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Singularities of Monotone Vector Fields and an Extragradient-type Algorithm

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Abstract. Bearing in mind the notion of monotone vector field on Riemannian manifolds, see [12–16], we study the set of their singularities and for a particular class of manifolds develop an extragradient-type algorithm convergent to singularities of such vector fields. In particular, our method can be used for solving nonlinear constrained optimization problems in Euclidean space, with a convex objective function and the constraint set a constant curvature Hadamard manifold. Our paper shows how tools of convex analysis on Riemannian manifolds can be used to solve some nonconvex constrained problem in a Euclidean space.

Key words. extragradient algorithm, global optimization, Hadamard manifold, monotone vector field.

1. Introduction

Extensions of concepts and techniques for optimization in $\mathbb{R}^n$ to Riemannian manifolds are natural. It has been done frequently in the last few years, with theoretical objectives and also in order to obtain effective algorithms, see [4, 6, 7, 18, 19, 23] and [25]. By using these extensions it is possible to solve some nonconvex constrained optimization problems in Euclidean spaces, after rewriting them as convex problems on Riemannian manifolds.

The extragradient algorithm was proposed first by G.M. Korpelevich, see [11]. In [8], this method was used for solving variational inequality problems with continuous point-to-point monotone operators and convex constraint set. It was improved in [10]. In [22] were reported preliminary computational experience of a modification of the extragradient algorithm. It was extended to solving problems with monotone point-to-set maps in [2] and in [9]. In this work we recall the concept of monotone vector fields in Riemannian manifolds, see [12–16], discuss convex analysis on Hadamard manifolds and propose an extragradient-type method for finding singularities of monotone vector fields on Hadamard

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manifolds (i.e. complete, simply connected Riemannian manifolds of nonpositive curvature). It will be also shown how tools of convex analysis on Hadamard manifolds can solve non-convex constrained problems in Euclidean spaces. To illustrate this remarkable fact an example will be given.

In Section 2 we introduce some basic concepts, important for our analysis. In Section 3 we discuss some topics of convex analysis on Hadamard manifolds, particularly projection mappings onto convex sets. In Section 4 we present the definition of monotone vector fields on Riemannian manifolds, see [12–16], and prove some important properties.

In Section 5 we recall the notion of variational inequalities on Hadamard manifolds, some existence and uniqueness theorems, and analyze the properties of the solution set of a variational inequality.

In Section 6 we prove some existence and uniqueness theorems for the singularities of vector fields and analyze the properties of the singularity set.

Finally, in Section 7 we propose the extragradient algorithm for finding singularities of such vector fields and perform a complete convergence analysis, i.e., we prove that the sequence generated by our algorithm converges to a singularity of the field when the manifold is a constant curvature Hadamard one and the field does have singularities.

2. Basics Concepts

In this section we present some basic notation, definitions and properties of Riemannian manifolds. They can be found in any introductory book on Riemannian Geometry, see [3] and [20]. Throughout this paper, all manifolds are smooth, connected and paracompact, and all functions and vector fields are smooth. However, if differentiation is not needed all the results remain true for arbitrary noncontinuous vector fields.

Given a manifold M, denote by $\mathcal{X}(M)$ the space of vector fields over M, by T_pM the tangent space of M at p and by $\mathcal{F}(M)$ the ring of functions over M. The manifold M can be always endowed with a Riemannian metric $\langle\cdot,\cdot\rangle$, with corresponding norm denoted by $\|\cdot\|$, to become a *Riemannian* manifold. The length (with respect to the metric $\langle\cdot,\cdot\rangle$) of a piecewise smooth curve γ: $[a,b]\to M$ joining points p and q in M, i.e., such that $\gamma(a)=p$ and $\gamma(b)=q$, is defined by $L(\gamma)=\int_a^b\|\gamma'(t)\|\mathrm{d}t$. Minimizing this length functional over the set of all such curves joining arbitrary points p and q in M we obtain a distance function $(p,q)\mapsto d(p,q)$ which induces the original topology of M. The metric induces a map $f\in\mathcal{F}(M)\mapsto \operatorname{grad} f\in\mathcal{X}(M)$ which associates to each f its *gradient* via the rule $\langle \operatorname{grad} f, X\rangle=\mathrm{d}f(X)$, $X\in\mathcal{X}(M)$. Let ∇ be the Levi-Civita connection associated to $(M,\langle,\rangle)$. If γ is a curve joining points p and q in M, then, for each $t\in[a,b]$, ∇ induces an isometry (relative to $\langle,\rangle$) $P_\gamma(t)$: $T_pM\to T_{\gamma(t)}M$, the so-called *parallel transport* along γ from x to $\gamma(t)$. When the reference to a curve joining p and q is not necessary, we use the notation $P_{p,q}$. We say that

γ is a *geodesic* when $\nabla_{\gamma'}\gamma'=0$, in this case $\|\gamma'\|$ is constant. We say that γ is *normalized* if $\|\gamma'\|=1$. The restriction of a geodesic to a closed bounded interval is called a *geodesic segment*. A geodesic segment joining p and q in M is said to be *minimal* if its length equals $d(p,q)$. Let γ: $[a,b]\to M$ be a normalized geodesic segment. A *differentiable variation* of γ is, by definition, a differentiable mapping α: $[a,b]\times(-\varepsilon,\varepsilon)\to M$ satisfying $\alpha(t,0)=\gamma(t)$. The vector field along γ, defined by $V(t)=\frac{\partial\alpha}{\partial s}(t,0)$ is called the *variational vector field* of α. The *first variational formula* of arc length on α is then given as follows:

$$L'(\gamma):=\frac{\mathrm{d}}{\mathrm{d}s}L(c_s)\bigg|_{s=0}=\langle V,\gamma'\rangle|_a^b, \tag{2.1}$$

where $c_s(t)=\alpha(t,s)$ with $s\in(-\varepsilon,\varepsilon)$. A Riemannian manifold is *complete* if geodesics are defined for any values of t. Hopf-Rinow's Theorem, see [3], asserts that if this is the case, then any pair of points in M, say p and q, can be joined by a (not necessarily unique) minimal geodesic segment. Moreover, (M,d) is a complete metric space and bounded and closed subsets are compact. *In this paper, all manifolds are assumed to be complete*. The *exponential map* $\exp_p$: $T_pM\to M$ is defined by $\exp_x v=\gamma_v(1,x)$, where $\gamma(\cdot)=\gamma_v(\cdot,p)$ is the geodesic starting at the point p with velocity v. In this case, it is easy to see that $\exp_p tv=\gamma_v(t,p)$ for any values of t.

