This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the ICC 2008 proceedings.

Network Coding for Two-Way Relay Channels Using Lattices

Ihn-Jung Baik and Sae-Young Chung
School of EECS, KAIST, Daejeon, Korea
Email : ihn_jung@kaist.ac.kr, sychung@ee.kaist.ac.kr

***Abstract*—In this paper, we propose a network coding using a lattice for the two-way relay channel with two nodes communicating bidirectionally through a relay, which we call modulo-and-forward (MF). Our scheme extends the network coding in the binary channel to the Gaussian channel case, where XOR in the binary case is replaced by $\mathrm{mod}\Lambda$ for the Gaussian case, where Λ is a high-dimensional lattice whose shaping gain is close to optimal. If the relay node re-transmits the received signal after the $\mathrm{mod}\,\Lambda$ operation, we can reduce the complexity compared to decode-and-forward (DF) and can get a better power efficiency compared to amplify-and-forward (AF). When the transmission powers of two nodes are different, we use superposition coding and partial decoding at the relay node. Finally, we plot and compare the sum rates of three different schemes, i.e., AF, DF, and MF. We show that by applying the proposed scheme, we can get better performance than AF and DF schemes under some conditions.**

I. Introduction

In wireless communication, if the distance between a transmitter and a receiver is large or the channel condition between them is bad, the probability of error increases and the total throughput decreases. To solve this problem, we can place a relay node between the transmitter and receiver, which can perform amplify-and-forward (AF), decoding-and-forward (DF) or compress-and-forward (CF) [6],[7]. In AF scheme, the relay node just amplifies the received signal from the transmitter and retransmits the amplified signal to its destination. Thus the complexity is low but the achievable rate can be affected by the amplification of noise. In DF scheme, the rate can be higher but the decoding complexity is high. After decoding the message, the relay node re-encodes it to transmit to the destination. Furthermore, there can be a performance loss due to the fact that the signals have to be decoded at the relay node, which will be discussed in this paper. To overcome the problems of the two schemes, we propose a new scheme which is doing $\mathrm{mod}\Lambda$ operation at the relay node. In this scheme, we can lower the complexity compared to DF because it only decodes the received signal partially. Also, it is more efficiency compared to AF in terms of power.

Two-way channel [1] was introduced by Shannon in 1961 which is depicted in Fig. 1. In Fig. 1, Nodes a and b are transmitting their signals to each other. To help the transmission of the two nodes, a relay node is located between two nodes, which is called the two-way relay channel (TRC) [10], [2], [4], [9]. In [4], [5], lattice strategies were used to construct DF schemes. Our MF scheme is different from [4],

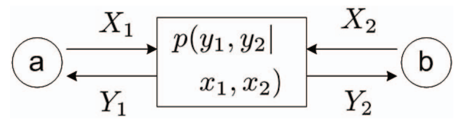


Fig. 1. Two-way channel model

[5] in that no decoding at the relay node for the lattice coded parts is performed. In [11], integer lattices were used for TRC, which was generalized to more general lattices in [12]. This paper generalizes [12] to the case when the transmitters have different powers.

II. Two-way relay channel

A. Two-way relay channel

In the TRC, two nodes exchange their signals with help of a relay node. Fig. 2 depicts the simple process of message passing between Nodes a and b. In this paper, we assume all nodes are half duplex, i.e. they do not transmit and receive simultaneously. Messages of Nodes a and b are delivered to Nodes b and a, respectively, in two phases. In the first phase, Nodes a and b transmit their signals to the relay node at the same time. Because we assume half duplex mode, there is no direct path between Nodes a and b. After receiving the signals, the relay node performs an appropriate operation and broadcasts the resulting signal to Nodes a and b in the second phase. At each node, their own signal is treated as interference but it can be canceled because they know what they sent before.

B. Amplify-and-Forward

This scheme was analyzed in [2] for the TRC. In Fig. 2, the signals transmitted from Nodes a and b are X_a, X_b with power P_a, P_b, respectively and the signal transmitted from the relay node is X_R with power P_R. Without loss of generality, we assume Node a's transmit power P_a is greater than or equal to Node b's transmit power P_b. The received signal of the relay node is Y_R and Y_a and Y_b are received signals of Nodes a and b, respectively. Gaussian noises at Nodes a and b and at the relay are denoted by Z_a, Z_b and Z_R, respectively and power of all noise is set to 1.

