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Absolute and convective instabilities of waves on unbounded and large bounded domains

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Abstract

Instabilities of nonlinear waves on unbounded domains manifest themselves in different ways. An absolute instability occurs if the amplitude of localized wave packets grows in time at each fixed point in the domain. In contrast, convective instabilities are characterized by the fact that even though the overall norm of wave packets grows in time, perturbations decay locally at each given point in the unbounded domain: wave packets are convected towards infinity. In experiments as well as in numerical simulations, bounded domains are often more relevant. We are interested in the effects that the truncation of the unbounded to a large but bounded domain has on the aforementioned (in)stability properties of a wave. These effects depend upon the boundary conditions that are imposed on the bounded domain. We compare the spectra of the linearized evolution operators on unbounded and bounded domains for two classes of boundary conditions. It is proved that periodic boundary conditions reproduce the point and essential spectrum on the unbounded domain accurately. Spectra for separated boundary conditions behave in quite a different way: firstly, separated boundary conditions may generate additional isolated eigenvalues. Secondly, the essential spectrum on the unbounded domain is in general not approximated by the spectrum on the bounded domain. Instead, the so-called absolute spectrum is approximated that corresponds to the essential spectrum on the unbounded domain calculated with certain optimally chosen exponential weights. We interpret the difference between the absolute and the essential spectrum in terms of the convective behavior of the wave on the unbounded domain. In particular, it is demonstrated that the stability of the absolute spectrum implies convective instability of the wave, but that convectively unstable waves can destabilize under domain truncation. The theoretical predictions are compared with numerical computations. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

We are interested in the stability properties of nonlinear waves such as fronts and pulses on unbounded and bounded domains. On unbounded domains, an instability can manifest itself in different ways. The physics literature distinguishes between two different kinds of instability, namely, absolute and convective instabilities. An absolute

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instability occurs if perturbations grow in time at every fixed point in the domain. Convective instabilities are characterized by the fact that, even though the overall norm of the perturbation grows in time, perturbations decay locally at every fixed point in the unbounded domain; in other words, the growing perturbation is transported, or convected, towards infinity. In experiments as well as in numerical simulations, bounded domains are often more relevant. From a physical point of view, it is then interesting and important to understand how absolute and convective instabilities manifest themselves on large bounded domains under various boundary conditions. A possible conclusion would be that convective instabilities disappear on bounded domains, while absolute instabilities persist. It turns out, however, that there are convective instabilities that survive the truncation to a bounded domain.

Understanding the spectral properties of waves under domain truncation amounts to identifying and capturing those instabilities that survive domain truncation, and to calculating and comparing the spectra of the relevant linearized operators on such domains. These are the issues we set out to explore in this article. Our main result establishes that it is not absolute and convective instabilities but what we call *remnant* and *transient* instabilities, see below, that determine the spectral (in)stability of waves under domain truncation. Before we explain these instabilities in more detail and outline our approach, we comment more on our motivation to study these issues.

Physical situations in which the aforementioned issues are relevant include, for instance, fluid flows in finite containers [9,43] and the break-up of spiral waves as observed in experiments [28] and numerical simulations [5,6,42]. In open flows, the difference between absolute and convective instabilities is important; this problem has been studied intensively for modulation equations such as the complex Ginzburg–Landau equation; see, for instance [3,13,14,43]. Part of our motivation comes from the break-up of spiral waves in two-dimensional excitable and oscillatory media [5,6]. Spirals can break-up either near the core or else in the far-field; the difference between these instabilities is the direction towards which unstable eigenmodes convect and transport. An interesting issue is to predict these instabilities, and the direction of transport, from spectral properties of the asymptotic wave trains of the spiral; this will be discussed in a forthcoming paper using the techniques introduced here.

A second reason for investigating the behavior of spectra under domain truncation is the fact that it is in general quite difficult to calculate the spectrum of the linearization about a given nonlinear wave analytically. Thus, one has to resort to numerical techniques which typically require that the unbounded domain is replaced by a bounded domain, supplemented with appropriate boundary conditions. There is then, however, no guarantee that the true spectrum on the unbounded domain is recovered as domain truncation is not a regular perturbation. In particular, the spectrum on the bounded domain may well depend upon the choice of boundary conditions.

1.1. Different instability mechanisms on unbounded domains

We begin by reviewing the different instability mechanisms that we are interested in on the unbounded domain $\mathbb{R}$. As mentioned above, absolute instabilities occur if perturbations grow in time at every fixed point in the domain. Convective instabilities are characterized by the fact that perturbations decay locally at every fixed point in the unbounded domain even though the overall norm of the perturbation grows in time.

There are, however, other ways of differentiating between instabilities on unbounded domains. We refer to the situation where every unstable mode travels to either left or right but not simultaneously to the left and right as a *transient* instability. Note that a convective instability allows waves to split into two wave packets that travel simultaneously to the left and right. In contrast, transiently unstable modes have a preferred direction of transport. We expect that transiently unstable waves are convectively unstable but not necessarily vice versa.

We outline how convective and transient instabilities can be captured mathematically on the unbounded domain $\mathbb{R}$. Suppose that we linearize a certain partial differential equation (PDE) about a pulse, say. We then investigate the resulting linear PDE operator $\mathcal{L}$ on the real line using the space $L^2(\mathbb{R})$ with norm $\|\cdot\|$. The spectrum of the operator $\mathcal{L}$ is the disjoint union of two sets: the point spectrum that consists of all isolated eigenvalues with finite multiplicity, and its complement which we refer to as the essential spectrum. If part of the essential spectrum lies in the right half-plane, then there is typically a continuum of unstable modes present. The essential spectrum can be computed using the dispersion relation $d(\lambda, \nu) = 0$ that relates temporal eigenvalues λ and spatial eigenvalues ν: the dispersion relation is calculated by substituting $u(x, t) = \exp(\lambda t + \nu x)$ into the PDE $u_t = \mathcal{L}_\infty u$, which is the linearization about the asymptotic rest state of the pulse. We remark that our notation of dispersion relation differs slightly from the physics convention where λ and ν are replaced by $\mathrm{i}\omega$ and $-\mathrm{i}k$, respectively.

In certain cases, the essential spectrum induces a convective instability. Suppose that part of the essential spectrum lies in the right half-plane. In many situations, it can be shown that a wave is convectively unstable if the dispersion relation $d(\lambda, \nu)$ does not have any double roots in ν for λ in the closed right half-plane; see [3,9] and the references therein. A wave becomes absolutely unstable if a temporal eigenvalue λ for which the dispersion relation has a spatial double root crosses into the right half-plane.

To describe transient instabilities, it is convenient to introduce exponential weights; see [40]: for any given real number η, define a new norm $\|\cdot\|_\eta$ by

$$\|u\|_\eta^2 = \int_{-\infty}^{\infty} |\mathrm{e}^{\eta x} u(x)|^2 \,\mathrm{d}x,$$

and denote by $L^2_\eta(\mathbb{R})$, equipped with the norm $\|\cdot\|_\eta$, the space of functions $u(x)$ for which $\mathrm{e}^{\eta x}u(x)$ is in $L^2(\mathbb{R})$. Note that the norms $\|\cdot\|_\eta$ for different values of η are not equivalent to each other. We then consider $\mathcal{L}$ as an operator on $L^2_\eta(\mathbb{R})$ and compute its spectrum using the new norm $\|\cdot\|_\eta$ for appropriate values of η. The key is that, for $\eta > 0$, the norm $\|\cdot\|_\eta$ penalizes perturbations at $+\infty$, while it tolerates perturbations (which may in fact grow exponentially with any rate less than η) at $-\infty$. Thus, if an instability is of transient nature so that it manifests itself by modes that travel towards $-\infty$, then the essential spectrum calculated in the norm $\|\cdot\|_\eta$ should move to the left as $\eta > 0$ increases. Indeed, as the perturbations travel towards $-\infty$, they are multiplied by $\mathrm{e}^{\eta x}$ which is small as $x \to -\infty$ and therefore reduces their growth or even causes them to decay. Exponential weights have been used to study a variety of problems posed on the real line such as reaction–diffusion operators [40], conservative systems such as the KdV equation [32], and generalized Kuramoto–Sivashinsky equations that describe thin films [11,12].

As mentioned above, convective and transient instabilities are not identical: an example of a convectively unstable wave that is not transiently unstable is given in Example 2 in Section 3.3. What happens in this example is that perturbations travel to both $+\infty$ and $-\infty$ at the same time. Such instabilities cannot be removed by exponential weights since we would need $\eta > 0$ to get rid of modes travelling to the left but $\eta < 0$ to handle the modes that travel to the right. This might seem to be a minor point but is in fact of importance when the domain is truncated to a bounded interval; see below. We refer to Fig. 1, where we illustrate absolute as well as transient and convective instabilities.

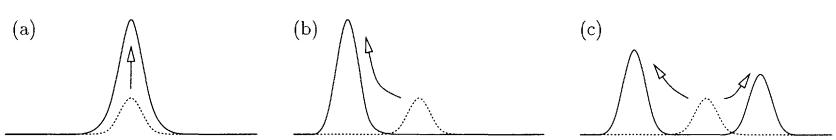

Fig. 1. The dotted waves are the initial data $u_0(x)$ to the linearized equation $u_t = \mathcal{L}u$, while the solid waves $u(x, t)$ represent the solution at a fixed positive time t; the horizontal axis is x, the vertical axis corresponds to the value of $u(x, t)$ at x. (a) Illustrates an absolute instability where the solution to the linearized equation grows without bounds at each given point x in space as time tends to ∞. (b) Illustrates transient instabilities: the solution $u(x, t)$ grows but also travels in *one* direction so that $u(x, t)$ actually decays for each fixed value of x as $t \to \infty$. The operator $\mathcal{L}$ would have stable spectrum in the norm $\|\cdot\|_\eta$ for a certain $\eta > 0$. (c) Shows a convectively unstable pattern that is not transiently unstable: the solution $u(x, t)$ consists of two waves that grow while travelling in opposite directions. Such an instability cannot be stabilized by using the norm $\|\cdot\|_\eta$. Typically, the group velocities of the two waves would differ in modulus as shown here.

Finally, we say that a wave is *remnantly* unstable if the spectrum of $\mathcal{L}$, computed in the space $L^2_\eta(\mathbb{R})$, is unstable for any choice of η. Thus, remnantly unstable modes are modes which are not affected by exponential weights. We can capture remnant and absolute instabilities by computing what we call the absolute spectrum: roughly speaking, the absolute spectrum Σ_{abs} is defined as the set of complex numbers λ for which the resolvent $\mathcal{L} - \lambda$ is not invertible in $L^2_\eta(\mathbb{R})$ for any choice of η; see Section 3.2 for a more precise definition. In fact, the absolute spectrum can be computed using only the asymptotic coefficients of the linear operator $\mathcal{L}$, that is, it depends only upon the asymptotic rest states of the underlying wave. The absolute spectrum captures remnant instabilities: the absolute spectrum moves into the right half-plane if, and only if, the wave experiences a remnant instability. As the absolute spectrum contains any points λ for which the dispersion relation has double spatial roots, we can also use it to capture absolute instabilities. Such temporal eigenvalues λ correspond to unstable eigenmodes with zero group velocity. Absolute and remnant instabilities would be identical if the rightmost unstable temporal eigenvalue λ in the absolute spectrum always corresponds to an eigenmode with zero group velocity; there is, however, the possibility that the most unstable eigenmode in the absolute spectrum has non-zero group velocity; see Examples 2 and 3 in Section 3.3.

In summary, upon using exponentially weighted norms, the essential spectrum may move to a different location. The new location of the essential spectrum is determined by a balance between the growth in amplitude and the speed of advection associated with each eigenmode on the one hand and the rate η that is being used in the weight on the other hand.

1.2. Instabilities on large bounded domains

Next, we consider the relevant PDE operator on a large but bounded interval with appropriate boundary conditions. The distinction between point and essential spectrum then disappears. We may, however, define the extrapolated essential spectral set $\Sigma^{\text{e}}_{\text{ext}}$ that consists of all complex numbers that are approached by infinitely many eigenvalues as the interval approaches the entire real line. In other words, rather than investigating how the essential spectrum breaks up under domain truncation, we consider the inverse problem by determining the asymptotic location of eigenvalues on the bounded interval as the domain size tends to infinity. The first result, Theorem 4 in Section 5.2, demonstrates that the essential spectrum Σ_{ess} and the extrapolated essential spectral set $\Sigma^{\text{e}}_{\text{ext}}$ are equal to each other provided we use periodic boundary conditions; this requires that the nonlinear wave is a pulse and not a front. In other words, with periodic boundary conditions, the essential spectrum of pulses is well approximated under domain truncation. The second result, Theorem 5 in Section 5.3, shows that the sets Σ_{ess} and $\Sigma^{\text{e}}_{\text{ext}}$ are in general *different* if separate boundary conditions are used. Thus, no matter how large we choose the interval length, the resulting spectrum will never be close to the spectrum on the real line. In fact, we demonstrate that, for separated boundary conditions, the extrapolated essential spectral set $\Sigma^{\text{e}}_{\text{ext}}$ is typically equal to the absolute spectrum Σ_{abs}. The reason that the spectrum on the real line is not well approximated by the spectrum on bounded intervals is related to the existence of transiently unstable eigenmodes. We had seen that we can shift the transient part of the essential spectrum by using exponential weights. As mentioned above, this amounts to changing the underlying function space as these norms are not equivalent to the standard L^2-norm. On bounded intervals, however, these norms are all equivalent to each other since $\text{e}^{\eta x}$ is then bounded away from zero and from infinity. Thus, we expect that, if the operator is stable in an exponentially weighted norm, then it should also be stable on large bounded intervals. In other words, even if it is unstable on the real line, it will be stable on bounded intervals provided it is also stable on the real line considered with exponential weights. Heuristically, transiently unstable eigenmodes transport perturbations towards either $+\infty$ or $-\infty$; on bounded intervals, the perturbations then disappear through the boundary. This also explains why the essential spectrum is recovered upon using periodic boundary conditions: the transient modes transport perturbations towards the boundary as before but they get fed in on the other endpoint of the interval through the boundary conditions. It also explains why convective instabilities may lead to instabilities on bounded domains: if there are unstable eigenmodes that transport perturbations to the left and other modes that transport to the right, then these modes may couple through the boundary conditions; even for separated boundary conditions, this may lead to an instability on bounded intervals; see Example 2 in Section 3.3.

An interesting consequence of these remarks is a characterization of the so-called pseudo-spectrum. Roughly speaking, for arbitrary small $\epsilon > 0$, the ϵ-pseudo-spectrum of an operator $\mathcal{L}$ on a bounded domain consists, by definition, of all complex numbers such that the associated resolvent $(\mathcal{L}-\lambda)^{-1}$ has norm larger than $1/\epsilon$. It has been used in linear numerical analysis; see, for instance [44] for more background information. Our results imply that the pseudo-spectrum on large domains typically interpolates between the absolute spectrum Σ_{abs} and the essential spectrum Σ_{ess}: for fixed interval length L, the ϵ-pseudo-spectrum of $\mathcal{L}$ approaches the absolute spectrum as $\epsilon \to 0$; on the other hand, for fixed $\epsilon > 0$, the pseudo-spectrum converges, as $L \to \infty$, to an open set whose closure contains the essential spectrum, see Section 4.4. The reason is that the resolvent on the real line is invertible only in some exponentially weighted norm. Even though all these norms are equivalent on bounded intervals, their equivalence constants approach infinity exponentially fast in terms of the interval length. Thus, the norm of the resolvents also grows exponentially in terms of the interval length. For the constant-coefficient convection–diffusion operator $u_{xx} + cu_x$, this fact has been established in [34].

It remains to consider the effects of domain truncations on isolated eigenvalues. Again, we have to distinguish between periodic and separated boundary conditions. In Theorem 2 in Section 4.2, we prove that eigenvalues persist with their multiplicity under periodic boundary conditions. Furthermore, all the eigenvalues for the operator on the bounded interval originate from eigenvalues on the real line. Thus, periodic boundary conditions recover not only the essential spectrum but also the point spectrum accurately.