A complete, simply connected Riemannian manifold of nonpositive curvature is called a Hadamard manifold. *From now on M will denote a Riemannian manifold and H will denote a Hadamard manifold*. Hadamard's Theorem, see [3], asserts that the topological and differential structure of a Hadamard manifolds coincide with those of an Euclidean space of the same dimension. More precisely, at any point $p\in M$, the exponential map $\exp_p$: $T_pM\to M$ is a diffeomorphism. Furthermore, Hadamard manifolds have some geometrical properties similar to some well-known geometrical properties of Euclidean spaces. We mention now another property of Hadamard manifolds, similar to a property of the Euclidean space, which will be used in the sequel.

A *geodesic triangle* $\Delta(p_1p_2p_3)$ of a Riemannian manifold is the set consisting of three distinct points p_1, p_2 and p_3 called the *vertices* and three minimizing geodesic segments γ_{i+1} joining p_{i+1} to p_{i+2} called the *sides*, where $i=1,2,3$ (mod 3).

THEOREM 2.1. *Let $\Delta(p_1p_2p_3)$ be a geodesic triangle in H. Denote by γ_{i+1}: $[0,l_{i+1}]\to H$ geodesic segment joining p_{i+1} to p_{i+2} and set $l_{i+1}:=L(\gamma_{i+1})$, $\theta_{i+1}=\measuredangle(\gamma'_{i+1}(0),-\gamma'_i(l_i))$, where $i=1,2,3$ (mod 3). Then*

$$\theta_1+\theta_2+\theta_3\leqslant\pi, \tag{2.2}$$

$$l_{i+1}^2+l_{i+2}^2-2l_{i+1}l_{i+2}\cos\theta_{i+2}\leqslant l_i^2. \tag{2.3}$$

and

$$l_{i+1}\cos\theta_{i+2}+l_i\cos\theta_i \geqslant l_{i+2}. \tag{2.4}$$

Proof. Inequalities (2.2) and (2.3) are proved in [20] Proposition 4.5, p. 223. Inequality (2.4) is a consequence of (2.3). □

3. Projections onto Convex Set

In this section, we summarize some basic notation, definitions and results on convex analysis in Hadamard manifolds, which will be useful in the sequel, see [1, 19–21, 25].

The set $C\subset M$ is said to be *convex* if contains a geodesic segment γ whenever it contains the end points of γ, that is, $\gamma((1-t)a+tb)$ is in C whenever $x'=\gamma(a)$ and $x=\gamma(b)$ are in C, and $t\in[0,1]$. *From now on C will denote a nonempty, closed and convex set* in a Riemannian manifold. For any $p'\in H$ and $C\subset H$, there exists a unique $p\in C$ such that $d(p',p)\leqslant d(p',q)$ for all $q\in C$, that unique point is called the *projection* of p' onto the convex set C and is denoted as $\Pi_C(p')$. The next result gives a characterization of the projection Π_C.

PROPOSITION 3.1. *For any $p'\in H$, there exists a unique projection $\Pi_C(p')$. Furthermore, the following inequality holds for all $p\in C$:*

$$\langle \exp^{-1}_{\Pi_C(p')}p', \exp^{-1}_{\Pi_C(p')}p\rangle \leqslant 0. \tag{3.1}$$

Proof. See [27]. □

COROLLARY 3.1. *Let H be of constant curvature. Given $p'\in H$ and $s\in T_{p'}H$, the set*

$$L_{p',s}=\left\{p\in H : \langle \exp^{-1}_{p'}p, s\rangle \leqslant 0\right\}. \tag{3.2}$$

is convex.

Proof. Take p and q in $L_{p',s}$, $p\neq q$. We will prove that the geodesic segment γ: $[0,1]\to H$ starting at p and ending at q (i.e., $\gamma(0)=p$ and $\gamma(1)=q$) is contained in $L_{p',s}$, i.e., $\gamma(t)\in L_{p',s}$ for all $t\in[0,1]$. Since H is of constant curvature there is a two dimensional total geodesic submanifold N of H containing p', p and q. Hence, $\exp^{-1}_{p'}\gamma(t)$ is contained in the two dimensional pointed convex cone of $T_{p'}H$ spanned by $\exp^{-1}_{p'}p$ and $\exp^{-1}_{p'}q$. It follows that, for each $t\in[0,1]$, there are some $a,b\geqslant 0$ such that $\exp^{-1}_{p'}\gamma(t)=a\exp^{-1}_{p'}p+b\exp^{-1}_{p'}q$. Since $p,q\in L_{p',s}$, we have that $\langle\exp^{-1}_{p'}\gamma(t),s\rangle=a\langle\exp^{-1}_{p'}p,s\rangle+b\langle\exp^{-1}_{p'}q,s\rangle\leqslant 0$. Then, by (3.2), $\gamma(t)\in L_{p',s}$ for all $t\in[0,1]$. □

The set $L_{p',s}$ plays a fundamental role in the convergence proof of the Extragradient Method. Observe that the boundary of $L_{p',s}$ is

$$\partial L_{p',s} = \{p \in H : \langle \exp_{p'}^{-1} p, s\rangle = 0\}. \tag{3.3}$$

and if $p \notin L_{p',s}$ then $\Pi_{L_{p',s}}(p) \in \partial L_{p',s}$.

4. Monotone Vector Fields

In this section we recall the concept of monotone vector fields on Riemannian manifolds, see [12–16] and [5].

DEFINITION 1. A vector field X on M is said to be *monotone* if for all distinct p and q in M

$$\langle \gamma'(0), P_{p,q}^{-1} X(q) - X(p)\rangle \geqslant 0, \tag{4.1}$$

where γ is a geodesic, linking p and q, $\gamma(0) = p$. If (4.1) is satisfied with strict inequality for all p and q, then X is said to be *strictly monotone*. The vector field X is strongly monotone if there is some λ such that for all p, $q \in M$

$$\langle \gamma'(0), P_{p,q}^{-1} X(q) - X(p)\rangle \geqslant \lambda l(\gamma)^2, \tag{4.2}$$

where γ is a geodesic segment joining p to q, with $\gamma(0) = p$ and $\gamma(1) = q$.