In the first phase, the relay node receives the superposition of signals transmitted from Nodes a and b and the Gaussian noise.

$$Y_R = X_a + X_b + Z_R \tag{1}$$

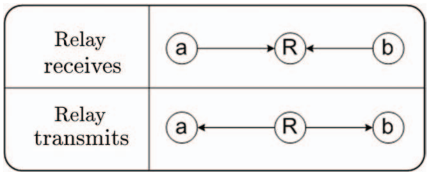


Fig. 2. Two-way relay channel using two phases [10], [2], [4], [9]

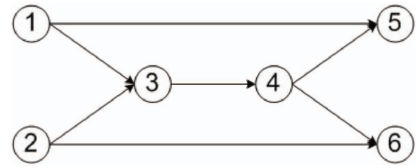


Fig. 3. Network coding model

The relay node amplifies the received signal by multiplying an amplifying factor γ.

$$X_R = \gamma Y_R \tag{2}$$

We can satisfy the relay power constraint by choosing

$$\gamma = \sqrt{\frac{P_R}{P_a + P_b + 1}} \; . \tag{3}$$

In the second phase, the relay node transmits X_R to Nodes a and b. We first focus on the message from Node a to Node b. The received signal of Node b is

$$\begin{aligned} Y_b &= X_R + Z_b \\ &= \gamma Y_R + Z_b \\ &= \gamma(X_a + X_b + Z_R) + Z_b \; . \end{aligned} \tag{4}$$

Since Node b already knows the signal X_b sent by itself in the first phase, X_b can be removed from Y_b. After canceling the known part of the signal, the new signal can be represented as

$$Y_b' = \gamma(X_a + Z_R) + Z_a \; . \tag{5}$$

From the received signal of Node b we can get the achievable rate of the AF scheme, which is

$$R_a = \frac{1}{2}\log_2\left(1 + \frac{P_R P_a}{P_R + P_a + P_b + 1}\right) \; . \tag{6}$$

Similarly, the achievable rate from Node b to Node a is

$$R_b = \frac{1}{2}\log_2\left(1 + \frac{P_R P_b}{P_R + P_a + P_b + 1}\right) \; . \tag{7}$$

C. Decode-and-Forward

In this section, we introduce the DF scheme for the two-way relay channel. In the first phase, Nodes a and b transmit their signals to the relay node simultaneously thus it is a multiple access channel (MAC) and we can get the achievable rate region for the Gaussian case as follows assuming Gaussian signaling.

$$R_a \le I(X_a; Y|X_b) = \frac{1}{2}\log(1 + P_a) \tag{8}$$

$$R_b \le I(X_b; Y|X_a) = \frac{1}{2}\log(1 + P_b) \tag{9}$$

$$R_a + R_b \le I(X_a, X_b; Y) = \frac{1}{2}\log(1 + P_a + P_b) \tag{10}$$

The relay node decodes two signals separately and re-encodes them. The rate of each node can be different because of the difference in the transmit power. Thus we split the message of Node a having a higher transmit power than Node b into two parts, one having the same rate as the rate of Node b and the other having the rest similarly as in [4]. The first part of the message of Node a and the message of Node b are XORed and encoded using a Gaussian codebook and then the second part of the message of Node a is added to the resulting signal. In the second phase, the relay node transmits the re-encoded signal to Nodes a and b and this is a broadcast channel with a degraded message set. Since the channels from the relay to Nodes a and b have the same noise variance, we can get the following achievable rate region for the channel.

$$R_a \le \frac{1}{2}\log(1 + P_R) \tag{11}$$

$$R_b \le \frac{1}{2}\log(1 + P_R) \tag{12}$$

The achievable rates should be supported by both phases, so the achievable rate region for the end-to-end transmission is

$$R_a \le \min\left\{\frac{1}{2}\log(1 + P_a), \frac{1}{2}\log(1 + P_R)\right\} \tag{13}$$