The case of separated boundary conditions is again quite different. There are three issues that have to be dealt with. Firstly, given an isolated eigenvalue on the real line, we may ask for its persistence when truncating to a bounded interval. Secondly, additional eigenvalues could be created through the boundary conditions. The third issue is as follows. We have seen that the essential spectrum may shift upon using exponential weights. In the region between the original and the shifted essential spectrum, additional eigenvalues may arise. The associated eigenfunctions are bounded in the $\|\cdot\|_\eta$ norm that was used to shift the spectrum but are unbounded in the original L^2-norm. It is then possible that these new eigenvalues, which are often referred to as resonance poles, persist upon domain truncation. The reason is that the exponential weights do not matter on any bounded interval. All these issues are taken care of in Theorem 2 in Section 4.3.2. Resonance poles indeed persist under domain truncation in addition to eigenvalues of the operator in the original L^2-norm. Furthermore, it is possible that additional eigenvalues are created through the boundary conditions, and we give precise conditions on when this phenomenon occurs and how many eigenvalues are created.

We remark that we do not give asymptotic expansions of isolated eigenvalues in the interval length L of the underlying bounded interval as $L \to \infty$. Such expansions can, however, be obtained using the approach utilized here; see, for instance, [37] for expansions of eigenvalues for the linearization about a pulse under periodic boundary conditions.

The main techniques that we use to demonstrate the persistence of eigenvalues are the Evans function [1] for bounded intervals [19,21] applied with exponential weights [40]. Domain truncation for the absolute and essential spectrum are investigated using exponential dichotomies [16,29,33,35]; implicitly, we also use extensions of the Evans function across the essential spectrum [25]. We emphasize that we prove our results for λ in bounded subsets of the complex plane. Thus, we do not establish resolvent estimates for large λ; in particular examples, such estimates are typically obtained on a case-to-case basis.

Finally, we mention related results. In [8], the persistence of eigenvalues under domain truncation has been investigated for reaction–diffusion operators under periodic and separated boundary conditions. The authors also provided error estimates for the dependence of the eigenvalues on the interval length L. The results established in [8] apply only to eigenvalues that are to the right of the essential spectrum; resonance poles or the behavior of the essential spectrum itself were not discussed. A general reference for boundary-value problems is [4].

In addition, there is tremendous amount of articles in the physics literature where absolute and convective instabilities have been investigated; see, for instance, [3,43] to name but two. In many of these articles, absolute and convective instabilities were investigated for the complex Ginzburg–Landau equation. The results typically characterize the onset to instability by the crossing of a double root of the dispersion relation through the imaginary axis into the right half-plane. As we have already mentioned above, this criterion is in general not correct even though it gives the right answer in almost all the cases we are aware of. Our contribution is firstly the correct criterion for instability on large bounded domains through the notions of remnant and transient instabilities and, secondly, a characterization of the entire spectrum, and not only of the double roots of the dispersion relation, on bounded domains. This allows for a comparison of numerical calculations with theoretical predictions. In addition, the systematic use of exponentially weighted norms allows us to predict the absolute or convective nature of instabilities including the direction of transport.

This article is organized as follows. The set-up and most of the relevant definitions are given in Section 2. In Section 3, we introduce the various notions of spectrum that we shall use. The behavior of point spectrum under domain truncation is discussed in Section 4. Section 5 contains the results for the essential spectrum. Numerical simulations for the KdV equation and the Gray–Scott model are presented in Section 6. Section 7 contains conclusions and discussion of open problems.

2. Operators, boundary conditions, and exponential dichotomies

In this section, we introduce our precise set-up as well as all necessary definitions that we shall use.

2.1. *The coefficient matrix*

Throughout this article, we assume that $A(x;\lambda) \in \mathbb{R}^{N\times N}$ is a matrix-valued function of $(x,\lambda) \in \mathbb{R}\times\mathbb{C}$ of the form

$$A(x;\lambda) = \hat{A}(x) + \lambda B(x).$$

Most of our results are valid for more general $A(x;\lambda)$; we note, however, that eigenvalue problems arising from evolutionary equations are typically of the above type. We assume that $A(x;\lambda)$ satisfies the following hypothesis.

Hypothesis 1. The matrices $A(x;\lambda) \in \mathbb{R}^{N\times N}$ are smooth in $x \in \mathbb{R}$ and analytic in $\lambda \in \mathbb{C}$. Furthermore, the following conditions are met.

1. *Asymptotically constant-coefficients*: There are positive constants K and θ, independent of x and λ, and matrices $A_\pm(\lambda)$ that depend analytically on λ such that
$$\|A(x;\lambda) - A_\pm(\lambda)\| \le K\,\mathrm{e}^{-\theta|x|}$$
as $x \to \pm\infty$.
2. *Well-posedness*: There is a number $\rho > 0$ and an integer $i_\infty \in \mathbb{N}$, such that for all λ with $\mathrm{Re}\,\lambda \ge \rho$, the asymptotic matrices $A_\pm(\lambda)$ are hyperbolic (i.e. they have no spectrum on $\mathrm{i}\mathbb{R}$), and the dimension of their generalized unstable eigenspaces is equal to i_∞.

The second condition is satisfied for eigenvalue problems that arise from evolution equations. It guarantees that the essential spectrum is to the left of the line $\mathrm{Re}\,\lambda = \rho$ for some finite number ρ.

We emphasize that most of our results hold in more generality; for instance, it suffices that $A(x;\lambda)$ is asymptotically periodic in x.

Throughout this paper, we label eigenvalues of $A_\pm(\lambda)$ according to their real part, and repeated with their multiplicity, i.e.

$$\mathrm{Re}\,\nu_1^\pm(\lambda) \geq \mathrm{Re}\,\nu_2^\pm(\lambda) \geq \cdots \geq \mathrm{Re}\,\nu_{N-1}^\pm(\lambda) \geq \mathrm{Re}\,\nu_N^\pm(\lambda). \tag{2.1}$$

In particular, choose λ such that $\mathrm{Re}\,\lambda$ is large. Using Hypothesis 1(2), we see that

$$\mathrm{Re}\,\nu_1^\pm(\lambda) \geq \cdots \geq \mathrm{Re}\,\nu_{i_\infty}^\pm(\lambda) > 0 > \mathrm{Re}\,\nu_{i_\infty+1}^\pm(\lambda) \geq \cdots \geq \mathrm{Re}\,\nu_N^\pm(\lambda).$$

We refer to λ and $\nu_j^\pm$ as the *temporal* and *spatial* eigenvalues, respectively. For every fixed temporal eigenvalue λ, the spatial eigenvalues $\nu_j^\pm$ are the roots of the characteristic polynomials

$$d_\pm(\nu,\lambda) = \det[\nu - A_\pm(\lambda)]$$

of $A_\pm(\lambda)$. The dependence between spatial and temporal eigenvalues is commonly referred to as the dispersion relation, where $\omega = -\mathrm{i}\lambda$ is considered as a function of $k = \mathrm{i}\nu$.

2.2. *The operator on the real line*

On the unbounded real line $\mathbb{R}$, we consider the family $\mathcal{T}$ of linear operators

$$\mathcal{T}(\lambda): H^1(\mathbb{R},\mathbb{C}^N) \to L^2(\mathbb{R},\mathbb{C}^N), \qquad u \mapsto \frac{\mathrm{d}u}{\mathrm{d}x} - A(\cdot;\lambda)u \tag{2.2}$$

for $\lambda \in \mathbb{C}$, where H^1 is the usual Sobolev space of L^2-functions that have a weak derivative which is in L^2.

As mentioned in Section 1, it is often convenient to consider the operators $\mathcal{T}(\lambda)$ on exponentially weighted function spaces; see [40,31]. Thus, for arbitrary $\eta = (\eta_-,\eta_+) \in \mathbb{R}^2$, we set

$$\|v(\cdot)\|_{L^2_\eta}^2 = \int_{-\infty}^0 |v(\xi)\exp(\eta_-\xi)|^2\,\mathrm{d}\xi + \int_0^\infty |v(\xi)\exp(\eta_+\xi)|^2\,\mathrm{d}\xi, \qquad \|v(\cdot)\|_{H^1_\eta}^2 = \|v(\cdot)\|_{L^2_\eta}^2 + \|v_x(\cdot)\|_{L^2_\eta}^2.$$

We may then consider the operator

$$\mathcal{T}^\eta(\lambda): H^1_\eta(\mathbb{R},\mathbb{C}^N) \to L^2_\eta(\mathbb{R},\mathbb{C}^N), \qquad v \mapsto \frac{\mathrm{d}v}{\mathrm{d}x} - A(\cdot;\lambda)v. \tag{2.3}$$

For any function v defined on $\mathbb{R}$, let

$$(J_\eta v)(x) = \begin{cases} \exp(\eta_- x)v(x) & \text{for } x < 0, \\ \exp(\eta_+ x)v(x) & \text{for } x > 0. \end{cases}$$

The maps

$$J_\eta : H^1_\eta \to H^1, \quad v \mapsto J_\eta v \quad \text{and} \quad J_\eta : L^2_\eta \to L^2, \quad v \mapsto J_\eta v$$

are then linear isomorphism, and the operator $\mathcal{T}^\eta(\lambda)$ can be written as

$$\tilde{\mathcal{T}}^\eta(\lambda): H^1(\mathbb{R},\mathbb{C}^N) \to L^2(\mathbb{R},\mathbb{C}^N), \qquad u \mapsto \frac{\mathrm{d}u}{\mathrm{d}x} - A(\cdot;\lambda)u - \eta(\cdot)u, \tag{2.4}$$

where

$$\eta(x)=\begin{cases}\eta_- & \text{for } x<0,\\ \eta_+ & \text{for } x>0.\end{cases} \tag{2.5}$$

Indeed, we have

$$\tilde{\mathcal{T}}^\eta(\lambda)=J_\eta\mathcal{T}^\eta(\lambda)J_\eta^{-1}.$$

In the following, we omit the tilde and denote both operators by $\mathcal{T}^\eta(\lambda)$.

2.3. *The operators on the bounded interval* $(-L, L)$

Alternatively, we could consider the operators on the bounded interval $(-L, L)$ for large numbers L. In this case, we introduce boundary conditions at the endpoints of the interval. For periodic boundary conditions, a suitable function space is

$$H^1_{\text{per}}((-L,L),\mathbb{C}^N)=H^1((-L,L),\mathbb{C}^N)\cap\{u;\,u(-L)=u(L)\},$$

and we consider the operator

$$\mathcal{T}^{\text{per}}_L(\lambda):H^1_{\text{per}}((-L,L),\mathbb{C}^N)\to L^2((-L,L),\mathbb{C}^N),\qquad u\mapsto\frac{\mathrm{d}u}{\mathrm{d}x}-A(\cdot;\lambda)u. \tag{2.6}$$

Separated boundary conditions can be realized by choosing appropriate subspaces Q_+ and Q_- of $\mathbb{C}^N$. We assume that these subspaces satisfy the following hypothesis.

Hypothesis 2 (*Separated boundary conditions*). We assume that

$$\dim Q_-=i_\infty,\qquad \dim Q_+=N-i_\infty,$$

where the asymptotic Morse index i_∞ has been introduced in Hypothesis 1.

The correct function space for separated boundary conditions is then given by

$$H^1_{\text{sep}}((-L,L),\mathbb{C}^N)=H^1((-L,L),\mathbb{C}^N)\cap\{u;\,u(-L)\in Q_- \text{ and } u(L)\in Q_+\},$$

and we are interested in the operator

$$\mathcal{T}^{\text{sep}}_L(\lambda):H^1_{\text{sep}}((-L,L),\mathbb{C}^N)\to L^2((-L,L),\mathbb{C}^N),\qquad u\mapsto\frac{\mathrm{d}u}{\mathrm{d}x}-A(\cdot;\lambda)u. \tag{2.7}$$

Example 1. Consider the convection–diffusion problem $U_t=U_{xx}+cU_x$ together with the associated eigenvalue problem $\lambda U=U_{xx}+cU_x$. Upon writing the eigenvalue problem as a first-order system, we see that $N=2$ and

$$\hat{A}=\begin{pmatrix}0&1\\0&-c\end{pmatrix},\qquad B=\begin{pmatrix}0&0\\1&0\end{pmatrix},\qquad A(x;\lambda)=\hat{A}+\lambda B, \tag{2.8}$$

such that $u=(u_1,u_2)^{\mathrm{T}}=(U,U_x)^{\mathrm{T}}\in\mathbb{R}^2$. Dirichlet and Neumann boundary conditions are given by $U(\pm L)=0$ and $U_x(\pm L)=0$, respectively, and can be realized using the subspaces $Q^{\text{Dir}}_\pm=\text{span}\{(0,1)^{\mathrm{T}}\}$ and $Q^{\text{Neu}}_\pm=\text{span}\{(1,0)^{\mathrm{T}}\}$.

Note that, for separated boundary conditions, the integer i_∞ is singled out as the number of boundary conditions at the right endpoint of the interval $(-L, L)$; observe that the number of boundary conditions at $x = \pm L$ is the codimension of $Q_\pm$. Furthermore, we emphasize that exponential weights do not affect separated boundary conditions but that they change periodic boundary conditions.

2.4. Exponential dichotomies

The main tool that we use below to investigate the spectral properties of the family $\mathcal{T}$ are exponential dichotomies of the associated ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}u = A(x;\lambda)u \tag{2.9}$$

for $u \in \mathbb{C}^N$.

Definition 2.1 (*Exponential dichotomies*). Let $I = \mathbb{R}^+$, $\mathbb{R}^-$ or $\mathbb{R}$, and fix $\lambda_* \in \mathbb{C}$. We say that (2.9), with $\lambda = \lambda_*$ fixed, has an exponential dichotomy on I if there exist positive constants K, κ^{s} and κ^{u} and a family of projections $P(x)$ defined and continuous for $x \in I$ such that the following is true.

1. For any fixed $y \in I$ and $u_0 \in \mathbb{C}^N$, there exists a solution $\varphi^{\mathrm{s}}(x, y)u_0$ of (2.9) with initial value $\varphi^{\mathrm{s}}(y, y)u_0 = P(y)u_0$ for $x = y$, and we have

$$|\varphi^{\mathrm{s}}(x, y)| \le K \exp(-\kappa^{\mathrm{s}}|x - y|)$$

for all $x \ge y$ with $x, y \in I$.
2. For any fixed $y \in I$ and $u_0 \in \mathbb{C}^N$, there exists a solution $\varphi^{\mathrm{u}}(x, y)u_0$ of (2.9) with initial value $\varphi^{\mathrm{u}}(y, y)u_0 = (1 - P(y))u_0$ for $x = y$, and we have

$$|\varphi^{\mathrm{u}}(x, y)| \le K \exp(-\kappa^{\mathrm{u}}|x - y|)$$

for all $x \le y$ with $x, y \in I$.
3. The solutions $\varphi^{\mathrm{s}}(x, y)u_0$ and $\varphi^{\mathrm{u}}(x, y)u_0$ satisfy

$$\begin{aligned} &\varphi^{\mathrm{s}}(x, y)u_0 \in \mathbf{R}(P(x)) \quad \text{for all } \ x \ge y \quad \text{with } \ x, y \in I, \\ &\varphi^{\mathrm{u}}(x, y)u_0 \in \mathbf{N}(P(x)) \quad \text{for all } \ x \le y \quad \text{with } \ x, y \in I. \end{aligned}$$

The (x-independent) dimension of $\mathbf{N}(P(x))$ is referred to as the Morse index $i(\lambda_*)$ of the exponential dichotomy on I. If (2.9) has exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$, the associated Morse indices are denoted by $i_+(\lambda_*)$ and $i_-(\lambda_*)$, respectively.

The existence of exponential dichotomies of (2.9) is related to hyperbolicity of the asymptotic matrices $A_\pm(\lambda)$; recall that $A(x;\lambda)$ converges to $A_\pm(\lambda)$ as $x \to \pm\infty$. If $A_\pm(\lambda)$ is hyperbolic, then we denote by $E_\pm^{\mathrm{s,u}}(\lambda)$ the associated stable and unstable eigenspaces. Furthermore, we denote the spectral projections of $A_\pm(\lambda)$ associated with the stable and unstable eigenvalues by $P_\pm^{\mathrm{s}}(\lambda)$ and $P_\pm^{\mathrm{u}}(\lambda)$, respectively.