In Hadamard manifolds, inequality (4.1) is equivalent to

$$\langle \exp_p^{-1} q, P_{p,q}^{-1} X(q) - X(p)\rangle \geqslant 0. \tag{4.3}$$

and (4.2) to

$$\langle \exp_p^{-1} q, P_{p,q}^{-1} X(q) - X(p)\rangle \geqslant \lambda d(p,q)^2. \tag{4.4}$$

The previous definition extends to Riemannian manifolds the concept of monotone operators in $\mathbb{R}^n$.

PROPOSITION 4.1. *The vector field X on M is monotone if and only if the function $\varphi_{(X,\gamma)}\colon \mathbb{R} \to \mathbb{R}$ defined by*

$$\varphi_{(X,\gamma)}(t) = \langle \gamma'(t), X(\gamma(t))\rangle. \tag{4.5}$$

is increasing for all geodesic γ in M.

Proof. Let γ be a geodesic and X a vector field in M. Let $\varphi = \varphi_{(X,\gamma)}$. Let t_1 and t_2 be two different points in $\mathbb{R}$. Define $\alpha(t) = \gamma(t_1 + t(t_2 - t_1))$, a reparametrization of γ. Let $q=\gamma(t_1)$, $p=\gamma(t_2)$ and $P_{q,p}$ be the parallel transport from q to p along α. Since $P_{q,p}$ is an isometry, we have

$$\begin{aligned}(t_2-t_1)(\varphi(t_2)-\varphi(t_1)) &= (t_2-t_1)(\langle\gamma'(t_2),X(\gamma(t_2))\rangle-\langle\gamma'(t_1),X(\gamma(t_1))\rangle)\\ &= \langle\alpha'(1),X(p)\rangle-\langle\alpha'(0),X(q)\rangle\\ &= \langle P_{q,p}^{-1}\alpha'(1),P_{q,p}^{-1}X(p)\rangle-\langle\alpha'(0),X(q)\rangle\\ &= \langle\alpha'(0),P_{q,p}^{-1}X(p)-X(q)\rangle.\end{aligned}$$

Monotonicity of X implies

$$(t_2-t_1)(\varphi(t_2)-\varphi(t_1))=\langle\alpha'(0),P_{q,p}^{-1}X(p)-X(q)\rangle\geqslant 0.$$

Therefore, $\varphi_{(X,\gamma)}$ is increasing for all geodesic γ.

Reciprocally, let $p,q\in M$ two different points, and γ a geodesic linking q and p. Let $P_{q,p}$ be the parallel transport from q to p along γ. Then,

$$\begin{aligned}\langle\gamma'(0),P_{q,p}^{-1}X(p)-X(q)\rangle &= \langle P_{q,p}^{-1}\gamma'(1),P_{q,p}^{-1}X(p)\rangle-\langle\gamma'(0),X(q)\rangle\\ &= \langle\gamma'(1),X(\gamma(1))\rangle-\langle\gamma'(0),X(\gamma(0))\rangle\\ &= (1-0)(\varphi(1)-\varphi(0)).\end{aligned}$$

Since $\varphi_{(X,\gamma)}$ is increasing, we have

$$\langle\gamma'(0),P_{q,p}^{-1}X(p)-X(q)\rangle=\varphi_{(X,\gamma)}(1)-\varphi_{(X,\gamma)}(0)\geqslant 0.$$

Thus, X is monotone. □

The equivalence in Proposition 4.1 will free us from some algebraic manipulations which are necessary when Definition 1 is used. Besides, it makes explicit the geometric meaning in Definition 1. The proof of the following proposition, see [24, 26] and [19], will exemplify this point.

PROPOSITION 4.2. *The function* $f\colon M\to\mathbb{R}$ *is convex if and only if its gradient field,* grad f, *is monotone.*

Proof. Consider the function $\psi=f\circ\gamma\colon\mathbb{R}\to\mathbb{R}$, where γ is a geodesic of M. Then ψ is convex if and only if ψ' is monotone. By the chain rule of differentiation and (4.5), $\psi'=\langle\gamma',\operatorname{grad} f\circ\gamma\rangle=\varphi_{(\operatorname{grad} f,\gamma)}$. Thus, $\psi=f\circ\gamma$ is convex if and only if $\varphi_{(\operatorname{grad} f,\gamma)}$ is monotone. Therefore, by Proposition 4.1, f is convex if and only if grad f is monotone. □

Proposition 4.2 provides a class of monotone vector fields, namely, those which are gradients of convex functions. For more relations between different kinds of generalized convex functions and generalized monotone vector fields on Riemannian manifolds see [12]. For examples of monotone vector field which are not of gradient type, see [14–16].

5. Variational Inequalities on Hadamard Manifolds

In this section we recall some results of [17], that are an important first step in developing algorithms to solve variational inequalities on Hadamard manifolds.

Let X be a vector field on $C \subset H$. Then, the problem

$$\text{find } p \in C \quad \text{such that} \quad \langle X(p), \exp_p^{-1} q \rangle \geqslant 0 \quad \text{for all } q \in C. \tag{5.1}$$

is called a variational inequality on C.

THEOREM 5.1. *If X is continuous and C is compact, then the variational inequality* (5.1) *has a solution.*

Proof. See [17]. □

COROLLARY 5.1. *Let p be a solution of* (5.1) *and suppose that $p \in \mathrm{int}(C)$, the interior of C. Then $X(p) = 0$.*

Proof. See [17]. □

The following theorem gives a necessary and sufficient condition for the existence of solutions. Given a point $o \in H$, we set $C_R = C \cap \Sigma_R$, where Σ_R is the closed geodesic ball of radius R and center o. Returning to our X, we notice that, by the previous theorem, there exist at least one

$$p_R \in C_R : \langle X(p_R), \exp_{p_R}^{-1} q \rangle \geqslant 0 \quad \text{for } q \in C_R, \tag{5.2}$$

whenever $C_R \neq \emptyset$.