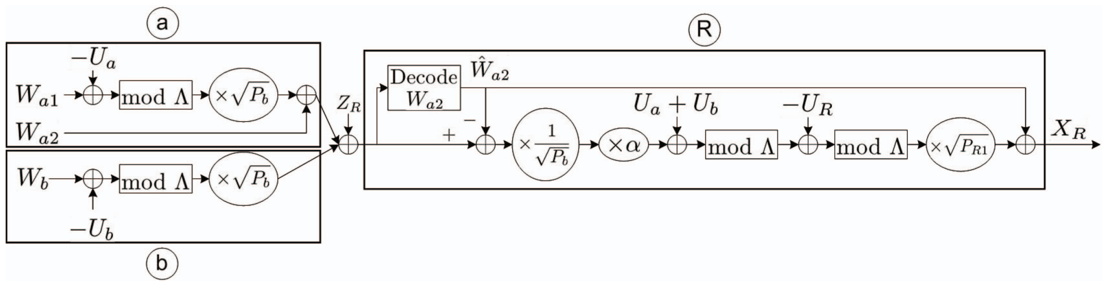
$$R_b \le \min\left\{\frac{1}{2}\log(1 + P_b), \frac{1}{2}\log(1 + P_R)\right\} \tag{14}$$

$$R_a + R_b \le \frac{1}{2}\log(1 + P_a + P_b). \tag{15}$$

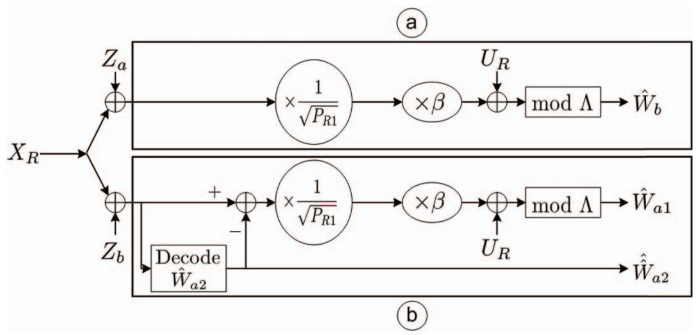
D. Modulo-and-Forward

In this section, we explain our proposed scheme in detail. This scheme is a generalization of the network coding example [8] for binary channel to the Gaussian channel. Fig. 3 shows an example of the binary network coding in [8]. Nodes 1 and 2 intend to send their signals X_1 and X_2 to Nodes 6 and 5, respectively. At Node 3, the signals from Nodes 1 and 2 are added by $\bigoplus$(XOR) operation and then the added signal is transmitted from Node 4 to Nodes 5 and 6. Node 5 already knows X_1 because of the direct path between Nodes 1 and 5. Also, Node 6 knows the signal from Node 2. To decode the intended signal X_1 at Node 6, again use $\bigoplus$ operation. That is, by $X_2 \bigoplus (X_1 \bigoplus X_2)$ operation, we can decode X_1. Also, Node 5 can decode X_2 similarly. We explain how this binary network coding using XOR operation is extended to the Gaussian case.

In our scheme, we use the lattice scheme in [3] as a building block, which achieves the Gaussian channel capacity asymptotically. The end-to-end message passing is depicted in Fig. 4. We assume that $P_a \ge P_b$ as before and use superposition coding at Node a as was done in [4]. As seen in Fig. 4, at Node a, a fraction of power equal to P_b is assigned to message W_{a1} and the rest of power $P_a - P_b$ is assigned to a coded message W_{a2} using a Gaussian codebook. Both



(a) First phase



(b) Second phase

Fig. 4. Channel model for proposed scheme

messages W_{a1} and W_b are lattice coded using lattice Λ [3] and multiplied by a power factor $\sqrt{P_b}$, where the power (second moment) of Λ, i.e., the power of a signal uniform over the fundamental Voronoi region of Λ, is assumed to be 1. Also, we assume that the messsages are uniformly distributed in the fundamental Voronoi region of Λ. At Nodes a and b, random dithers U_a and U_b are subtracted from the messages. The dithers are uniformly distributed in a Voronoi region of Λ, which are assumed to be known by both the transmitter and receiver using a pseudo random number generator. These dithers guarantee that the transmitted signals X_{a1} and X_b are independent from the messages W_{a1} and W_b, respectively. After dithering, the resulting signals pass through $\bmod \Lambda$. We assume a high-dimensional lattice whose shaping gain is close to optimal. This can minimize the power of transmitted signals X_a and X_b. In the first phase, Nodes a and b transmit their signals X_a and X_b to the relay node and the relay node receives signal Y_R, which is summarized below.