Statement 1 (Coppel [16]). *Fix $\lambda_* \in \mathbb{C}$. Eq.* (2.9) *has an exponential dichotomy on $\mathbb{R}^+$ if and only if, the matrix $A_+(\lambda_*)$ is hyperbolic. In this case, the Morse index $i_+(\lambda_*)$ is equal to the dimension* dim $E_+^{\mathrm{u}}(\lambda)$ *of the generalized unstable eigenspace of $A_+(\lambda_*)$. The same statements are true on $\mathbb{R}^-$ with $A_+(\lambda_*)$ replaced by $A_-(\lambda_*)$.*

Finally, (2.9) *has an exponential dichotomy on* $\mathbb{R}$ *if, and only if, it has exponential dichotomies on* $\mathbb{R}^+$ *and* $\mathbb{R}^-$ *with projections* $P_\pm(x)$ *so that* $\mathbf{R}(P_+(0)) \oplus \mathbf{N}(P_-(0)) = \mathbb{C}^N$*; this requires in particular that the Morse indices* $i_+(\lambda_*)$ *and* $i_-(\lambda_*)$ *are equal.*

In fact, we can say more about the asymptotic behavior of exponential dichotomies and their dependence on λ. We denote by $\varphi(x, y; \lambda)$ the evolution operator to (2.9) with initial time y. Recall that we ordered the eigenvalues $\nu_j^\pm(\lambda)$ of $A_\pm(\lambda)$; see (2.1). Let $U_\delta(\lambda_*)$ be the ball in $\mathbb{C}$ with center λ_* and radius δ.

Theorem 1 ([16,33,35]). *Fix* $\lambda_* \in \mathbb{C}$ *and assume that* $A_+(\lambda_*)$ *is hyperbolic. There are then numbers* κ_+^{s}, κ_+^{u} *and* $\delta > 0$ *so that, with* $i_+ = i_+(\lambda_*)$,

$$\mathrm{Re}\, \nu_{i_+}^+(\lambda) > \kappa_+^{\mathrm{u}} > 0 > -\kappa_+^{\mathrm{s}} > \mathrm{Re}\, \nu_{i_++1}^+(\lambda)$$

for all $\lambda \in U_\delta(\lambda_*)$*. Furthermore, there is a* $K \geq 1$ *so that, for every* $\lambda \in U_\delta(\lambda_*)$*, there are evolution operators* $\varphi_+^{\mathrm{s}}(x, y; \lambda)$ *and* $\varphi_+^{\mathrm{u}}(x, y; \lambda)$*, defined for* $x, y \geq 0$ *and analytic in* λ*, such that, for* $x, y \in \mathbb{R}^+$,

$$\varphi(x, y; \lambda) = \varphi_+^{\mathrm{s}}(x, y; \lambda) + \varphi_+^{\mathrm{u}}(x, y; \lambda), \qquad \|\varphi_+^{\mathrm{s}}(x, y; \lambda)\| \leq K \exp(-\kappa_+^{\mathrm{s}}|x-y|), \quad x \geq y,$$

$$\|\varphi_+^{\mathrm{u}}(x, y; \lambda)\| \leq K \exp(-\kappa_+^{\mathrm{u}}|x-y|), \quad y \geq x, \qquad \|\varphi_+^{\mathrm{u}}(x, x; \lambda) - P_+^{\mathrm{u}}(\lambda)\| \leq K(\mathrm{e}^{-\theta|x|} + \exp(-\sigma_+|x|)),$$

where θ *appeared in Hypothesis* 1 *and* $\sigma_+ = \kappa_+^{\mathrm{s}} + \kappa_+^{\mathrm{u}}$ *is a lower bound for the gap, in the real part, between the stable and unstable spectral sets of* $A_+(\lambda)$.

The matrices $\varphi_+^{\mathrm{s}}(x, x; \lambda)$ *and* $\varphi_+^{\mathrm{u}}(x, x; \lambda)$ *are complementary projections, and we define the subspaces* $E_+^{\mathrm{s}}(x; \lambda) = \mathbf{R}(\varphi_+^{\mathrm{s}}(x, x; \lambda))$ *of dimension* $N - i_+$ *and* $E_+^{\mathrm{u}}(x; \lambda) = \mathbf{R}(\varphi_+^{\mathrm{u}}(x, x; \lambda))$ *of dimension* i_+*. For any subspace* $\hat{E}$ *of* $\mathbb{C}^N$ *with* $E_+^{\mathrm{s}}(0; \lambda) \oplus \hat{E} = \mathbb{C}^N$*, there is a constant C such that*

$$\mathrm{dist}(\varphi(x, 0; \lambda)\hat{E}, E_+^{\mathrm{u}}(\lambda)) \leq C[\mathrm{e}^{-\theta|x|} + \exp(-\sigma_+|x|)], \quad x \geq 0,$$

where $E_+^{\mathrm{u}}(\lambda) = \mathbf{R}(P_+^{\mathrm{u}}(\lambda))$. (*We refer to Section* 4.1 *for a definition of the distance between subspaces.*)

Finally, $\varphi_+^{\mathrm{s}}(x, y; \lambda)$ *and* $\varphi_+^{\mathrm{u}}(x, y; \lambda)$ *are unique up to the choice of* $E_+^{\mathrm{u}}(0; \lambda)$*: any other analytic choice of a complement of* $E_+^{\mathrm{s}}(0; \lambda)$ *leads to evolution operators with the above properties. Furthermore, these evolution operators are exponential dichotomies in the sense of Definition* 2.1.

The same statements are true on $\mathbb{R}^-$ *with* $A_+(\lambda)$ *replaced by* $A_-(\lambda)$*. Furthermore, if* (2.9) *has an exponential dichotomy on* $\mathbb{R}$*, then the operators defined above can be chosen to be analytic in* λ *for all* $x, y \in \mathbb{R}$.

Proof. The proofs can be found in [35, Section 1.1] and [33, Sections 2.2 and 3.4]. □

Remark 2.1. *We emphasize that the above results can be extended to the case where the asymptotic matrix* $A_+(\lambda)$ *has also spectrum on the imaginary axis. The evolution operator can then be written as the sum* $\varphi(x, y; \lambda) = \varphi_+^{\mathrm{s}}(x, y; \lambda) + \varphi_+^{\mathrm{c}}(x, y; \lambda) + \varphi_+^{\mathrm{u}}(x, y; \lambda)$ *of evolution operators that depend analytically on* λ*. The operators* φ_+^{s} *and* φ_+^{u} *satisfy the same estimates as before, while we have in addition*

$$\|\varphi_+^{\mathrm{c}}(x, y; \lambda)\| \leq K \exp(\kappa_+^{\mathrm{c}}|x-y|), \quad x, y \geq 0$$

for fixed $0 < \kappa_+^{\mathrm{c}} < \min\{\kappa_+^{\mathrm{s}}, \kappa_+^{\mathrm{u}}\}$*. This statement can be proved by applying Theorem* 1 *twice to* (2.9) *with A replaced by* $A + \eta$ *for* $\eta > 0$ *and* $\eta < 0$ *close to zero, respectively. We omit the details and instead refer to* [38].

The following theorem proved by Palmer relates Fredholm properties of the operator $\mathcal{T}(\lambda)$ to properties pertaining to the existence of dichotomies of (2.9).

Statement 2 (Palmer [29,30]). *Fix* $\lambda \in \mathbb{C}$. *If* (2.9) *has exponential dichotomies on* $\mathbb{R}^+$ *and* $\mathbb{R}^-$, *then* $\mathcal{T}(\lambda)$ *is Fredholm with index* $i_-(\lambda) - i_+(\lambda)$. *Conversely, if* $\mathcal{T}(\lambda)$ *is Fredholm, then* (2.9) *has exponential dichotomies on* $\mathbb{R}^+$ *and* $\mathbb{R}^-$ *with associated Morse indices* $i_+(\lambda)$ *and* $i_-(\lambda)$, *respectively, and the difference* $i_-(\lambda) - i_+(\lambda)$ *is the Fredholm index of* $\mathcal{T}(\lambda)$. *Finally,* $\mathcal{T}(\lambda)$ *is invertible if, and only if,* (2.9) *has an exponential dichotomy on* $\mathbb{R}$. *If* $\mathcal{T}$ *is invertible, we denote by* $i = i_+ = i_-$ *the spatial Morse index of* $\mathcal{T}$ *that is given by the dimension* $\dim E^{\mathrm{u}}(0;\lambda)$ *of the unstable subspace* $E^{\mathrm{u}}(0;\lambda)$ *of the associated dichotomy*.

As a consequence of Statement 2 and the above discussion, $\mathcal{T}^{\eta}(\lambda)$ is Fredholm if, and only if, the matrices $A_{\pm}(\lambda) + \eta_{\pm}$ are both hyperbolic. The Fredholm index is then given by the difference of the dimensions of the generalized unstable eigenspaces of $A_{\pm}(\lambda) + \eta_{\pm}$.

3. Spectra on the unbounded real line

3.1. Point and essential spectrum

We consider the family of operators $\mathcal{T}$ with parameter λ. The spectrum of the operator $\mathcal{T}(\lambda)$ for fixed λ is of no interest to us; instead, we consider the so-called B-spectrum, see [23, Chapter IV], of $(\mathrm{d}/\mathrm{d}x) - \hat{A}(x)$.

Definition 3.2 (*Spectrum*). We say that λ is in the spectrum Σ of $\mathcal{T}$ if $\mathcal{T}(\lambda)$ is not invertible. We say that $\lambda \in \Sigma$ is in the point spectrum Σ_{pt} of $\mathcal{T}$, or alternatively that $\lambda \in \Sigma$ is an eigenvalue of $\mathcal{T}$, if $\mathcal{T}(\lambda)$ is a Fredholm operator with index zero. The complement $\Sigma \setminus \Sigma_{\mathrm{pt}} =: \Sigma_{\mathrm{ess}}$ is called the essential spectrum.

Example 1 (*continued*). We decompose $A(x;\lambda) = \hat{A} + \lambda B$ as in (2.8). The spectrum of $\mathcal{T}$ then coincides with the spectrum of the associated elliptic differential operator $\mathcal{L} = (\mathrm{d}^2/\mathrm{d}x^2) + c(\mathrm{d}/\mathrm{d}x)$.

In particular, $\lambda \notin \Sigma_{\mathrm{ess}}$ if, and only if, (2.9) has exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$ with equal Morse index. The essential spectrum is determined by the asymptotic matrices $A_{\pm}(\lambda)$: $\mathcal{T}(\lambda)$ is Fredholm if, and only if, the spectra of $A_+(\lambda)$ and $A_-(\lambda)$ are disjoint from the imaginary axis; the Morse indices $i_{\pm}(\lambda)$ are given by the dimensions of the unstable eigenspaces of $A_{\pm}(\lambda)$.

For any λ in the point spectrum, we define the multiplicity of λ as follows. Recall that $A(x;\lambda)$ is of the form $A(x;\lambda) = \hat{A}(x) + \lambda B(x)$. Suppose that λ is in the point spectrum of $\mathcal{T}$, where

$$\mathcal{T}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}x} - \hat{A}(x) - \lambda B(x),$$

so that $\mathbf{N}(\mathcal{T}(\lambda)) = \mathrm{span}\{u_1(x)\}$. We say that λ has multiplicity ℓ if there are functions $u_j(x)$ for $j = 2, \dots, \ell$, so that

$$\frac{\mathrm{d}}{\mathrm{d}x} u_j = (\hat{A}(x) + \lambda B(x)) u_j + B(x) u_{j-1}$$

for $j = 2, \dots, \ell$ but no solution to

$$\frac{\mathrm{d}}{\mathrm{d}x} u = (\hat{A}(x) + \lambda B(x)) u + B(x) u_{\ell}.$$

Here, we assumed that the functions u_j belong to the same function space that may include boundary conditions. Finally, we say that an arbitrary eigenvalue λ of $\mathcal{T}$ has multiplicity ℓ if the sum of the multiplicities of a maximal set of linearly independent elements in $\mathbf{N}(\mathcal{T}(\lambda))$ is equal to ℓ.

Next, we discuss stability in exponentially weighted spaces. Choose λ to the right of the essential spectrum. As before, we label eigenvalues of $A_\pm(\lambda)$ according to their real part and repeated with their multiplicity. For any λ to the right of the essential spectrum, we then have

$$\operatorname{Re} \nu^\pm_{i_\infty}(\lambda) > 0 > \operatorname{Re} \nu^\pm_{i_\infty+1}(\lambda)$$

due to Hypothesis 1(2). These inequalities are satisfied upon varying λ until λ touches the boundary of the essential spectrum where at least one of these eigenvalues crosses the imaginary axis. Using exponential weights $\eta = (\eta_-, \eta_+)$ has the effect of replacing the matrices $A_\pm(\lambda)$ by $A_\pm(\lambda) + \eta_\pm$. Thus, λ is to the right of the essential spectrum of $\mathcal{T}^\eta$ provided the eigenvalues of $A_\pm(\lambda)$ satisfy

$$\operatorname{Re} \nu^\pm_{i_\infty}(\lambda) > -\eta_\pm > \operatorname{Re} \nu^\pm_{i_\infty+1}(\lambda).$$

We give several different notions of stability, and begin with spectral stability.

Definition 3.3 (*Spectral stability*). We say that $\mathcal{T}$ is stable if Σ is contained in the open left half-plane. We say that $\mathcal{T}$ is unstable if part of its spectrum Σ lies in the closed right half-plane.

The next definition measures stability up to exponential weights. We restrict the allowed set of exponential weights to make them compatible with the asymptotic behavior of spatial eigenvalues for large $\lambda > 0$ as expressed in Hypothesis 1(2).

Definition 3.4 (*Transient and remnant instability*). Suppose that Σ_{ess} is not contained in the open left half-plane. We then say that $\mathcal{T}$ is transiently unstable if there are exponential weights $\eta(\lambda)$ such that $\mathcal{T}^{\eta(\lambda)}(\lambda)$ is invertible with spatial Morse index i_∞ for every λ in the closed right half-plane; we say that $\mathcal{T}$ is remnantly unstable if it is not transiently unstable.

We refer to Statement 2 in Section 2.4 for the definition of the spatial Morse index.

We emphasize that the weight η that we use to invert $\mathcal{T}^\eta(\lambda)$ may depend upon λ; in other words, we do not require that the spectrum of $\mathcal{T}^\eta$ lies in the open left half-plane for some choice of η.

Note that, despite its name, we really consider a transient instability as some kind of stability: if a wave is transiently unstable, it is stable in an exponentially weighted norm. In such a norm, unstable modes that travel sufficiently fast in one preferred direction are considered to be stable.

In Section 3.3, we shall compare transient instabilities with convective instabilities. Convective instabilities are related to the absence, in the right half-plane, of temporal eigenvalues λ that correspond to certain spatial double roots ν of the dispersion relation $d_\pm(\lambda, \nu) = 0$; this latter condition typically implies pointwise stability.

Example 1 (*continued*). Without exponential weights, we have

$$A_+ = A_- = \begin{pmatrix} 0 & 1 \\ \lambda & -c \end{pmatrix},$$

and the associated eigenvalues ν satisfy $\nu^2 + c\nu - \lambda = 0$, i.e. $\nu_{1,2} = -\frac{1}{2}c \pm (\frac{1}{4}c^2 + \lambda)^{1/2}$. Therefore, $\nu \in \mathrm{i}\mathbb{R}$ if, and only if, $\lambda = -k^2 + \mathrm{i}ck$ for some $k \in \mathbb{R}$; by the arguments above, this gives the essential spectrum $\Sigma_{\mathrm{ess}} = \{-k^2 + \mathrm{i}ck;\, k \in \mathbb{R}\}$ with "eigenfunctions" $\mathrm{e}^{\mathrm{i}kx}(1, \mathrm{i}k)^{\mathrm{T}}$. Using weights induced by $\eta_+ = \eta_- = \eta$, the essential spectrum is shifted to $\lambda = -k^2 + \eta(\eta - c) + \mathrm{i}k(c - 2\eta)$ with "eigenfunctions" $\exp((\mathrm{i}k - \eta)x)(1, \mathrm{i}k - \eta)^{\mathrm{T}}$. This curve is shifted further to the left if $\eta^2 - \eta c$ is minimal. Thus, $\eta = \frac{1}{2}c$ gives the optimal weight, and we have $\operatorname{Re}\lambda \le -\frac{1}{4}c^2$. This corresponds to the point where the characteristic polynomial $\nu^2 + c\nu - \lambda$ has a double root in the complex plane. Note that the convection term cu_x with c positive has the effect that localized initial conditions travel to the left. Exponential weights with η positive are compatible with such temporal behavior as these weights penalize solutions that travel to the right and favor solutions that travel to the left.