THEOREM 5.2. *Suppose that X is a continuous vector field on C. A necessary and sufficient condition so that there should be a solution to* (5.1), *is that there should exist an $R > 0$ so that a solution $p_R \in C_R$ of* (5.2) *satisfies*

$$d(o, p_R) < R. \tag{5.3}$$

Proof. See [17]. □

From this theorem, we may deduce a sufficient condition for existence. This condition is useful and introduces the notion of coerciveness.

COROLLARY 5.2. *Suppose that X is a continuous vector field on C. Let the vector field X on C satisfy*

$$\frac{\langle X(p), \exp_p^{-1} p_0 \rangle + \langle X(p_0), \exp_{p_0}^{-1} p \rangle}{d(p_0, p)} \to -\infty \quad \text{as } d(o,p) \to +\infty, \; p \in C, \tag{5.4}$$

for some $p_0 \in C$. Then, there exists a solution to (5.1).

Proof. See [17]. □

Generally, the solution of the variational inequality is not unique. There is, however, a natural condition which insures uniqueness. Suppose that $p, p' \in C$ are two distinct solutions to (5.1). Then,

$$p \in C: \langle X(p), \exp_p^{-1} q\rangle \geqslant 0, \quad q \in C,$$
$$p' \in C: \langle X(p'), \exp_{p'}^{-1} q\rangle \geqslant 0, \quad q \in C,$$

so, setting $q = p'$ in the first inequality, $q = p$ in the second, and adding the two we, obtain

$$\langle X(p), \exp_p^{-1} p'\rangle + \langle X(p'), \exp_{p'}^{-1} p\rangle \geqslant 0.$$

Hence, a natural condition for uniqueness is that

$$\langle X(p), \exp_p^{-1} q\rangle + \langle X(q), \exp_q^{-1} p\rangle < 0, \tag{5.5}$$

whenever $p, q \in C$, $p \neq q$. Which it is easy to see that it is equivalent to the strict monotonicity of X. Indeed, X strictly monotone implies

$$\begin{aligned}
&\langle X(p), \exp_p^{-1} q\rangle + \langle X(q), \exp_q^{-1} p\rangle \\
&\quad = \langle P_{p,q} X(p), P_{p,q}^{-1} \exp_p^{-1} q\rangle + \langle X(q), \exp_q^{-1} p\rangle \\
&\quad = \langle P_{p,q} X(p), -\exp_q^{-1} p\rangle + \langle X(q), \exp_q^{-1} p\rangle \\
&\quad = -\langle P_{p,q} X(p), \exp_q^{-1} p\rangle - \langle -X(q), \exp_q^{-1} p\rangle \\
&\quad = -\langle P_{p,q} X(p) - X(q), \exp_q^{-1} p\rangle \\
&\quad < 0,
\end{aligned}$$

whenever $p,\ q \in C,\ p \neq q$. The converse implication can be proved similarly.

DEFINITION 2. Condition (5.4) of Corollary 5.2 is a *coerciveness* condition.

EXAMPLE 5.1 (of a coercive vector field). Let $p_0 \in M$. It is easy to verify that vector field X on M defined by $X(p) = -\exp_p^{-1} p_0$ is coercive.

DEFINITION 3. Suppose that X is a vector field on C. X is called *hemi-continuous* if for every geodesic γ: $[0,1] \to C$, and $w \in T_{\gamma(0)} M$ the function $t \mapsto \langle P_\gamma^{-1}(t) X(\gamma(t)), w\rangle$ is continuous, where $P_\gamma^{-1}(t)$ is the parallel transport along γ from $\gamma(t)$ to $\gamma(0)$.

LEMMA 5.1. *Suppose that X is a hemicontinuous, monotone vector field on C. Then, p satisfies*

$$p \in C : \langle X(p), \exp_p^{-1} q \rangle \geqslant 0 \quad \text{for any } q \in C \tag{5.6}$$

if and only if it satisfies

$$p \in C : \langle X(q), \exp_q^{-1} p \rangle \leqslant 0 \quad \text{for any } q \in C. \tag{5.7}$$

Proof. See [17]. □

THEOREM 5.3. *Suppose that X is a continuous, monotone and coercive vector field on C. Then, variational inequality* (5.6) *has a solution and for every $q \in C$ the set* $\exp_q^{-1} S$ *is closed and convex, where S is the solution set of* (5.6).

Proof. See [17]. □

6. Singularities of Monotone Vector Fields

Let be $X \in \mathcal{X}(M)$. We recall that $p \in M$ is called a *singularity* of X if X vanishes at p, i.e., $X(p) = 0$. Using the definition it is easy to prove that strict monotone vector fields has at most one singularity.

6.1. EXISTENCE OF SINGULARITIES

THEOREM 6.1. *If $X \in \mathcal{X}(H)$ is strongly monotone, then X has at least one singularity.*

Proof. The inequality (4.4) implies that

$$\begin{aligned} &\langle X(p), \exp_p{}^{-1} q \rangle + \langle X(q), \exp_q^{-1} p \rangle \\ &\quad = \langle P_{p,q} X(p), P_{p,q}^{-1} \exp_p^{-1} q \rangle + \langle X(q), \exp_q^{-1} p \rangle \\ &\quad = \langle P_{p,q} X(p), -\exp_q^{-1} p \rangle + \langle X(q), \exp_q^{-1} p \rangle \\ &\quad = -\langle P_{p,q} X(p), \exp_q^{-1} p \rangle - \langle -X(q), \exp_q^{-1} p \rangle \\ &\quad = -\langle P_{p,q} X(p) - X(q), \exp_q^{-1} p \rangle \\ &\quad \leqslant -\lambda d^2(p,q). \end{aligned}$$

Therefore

$$\frac{\langle X(p), \exp_p^{-1} q \rangle + \langle X(q), \exp_q^{-1} p \rangle}{d(p,q)} \leqslant -\lambda d(p,q).$$

Now by using Corollary 5.1 and Corollary 5.2 with $C = H$ it follows that X has singularities in H. □

6.2. CONVEXITY OF THE SINGULARITY SET

It is known that the set of singularities of a monotone map A: $G \to \mathbb{R}^n$, where G is an open convex subset of $\mathbb{R}^n$, is convex, see [28] Theorem 32.C.(b). We generalize this result for Riemannian manifolds.