$$\begin{aligned}
X_a &= X_{a1} + W_{a2} = \sqrt{P_b}\left([W_{a1} - U_a] \bmod \Lambda\right) + W_{a2} \\
X_b &= \sqrt{P_b}\left([W_b - U_b] \bmod \Lambda\right) \\
Y_R &= X_a + X_b + Z_R
\end{aligned}$$

We first consider the message passing from Node a to Node b. At the relay node, W_{a2} is first decoded as $\hat{W}_{a2}$ and subtracted from the received signal Y_R and we get $Y'_R = X_{a1} + X_b + Z_R$. With a high probability $\hat{W}_{a2}$ will be equal to W_{a2} if the encoding rate of W_{a2} is below its capacity. We multiply the resulting signal by a minimum mean square error (MMSE) factor α that minimizes the effective noise [3] and the sum of the two dithers $U_a + U_b$ are added. Let K be the signal after the first $\bmod \Lambda$ operation at the relay node.

$$\begin{aligned}
K &= \left[\frac{\alpha}{\sqrt{P_b}} Y'_R + U_a + U_b\right] \bmod \Lambda \\
&= \left[W_{a1} - W_{a1} + \frac{\alpha}{\sqrt{P_b}} Y'_R + U_a + U_b\right] \bmod \Lambda \\
&= \left[W_{a1} + \left[-\frac{X_{a1}}{\sqrt{P_b}} - U_a\right] \bmod \Lambda + \frac{\alpha}{\sqrt{P_b}} Y'_R + U_a + U_b\right] \\
&\quad \bmod \Lambda \\
&= \left[W_{a1} - \frac{X_{a1}}{\sqrt{P_b}} - U_a + \frac{\alpha}{\sqrt{P_b}} Y'_R + U_a + U_b\right] \bmod \Lambda \\
&= \left[W_{a1} + \frac{X_{a1}}{\sqrt{P_b}}(\alpha - 1) + \frac{\alpha}{\sqrt{P_b}} Z_R + \frac{\alpha}{\sqrt{P_b}} X_b + U_b\right] \\
&\quad \bmod \Lambda
\end{aligned} \tag{16}$$

The third equality is satisfied by using the following equality.

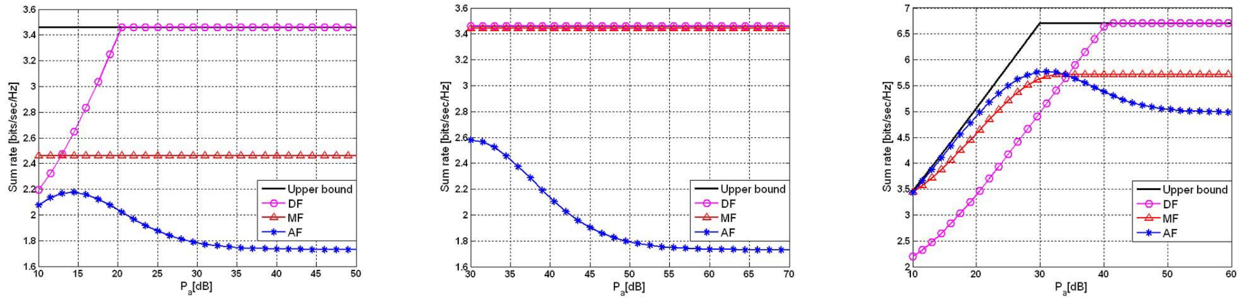
$$X_{a1} = \sqrt{P_b}\left([W_{a1} - U_a] \bmod \Lambda\right) \tag{17}$$

A random dither U_R is subtracted from K followed by $\bmod \Lambda$ and then scaled by $\sqrt{P_{R1}}$, which is denoted by X_{R1}. The decoded message $\hat{W}_{a2}$ is added to X_{R1} to generate X_R, i.e.,

$$\begin{aligned}
X_R &= X_{R1} + \hat{W}_{a2} \\
&= \sqrt{P_{R1}}\left([K - U_R] \bmod \Lambda\right) + \hat{W}_{a2}
\end{aligned} \tag{18}$$

