, Eq. (18)], we must account for the fact that a different convention for the Manakov equation is used in [4]. Thus, we need to replace γ with $8\gamma/9$. Furthermore, the NLI PSD in [4, Eq. (18)] is the summation over both polarizations, i.e., we here get $G_{\text{NLI}} = G_{x,p} + G_{y,p} = 2G_{x,p}$. Similarly, we here have $G_{\text{Tx}} = G_x + G_y = 2G_x$. (From [4, Eq. (13)], it seems like G_{Tx} is equal to G_x. However, this is not the intended meaning of G_{Tx}. We thank the authors of [4] for explaining this.) Finally, there is a difference in how the attenuation is defined. For the definition used here, we refer to (1) and (9). When these differences in notation are taken into account, the results become identical.

VI. Conclusion

We have presented a derivation of the power spectral density of the nonlinear interference under the assumptions that (i) the nonlinear effects are weak, (ii) the signal can be written as suggested in [2], and (iii) the signal–noise interaction can be neglected. Using these three assumptions, we obtain the PSD expression for a general system. Applying the result to a system consisting entirely of SMF and accounting for the differences in notation, we obtain a result identical to that presented in [2]–[4].

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