THEOREM 6.2. *Let (M, g) be a Riemannian manifold, G an open convex set of M and X a (smooth!) monotone vector field on G. Then the set of singularities of X is convex.*

Proof. If X has at most one fix point we have nothing to prove. Therefore suppose that X has at least two distinct singularities. Let p, $q \in M$; $p \neq q$ be two arbitrary singularities of X and γ: $[0, l] \to M$ be a unit speed geodesic arc beginning at p ($\gamma(0) = p$) and ending at q ($\gamma(1) = q$). We must prove that for all $s \in [0, l]$ $\gamma(s)$ is a singularity of X. Suppose that $X(\gamma(s)) \neq 0$. The monotonicity of X implies that

$$\langle X(\gamma(s)), \dot{\gamma}(s) \rangle = 0. \tag{6.1}$$

Let ϕ_t: $M \to M$ be the one parameter transformation group generated by X. Then, by [15] Theorem 3.8(i), ϕ_t is nonexpansive for $t < 0$. Since G is open $\gamma(s) \neq \phi_t(\gamma(s))$ and $\phi_t(\gamma(s)) \in G$ for $t < 0$ sufficiently small. Fix such a t_0. By (6.1) the trajectory $\phi_t(\gamma(s))$ is perpendicular to γ. Hence there is a nondegenerate geodesic triangle of vertices p, q, $\rho_s = \phi_{t_0}(\gamma(s))$. By the triangle inequality we have

$$d(p, \rho_s) + d(q, \rho_s) > d(p, q). \tag{6.2}$$

On the other hand the nonexpansivity of ϕ_{t_0} implies

$$d(p, \rho_s) + d(q, \rho_s) \leqslant d(p, \gamma(s)) + d(q, \gamma(s)) = d(p, q). \tag{6.3}$$

But (6.2) is in contradiction with (6.3). Hence we must have $X(\gamma(s)) = 0$. Thus for all $s \in [0, 1]$ $\gamma(s)$ is a singularity of X. □

Next, we give an example of a non-monotone problem in the usual sense, which becomes monotone in an appropriate metric. Set $\Omega = \{p = (p_1, p_2) \in \mathbb{R}^2 : p_2 > 0\}$. The vector field X: $\Omega \to \mathbb{R}^2$ given by

$$X(p_1, p_2) = (p_2 \sinh(p_1), 1 - \cosh(p_1))$$

is not monotone and has zeroes, namely, $X(0, p_2) = (0, 0)$ for all $p_2 > 0$.

Endowing Ω with the Riemannian metric defined by matrix $G = (g_{ij})$, where

$$g_{11}(p_1, p_2) = g_{22}(p_1, p_2) = \frac{1}{p_2}, \quad g_{12}(p_1, p_2) = g_{21}(p_1, p_2) = 0,$$

we obtain the upper half-plane model of the Hyperbolic space $\mathbb{H}^2$, that is a Hadamard manifold of constant curvature $K=-1$. The Christoffel Symbols of $\mathbb{H}^2$ are given by

$$\Gamma_{11}^1=\Gamma_{12}^2=\Gamma_{21}^2=\Gamma_{22}^1=0, \qquad \Gamma_{12}^1=\Gamma_{21}^1=\Gamma_{22}^2=-1/p_2 \quad \text{and} \quad \Gamma_{11}^2=1/p_2,$$

see [3]. Then, if $Y(p_1,p_2)=(a(p_1,p_2),b(p_1,p_2))$, it holds that

$$A_Y(p_1,p_2)=\frac{1}{p_2^2}\begin{pmatrix} \frac{\partial a}{\partial p_1}-\frac{1}{p_2}b & \frac{\partial a}{\partial p_2}-\frac{1}{p_2}a \\ \frac{\partial b}{\partial p_1}+\frac{1}{p_2}a & \frac{\partial b}{\partial p_2}-\frac{1}{p_2}b \end{pmatrix}$$

where $A_Y(p_1,p_2)Z=\nabla_Z Y$. Then,

$$A_X(p_1,p_2)=\begin{pmatrix} p_2\cos\text{h}(p_1)+\frac{1}{p_2}\left(\cos\text{h}(p_1)-1\right) & 0 \\ 0 & \frac{1}{p_2}\left(\cos\text{h}(p_1)-1\right) \end{pmatrix},$$

which is a diagonal positive semidefinite matrix. Then X is monotone in $\mathbb{H}^2$, see [16]. Henceforth, the problem

EXAMPLE 6.1.

$$\text{find } p\in\Omega \text{ such that } (p_2\sin\text{h}(p_1),\ 1-\cos\text{h}(p_1))=0.$$

is non-monotone in Ω and monotone in $\mathbb{H}^2$.

Note that the zeros of the monotone vector field X form a convex set with respect to the hyperbolic metric.

7. Extragradient Method

The problem from Example 6.1 can not be solved by the usual Extragradient algorithm, because the vector field is not monotone. In this section we extend the extragradient-method so that it can be applied to the problem of finding singularities of monotone vector fields in constant curvature Hadamard manifolds, in particular it solves the problem from Example 6.1, if there exist one.

We will prove that the sequence generated by our algorithm converges to a singularity of the vector field.

7.1. STATEMENT OF THE ALGORITHM

We study the following algorithm for finding singularities of a continuous monotone vector field X defined on a constant curvature Hadamard manifold H. The algorithm requires exogenous parameters δ, β^- and β^+ such that $0<\delta<1$, $0<\beta^-\leqslant\beta^+$, and an exogenous sequence $\{\beta_k\}$, satisfying $\beta^-\leqslant\beta_k\leqslant\beta^+$.

The Algorithm

1. Initialization. $p_0 \in H$.
2. Iterative step.
   (a) Stopping criterion. If $X(p_k) = 0$ then STOP, otherwise continue.
   (b) Selection of q_k and t_k
   Define the geodesic

$$\gamma_k(t) = \exp_{p_k} -tX(p_k). \tag{7.1}$$

and the function

$$\varphi_k(t) = \langle \gamma_k'(t), X(\gamma_k(t)) \rangle. \tag{7.2}$$

Compute

$$l_k = \min\{l \geqslant 0 : \varphi_k(2^{-l}\beta_k) \leqslant -\delta \|X(p_k)\|^2\}. \tag{7.3}$$

Set

$$t_k = 2^{-l_k}\beta_k. \tag{7.4}$$

and

$$q_k = \gamma_k(t_k). \tag{7.5}$$

   (c) Definition of p_{k+1}
   Define

$$L_k = \{p \in H : \langle \exp_{q_k}^{-1} p, X(q_k) \rangle \leqslant 0\}. \tag{7.6}$$

and

$$p_{k+1} = \prod\nolimits_{L_k}(p_k). \tag{7.7}$$

7.2. CONVERGENCE ANALYSIS

We start by establishing that our algorithm is well defined.