Note that the relay's power P_R is split into P_{R1} and P_{R2}, the powers of X_{R1} and $\hat{W}_{a2}$, respectively. At Node b, $\hat{W}_{a2}$ is decoded first and subtracted from the received signal. The resulting signal is

$$Y'_b = X_{R1} + Z_b \tag{19}$$



(a) $P_b = 10dB,\ P_R = 10dB$ (b) $P_b = 30dB,\ P_R = 10dB$ (c) $P_b = 10dB,\ P_R = 30dB$

Fig. 5. Comparison of the sum rates

This is normalized, i.e., multiplied by $\frac{1}{\sqrt{P_{R1}}}$, then the MMSE factor β is multiplied and the dither U_R is removed. The resulting signal is given by

$$\begin{aligned}\hat{W}_{a1} &= \left[\frac{\beta}{\sqrt{P_{R1}}}Y'_b + U_R\right] \bmod \Lambda \\ &= \left[K - K + \frac{\beta}{\sqrt{P_{R1}}}Y'_b + U_R\right] \bmod \Lambda \\ &= \left[K + \left[-\frac{X_{R1}}{\sqrt{P_{R1}}} - U_R\right] \bmod \Lambda + \frac{\beta}{\sqrt{P_{R1}}}Y'_b + U_R\right] \\ &\quad \bmod \Lambda \\ &= \left[K - \frac{X_{R1}}{\sqrt{P_{R1}}} - U_R + \frac{\beta}{\sqrt{P_{R1}}}Y'_b + U_R\right] \bmod \Lambda \\ &= \left[K + \frac{X_{R1}}{\sqrt{P_{R1}}}(\beta - 1) + \frac{\beta}{\sqrt{P_{R1}}}Z_b\right] \bmod \Lambda, \end{aligned} \tag{20}$$

where the derivation is similar to that of (16). Also, Node b already knows X_b and U_b so they can be removed. As a result, we get

$$\hat{W}'_{a1} = [W_{a1} + Z_{eff}] \bmod \Lambda. \tag{21}$$

Except the intended message W_{a1}, all other parameters are treated as the effective noise

$$Z_{eff} = \left[\frac{X_{a1}}{\sqrt{P_b}}(\alpha - 1) + \frac{\alpha}{\sqrt{P_b}}Z_R + \frac{X_{R1}}{\sqrt{P_{R1}}}(\beta - 1) + \frac{\beta}{\sqrt{P_{R1}}}Z_b\right] \bmod \Lambda. \tag{22}$$

We can maximize the total rate by minimizing the power of the effective noise that is a function of α and β.

$$\mathbf{E}[|Z_{eff}|^2] = (\alpha - 1)^2 + \frac{\alpha^2}{P_b} + (\beta - 1)^2 + \frac{\beta^2}{P_{R1}} \tag{23}$$

The optimal α and β are $\frac{P_b}{P_b+1}$ and $\frac{P_{R1}}{P_{R1}+1}$, respectively and the corresponding effective noise power is

$$\mathbf{E}[|Z_{eff}|^2] = \frac{1}{P_b + 1} + \frac{1}{P_{R1} + 1}. \tag{24}$$

If we assume that the shaping gain of Λ tends to $\frac{\pi e}{6}(\sim 1.53\text{dB})$ as n goes to infinity, which is optimal , the achievable rate R_{a1} of message W_{a1} in the limit is

$$\begin{aligned} R_{a1} &= \frac{1}{2}\log_2\left(\frac{1}{\frac{1}{P_b+1} + \frac{1}{P_{R1}+1}}\right) \\ &= \frac{1}{2}\log_2\left(1 + \frac{P_bP_{R1} - 1}{P_b + P_{R1} + 2}\right). \end{aligned} \tag{28}$$

The achievable rate of message W_{a2} is

$$R_{a2} = \min\left\{\frac{1}{2}\log\left(1 + \frac{P_a - P_b}{2P_b + 1}\right), \frac{1}{2}\log\left(1 + \frac{P_R - P_{R1}}{P_{R1} + 1}\right)\right\}. \tag{29}$$