Finally, we remark that the point spectrum is often defined as the set of all isolated eigenvalues with finite multiplicity, i.e. as the set $\tilde{\Sigma}_{\rm pt}$ of those λ for which $\mathcal{T}(\lambda)$ is Fredholm with index zero, the null space of $\mathcal{T}(\lambda)$ is non-trivial, and $\mathcal{T}(\tilde{\lambda})$ is invertible for all $\tilde{\lambda}$ in a small neighborhood of λ.

The sets $\Sigma_{\rm pt}$ and $\tilde{\Sigma}_{\rm pt}$ differ in the following way. The set of λ for which $\mathcal{T}(\lambda)$ is Fredholm with index zero is open. Take a connected component $\mathcal{C}$ of this set, then the following alternative holds. Either $\mathcal{T}(\lambda)$ is invertible for all but a discrete set of elements in $\mathcal{C}$, or else $\mathcal{T}(\lambda)$ has a non-trivial null space for all $\lambda \in \mathcal{C}$. We assume that the latter case does not occur.

Hypothesis 3 (*Isolated eigenvalues*). Eigenvalues in $\mathbb{C} \setminus \Sigma_{\rm ess}$ are isolated with finite multiplicity.

In the connected component of $\mathbb{C} \setminus \Sigma_{\rm ess}$ that contains large positive real numbers, eigenvalues are typically isolated; see, for instance [1] for the relevant argument.

3.2. Absolute spectrum

On bounded domains with separated boundary conditions, it is not the essential spectrum but what we call the absolute spectrum that is important. We remark that the absolute spectrum is not a "spectrum" in that it is not defined as the set of complex numbers for which a certain operator is not invertible; nevertheless, the absolute spectrum gives information about the spectra of certain operators.

Definition 3.5 (*Absolute spectrum*). The subset $\Sigma^+_{\rm abs}$ of $\mathbb{C}$ consists exactly of those λ for which $\mathrm{Re}\,\nu^+_{i_\infty}(\lambda) = \mathrm{Re}\,\nu^+_{i_\infty+1}(\lambda)$. Analogously, λ is in $\Sigma^-_{\rm abs}$ if, and only if, $\mathrm{Re}\,\nu^-_{i_\infty}(\lambda) = \mathrm{Re}\,\nu^-_{i_\infty+1}(\lambda)$. Finally, we say that λ is in the absolute spectrum $\Sigma_{\rm abs}$ of $\mathcal{T}$ if λ is in $\Sigma^+_{\rm abs}$ or in $\Sigma^-_{\rm abs}$ (or in both).

In other words, if $\lambda \notin \Sigma_{\rm abs}$, then there are numbers $\eta_\pm$ such that $\mathrm{Re}\,\nu^\pm_{i_\infty}(\lambda) > -\eta_\pm > \mathrm{Re}\,\nu^\pm_{i_\infty+1}(\lambda)$. In particular, if we ignore point spectrum, then $\mathcal{T}$ is transiently unstable if, and only if, its absolute spectrum is contained in the open left half-plane.

In particular, for constant-coefficient matrices $A(x;\lambda) = A_\infty(\lambda)$, we have that $\mathcal{T}$ is transiently unstable if, and only if, its absolute but not its essential spectrum is contained in the open left half-plane.

Example 1 (*continued*). Recall that

$$A_+ = A_- = \begin{pmatrix} 0 & 1 \\ \lambda & -c \end{pmatrix}$$

with spatial eigenvalues $\nu_{1,2} = -\frac{1}{2}c \pm (\frac{1}{4}c^2 + \lambda)^{1/2}$. Thus, we have $\Sigma_{\rm abs} = (-\infty, -\frac{1}{4}c^2]$ since then $\mathrm{Re}\,\nu_1 = \mathrm{Re}\,\nu_2$. In particular, we have $\Sigma_{\rm ess} \neq \Sigma_{\rm abs}$ except when $c = 0$. The absolute eigenmodes for the absolute spectrum are $\exp((-\frac{1}{2}c + \mathrm{i}k)x)(1, -\frac{1}{2}c + \mathrm{i}k)^{\rm T}$, where $k \in \mathbb{R}$. Growing exponentially as $x \to -\infty$, they reflect the transport to the left that is induced by the linear drift term cu_x.

Typically, we expect that $\Sigma_{\rm abs} \neq \Sigma_{\rm ess}$. One exception is reversible systems that admit a symmetry $x \mapsto -x$. In this case, whenever a spatial eigenvalue crosses from right to left, then, by symmetry, another spatial eigenvalue crosses simultaneously from left to right. Thus, for reversible systems, we expect that $\Sigma_{abs} = \Sigma_{ess}$. An example is the diffusion operator u_{xx} without convection; see above with $c = 0$.

3.3. *Convective instability and pointwise decay*

Convective instability is defined as follows. As before, we label the eigenvalues of $A_\pm(\lambda)$ according to their real part so that

$$\mathrm{Re}\,\nu_1^\pm(\lambda) \geq \cdots \geq \mathrm{Re}\,\nu_{i_\infty}^\pm(\lambda) \geq \mathrm{Re}\,\nu_{i_\infty+1}^\pm(\lambda) \geq \cdots \geq \mathrm{Re}\,\nu_N^\pm(\lambda).$$

We denote by ρ_{db} the largest real number such that there exists a $\lambda_* \in \mathbb{C}$ with $\mathrm{Re}\,\lambda_* = \rho_{db}$ so that $\nu_{i_\infty}^+(\lambda_*) = \nu_{i_\infty+1}^+(\lambda_*)$ or $\nu_{i_\infty}^-(\lambda_*) = \nu_{i_\infty+1}^-(\lambda_*)$. Note that λ_* always corresponds to a spatial double root ν of one of the dispersion relations $d_\pm(\lambda, \nu) = 0$ (recall that replacing $\lambda = \mathrm{i}\omega$ and $\nu = -\mathrm{i}k$ with spatio-temporal behavior $\exp(\mathrm{i}(\omega t - kx))$ gives the standard form of the linear dispersion relation at the asymptotic states, with group velocity $\mathrm{d}\omega/\mathrm{d}k = -\mathrm{d}\lambda/\mathrm{d}\nu$). The above criterion on the double root, namely that is has to involve the spatial eigenvalues with index i_∞ and $i_\infty + 1$, is often called the pinching condition; see [9].

Definition 3.6 (*Convective and absolute instability*). Suppose that Σ_{ess} is not contained in the open left half-plane. We then say that $\mathcal{T}$ is convectively unstable if $\rho_{db} < 0$, while we say that $\mathcal{T}$ is absolutely unstable if $\rho_{db} \geq 0$.

We shall see below that convective instability sometimes implies pointwise stability: perturbations grow in function space but decay pointwise for each fixed x. In other words, they are convected away. The different spectra that we used as well as their characterization in terms of the asymptotic matrices are illustrated in Fig. 2.

The next example demonstrates that even for constant matrices $A(\lambda)$, the operator may be remnantly unstable but not absolutely unstable: this means that, even though there are no double spatial eigenvalues for λ in the closed right half-plane, we cannot move the temporal spectrum into the left half-plane by using exponential weights.

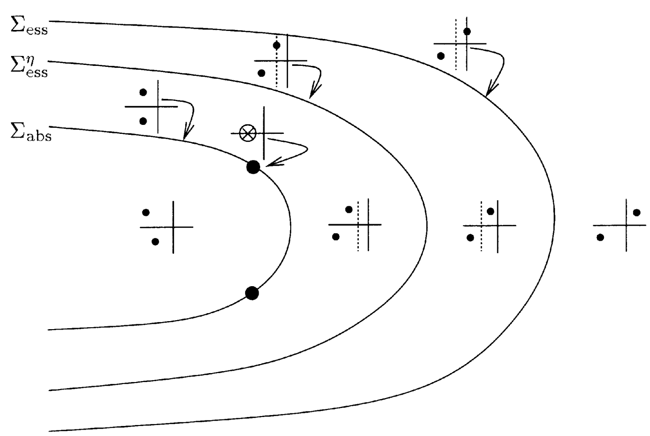

Fig. 2. A schematic picture of the various spectra that we defined and their relationship to the spatial spectra of the matrix $A_+(\lambda) = A_-(\lambda)$ that we plotted as inlets. The essential spectrum of $\mathcal{T}^\eta$ is denoted by Σ_{ess}^η; the dotted line in the spatial spectra consists of all spatial complex numbers with real part $-\eta$. The two circles in the absolute spectrum mark the temporal eigenvalues that correspond to spatial double roots.

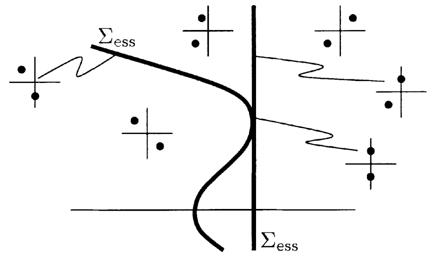

Fig. 3. The thick curves correspond to the essential spectrum of the operator (3.1) while the inlets represent the spatial spectra in the different regions. The two spectral curves intersect at $\lambda = \mathrm{i}$. At this point, the spatial eigenvalues on the imaginary axis are $\nu_1 = \mathrm{i}$ and $\nu_2 = -\mathrm{i}$ which are not equal. Hence, there are no double spatial eigenvalues ν for λ on (or to the right of) the imaginary axis, even though $\lambda = \mathrm{i}$ is in the absolute spectrum.

Example 2. Consider the operator $\mathcal{L}$

$$\mathcal{L}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} \partial_x U_1 \\ -(\partial_x^2+1)^2 U_2 - \partial_x U_2 \end{pmatrix} \tag{3.1}$$

as well as the associated eigenvalue problem

$$\partial_x U_1 = \lambda U_1, \qquad -(\partial_x^2+1)^2 U_2 - \partial_x U_2 = \lambda U_2.$$

Eigenvalues ν of the spatial dynamics solve the characteristic equation

$$(\nu-\lambda)((\nu^2+1)^2+\nu+\lambda) = 0.$$

Double roots occur if ν is a double root of one of the factors or if the roots of the two factors coincide. It is not hard to verify that all double roots that arise as collisions of unstable eigenvalues ν_1 and stable eigenvalues ν_2 occur at values of λ in the open left half-plane; see Fig. 3. On the other hand, the essential spectrum cannot be pushed into the open left half-plane by means of exponential weights since the different signs of the transport terms in the two components of $\mathcal{L}$ would always lead to an instability in one of the two components. Therefore, adding ϵU_1 to the first component of $\mathcal{L}$ and ϵU_2 to the second produces an instability which, for $\epsilon > 0$ sufficiently small, does not disappear when introducing exponential weights even though all relevant spatial double roots occur for λ in the open left half-plane. As we shall see below, this instability is also present on any large bounded interval provided we couple the two components appropriately through the boundary conditions; in fact, generic choices of the boundary conditions will produce such an instability.

Example 3. The same phenomenon can be observed in a Turing–Hopf instability of a reaction–diffusion system provided a small drift term is added to destroy the reflection symmetry. In a Turing–Hopf instability, the first unstable modes are travelling waves of the form $\sin(\omega t - kx)$ and $\sin(\omega t + kx)$ with non-zero k and ω. One of these modes travel to the left, the other one to the right. The superposition of these waves is a standing wave $\sin(\omega t)\cos(kx)$ that corresponds to a double root in the dispersion relation; in other words, the most unstable eigenmodes have zero group velocity. Adding a linear drift term cu_x to the equation transforms these eigenmodes into $\sin(\omega t - (k-c)x)$ and $\sin(\omega t + (k+c)x)$, respectively, which have non-zero group velocity. The spatial eigenvalues $\pm\mathrm{i}(k-c)$ and $\pm\mathrm{i}(k+c)$ are non-resonant, and the system is therefore convectively unstable. On the other hand, the presence of waves that travel to the left and to the right shows that the instability cannot be suppressed in exponentially weighted spaces; hence, the operator is not transiently unstable.

We conclude this section with a brief digression on pointwise stability; we refer to [9] for more details and references regarding this topic. Suppose that $\mathcal{T}(\lambda)$ is invertible so that (2.9) has an exponential dichotomy on $\mathbb{R}$ with evolution operators $\varphi^{\mathrm{u,s}}(x, y; \lambda)$. We can then construct the Green's function $G(x, y; \lambda)$ of the operator $\mathcal{T}(\lambda)$ in the following fashion. The solution of $\mathcal{T}(\lambda)u = h$ is given explicitly by

$$u(x) = \int_{\mathbb{R}} G(x, y; \lambda)h(y)\,\mathrm{d}y,$$

where

$$G(x, y; \lambda) = \begin{cases} -\varphi^{\mathrm{u}}(x, y; \lambda) & \text{for } x < y, \\ \varphi^{\mathrm{s}}(x, y; \lambda) & \text{for } x > y. \end{cases}$$

Using the Green's function, the solution of the linear initial-value problem of

$$\partial_x u - A(x; \partial_t)u = 0 \tag{3.2}$$

can be constructed via Laplace transform in t. Recall that $A(x; \lambda)$ is given by

$$A(x; \lambda) = \hat{A}(x) + \lambda B(x),$$

and define

$$u(x, t) = -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \mathrm{e}^{\lambda t} \int_{\mathbb{R}} G(x, y; \lambda)B(y)u_0(y)\,\mathrm{d}y\,\mathrm{d}\lambda,$$

where the contour Γ is to the right of Σ; the precise shape of Γ depends on the type of the problem. For parabolic problems, Γ can be chosen to include a sector of the left λ-half-plane; for hyperbolic problems, Γ is a vertical line, and the integral is understood to be the principal value. Under reasonable convergence assumptions, and under certain compatibility conditions on $u_0(x)$, the function $u(x, t)$ then satisfies the PDE (3.2) with initial data $u(x, 0) = u_0(x)$.

If the contour Γ can be deformed continuously into a contour that is contained in the left half-plane without changing the value of the integral, the zero-solution is pointwise stable. This follows, for instance, from the Riemann–Lebesgue lemma

$$|u(x, t)| \leq \frac{1}{2\pi} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega t} G(x, y; \mathrm{i}\omega)B(y)u_0(y)\,\mathrm{d}\omega\,\mathrm{d}y \right| \to 0$$

as $t \to \infty$.

To deform the contour, we need analyticity of G in λ for λ in the right complex half-plane and suitable decay estimates for large values of λ. Typically, stability, or at least convective instability of the essential spectrum, is necessary for analyticity of G in λ, since multiple eigenvalues typically create branch points of G. Sufficient conditions are given by convective instability together with the absence of embedded point spectrum. However, an additional condition, known as the Gap lemma, is needed in order to be able to continue the Green's function into regions where hyperbolicity (and exponential dichotomies) is lacking. Roughly speaking, the Gap lemma states that analytic continuation is possible if the exponential convergence of the coefficients of $A(x; \lambda)$ is faster than the lack of hyperbolicity that is created by the unstable and stable part of the overlapping spectrum of $A_{\pm}(\lambda)$; see [22,25]. Necessary and sufficient conditions do not seem to be known.

4. Persistence of isolated eigenvalues and resonance poles with finite multiplicity

In this section, we begin our investigation of the spectrum of $\mathcal{T}_L$ on the bounded interval $(-L, L)$; see (2.6) and (2.7). The goal is to characterize the spectra of $\mathcal{T}_L$ for large values of L. Before we continue, we point out that Fredholm properties no longer classify the spectrum.

Lemma 4.1. *The operators $\mathcal{T}_L(\lambda)$ on the bounded interval $(-L, L)$ with periodic or separated boundary conditions are Fredholm with index zero for all λ.*

Proof. This can be readily seen by considering $\mathcal{T}_L(\lambda)$ as a compact perturbation of $\mathrm{d}/\mathrm{d}x$ with periodic or separated boundary conditions. □

Hence, it suffices to locate eigenvalues of $\mathcal{T}_L$. We begin by studying the persistence of eigenvalues and resonance poles under domain truncation. In addition, we show that separated boundary conditions can sometimes generate additional eigenvalues on $(-L, L)$.