LEMMA 7.1. *If $X(p_k) \neq 0$ then t_k is well defined, i.e., the Armijo search for t_k in* (7.3) *is finite.*

Proof. Since X is monotone and continuous, and the parallel transport is an isometry, there exits $\tau > 0$ such that

$$\varphi_k(t) = \langle \gamma_k'(t), X(\gamma_k(t)) \rangle \leqslant -\delta \|X(p_k)\|^2, \quad \forall t \in (0, \tau), \tag{7.8}$$

because $\gamma_k'(t) = -P_{p_k \gamma_k(t)} X(p_k)$. Take l_k such that $\beta_k 2^{-l_k} \in (0, \tau)$. Then $\gamma_k(\beta_k 2^{-l_k})$ satisfies (7.8). □

In a complete metric space (M, d), the sequence $\{p_k\} \subset M$ is said to be *Féjer convergent* to the nonempty set $U \subset M$ when

$$d(p_{k+1}, y) \leqslant d(p_k, y). \tag{7.9}$$

for all $y \in U$ and $k \geqslant 0$.

LEMMA 7.2. *In a complete metric space, (M, d) if $\{p_k\} \subset M$ is Féjer convergent to a nonempty set $U \subset M$, then $\{p_k\}$ is bounded. If furthermore a cluster point p of $\{p_k\}$ belongs to U then* $\lim_{k\to+\infty} p_k = p$.

Proof. Take $p \in U$. Inequality (7.9) implies $d(p_k, p) \leqslant d(p_0, p)$ for all k. Therefore $\{p_k\}$ is bounded. Take a subsequence $\{p_{k_j}\}$ of $\{p_k\}$ such that $\lim_{k\to+\infty} p_{k_j} = p$. By (7.9), the sequence of positive numbers $\{d(p_k, p)\}$ is decreasing and it has a subsequence, namely $\{d(p_{k_j}, p)\}$, which converges to 0. Thus, the whole sequence converges to 0, i.e., $\lim_{k\to+\infty} d(p_k, p) = 0$, implying $\lim_{k\to+\infty} p_k = p$. □

From now on $S = \{p \in H : X(p) = 0\}$ is the set of singularities of X. Next we will prove that, if S is nonempty, then the sequence $\{p_k\}$ generated by the procedure (7.1)–(7.7) is Féjer convergent to S.

LEMMA 7.3. *Suppose that the Algorithm doesn't stop at iteration k. If $S \neq \emptyset$, then for all $p_* \in S$*

$$d(p_{k+1}, p_*) < d(p_k, p_*). \tag{7.10}$$

Proof. Define

$$L_k = \{p \in H : \langle \exp_{p_k}^{-1} p, X(p_k)\rangle \leqslant 0\}. \tag{7.11}$$

Since H is a constant curvature Hadamard manifold, by Corollary 3.2, L_k is convex. If p_* is a singularity of X then, by (4.3), $\langle \exp_{p_k}^{-1} p_*, X(p_k)\rangle \leqslant 0$, i.e., $p_* \in L_k$. Observe that $\partial L_k = \{p \in H : \langle \exp_{p_k}^{-1} p, X(p_k)\rangle = 0\}$ is the boundary of L_k and if $p \notin L_k$ then $\Pi_{L_k}(p) \in \partial L_k$.

If the Algorithm doesn't stop at iteration k, then $p_k \notin L_k$, $p_{k+1} = \prod_{L_k}(p_k)$ and $d(p_{k+1}, p_k) \neq 0$. Take $p_* \in S$. By Proposition 3.1,

$$\langle \exp_{p_{k+1}}^{-1} p_*, \exp_{p_{k+1}}^{-1} p_k\rangle \leqslant 0. \tag{7.12}$$

Consider the geodesic triangle $\Delta(p_k p_{k+1} p_*)$. By Theorem 2.1

$$d^2(p_k, p_*) \geqslant d^2(p_{k+1}, p_*) + d^2(p_{k+1}, p_k) - \\ -2d(p_{k+1}, p_*) d(p_{k+1}, p_k) \cos(\theta), \tag{7.13}$$

where $\theta = \measuredangle(\exp_{p_{k+1}}^{-1} p_*, \exp_{p_{k+1}}^{-1} p_k)$. The inequality (7.13) implies that

$$d^2(p_k, p_*) \geqslant d^2(p_{k+1}, p_*) + d^2(p_{k+1}, p_k) - \langle \exp_{p_{k+1}}^{-1} p_*, \exp_{p_{k+1}}^{-1} p_k\rangle. \tag{7.14}$$

It follows from (7.12) and (7.14) that

$$d^2(p_k,p_*)\geqslant d^2(p_{k+1},p_*)+d^2(p_{k+1},p_k).$$

Since $d(p_{k+1},p_k)\neq 0$, we get the statement of the lemma. □

We present next the main convergence result on our algorithm.

THEOREM 7.1. *Suppose that* $S\neq\emptyset$. *Then, either the sequence* $\{p_k\}$ *generated by procedure* (7.1)–(7.7) *stops at some iteration* k, *in which case* $p_k\in S$, *or* $\{p_k\}$ *converges, and its limit belongs to* S.