The total achievable rate is given by

$$R_a = R_{a1} + R_{a2}. \tag{30}$$

The achievable rate of the message from Node b to Node a can be obtained similarly as follows

$$R_b = \frac{1}{2}\log_2\left(1 + \frac{P_bP_{R1} - 1}{P_b + P_{R1} + 2}\right). \tag{31}$$

III. Comparison

In this section, we compare AF, DF, and our schemes. First, we compare the sum rates of three schemes and an upper bound for the two-way relay channel. An upper bound for this channel is given in [4]

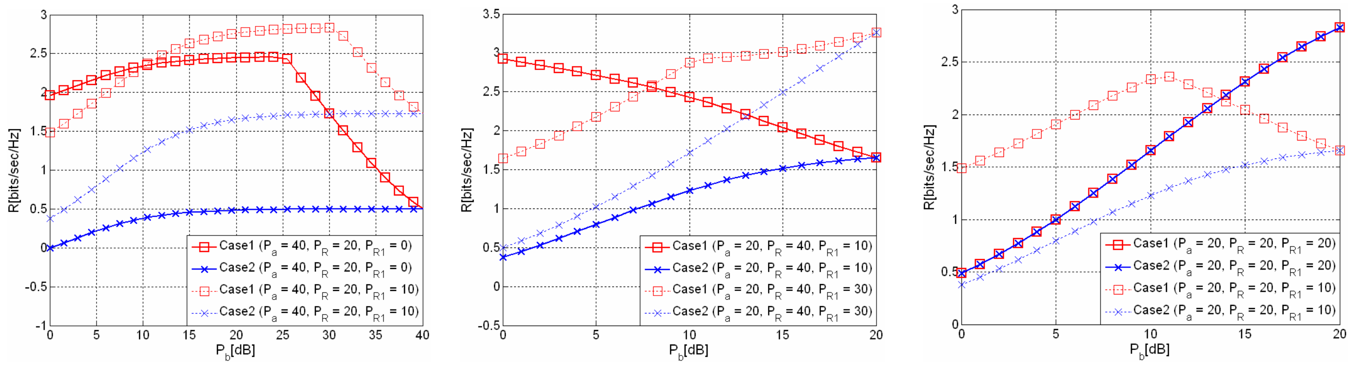
$$I(X_a; Y_b) = \min\{I(X_a; Y_R|X_b), I(X_R; Y_b)\} \tag{32}$$

For the Gaussian channel, the upper bound is

$$R_a = \min\left\{\frac{1}{2}\log_2(1 + P_a), \frac{1}{2}\log_2(1 + P_R)\right\} \tag{33}$$

$$R_b = \min\left\{\frac{1}{2}\log_2(1 + P_b), \frac{1}{2}\log_2(1 + P_R)\right\} \tag{34}$$

For each scheme, sum rates are given by (25), (26), (27), where R_{AF}, R_{DF}, and R_{MF} are sum rates of AF, DF, and MF, respectively. Fig. 5 shows the comparison of the sum rates. We change the transmit power of Node a while fixing the transmit power of Node b. At the relay node, we set the fraction of power for the signal X_{R1} to be the optimal fraction of the total transmit power of the relay node. In Fig. 5 (a),(b), the sum rate of DF is almost always greater than the other two schemes and the sum rate of MF is always greater than AF scheme. In Fig. 5 (a), when P_a is low, MF gives the best



(a) (b) (c)

Fig. 6. Comparison of Case 1 and Case 2

$$R_{AF} = \frac{1}{2}\log_2\left(1 + \frac{P_R P_a}{P_R + P_a + P_b + 1}\right) + \frac{1}{2}\log_2\left(1 + \frac{P_R P_b}{P_R + P_a + P_b + 1}\right) \quad (25)$$

$$R_{DF} = \min\left[\min\left\{(\frac{1}{2}\log(1+P_a), \frac{1}{2}\log(1+P_R)\right\} + \min\left\{\frac{1}{2}\log(1+P_b), \frac{1}{2}\log(1+P_R)\right\},\ \frac{1}{2}\log(1+P_a+P_b)\right] \quad (26)$$

$$R_{MF} = \log_2\left(1 + \frac{P_b P_{R1} - 1}{P_b + P_{R1} + 2}\right) + \min\left\{\frac{1}{2}\log\left(1 + \frac{P_a - P_b}{2P_b + 1}\right), \frac{1}{2}\log\left(1 + \frac{P_R - P_{R1}}{P_{R1} + 1}\right)\right\} \quad (27)$$

sum rate. However, in Fig. 5 (c), the AF scheme can be better than MF when P_a is low because MF is affected by more self noise from X_{a1} and X_b. When $P_b \geq P_R$, as P_{R1} increases, the achievable rate for the MF is almost always greater than AF because MF is more power efficient than AF due to the modulo operation. Note that in Fig. 5 (c), DF can be worse than AF and MF due to the overhead of decoding of both users' signals.