Our strategy is to use various versions of the Evans function. Each Evans function is designed to track isolated eigenvalues with finite multiplicity of one of the operators that we are interested in. We shall then show that the Evans functions defined for bounded intervals are small perturbations of the Evans function that is associated with the entire real line. Since all these functions are analytic, we can then conclude that eigenvalues persist with their multiplicity.

Throughout the remainder of this paper, we denote by $U_\delta(\lambda_*)$ the ball in $\mathbb{C}$ with center λ_* and radius δ.

4.1. Evans functions

Let $E_\pm(\lambda)$ be two subspaces of $\mathbb{C}^N$ that depend analytically on λ such that $n_- + n_+ = N$, where $n_\pm := \dim E_\pm(\lambda)$ is independent of λ. Choose vectors $v_1^\pm(\lambda), \dots, v_{n_\pm}^\pm(\lambda)$ such that

$$E_\pm(\lambda) = \mathrm{span}\{v_1^\pm(\lambda), \dots, v_{n_\pm}^\pm(\lambda)\},$$

and $v_j^\pm(\lambda)$ is analytic in λ for all j; this is possible due to [26, Chapter II.4.2]. We then define

$$E_-(\lambda) \wedge E_+(\lambda) := \det[v_1^-(\lambda), \dots, v_{n_-}^-(\lambda), v_1^+(\lambda), \dots, v_{n_+}^+(\lambda)] \in \mathbb{C}.$$

Note that this function is analytic in λ. In addition, its zeros and the order of its zeros do not depend on the choice of the bases; in fact, any two such functions differ by a product with a non-zero analytic complex-valued function. In this sense, the function depends only on the two subspaces.

Remark 4.2. *We shall often use the following argument: if $E_1(\lambda)$ and $E_2(\lambda)$ are two subspaces of $\mathbb{C}^N$ that depend analytically on $\lambda \in U_{2\delta}(\lambda_*)$ so that $\dim E_1(\lambda) + \dim E_2(\lambda) = N$, then either $\dim(E_1(\lambda) \cap E_2(\lambda)) > 0$ for all $\lambda \in U_{2\delta}(\lambda_*)$ or else $E_1(\lambda) \oplus E_2(\lambda) = \mathbb{C}^N$ for all $\lambda \in U_\delta(\lambda_*)$ except for at most finitely many $\lambda \in U_\delta(\lambda_*)$. This statement follows immediately from analyticity of the determinant $E_1(\lambda) \wedge E_2(\lambda)$ in $\lambda \in U_{2\delta}(\lambda_*)$.*

The following remark, which will be used repeatedly below, shows that the dependence of D on the subspaces $E_\pm$ is continuous in an appropriate sense: we say that two k-dimensional subspaces E and $\hat{E}$ of $\mathbb{C}^N$ are ϵ-close provided $|e - \hat{e}| \le \epsilon$ for all unit vectors $e \in E$ and $\hat{e} \in \hat{E}$.

Remark 4.3. *Suppose that for some $\lambda_* \in \mathbb{C}$,*

$$D(\lambda) := E_-(\lambda) \wedge E_+(\lambda) = (\lambda - \lambda_*)^\ell + \mathrm{O}(|\lambda - \lambda_*|^{\ell+1})$$

for some $\ell \geq 0$. *Suppose that* $\hat{E}_\pm(\lambda)$ *and* $E_\pm(\lambda)$ *are* ϵ*-close to each other, with* ϵ *sufficiently small, uniformly for all* λ *near* λ_*. *By Rouché's theorem, we then have that*

$$\hat{D}(\lambda) = \hat{E}_-(\lambda) \wedge \hat{E}_+(\lambda)$$

has ℓ *zeros* (*counted with multiplicity*) *near* λ_*, *and these zeros are* $\epsilon^{1/\ell}$*-close to* λ_*.

Assume that $D(\lambda)$ is an analytic function. We denote by $\mathrm{ord}(\lambda_*, D)$ the order of λ_* as a zero of $D(\lambda)$. If the order is finite, then it is equal to the winding number of $D(\lambda)$ about any sufficiently small circle in $\mathbb{C}$ that is centered at λ_*.

4.2. Periodic boundary conditions

We begin by investigating the behavior of eigenvalues under domain truncation for periodic boundary conditions. We demonstrate that eigenvalues persist with their multiplicity without any additional assumptions and that no additional eigenvalues are created.

Throughout this section, we assume that $A_\pm(\lambda)$ are equal to each other, and denote $A_\pm(\lambda) = A_0(\lambda)$.

Our proofs are based upon the Evans function. Eigenvalues of $\mathcal{T}$ can be found by seeking bounded solutions to

$$u' = A(x; \lambda)u. \tag{4.1}$$

For $\lambda \notin \Sigma_{\text{ess}}$, the asymptotic matrix $A_0(\lambda)$ is hyperbolic. Eq. (4.1) then has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$, and we denote the associated x-dependent stable and unstable subspaces defined for $x \in \mathbb{R}^+$ and $x \in \mathbb{R}^-$ by $E_+^{\mathrm{s,u}}(x; \lambda)$ and $E_-^{\mathrm{s,u}}(x; \lambda)$, respectively, see Section 2.4. Thus, for every $\lambda \notin \Sigma_{\text{ess}}$, we can define the Evans function

$$D_\infty(\lambda) = E_-^{\mathrm{u}}(0; \lambda) \wedge E_+^{\mathrm{s}}(0; \lambda). \tag{4.2}$$

Note that the dimension of the subspaces that appear in the wedge products in (4.2) add up to N due to the assumption on λ. It has been proved in [1,20] that $\mathrm{ord}(\lambda_*, D_\infty)$ is equal to the multiplicity of λ_* as an eigenvalue of $\mathcal{T}$.

Next, for every $\lambda \notin \Sigma_{\text{ess}}$, we define

$$D_{\text{per}}(\lambda) = \det[\varphi(0, -L; \lambda) - \varphi(0, L; \lambda)], \tag{4.3}$$

where $\varphi(x, y; \lambda)$ is the λ-dependent linear evolution operator to (4.1) with initial time y. It has been proved in [18] that λ_* is an eigenvalue of $\mathcal{T}_L^{\text{per}}$ with multiplicity ℓ if, and only if, λ_* is a zero of $D_{\text{per}}(\lambda)$ of order ℓ.

Theorem 2 (Periodic boundary conditions). *Assume that* $\lambda_* \notin \Sigma_{\text{ess}}$ *and that* $\mathrm{ord}(\lambda_*, D_\infty) = \ell$ *for some* $\ell \geq 0$. *For every small* $\delta > 0$, *there is then an* $L_* > 0$ *such that* $\mathcal{T}_L^{\text{per}}$ *has precisely* ℓ *eigenvalues* (*counted with multiplicity*) *in the* δ*-neighborhood* $U_\delta(\lambda_*)$ *of* λ_* *in* $\mathbb{C}$ *for every* $L \geq L_*$. *For* $\ell > 0$, *these eigenvalues are* $\exp(-2\kappa L/\ell)$*-close to* λ_* *for all* $L \geq L_*$. *Here,* $\kappa = \min\{\kappa^{\mathrm{u}}, \kappa^{\mathrm{s}}\}$ *is the distance of the spectrum of* $A_0(\lambda_*)$ *from the imaginary axis.*

The statement regarding the persistence of eigenvalues with their multiplicity has been proved first in [19] using a topological construction that involved Chern numbers. For the sake of completeness, we include a shorter proof that illustrates in addition that the eigenvalues on the unbounded and the bounded interval are exponentially close; see also [8].

Proof. The strategy is to show that $D_\infty(\lambda)$ and $D_{\text{per}}(\lambda)$ are $\mathrm{e}^{-2\kappa L}$-close to each other for all λ close to λ_*. Recall that

$$D_{\text{per}}(\lambda) = \det[\varphi(0, -L; \lambda) - \varphi(0, L; \lambda)].$$

Since $\lambda_* \notin \Sigma_{\rm ess}$, the matrices $A_0(\lambda)$ are hyperbolic for λ close to λ_*, and we denote their stable and unstable eigenspaces by $E_0^{\rm s}(\lambda)$ and $E_0^{\rm u}(\lambda)$, respectively. Let $i_0 = \dim E_0^{\rm u}(\lambda_*)$. Recall that $E_-^{\rm u}(x;\lambda)$ and $E_+^{\rm s}(x;\lambda)$ consist of precisely those solutions to

$$u' = A(x;\lambda)u$$

that converge to zero as $x \to -\infty$ and $x \to \infty$, respectively. The Evans function $D_\infty(\lambda)$ measures non-trivial intersections of these subspaces evaluated at $x = 0$; see (4.2) and Section 4.1. To capture these intersections, if they exist, we choose analytic bases $\{v_i^-(\lambda)\}_{i=1,\dots,i_0}$ and $\{v_i^+(\lambda)\}_{i=i_0+1,\dots,N}$ of $E_-^{\rm u}(0;\lambda)$ and $E_+^{\rm s}(0;\lambda)$, respectively.

We shall use that, due to Theorem 1, the spaces $E_+^{\rm s,u}(L;\lambda)$ and $E_-^{\rm s,u}(-L;\lambda)$ are $e^{-\hat\theta L}$-close to $E_0^{\rm s,u}(\lambda)$, where $\hat\theta = \min\{\theta, \kappa^{\rm s}+\kappa^{\rm u}\}$.

For every i with $1 \le i \le i_0$, there are then unique vectors $w_i^+(\lambda) \in E_+^{\rm u}(L;\lambda)$ and $w_i^-(\lambda) \in E_-^{\rm s}(-L;\lambda)$ such that

$$\varphi(-L,0;\lambda)v_i^-(\lambda) = w_i^+(\lambda) - w_i^-(\lambda),$$

since $E_+^{\rm u}(L;\lambda)$ and $E_-^{\rm s}(-L;\lambda)$ are $e^{-\hat\theta L}$-close to $E_0^{\rm u}(\lambda)$ and $E_0^{\rm s}(\lambda)$, respectively (see above), and since the direct sum of the latter two spaces is $\mathbb{C}^N$. Since $E_-^{\rm u}(-L;\lambda)$ is also $e^{-\hat\theta L}$-close to $E_0^{\rm u}(\lambda)$, see again above, we have

$$|w_i^+(\lambda)| \le |\varphi(-L,0;\lambda)v_i^-(\lambda)|, \qquad |w_i^-(\lambda)| \le e^{-\hat\theta L}|\varphi(-L,0;\lambda)v_i^-(\lambda)|.$$

We conclude that $w_i^+(\lambda)$ is of the order $\exp(-\kappa^{\rm u}L)$, while $w_i^-(\lambda)$ is of the order $\exp(-(\hat\theta+\kappa^{\rm u})L)$. Finally, for $1 \le i \le i_0$, we define

$$u_i(\lambda) := w_i^+(\lambda) = \varphi(-L,0;\lambda)v_i^-(\lambda) + w_i^-(\lambda).$$

Analogously, for indices i with $i_0+1 \le i \le N$, there are unique vectors $w_i^+(\lambda) \in E_+^{\rm u}(L;\lambda)$ and $w_i^-(\lambda) \in E_-^{\rm s}(-L;\lambda)$ such that

$$u_i(\lambda) := -w_i^-(\lambda) = -\varphi(L,0;\lambda)v_i^+(\lambda) - w_i^+(\lambda).$$

For $i_0+1 \le i \le N$, the vectors $w_i^-(\lambda)$ are of the order $\exp(-\kappa^{\rm s}L)$, while $w_i^+(\lambda)$ is of the order $\exp(-(\hat\theta+\kappa^{\rm s})L)$. Using the estimates above and the definition of $v_i^\pm(\lambda)$, it is not hard to verify that the vectors $u_i(\lambda)$ with $1 \le i \le N$ form a basis of $\mathbb{C}^N$. We conclude that

$$(\varphi(0,-L;\lambda) - \varphi(0,L;\lambda))u_i(\lambda) = \begin{cases} v_i^-(\lambda) + \varphi(0,-L;\lambda)w_i^-(\lambda) - \varphi(0,L;\lambda)w_i^+(\lambda), & 1 \le i \le i_0, \\ v_i^+(\lambda) + \varphi(0,L;\lambda)w_i^+(\lambda) - \varphi(0,-L;\lambda)w_i^-(\lambda), & i_0+1 \le i \le N, \end{cases}$$

where the terms involving $w_i^\pm(\lambda)$ are of the order $e^{-2\kappa L}$ with $\kappa = \min\{\kappa^{\rm u}, \kappa^{\rm s}\}$. On the other hand, we have

$$D_\infty(\lambda) = \det[v_1^-(\lambda), \dots, v_{i_0}^-(\lambda), v_{i_0+1}^+(\lambda), \dots, v_N^+(\lambda)].$$

Invoking Remark 4.3 then proves the statement. □

Upon using the results in [24], it follows from Sandstede [36] that the rate of convergence in the above theorem is optimal.

4.3. Separated boundary conditions

In this section, we investigate the behavior of eigenvalues under domain truncation for separated boundary conditions.

4.3.1. The set-up

Throughout this section, we fix an element λ_* that does not belong to the absolute spectrum $\Sigma_{\rm abs}$. Since λ_* is not in the absolute spectrum, we find weights $\eta = (\eta_-, \eta_+)$ such that the eigenvalues $\nu_j^\pm(\lambda)$ of $A_\pm(\lambda)$ satisfy

$$\mathrm{Re}\, \nu^\pm_{i_\infty}(\lambda) > -\eta_\pm > \mathrm{Re}\, \nu^\pm_{i_\infty+1}(\lambda)$$

for all λ in a small δ_*-neighborhood $U_{\delta_*}(\lambda_*)$ of λ_* with $U_{\delta_*}(\lambda_*) \cap \Sigma_{\rm abs} = \emptyset$. We fix these weights from now onwards and vary λ in the $U_{\delta_*}(\lambda_*)$. In particular, the operator $\mathcal{T}^\eta(\lambda)$ is Fredholm with index zero for any such λ, and the associated asymptotic Morse indices are both equal to i_∞. We then consider the equation

$$\frac{\mathrm{d}v}{\mathrm{d}x} = (A(x;\lambda) + \eta(x))v, \tag{4.4}$$

where $\eta(x) = \eta_+$ for $x > 0$ and $\eta(x) = \eta_-$ for $x < 0$; see (2.5).

Notation. Any quantity that refers to the weighted equation (4.4) has a superscript tilde ($\tilde{}$).

Thus, the evolution operator of (4.4) is denoted by $\tilde\varphi(x, y; \lambda)$. The asymptotic matrices $A_\pm(\lambda)+\eta_\pm$ are hyperbolic, and we denote by $\tilde E^{\rm s,u}_\pm(\lambda)$ their stable and unstable subspaces. Also, by hyperbolicity of the asymptotic matrices $A_\pm(\lambda) + \eta_\pm$, (4.4) has exponential dichotomies on $\mathbb{R}^\pm$ with x-dependent stable and unstable subspaces $\tilde E^{\rm s,u}_\pm(x;\lambda)$, and we can construct an analytic Evans function for $\mathcal{T}^\eta$ by

$$\tilde D_\infty(\lambda) = \tilde E^{\rm u}_-(0;\lambda) \wedge \tilde E^{\rm s}_+(0;\lambda)$$

for $\lambda \in U_{\delta_*}(\lambda_*)$. We also define

$$D_{\rm sep}(\lambda) = \tilde\varphi(0,-L;\lambda)Q_- \wedge \tilde\varphi(0,L;\lambda)Q_+, \quad \tilde D_-(\lambda) = Q_- \wedge \tilde E^{\rm s}_-(\lambda), \quad \tilde D_+(\lambda) = Q_+ \wedge \tilde E^{\rm u}_+(\lambda). \tag{4.5}$$

Note that the dimension of the subspaces that appear in the wedge products in (4.5) add up to N due to the assumption on λ. This is the set-up that we use below.

We point out that the Evans function $D_{\rm sep}$ does not depend upon the choice of the weight. Indeed, solutions $u(x)$ to the original equation (4.1) and $v(x)$ to (4.4) only differ by multiplication by the scalar $\mathrm{e}^{\eta(x)x}$. Thus, the direction of solutions is not changed, and in particular, the x-evolution of subspaces is independent of the weight.