Proof. It follows from (7.2)–(7.5) that

$$\langle P_{p_k,q_k}(-X(p_k)),X(q_k)\rangle\leqslant-\delta\|X(p_k)\|^2.$$

Multiplying this inequality by t_k, we get

$$\langle P_{p_k,q_k}(-t_kX(p_k)),X(q_k)\rangle\leqslant-\delta\|t_kX(p_k)\|\|X(p_k)\|.$$

Then, by (7.1) and (7.5),

$$\langle P_{p_k,q_k}(-t_kX(p_k)),X(q_k)\rangle\leqslant-\delta d(p_k,q_k)\|X(p_k)\|. \tag{7.15}$$

Since $\{p_k\}$ is Féjer convergent to $S\neq\emptyset$ and X is continuous, by using the definition of q_k it follows that $\{p_k\}$, $\{X(p_k)\}$, $\{q_k\}$ and $\{X(q_k)\}$ are bounded. Take a convergent subsequence $\{p_{k_j}\}$ of $\{p_k\}$ such that $\lim_{j\to\infty}p_{k_j}=\hat{p}$ and set $\hat{q}=\lim_{j\to\infty}q_{k_j}$. In the proof of Lemma 7.2, we proved that $d^2(p_{k+1},p_k)\leqslant d^2(p_k,p_*)-d^2(p_{k+1},p_*)$, for all $p_*\in S$. Then, $\lim_{j\to\infty}p_{k_j}=\hat{p}$ implies that $\lim_{j\to\infty}p_{k_j+1}=\hat{p}$. By (7.15),

$$\langle P_{p_{k_j},q_{k_j}}(-t_{k_j}X(p_{k_j})),X(q_{k_j})\rangle\leqslant-\delta d(p_{k_j},q_{k_j})\|X(p_{k_j})\|. \tag{7.16}$$

By (7.1) and (7.5)

$$\langle P_{p_{k_j},q_{k_j}}(-t_{k_j}X(p_{k_j})),X(q_{k_j})\rangle=\langle \exp^{-1}_{q_{k_j}}p_{k_j},X(q_{k_j})\rangle. \tag{7.17}$$

By definition of p_{k_j+1}

$$\langle \exp^{-1}_{q_{k_j}}p_{k_j+1},X(q_{k_j})\rangle=0. \tag{7.18}$$

Since $\lim_{j\to\infty}p_{k_j}=\hat{p}$ and $\lim_{j\to\infty}p_{k_j+1}=\hat{p}$, we get, using (7.17) and (7.18),

$$\begin{aligned}\lim_{j\to\infty}\langle P_{p_{k_j},q_{k_j}}(-t_{k_j}X(p_{k_j})),X(q_{k_j})\rangle &= \lim_{j\to\infty}\langle \exp^{-1}_{q_{k_j}}p_{k_j},X(q_{k_j})\rangle\\ &= \lim_{j\to\infty}\langle \exp^{-1}_{q_{k_j}}p_{k_j+1},X(q_{k_j})\rangle\\ &= 0.\end{aligned}$$

Then, (7.16) implies that

$$\lim_{j\to\infty} d(p_{k_j}, q_{k_j}) = d(\hat{p}, \hat{q}) = 0. \tag{7.19}$$

or

$$\lim_{j\to\infty} \|X(p_{k_j})\| = \|X(\hat{p})\| = 0. \tag{7.20}$$

Suppose that (7.19) holds. Then, is also holds that

$$\lim_{j\to\infty} t_{k_j} \|X(p_{k_j})\| = \lim_{j\to\infty} d(p_{k_j}, q_{k_j}) = d(\hat{p}, \hat{q}) = 0.$$

We consider again two cases. Either (7.20) holds or $\lim_{j\to\infty} t_{k_j} = 0$. Suppose that $\lim_{j\to\infty} t_{k_j} = 0$. Then, there exits $\tilde{j}$ such that $l_{k_j} > 2$, for all $j \geqslant \tilde{j}$.
Define the sequence $\{\tilde{q_{k_j}}\}$ as $\tilde{q_{k_j}} = \exp_{p_{k_j}}(\gamma_{k_j}(2t_{k_j}))$. By (7.3),

$$\varphi_{k_j}(2t_{k_j}) > -\delta \|X(p_{k_j})\|^2. \tag{7.21}$$

Taking limits in (7.21) as j goes to ∞, we get $-\|X(\hat{p})\| \geqslant -\delta \|X(\hat{p})\|$. Since $0 < \delta < 1$, $\|X(\hat{p})\| = 0$, i.e., $\hat{p} \in S$, and by Lemma 7.2 $\lim_{k\to\infty} p_k = \hat{p}$. □

7.3. EXAMPLE

Let $M = \mathbb{H}^n$ where $\mathbb{H}^n$ is the n dimensional hyperbolic space of constant sectional curvature $K = -1$. Consider the following model for $\mathbb{H}^n$:

$$\mathbb{H}^n = \{\xi = (\xi^1, \ldots, \xi^n, \xi^{n+1}) \in \mathbb{R}^{n+1} : \xi^{n+1} > 0 \text{ and } \{\xi, \xi\} = -1\},$$

where for the vectors $\xi = (\xi^1, \ldots, \xi^{n+1})$, $\eta = (\eta^1, \ldots, \eta^{n+1}) \in \mathbb{R}^{n+1}$, $\{\xi, \eta\} = \xi^1\eta^1 + \cdots + \xi^n\eta^n - \xi^{n+1}\eta^{n+1}$. The metric of $\mathbb{H}^n$ is induced from the Lorentz metric $\{.,.\}$ of $\mathbb{R}^{n+1}$ and it will be denoted by the same symbol. Then a normalized geodesic γ_x of $\mathbb{H}^n$ starting from $x(\gamma_x(0) = x)$ will have the equation

$$\gamma_x(t) = (\cos\mathrm{h}t)x + (\sin\mathrm{h}t)v, \tag{7.22}$$

where $v = \dot{\gamma}_x(0) \in T_x\mathbb{H}^n$ is the tangent unit vector of γ in the starting point. We also have

$$\{u, x\} = 0, \tag{7.23}$$

for all $u \in T_x\mathbb{H}^n$. Equation (7.22) implies

$$\exp tv = (\cos\mathrm{h}t)x + (\sin\mathrm{h}t)v, \tag{7.24}$$

for any unit vector v and

$$\exp_x^{-1} y = \operatorname{arccosh}(-\{x,y\}) \frac{y+\{x,y\}x}{\sqrt{\{x,y\}^2-1}}, \tag{7.25}$$

for all $x, y \in \mathbb{H}^n$ and $v \in T_x\mathbb{H}^n$. Let X be a monotone vector field on $\mathbb{H}^n$,