We now show the benefit of performing superposition at Node a for MF. We compare two cases. Case 1 is when we separate the message W_a into W_{a1} and W_{a2} and send them both as described in Section II. Case 2 is when we do not send W_{a2} and send only the message W_{a1} with a lowered power P_b to match the power of W_{a1} and W_b. As seen in Fig. 6, the achievable rate of Case 1 is always greater than Case 2 because R_{a2} is zero in Case 2. In other words, in Case 2 the amount of power $P_a - P_b$ is wasted. In Fig. 6 (a)(c), the achievable rate of Case 1 decreases at certain point and the point is when the rate between Node a and the relay node becomes smaller than that between the relay node and Node b since the second phase becomes a bottleneck.

IV. Conclusion

We have proposed a new scheme for the two-way relay channel. The proposed scheme has less complexity compared to DF and has more power efficiency compared to AF. For the different transmission power of two nodes, we allocate the power difference between two nodes to a signal to be decoded at the relay node. We show that partial decoding can achieve a higher rate than a scheme without it. Also, we compare the sum rate of three schemes and show that the rate of proposed scheme is almost always greater than that of AF and sometimes better than that of DF.

Acknowledgment

This work has been supported in part by the Basic Research Program of the Korea Science and Engineering Foundation (R01-2006-000-11112-0).

References

[1] C. E. Shannon, "Two-way communication channels," *Proc. 4th Berkeley Symp. Math. Stat. and Prob.,* vol. 1, pp. 611-644, 1961.
[2] S. Katti, I. Maric, A. Goldsmith, D. Katabi, and M. Medard, "Joint relaying and network coding in wireless networks," in *IEEE Int. Symp. Inform. Theory (ISIT),* Nice, France, 2007.
[3] U. Erez and R. Zamir, "Achieving $\frac{1}{2}\log(1+\text{SNR})$ on the AWGN channel with lattice encoding and decoding," *IEEE Trans. Inf. Theory*, vol. 50, pp. 2293-2314, Oct. 2004
[4] R. Knopp, "Two-way wireless communication via a relay station," *GDR-ISIS meeting*, Mar. 2007.
[5] W. Nam, S.-Y. Chung, and Y. H. Lee, "Capacity bounds for two-way relay channels," in proc. *International Zurich Seminar,* Mar. 2008.
[6] T. M. Cover and A. E. Gamal, "Capacity theorems for the relay channel," *IEEE Trans. Inf. Theory*, vol. 25, pp. 572-584, Sep. 1979.
[7] G. Kramer, M. Gastpar, and P. Gupta, "Cooperative strategies and capacity theorems for relay networks," *IEEE Trans. Inf. Theory*, vol. 51, pp. 3037-3063, Sep. 2005.
[8] R. Ahlswede, N. Cai, S. Y. R. Li, and R. W. Yeung, "Network information flow," *IEEE Trans. Inf. Theory*, vol. 46, pp. 1204-1216, July 2000.
[9] H. Yang, K. Lee, J. Chun, "Zero-forcing based two-phase relaying," in *Proc. IEEE Int. Conf. Communications (ICC),* 2007.
[10] B. Rankov and A. Wittneben, "Achievable rate regions for the two-way relay channel," in *IEEE Int. Symp. Inf. Theory*, Seatle, USA, 2006.
[11] Y. Hao, D. Goeckel, Z.Ding, D. Towsley and K. K. Leung, "Achievable Rates of Physical Layer Network Coding Schemes on the Exchange Channel," in Military Communications Conference, (MILCOM-07), Orlando, FL, Oct. 2007.
[12] I.-J. Baik and S.-Y. Chung, "Network coding for two-way relay channels using lattices," in *Telecommunications review* vol. 17, 2007, pp. 1009-1021.