We briefly comment on the dependence of the other Evans functions on the choice of our weight. Through the separated boundary conditions, a canonical dimension, namely i_∞, is selected via the number of boundary conditions at the endpoints of the interval; see Hypothesis 2. The relevant information that we require is a spectral decomposition of the spatial spectrum $\Upsilon_\pm$ of the original asymptotic matrices $A_\pm(\lambda)$ into two spectral sets $\Upsilon^{\rm s,u}_\pm$ which is induced by a gap in the real part (i.e. the spectral sets $\Upsilon^{\rm s,u}_\pm$ are such that $\mathrm{Re}\,\nu^{\rm s}_\pm < \mathrm{Re}\,\nu^{\rm u}_\pm$ for any two elements $\nu^{\rm s}_\pm \in \Upsilon^{\rm s}_\pm$ and $\nu^{\rm u}_\pm \in \Upsilon^{\rm u}_\pm$); most importantly, the associated generalized "stable" and "unstable" eigenspaces have dimension i_∞ and $N-i_\infty$, respectively. One way of obtaining these eigenspaces is by introducing a weight so that the spectral decomposition is given by eigenvalues with negative and positive real part. Afterwards, this decomposition is extended dynamically to x-dependent subspaces of (4.4), and the resulting subspaces are then used to construct the Evans function. Again, weights allow us to construct these x-dependent subspaces by using exponential dichotomies. None of these constructions, however, depends really upon the weights: what we require is that we can distinguish solutions by their growth or decay rate (corresponding to the spectral gap) and that the spaces of initial data leading to these solutions have the correct dimension, namely the one selected by the boundary conditions. As we already mentioned, the x-evolution of subspaces does not depend on the weight.

4.3.2. The persistence and generation of eigenvalues and resonance poles

We discuss the persistence of eigenvalues under domain truncation as well as the potential generation of additional eigenvalues through the boundary conditions. Throughout this section, we use the set-up introduced in Section 4.3.1.

Eigenvalues of $\mathcal{T}_L^{\mathrm{sep}}$ can be found as zeros of the function $D_{\mathrm{sep}}(\lambda)$.

Lemma 4.2 (Ref. [21]). *Assume that $\lambda_* \notin \Sigma_{\mathrm{abs}}$, then λ_* is an eigenvalue of $\mathcal{T}_L^{\mathrm{sep}}$ with multiplicity ℓ if, and only if, λ_* is a zero of $D_{\mathrm{sep}}(\lambda)$ of order ℓ.*

Proof. For the case of eigenvalues of reaction–diffusion equations with separated boundary conditions, the proof can be found in [21, Proposition 4.1]. The proof for the more general situation considered here is the same, save for notation, and we omit it. □

For separated boundary conditions, eigenvalues and resonance poles persist with their multiplicity provided the boundary conditions satisfy appropriate transversality conditions. The situation where these conditions are violated is discussed below.

Lemma 4.3. *Assume that $\lambda_* \notin \Sigma_{\mathrm{abs}}$. Choose a weight η as in Section 4.3.1, and suppose that $\tilde{D}_\pm(\lambda_*) \neq 0$ and $\mathrm{ord}(\lambda_*, \tilde{D}_\infty) = \ell$ for some $\ell \geq 0$. For every small $\delta > 0$, there is then an $L_* > 0$ such that $\mathcal{T}_L^{\mathrm{sep}}$ has precisely ℓ eigenvalues (counted with multiplicity) in $U_\delta(\lambda_*)$ for every $L \geq L_*$. For $\ell > 0$, these eigenvalues are $\exp(-\sigma L/\ell)$-close to λ_* for all $L \geq L_*$, where $\sigma = \min\{\sigma_\pm\}$, and $\sigma_\pm = \mathrm{Re}\,\nu^\pm_{i_\infty}(\lambda_*) - \mathrm{Re}\,\nu^\pm_{i_\infty+1}(\lambda_*)$ are the spectral gaps of the matrices $A_\pm(\lambda_*)$.*

For reaction–diffusion systems, this lemma has been proved in [8] for λ_* to the right of the essential spectrum. Note that the rate of convergence proved in [8] is smaller than the rate that we establish here. The reason for the improved rate is that we can always balance the distance from the stable and unstable part of the spectrum to the imaginary axis by adjusting the weights. In contrast to the case of periodic boundary conditions, this does not change the boundary conditions. Again, our rate is optimal, see [36], except when the boundary conditions Q_- and Q_+ happen to coincide with the unstable and stable subspaces $\tilde{E}^{\mathrm{u}}_-(\lambda_*)$ and $\tilde{E}^{\mathrm{s}}_+(\lambda_*)$, respectively.

Proof. The proof is similar to the one given above for periodic boundary conditions. We consider the weighted equation (4.4) and use the notation introduced in Section 4.3.1, recall that

$$D_{\mathrm{sep}}(\lambda) = \tilde{\varphi}(0, -L; \lambda) Q_- \wedge \tilde{\varphi}(0, L; \lambda) Q_+.$$

Choose an analytic basis $\{v_j^+(\lambda)\}_{j=1,\dots,N-i_\infty}$ of $\tilde{E}^{\mathrm{s}}_+(0;\lambda)$. Since $\tilde{D}_+(\lambda) \neq 0$ for all λ close to λ_* by assumption, we have

$$Q_+ \oplus \tilde{E}^{\mathrm{u}}_+(\lambda) = \mathbb{C}^N.$$

Since $\tilde{E}^{\mathrm{s}}_+(L;\lambda)$ and $\tilde{E}^{\mathrm{u}}_+(L;\lambda)$ converge to $\tilde{E}^{\mathrm{s}}_+(\lambda)$ and $\tilde{E}^{\mathrm{u}}_+(\lambda)$, respectively, exponentially fast as $L \to \infty$, see Theorem 1, there are unique vectors $w_j^+(\lambda) \in \tilde{E}^{\mathrm{u}}_+(L;\lambda)$ such that

$$Q_+ = \mathrm{span}\{\tilde{\varphi}(L, 0; \lambda) v_j^+(\lambda) + w_j^+(\lambda);\ j = 1, \dots, i_\infty\}.$$

As in Theorem 2, we obtain that

$$\tilde{\varphi}(0, L; \lambda) Q_+ = \mathrm{span}\{v_j^+(\lambda) + \tilde{\varphi}(0, L; \lambda) w_j^+(\lambda);\ j = 1, \dots, i_\infty\}.$$

As a consequence, $\tilde{\varphi}(0, L; \lambda)Q_+$ and $\tilde{E}^{\mathrm{s}}_+(0; \lambda)$ are $\exp(-\sigma_+ L)$-close to each other. By the same argument, we have that $\tilde{\varphi}(0, -L; \lambda)Q_-$ and $\tilde{E}^{\mathrm{u}}_-(0; \lambda)$ are $\exp(-\sigma_- L)$-close to each other. Since

$$\tilde{D}_\infty(\lambda) = \tilde{E}^{\mathrm{u}}_-(0; \lambda) \wedge \tilde{E}^{\mathrm{s}}_+(0; \lambda),$$

the statements of the lemma follow from Remark 4.3 and Lemma 4.2. □

Remark 4.4. *In the set-up of Lemma* 4.3, *we have that an eigenfunction $u(x)$ to the original equation* (4.1) *on the interval $(-L, L)$ typically satisfies*

$$|u(-L)| \approx \exp(-\mathrm{Re}\, \nu^-_{i_\infty}(\lambda)L), \qquad |u(L)| \approx \exp(\mathrm{Re}\, \nu^+_{i_\infty+1}(\lambda)L).$$

In particular, the convective properties of resonance poles manifest themselves via the growth of the associated eigenmodes at $x = \pm L$ depending on the direction of transport. The remark is a consequence of the proof of Lemma 4.3.

Next, we investigate eigenvalues that are created by separated boundary conditions near points where either $\tilde{D}_+$ or $\tilde{D}_-$ vanishes.

Lemma 4.4. *Assume that $\lambda_* \notin \Sigma_{\mathrm{abs}}$. Choose a weight η as described in Section* 4.3.1. *Suppose that $\tilde{D}_-(\lambda_*) \neq 0$, $\tilde{D}_\infty(\lambda_*) \neq 0$ and $\mathrm{ord}(\lambda_*, \tilde{D}_+) = \ell$ for some $\ell > 0$. For every $\delta > 0$ sufficiently small, there is then an $L_* > 0$ such that $\mathcal{T}^{\mathrm{sep}}_L$ has precisely ℓ eigenvalues (counted with multiplicity) in $U_\delta(\lambda_*)$ for every $L \geq L_*$. In addition, these eigenvalues are $\exp(-\alpha_+ L/\ell)$-close to λ_* for all $L \geq L_*$. Here, $\alpha_+ = \min\{\theta, \sigma_+\}$ where θ appears in Hypothesis* 1, *and σ_+ has been introduced in Lemma* 4.3.

We have an analogous lemma in the case that $\tilde{D}_+(\lambda_*) \neq 0$, $\tilde{D}_\infty(\lambda_*) \neq 0$ and $\mathrm{ord}(\lambda_*, \tilde{D}_-) = \ell$.

Proof. The general set-up is as in Section 4.3.1. We write

$$D_{\mathrm{sep}}(\lambda) = \tilde{\varphi}(0, -L; \lambda)Q_- \wedge \tilde{\varphi}(0, L; \lambda)Q_+ = \det[\tilde{\varphi}(0, L; \lambda)] \times (\tilde{\varphi}(L, -L; \lambda)Q_- \wedge Q_+),$$

and note that it suffices to determine the number of zeros of the function

$$\tilde{\varphi}(L, -L; \lambda)Q_- \wedge Q_+$$

since $\det[\tilde{\varphi}(0, L; \lambda)] \neq 0$ for all L and λ. Since $Q_- \oplus \tilde{E}^{\mathrm{s}}_-(\lambda_*) = \mathbb{C}^N$ by assumption, it follows that $\tilde{\varphi}(0, -L; \lambda)Q_-$ is $\exp(-\sigma_- L)$-close to $\tilde{E}^{\mathrm{u}}_-(0; \lambda)$ uniformly for λ close to λ_*; see the proof of Lemma 4.3. Since $\tilde{D}_\infty(\lambda_*) \neq 0$, we see that $\tilde{E}^{\mathrm{u}}_-(0; \lambda) \oplus \tilde{E}^{\mathrm{s}}_+(0; \lambda) = \mathbb{C}^N$. Hence, we can conclude that $\tilde{\varphi}(L, -L; \lambda)Q_-$ is $\exp(-\sigma_+ L)$-close to $\tilde{E}^{\mathrm{u}}_+(L; \lambda)$, and therefore $\exp(-\min\{\theta, \sigma_+\}L)$-close to $\tilde{E}^{\mathrm{u}}_+(\lambda)$ uniformly in λ; see the arguments in the proof of Lemma 4.3. Since $\tilde{D}_+(\lambda) = Q_+ \wedge \tilde{E}^{\mathrm{u}}_+(\lambda)$, the statements of the lemma follow again from Remark 4.3 and Lemma 4.2. □

The general case where $\tilde{D}_\pm$ and $\tilde{D}_\infty$ vanish for the same value of λ is treated in the following theorem.

Theorem 3 (Separated boundary conditions). *Assume that $\lambda_* \notin \Sigma_{\mathrm{abs}}$. Choose a weight η as described in Section* 4.3.1. *Suppose that $\mathrm{ord}(\lambda_*, \tilde{D}_\pm) = \ell_\pm$ and $\mathrm{ord}(\lambda_*, \tilde{D}_\infty) = \ell_\infty$ for some $\ell_\pm$ and ℓ_∞. For every small $\delta > 0$, there is then an $L_* > 0$ such that $\mathcal{T}^{\mathrm{sep}}_L$ has precisely $\ell_- + \ell_+ + \ell_\infty$ eigenvalues (counted with multiplicity) in a δ-neighborhood of λ_* for every $L \geq L_*$. If either $\ell_\pm = 0$ or $\ell_\infty = 0$, then error estimates for the eigenvalues on $(-L, L)$ are given in Lemmas* 4.3 *and* 4.4, *respectively*.

Proof. Recall that, if $\tilde{D}_\pm(\lambda) \neq 0$, then $D_{\rm sep}(\lambda)$ and $\tilde{D}_\infty(\lambda)$ are $e^{-\sigma L}$-close to each other uniformly in L and λ; see the proof of Lemma 4.3. Hence, there is a $\delta_* > 0$ such that, for every $0 < \delta < \delta_*$, there are numbers L_* and $\epsilon_* > 0$ with $|D_{\rm sep}(\lambda)| > \epsilon_*$ for all $L \geq L_*$ and all λ with $|\lambda - \lambda_*| = \delta$. In particular, the number of zeros of $D_{\rm sep}$ inside the δ-neighborhood $U_\delta(\lambda_*)$ is independent of L for $L \geq L_*$. In the following, we fix such a δ and the associated $\epsilon_* > 0$.

Next, choose a subspace $\hat{Q}_+$ so close to Q_+ that $|\tilde{D}_+(\lambda) - \hat{D}_+(\lambda)| < \frac{1}{2}\epsilon_*$ for λ with $|\lambda - \lambda_*| = \delta$ but such that $\hat{Q}_+ \oplus \tilde{E}^{\rm u}_+(\lambda_*) = \mathbb{C}^N$. Here, $\hat{D}_+(\lambda) = \tilde{Q}_+ \wedge \tilde{E}^{\rm u}_+(\lambda)$. Hence, the number of zeros of $\hat{D}_+(\lambda)$ inside $U_\delta(\lambda_*)$ is also equal to ℓ_+.

Similarly, choose a subspace $\hat{Q}_-$ with analogous properties; in addition, we require that $\hat{Q}_-$ is chosen such that $\hat{D}_-(\lambda) \neq 0$ whenever $\hat{D}_+(\lambda) = 0$ for $\lambda \in U_\delta(\lambda_*)$. Such a choice is clearly possible since $\hat{D}_+(\lambda)$ has only finitely many zeros in $U_\delta(\lambda_*)$.

As a consequence of the above arguments, there is a number L_* that depends on δ and the above choices of $\hat{Q}_\pm$ such that $D_{\rm sep}(\lambda)$ and $\hat{D}_{\rm sep}(\lambda)$ are $\exp(-\sigma L)$-close to each other for all λ with $|\lambda - \lambda_*| = \delta$ and all $L \geq \tilde{L}_*$; indeed, both functions are $e^{-\sigma L}$-close to $\tilde{D}_\infty(\lambda)$. Thus, $D_{\rm sep}(\lambda)$ and $\hat{D}_{\rm sep}(\lambda)$ have the same number of zeros in $U_\delta(\lambda_*)$ for all L sufficiently large. Due to Lemmas 4.3 and 4.4, $\hat{D}_{\rm sep}(\lambda)$ has precisely $\ell_- + \ell_+ + \ell_\infty$ zeros in $U_\delta(\lambda_*)$ since $\tilde{D}_\infty(\lambda)$ has not changed, and $\hat{D}_\pm(\lambda)$ and $\tilde{D}_\infty(\lambda)$ have no common zeros by construction. This completes the proof. □

4.4. Resolvent estimates for periodic and separated boundary conditions

In this section, we establish estimates for the inverse of the operator $\mathcal{T}_L(\lambda)$ posed on the interval $(-L, L)$.

For periodic boundary conditions, it is a consequence of the results presented in [8,27] that the inverse of $\mathcal{T}_L^{\rm per}(\lambda)$ is bounded uniformly in L for λ away from the point and essential spectrum of $\mathcal{T}$ posed on $\mathbb{R}$. Alternatively, the proofs given below for separated boundary conditions can be adapted in a straightforward fashion to the case of periodic boundary conditions.