$$L_x = \{y \in \mathbb{H}^n : \{X(x), \exp_x^{-1} y\} \leqslant 0\}$$

and

$$M_x = \{y \in \mathbb{H}^n : \{X(x), \exp_x^{-1} y\} = 0\}.$$

Then equations (7.25) and (7.23) imply that

$$L_x = \{y \in \mathbb{H}^n : \{X(x), y\} \leqslant 0\} \tag{7.26}$$

and

$$M_x = \{y \in \mathbb{H}^n : \{X(x), y\} = 0\}. \tag{7.27}$$

Let $z \in \mathbb{H}^n$. Then, $y = \Pi_{L_x} z$ iff $y \in M_x$ and

$$\{\exp_y^{-1} z, f\} = 0, \tag{7.28}$$

for all unit vectors $f \in T_y M_x$. If $f \in T_y \mathbb{H}^n$ is a unit vector then $f \in T_y M_x$ iff $f = \dot{\gamma}_y(0)$ where γ_y is a normalized geodesic in M_x starting from y, i.e., $\gamma_y(t) = (\cos h t) y + (\sin h t) f$ and $\{X(x), \gamma_y(t)\} = 0$. But (7.27) implies that $\{X(x), \gamma_y(t)\} = 0$ iff $\{X(x), f\} = 0$. Hence $f \in T_y M_x$ iff

$$\{X(x), f\} = 0. \tag{7.29}$$

On the other hand (7.25) implies that (7.28) is equivalent to

$$\{z, f\} = 0. \tag{7.30}$$

Since $y \in M_x$ and $f \in T_y \mathbb{H}^n$ we have

$$\{X(x), y\} = 0. \tag{7.31}$$

and

$$\{y, f\} = 0. \tag{7.32}$$

Now consider the case where $n = 2$. For every $a \in \mathbb{R}^3$ denote by a_1, a_2, a_3 the components of a. Hence $a = (a_1, a_2, a_3)$. Let $\tilde{a} = (a_1, a_2, -a_3)$. It is a straightforward computation to check that

$$\widetilde{\tilde{a} \times \tilde{b}} = -a \times b, \tag{7.33}$$

where '$\times$' is the vector product of $\mathbb{R}^3$. Denote by $\langle \cdot, \cdot \rangle$ the canonical scalar product of $\mathbb{R}^3$. It is easy to see that

$$\{a, b\} = \langle a, \tilde{b} \rangle. \tag{7.34}$$

for all $a, b \in \mathbb{R}^3$. By using (7.34), (7.29) and (7.30) it is easy to see that

$$f = \alpha_1 \tilde{z} \times \widetilde{X(x)}, \tag{7.35}$$

for some $\alpha_1 \in \mathbb{R}$. Similarly by using (7.34), (7.31) and (7.32)

$$y = \alpha_2 \tilde{f} \times \widetilde{X(x)}, \tag{7.36}$$

for some $\alpha_2 \in \mathbb{R}$. Equations (7.35), (7.36) and (7.33) imply that

$$y = \alpha (X(x) \times z) \times \widetilde{X(x)}, \tag{7.37}$$

for some $\alpha \in \mathbb{R}$. Hence by using the Gibbs formula and (7.34)

$$y = \alpha(\{X(x), X(x)\} z - \{z, X(x)\} X(x)),$$

or equivalently

$$\prod\nolimits_{L_x} z = \alpha(\{X(x), X(x)\} z - \{z, X(x)\} X(x)), \tag{7.38}$$

for some $\alpha \in \mathbb{R}$. α can be determined by the condition $\{y, y\} = -1$. Thus

$$\alpha = \pm \frac{1}{\sqrt{\{X(x), X(x)\}(1 + \{z, X(x)\}^2)}}, \tag{7.39}$$

where the sign $\pm$ is chosen so that $y^{n+1} > 0$. By using the notations of Section 7 and (7.38) we have that

$$p_{k+1} = \alpha_k(\{X(q_k), X(q_k)\} p_k - \{p_k, X(q_k)\} X(q_k)),$$

where by (7.39)

$$\alpha_k = \pm \frac{1}{\sqrt{\{X(q_k), X(q_k)\}(1 + \{p_k, X(q_k)\}^2)}},$$

by (7.24)

$$\begin{aligned} q_k &= (\cos\mathrm{h} t_k) p_k - (\sin\mathrm{h} t_k) X(p_k), \\ t_k &= 2^{-l_k} \beta_k, \\ l_k &= \min\{l \geqslant 0 : \varphi_k(2^{-l} \beta_k) \leqslant -\delta \{X(p_k), X(p_k)\}\}, \end{aligned}$$

and

$$\varphi_k(t) = \{(\sin \mathrm{h} t)p_k - (\cos \mathrm{h} t)X(p_k), X((\cos \mathrm{h} t)p_k - (\sin \mathrm{h} t)X(p_k))\},$$

where the sign $\pm$ is chosen so that $p_{k+1}^{n+1} > 0$. If X has fixed points than $(p_k)_{k\in\mathbb{N}}$ converges to a singularity of f. In [17] it is proved that the vector field $V(x) = (-x_1x_3, -x_2x_3, 1-x_3^2)$ is a monotone vector field on $\mathbb{H}^2$. The only singularity of V is (0,0,1). Hence for $X = V$ the extragradient method converges to (0,0,1).

8. Final Remarks

In [2], [8–11] and [22] extragradient-type algorithms are proposed for solving monotone variational inequality problems in $\mathbb{R}^n$. Our algorithm can be used for solving any constrained problem in $\mathbb{R}^n$ with monotone vector field (in the sense defined in Section 4), having a constant curvature Hadamard manifold as constraint set. It extends the classical extragradient-type algorithms cited above, since an Euclidean space $\mathbb{R}^n$ is a constant curvature Hadamard manifold. We remark that we use some techniques in the definition and convergence proof of our algorithm which impose restrictions on the manifold, namely non-positive constant curvature. The same proofs can be given for an arbitrary two dimensional Hadamard manifold. However, finding all the manifolds on which our method works remains an open problem. More precisely, it is desirable to determine in which manifolds, other than constant curvature (or two dimensional) Hadamard ones, it is possible to define an extragradient method and establish its convergence to a singularity of the vector field. Also, it remains as open problem the formulation of an extragradient-type algorithm to solve variational inequalities as defined in (5.1).

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