We therefore concentrate on the case of separated boundary conditions. Our main result in this section is that, under certain assumptions which are stated below, the norm of the inverse of $\mathcal{T}_L^{\rm sep}(\lambda)$ grows exponentially in L for any fixed λ for which $i_+(\lambda)$ or $i_-(\lambda)$ differs from i_∞; recall that $i_\pm(\lambda)$ are the asymptotic Morse indices of the matrices $A_\pm(\lambda)$. Roughly speaking, the inverse of $\mathcal{T}_L^{\rm sep}(\lambda)$ grows exponentially for every λ that is to the left of the boundary of the essential spectrum while being close to it: note that the boundary of the essential spectrum is given as the union of algebraic curves. For the operator $u_{xx} + u_x$, this fact has been proved in [34]. Besides its importance for the stability and convergence of numerical algorithms for the computation of spectra, the exponential growth of the resolvent in the region to the left of the essential spectrum has the following interesting consequence: suppose that $\mathcal{T}_L^{\rm sep}$ corresponds to the eigenvalue problem of the linearization about a certain nonlinear wave that is only transiently unstable. Since the resolvent grows as the interval length increases, the sensitivity of the wave with respect to small initial perturbations also increases. The large norm of the resolvent predicts a large constant in front of the exponential decay factor of the semigroup. Before the system picks up the exponential decay predicted from spectral information, there will be a long intermediate regime where solutions first grow in norm while travelling to one end of the domain, a phenomenon which is most easily illustrated in the pure convection problem $u_t = u_x + u$ with boundary condition $u(L) = 0$: localized initial conditions grow along characteristics $x = -t$ until they disappear through the boundary $x = -L$; in fact, the explicit solution is given by $u(x,t) = e^t u_0(x+t)$. With increasing sensitivity, stability then depends more and more on the nonlinear terms. We also refer to [44] for a discussion.

The remaining part of this section contains the precise statements of the relevant hypotheses and the results. Most of it is rather technical and can be skipped by the reader; we do not use any of these results in the following sections.

Throughout this section, we assume that $\lambda \notin \Sigma_{\mathrm{abs}}$. We begin by choosing weights $\eta = (\eta_-, \eta_+)$ so that $\mathrm{Re}\, \nu^{\pm}_{i_\infty}(\lambda) > -\eta_\pm > \mathrm{Re}\, \nu^{\pm}_{i_\infty+1}(\lambda)$; see Section 4.3.1. It is then a consequence of Theorem 3 that $\mathcal{T}_L^{\mathrm{sep}}(\lambda)$ is invertible for all $L \geq L_*$ if, and only if, $\tilde{D}_\pm(\lambda) \neq 0$ and $\tilde{D}_\infty(\lambda) \neq 0$. In this situation, we have to estimate the solution $u(x)$ to

$$\frac{\mathrm{d}u}{\mathrm{d}x} = A(x;\lambda)u + B(x)h(x), \quad u(\pm L) \in Q_\pm \tag{4.6}$$

on the interval $(-L, L)$ in terms of $h(x)$. The reason why we restrict to right-hand sides of the form $B(x)h(x)$ is that we are primarily interested in resolvent estimates for the underlying PDE operator that we had cast as a first-order operator. All of our results, however, are also true, and in fact easier to prove, in the case of general right-hand sides; see below.

Next, we consider the equation in the weighted space. We shall then establish estimates of the solution $v(x)$ to

$$\frac{\mathrm{d}v}{\mathrm{d}x} = (A(x;\lambda) + \eta(x))v + B(x)g(x), \quad v(\pm L) \in Q_\pm \tag{4.7}$$

on the interval $(-L, L)$ in terms of $g(x)$, where $\eta(x) = \eta_+$ for $x > 0$ and $\eta(x) = \eta_-$ for $x < 0$ has been chosen above. The functions u and v as well as h and g are then related via

$$u(x) = \mathrm{e}^{-\eta(x)x}v(x), \qquad g(x) = \mathrm{e}^{\eta(x)x}h(x). \tag{4.8}$$

Since $\tilde{D}_\infty(\lambda) \neq 0$, the equation

$$\frac{\mathrm{d}v}{\mathrm{d}x} = (A(x;\lambda) + \eta(x))v \tag{4.9}$$

has an exponential dichotomy on $\mathbb{R}$ with evolution operators $\tilde{\varphi}^{\mathrm{s}}(x, y; \lambda)$ and $\tilde{\varphi}^{\mathrm{u}}(x, y; \lambda)$ so that the estimates in Definition 2.1 are met for $I = \mathbb{R}$. In particular, we have

$$|\tilde{\varphi}^{\mathrm{s}}(L, 0; \lambda)| \leq K \exp(-\tilde{\kappa}^{\mathrm{s}}_+ L), \qquad |\tilde{\varphi}^{\mathrm{s}}(0, -L; \lambda)| \leq K \exp(-\tilde{\kappa}^{\mathrm{s}}_- L), \tag{4.10}$$

and the analogous estimates for $\tilde{\varphi}^{\mathrm{u}}$. The stable and unstable subspaces of the asymptotic matrices $A_\pm(\lambda) + \eta_\pm$ are denoted by $\tilde{E}^{\mathrm{s,u}}_\pm(\lambda)$. Similarly, the spectral projections of $A_\pm(\lambda) + \eta_\pm$ belonging to the stable and unstable spectral sets are denoted by $\tilde{P}^{\mathrm{s,u}}_\pm(\lambda)$.

The general solution to

$$v' = (A(x;\lambda) + \eta(x))v + B(x)g(x)$$

is given by

$$v(x) = \int_{-L}^{x} \tilde{\varphi}^{\mathrm{s}}(x, y; \lambda)B(y)g(y)\,\mathrm{d}y + \int_{L}^{x} \tilde{\varphi}^{\mathrm{u}}(x, y; \lambda)B(y)g(y)\,\mathrm{d}y + \tilde{\varphi}^{\mathrm{s}}(x, -L; \lambda)a^{\mathrm{s}}_- + \tilde{\varphi}^{\mathrm{u}}(x, L; \lambda)a^{\mathrm{u}}_+, \tag{4.11}$$

where $a^{\mathrm{s}}_- \in \tilde{E}^{\mathrm{s}}_-(\lambda)$ and $a^{\mathrm{u}}_+ \in \tilde{E}^{\mathrm{u}}_+(\lambda)$ are arbitrary.

It remains to satisfy the boundary conditions $v(\pm L) \in Q_\pm$. Since $\tilde{D}_\pm(\lambda) \neq 0$, we have

$$\tilde{E}^{\mathrm{u}}_+(\lambda) \oplus Q_+ = \mathbb{C}^n, \qquad \tilde{E}^{\mathrm{s}}_-(\lambda) \oplus Q_- = \mathbb{C}^n. \tag{4.12}$$

Hence, the boundary conditions are equivalent to the equation

$$
\begin{aligned}
&P(\tilde{E}^{\mathrm{u}}_{+}(\lambda), Q_{+})\left[\int_{-L}^{L}\tilde{\varphi}^{\mathrm{s}}(L, y; \lambda)B(y)g(y)\,\mathrm{d}y + \tilde{\varphi}^{\mathrm{s}}(L, -L; \lambda)a^{\mathrm{s}}_{-} + \tilde{\varphi}^{\mathrm{u}}(L, L; \lambda)a^{\mathrm{u}}_{+}\right] = 0,\\
&P(\tilde{E}^{\mathrm{s}}_{-}(\lambda), Q_{-})\left[-\int_{-L}^{L}\tilde{\varphi}^{\mathrm{u}}(-L, y; \lambda)B(y)g(y)\,\mathrm{d}y + \tilde{\varphi}^{\mathrm{s}}(-L, -L; \lambda)a^{\mathrm{s}}_{-} + \tilde{\varphi}^{\mathrm{u}}(-L, L; \lambda)a^{\mathrm{u}}_{+}\right] = 0, \qquad (4.13)
\end{aligned}
$$

where $P(X, Y)$ is the projection with range X and null space Y. By Theorem 1, $\tilde{\varphi}^{\mathrm{u}}(L, L; \lambda)$ and $\tilde{\varphi}^{\mathrm{s}}(-L, -L; \lambda)$ are $\mathrm{e}^{-\hat{\theta}L}$-close to the spectral projections $\tilde{P}^{\mathrm{u}}_{+}(\lambda)$ and $\tilde{P}^{\mathrm{s}}_{-}(\lambda)$, respectively, where $\hat{\theta} = \min\{\theta, \tilde{\kappa}^{\mathrm{u}}_{\pm} + \tilde{\kappa}^{\mathrm{s}}_{\pm}\}$. Exploiting (4.10), we get

$$
\begin{aligned}
&P(\tilde{E}^{\mathrm{u}}_{+}(\lambda), Q_{+})[\tilde{\varphi}^{\mathrm{s}}(L, -L; \lambda)a^{\mathrm{s}}_{-} + \tilde{\varphi}^{\mathrm{u}}(L, L; \lambda)a^{\mathrm{u}}_{+}]\\
&\quad = P(\tilde{E}^{\mathrm{u}}_{+}(\lambda), Q_{+})[\mathrm{O}(\exp(-(\tilde{\kappa}^{\mathrm{s}}_{-} + \tilde{\kappa}^{\mathrm{s}}_{+})L))a^{\mathrm{s}}_{-} + (\tilde{P}^{\mathrm{u}}_{+}(\lambda) + \mathrm{O}(\mathrm{e}^{-\theta L}))a^{\mathrm{u}}_{+}],\\
&\quad\quad \times P(\tilde{E}^{\mathrm{s}}_{-}(\lambda), Q_{-})[\tilde{\varphi}^{\mathrm{s}}(-L, -L; \lambda)a^{\mathrm{s}}_{-} + \tilde{\varphi}^{\mathrm{u}}(-L, L; \lambda)a^{\mathrm{u}}_{+}]\\
&\quad = P(\tilde{E}^{\mathrm{s}}_{-}(\lambda), Q_{-})[(\tilde{P}^{\mathrm{s}}_{-}(\lambda) + \mathrm{O}(\mathrm{e}^{-\theta L}))a^{\mathrm{s}}_{-} + \mathrm{O}(\exp(-(\tilde{\kappa}^{\mathrm{u}}_{-} + \tilde{\kappa}^{\mathrm{u}}_{+})L))a^{\mathrm{u}}_{+}].
\end{aligned}
$$

Using this equation, we see that (4.13) has a unique solution given by

$$
\begin{aligned}
\begin{pmatrix} a^{\mathrm{u}}_{+} \\ a^{\mathrm{s}}_{-} \end{pmatrix} &= (1 + \mathrm{O}(\mathrm{e}^{-\beta L}))\begin{pmatrix} 1 & \mathrm{O}(\exp(-(\tilde{\kappa}^{\mathrm{s}}_{-} + \tilde{\kappa}^{\mathrm{s}}_{+})L)) \\ \mathrm{O}(\exp(-(\tilde{\kappa}^{\mathrm{u}}_{-} + \tilde{\kappa}^{\mathrm{u}}_{+})L)) & 1 \end{pmatrix}\\
&\quad \times \begin{pmatrix} P(\tilde{E}^{\mathrm{u}}_{+}(\lambda), Q_{+})\int_{-L}^{L}\tilde{\varphi}^{\mathrm{s}}(L, y; \lambda)B(y)g(y)\,\mathrm{d}y \\ -P(\tilde{E}^{\mathrm{s}}_{-}(\lambda), Q_{-})\int_{-L}^{L}\tilde{\varphi}^{\mathrm{u}}(-L, y; \lambda)B(y)g(y)\,\mathrm{d}y \end{pmatrix}, \qquad (4.14)
\end{aligned}
$$

where $\beta = \min\{\hat{\theta}, \tilde{\kappa}^{\mathrm{s}}_{-} + \tilde{\kappa}^{\mathrm{s}}_{+} + \tilde{\kappa}^{\mathrm{u}}_{-} + \tilde{\kappa}^{\mathrm{u}}_{+}\}$, and we have the estimate

$$|a^{\mathrm{s}}_{-}| + |a^{\mathrm{u}}_{+}| \le C\|g\|$$

for some constant C that is independent of L for $L \ge L_*$.

It remains to relate the resolvent estimates for the v-equation (4.7) to resolvent estimates for the u-equation (4.6). If we can choose $\eta_{\pm} = 0$, then the above analysis demonstrates that the inverse of $\mathcal{T}^{\mathrm{sep}}_{L}(\lambda)$ is bounded uniformly in L. Indeed, note that the integral operators in (4.11) are uniformly bounded in L due to the exponential decay of the evolution operators $\tilde{\varphi}^{\mathrm{s,u}}$. Uniform bounds of the other two summands in (4.11) follow again from the bounds on a^{s}_{-} and a^{u}_{+} above. We summarize this result in the following proposition.

Proposition 1. *Assume that* $\lambda_* \notin \Sigma_{\mathrm{abs}}$ *and that* $\mathrm{Re}\,\nu^{\pm}_{i_\infty+1} < 0 < \mathrm{Re}\,\nu^{\pm}_{i_\infty}$. *Furthermore, we assume that* $D_{\pm}(\lambda_*) \neq 0$ *and* $D_{\infty}(\lambda_*) \neq 0$. *Then there are positive constants* δ, C *and* L_* *such that the inverse of* $\mathcal{T}^{\mathrm{sep}}_{L}(\lambda)$ *is bounded by* C *uniformly in* $L \ge L_*$ *for all* $\lambda \in U_{\delta}(\lambda_*)$.

If, on the other hand, we have to choose non-zero values for one or both of the rates $\eta_{\pm}$, then we expect that the inverse of $\mathcal{T}^{\mathrm{sep}}_{L}(\lambda)$ actually increases exponentially as L increases. The reason is that even though the unweighted and weighted norms are equivalent on $(-L, L)$, the equivalence constants grow exponentially in L.

We have

$$u(x) = \mathrm{e}^{-\eta(x)x}\left(\int_{-L}^{x} \tilde{\varphi}^{\mathrm{s}}(x, y; \lambda)\,\mathrm{e}^{\eta(y)y} B(y)h(y)\,\mathrm{d}y + \int_{L}^{x} \tilde{\varphi}^{\mathrm{u}}(x, y; \lambda)\,\mathrm{e}^{\eta(y)y} B(y)h(y)\,\mathrm{d}y \right.$$
$$\left. + \tilde{\varphi}^{\mathrm{s}}(x, -L; \lambda)a_-^{\mathrm{s}} + \tilde{\varphi}^{\mathrm{u}}(x, L; \lambda)a_+^{\mathrm{u}}\right), \tag{4.15}$$

where $(a_-^{\mathrm{s}}, a_+^{\mathrm{u}})$ are given by

$$\begin{pmatrix} a_+^{\mathrm{u}} \\ a_-^{\mathrm{s}} \end{pmatrix} = (1 + \mathrm{O}(\mathrm{e}^{-\beta L}))\begin{pmatrix} 1 & \mathrm{O}(\exp(-2\tilde{\kappa}^{\mathrm{s}}L)) \\ \mathrm{O}(\exp(-2\tilde{\kappa}^{\mathrm{u}}L)) & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} P(\tilde{E}_+^{\mathrm{u}}(\lambda), Q_+)\int_{-L}^{L} \tilde{\varphi}^{\mathrm{s}}(L, y; \lambda)\,\mathrm{e}^{\eta(y)y} B(y)h(y)\,\mathrm{d}y \\ -P(\tilde{E}_-^{\mathrm{s}}(\lambda), Q_-)\int_{-L}^{L} \tilde{\varphi}^{\mathrm{u}}(-L, y; \lambda)\,\mathrm{e}^{\eta(y)y} B(y)h(y)\,\mathrm{d}y \end{pmatrix}. \tag{4.16}$$

Here, $\tilde{\kappa}^{\mathrm{s,u}} = \min\{\tilde{\kappa}_\pm^{\mathrm{s,u}}\}$. First, we consider the case that the eigenvalues of $A_+(\lambda)$ satisfy the condition $\mathrm{Re}\,\nu_{i_\infty+1}^+ < \mathrm{Re}\,\nu_{i_\infty}^+ < 0$. Afterwards, we investigate the case $0 < \mathrm{Re}\,\nu_{i_\infty+1}^+ < \mathrm{Re}\,\nu_{i_\infty}^+$. The analogous cases for $A_-(\lambda)$ are handled in the same fashion; upon reversing the spatial variable $x \mapsto -x$, we end up with one of the aforementioned cases for the eigenvalues of $A_+(\lambda)$.

Thus, assume that $\mathrm{Re}\,\nu_{i_\infty+1}^+ < -\eta_+ < \mathrm{Re}\,\nu_{i_\infty}^+ < 0$ so that we have $\eta_+ > 0$.

Hypothesis 4. We assume that $\mathrm{Re}\,\nu_{i_\infty+1}^+ < \mathrm{Re}\,\nu_{i_\infty}^+ < 0$ and that $\mathrm{Re}\,\nu_{i_\infty}^+ < \mathrm{Re}\,\nu_{i_\infty-1}^+$. Furthermore, we assume that there is a vector $h_+ \in \mathbb{C}^n$ such that

$$(P(\tilde{E}_+^{\mathrm{u}}(\lambda), Q_+)\tilde{P}_+^{\mathrm{s}}(\lambda) - \tilde{P}_+^{\mathrm{u}}(\lambda))B_+h_+$$

has a non-zero component in the eigendirection of $A_+(\lambda)$ associated with the simple eigenvalue $\nu_{i_\infty}^+$, where we express vectors with respect to the basis that consist of (generalized) eigenvectors of $A_+(\lambda)$.

The above hypothesis can be interpreted as requiring that a certain transmission coefficient is non-zero. In the situation considered here, we have $\eta_+ > 0$ so that the rest state at $x = \infty$ sustains waves that travel to the left. The above hypothesis guarantees that the boundary condition at $x = L$ emits such waves: since the waves grow as they travel to the left, we expect that the resolvent grows as L increases.

Proposition 2. *Assume that Hypothesis* 4 *is met. The inverse of* $\mathcal{T}_L^{\mathrm{sep}}(\lambda)$ *grows exponentially with rate equal to or bigger than* $|\mathrm{Re}\,\nu_{i_\infty}^+|$.

The growth rate of the resolvent is not optimal; see Remark 4.5.

Proof. We define $h(x)$ by $h(x) = h_+$ for $L - \rho \le x \le L$ and zero otherwise. Hence, the integrands of the integrals above are zero whenever $x < L - \rho$. Furthermore, we have

$$\left|\int_{L-\rho}^{L} \tilde{\varphi}^{\mathrm{s}}(L, y; \lambda)\exp(\eta_+(y - L))B(y)h(y)\,\mathrm{d}y - \tilde{P}_+^{\mathrm{s}}(\lambda)B_+h_+\rho\right|$$
$$+ \left|\int_{L-\rho}^{L} \tilde{\varphi}^{\mathrm{u}}(L - \rho, y; \lambda)\exp(\eta_+(y - L))B(y)h(y)\,\mathrm{d}y - \tilde{P}_+^{\mathrm{u}}(\lambda)B_+h_+\rho\right| \le C(h_+)(\rho^2 + \mathrm{e}^{-\hat{\theta}L}) \tag{4.17}$$

uniformly in ρ and L, where $\hat{\theta} = \min\{\theta, \tilde{\kappa}^{\mathrm{u}}_{\pm} + \tilde{\kappa}^{\mathrm{s}}_{\pm}\}$. Using these expressions in (4.16), we obtain

$$a^{\mathrm{u}}_{+} = \exp(\eta_+ L)[P(\tilde{E}^{\mathrm{u}}_{+}(\lambda), Q_+)\tilde{P}^{\mathrm{s}}_{+}(\lambda)B_+ h_+ \rho + \mathrm{O}(\rho^2 + \mathrm{e}^{-\gamma L})], \tag{4.18}$$

where $\gamma = \min\{\hat{\theta}, \tilde{\kappa}^{\mathrm{u}}_{\pm}, \tilde{\kappa}^{\mathrm{s}}_{\pm}\}$. Evaluating (4.15) at $x = 0$, we get

$$\begin{aligned} u(0) = &-\exp(\eta_+ L)\tilde{\varphi}^{\mathrm{u}}(0, L-\rho; \lambda)\int_{L-\rho}^{L} \tilde{\varphi}^{\mathrm{u}}(L-\rho, y; \lambda)\exp(\eta_+(y-L))B(y)h(y)\,\mathrm{d}y \\ &+\tilde{\varphi}^{\mathrm{s}}(0, -L; \lambda)a^{\mathrm{s}}_{-} + \tilde{\varphi}^{\mathrm{u}}(0, L; \lambda)a^{\mathrm{u}}_{+}, \end{aligned}$$

and therefore, upon substituting (4.18),

$$\begin{aligned} u(0) = &\exp(\eta_+ L)\tilde{\varphi}^{\mathrm{u}}(0, L; \lambda)[(\tilde{P}^{\mathrm{u}}_{+}(\lambda) - P(\tilde{E}^{\mathrm{u}}_{+}(\lambda), Q_+)\tilde{P}^{\mathrm{s}}_{+}(\lambda))B_+ h_+ \rho + \mathrm{O}(\rho^2 + \mathrm{e}^{-\gamma L})] \\ &+\tilde{\varphi}^{\mathrm{s}}(0, -L; \lambda)a^{\mathrm{s}}_{-}. \end{aligned} \tag{4.19}$$

Since (4.9) has an exponential dichotomy on $\mathbb{R}$, the subspaces $\mathbf{R}(\tilde{\varphi}^{\mathrm{s}}(x, x; \lambda))$ and $\mathbf{N}(\tilde{\varphi}^{\mathrm{s}}(x, x; \lambda))$ have an angle that is bounded away from zero uniformly in x, and we may restrict to the first summand for a lower bound. Next, observe that $\tilde{\varphi}^{\mathrm{u}}(0, L; \lambda)$ satisfies (4.9). Therefore, $\exp(\eta_+ L)\tilde{\varphi}^{\mathrm{u}}(0, L; \lambda)$ is the evolution of the original u-equation, i.e. of (4.9) with $\eta = 0$. Exploiting Hypothesis 4, and using the results in [15, Section 3.8], it is then not hard to see that

$$|u(0)| \geq C\exp(|\mathrm{Re}\,\nu^{+}_{i_\infty}|L),$$

where $C > 0$ does not depend upon L. Thus, it follows that the inverse of $\mathcal{T}^{\mathrm{sep}}_{L}(\lambda)$ grows exponentially with rate equal to or bigger than $|\mathrm{Re}\,\nu^{+}_{i_\infty}|$. □

It remains to investigate the case $0 < \mathrm{Re}\,\nu^{+}_{i_\infty+1} < -\eta_+ < \mathrm{Re}\,\nu^{+}_{i_\infty}$ that leads to $\eta_+ < 0$.

Hypothesis 5. We assume that $0 < \mathrm{Re}\,\nu^{+}_{i_\infty+1} < \mathrm{Re}\,\nu^{+}_{i_\infty}$ and that $\mathrm{Re}\,\nu^{+}_{i_\infty+2} < \mathrm{Re}\,\nu^{+}_{i_\infty+1}$. Furthermore, we assume that there is a vector $h_+ \in \mathbb{C}^n$ such that $\tilde{P}^{\mathrm{s}}_{+}(\lambda)B_+ h_+$ has a non-zero component in the eigendirection of $A_+(\lambda)$ associated with the simple eigenvalue $\nu^{+}_{i_\infty+1}$, where we express vectors with respect to a basis that consists of (generalized) eigenvectors of $A_+(\lambda)$.

Here, we have $\eta_+ < 0$ so that the rest state at $x = \infty$ sustains waves that travel to the right while growing. The above hypothesis guarantees that these waves still grow when the boundary conditions at $x = L$ are imposed.

Proposition 3. *Assume that Hypothesis* 5 *is met. The inverse of* $\mathcal{T}^{\mathrm{sep}}_{L}(\lambda)$ *grows exponentially with rate equal to or bigger than* $|\mathrm{Re}\,\nu^{+}_{i_\infty+1}|$.

Proof. Define $h(x)$ by $h(x) = h_+$ for $L_+ - \rho \leq x \leq L_+$ and zero otherwise for some large L_+ that we specify below. Thus, any integrands that contain $h(x)$ are non-zero only for x between $L_+ - \rho$ and L_+. We have

$$\left|\int_{L_+-\rho}^{L_+} \tilde{\varphi}^{\mathrm{s}}(L_+, y; \lambda)\exp(\eta_+(y-L_+))B(y)h(y)\,\mathrm{d}y - \tilde{P}^{\mathrm{s}}_{+}(\lambda)B_+ h_+ \rho\right| \leq C(h_+)(\rho^2 + \mathrm{e}^{-\hat{\theta}L}) \tag{4.20}$$

uniformly in ρ and L. From (4.16), we obtain

$$|a^{\mathrm{s}}_{-}| \leq C(L_+, \rho, h_+). \tag{4.21}$$

Evaluating (4.15) at $x = L$, we get

$$u(L) = \exp(-\eta_+ L)\left(\tilde{\varphi}^{\rm s}(L, L_+; \lambda)\int_{L_+-\rho}^{L_+} \tilde{\varphi}^{\rm s}(L_+, y; \lambda)\exp(\eta_+ y)B(y)h(y)\,{\rm d}y \right.$$
$$\left. +\tilde{\varphi}^{\rm s}(L, -L; \lambda)a_-^{\rm s} + \tilde{\varphi}^{\rm u}(L, L; \lambda)a_+^{\rm u}\right),$$

and therefore, upon substituting (4.20) and (4.21),

$$u(L) = \exp(-\eta_+ L)(\exp(\eta_+ L_+)\tilde{\varphi}^{\rm s}(L, L_+; \lambda)[\tilde{P}_+^{\rm s}(\lambda)B_+ h_+$$
$$+{\rm O}(\rho^2 + {\rm e}^{-\hat{\theta}L} + \exp(-\tilde{\kappa}_-^{\rm s} L))] + \tilde{\varphi}^{\rm u}(L, L; \lambda)a_+^{\rm u}), \tag{4.22}$$

where the $\mathrm{O}(\cdots)$-term depends upon the choice of L_+ and h_+ but not on L. Again, it suffices to consider the norm of $u(L)$ in the stable components; see the proof of Proposition 2. Exploiting Hypothesis 4, and using the results in [15, Section 3.8], we see that

$$|u(L)| \geq C\exp(|\mathrm{Re}\,\nu_{i_\infty+1}^+|L)$$

upon choosing first L_+ large enough, and then L large compared to L_+. The constant C is strictly positive and does not depend upon L. □

Remark 4.5. *In fact, if η_+ and η_- have the same sign, then the inverse of $\mathcal{T}_L^{\rm sep}(\lambda)$ typically grows exponentially with a rate that is the sum of the rates established in the above propositions. If η_+ and η_- have opposite signs, then the resolvent typically grows exponentially with the larger of the rates that appear in the above propositions.*

5. The essential spectrum under truncation

In this section, which contains our main results, we investigate the fate of the essential spectrum when $\mathcal{T}$ is replaced by $\mathcal{T}_L^{\rm per}$ or $\mathcal{T}_L^{\rm sep}$. Recall that the spectrum of $\mathcal{T}_L$ on the bounded interval $(-L, L)$ consists of eigenvalues; see Lemma 4.1. Throughout this section, we assume that Hypotheses 1–3 are met.

5.1. Extrapolated essential spectral sets on bounded intervals

Rather than attempting to describe in detail how the essential spectrum breaks up and trying to track individual eigenvalues, we focus on the asymptotic shape of the set that consists of the accumulation points of eigenvalues of $\mathcal{T}_L$ as $L \to \infty$.

Definition 5.7 (*Extrapolated essential spectral set*). We say that λ_* is not in the extrapolated essential spectral set $\Sigma_{\rm ext}^{\rm e}$ of the family $\{T_L^{\rm sep}\}_L$ (or $\{T_L^{\rm per}\}_L$) if there exists a neighborhood $U(\lambda_*) \subset \mathbb{C}$ of λ_*, an integer ℓ and a positive number L_* such that $D_{\rm sep}$ (or $D_{\rm per}$) has at most ℓ zeros in $U(\lambda_*)$ for $L \geq L_*$.

Roughly speaking, the extrapolated essential spectral set consists of those points where infinitely many eigenvalues of $\mathcal{T}_L$ accumulate as $L \to \infty$. Note that the extrapolated essential spectral set of the family $\mathcal{T}_L$ as defined above is closed since its complement is open by definition.

Example 1 (*continued*). The essential spectrum $\Sigma_{\rm ess}$ of the operator $\mathcal{L}u = u_{xx} + cu_x$ on $\mathbb{R}$ is given by the curve $\lambda = -k^2 + c{\rm i}k$ for $k \in \mathbb{R}$. The spectrum of the operator $\mathcal{L}$ on the interval $(-L, L)$ with periodic boundary conditions is given by

$$\lambda=-\frac{\pi^2k^2}{L^2}+\mathrm{i}\frac{c\pi k}{L},\quad k\in\mathbb{Z}.$$

Thus, as $L\to\infty$, each point in Σ_{ess} is an accumulation point, and we have $\Sigma^{\text{e}}_{\text{ext}}=\Sigma_{\text{ess}}$. For Dirichlet or Neumann boundary conditions, however, we have $\Sigma^{\text{e}}_{\text{ext}}=(-\infty,-\frac{1}{4}c^2]$, and therefore $\Sigma^{\text{e}}_{\text{ext}}\neq\Sigma_{\text{ess}}$. Instead, we observe that $\Sigma^{\text{e}}_{\text{ext}}=\Sigma_{\text{abs}}$. It is instructive to check that the eigenfunctions of $\mathcal{T}^{\text{sep}}$ with Dirichlet or Neumann conditions converge, as $L\to\infty$, to the absolute eigenmodes of $\mathcal{T}$ that we computed in Section 3.2.

As we shall see in the next sections, the behavior of the essential spectrum in this example is rather typical.

5.2. Periodic boundary conditions

We assume that $A_+(\lambda)=A_-(\lambda)$ and denote these matrices by $A_0(\lambda)$. Furthermore, we impose periodic boundary conditions.

Proposition 4. *Under the above hypothesis, and Hypotheses* 1 *and* 3, *the spectrum of* $\mathcal{T}^{\text{per}}_L$ *satisfies* $\Sigma^{\text{e}}_{\text{ext}}\subset\Sigma_{\text{ess}}$.

Proof. It suffices to show that, if $\lambda\notin\Sigma_{\text{ess}}$, then there is a neighborhood $U\subset\mathbb{C}$ of λ and numbers $L_*>0$ and $\ell\geq 0$ such that $\mathcal{T}^{\text{per}}_L$ has at most ℓ eigenvalues in U for $L>L_*$. This, however, follows from Theorem 2. □

The example in the previous section suggests that the extrapolated spectral set $\Sigma^{\text{ext}}_{\text{e}}$ is in fact equal to Σ_{ess}. We show that this is indeed the case under the following assumption.

Hypothesis 6 (*Reducible essential spectrum*). The subset $\mathcal{S}_{\text{per}}$, defined below, of the essential spectrum Σ_{ess} is dense in Σ_{ess}. Here, $\lambda_*\in\mathcal{S}_{\text{per}}\subset\Sigma_{\text{ess}}$ provided $\text{spec}(A_0(\lambda_*))\cap\mathrm{i}\mathbb{R}=\{\mathrm{i}\omega(\lambda_*)\}$ with geometric and algebraic multiplicity equal to one and $(\mathrm{d}\omega/\mathrm{d}\lambda)|_{\lambda_*}\neq 0$, where $\mathrm{i}\omega(\lambda)$ is the eigenvalue of $A_0(\lambda)$ that is close to $\mathrm{i}\omega(\lambda_*)$ for λ close to λ_*.

It is important to note that the reducible essential spectrum $\mathcal{S}_{\text{per}}$ consists of regular curve segments.

Theorem 4. *If Hypotheses* 1, 3 *and* 6 *are met, then the spectrum of* $\mathcal{T}^{\text{per}}_L$ *satisfies* $\Sigma^{\text{e}}_{\text{ext}}=\Sigma_{\text{ess}}$.

Proof. Since $\Sigma^{\text{e}}_{\text{ext}}$ is closed, it suffices to show that $\lambda_*\in\mathcal{S}_{\text{per}}$ implies $\lambda_*\in\Sigma^{\text{e}}_{\text{ext}}$.

Thus, we fix some $\lambda_*\in\mathcal{S}_{\text{per}}$, and denote by $E^{\text{ss}}_0(\lambda_*)$, $E^{\text{c}}_0(\lambda_*)$ and $E^{\text{uu}}_0(\lambda_*)$ the stable, center and unstable eigenspaces of $A_0(\lambda_*)$. Exploiting Hypothesis 6, there are x-dependent subspaces $E^{\text{cs}}_+(x;\lambda_*)$ and $E^{\text{cu}}_-(x;\lambda_*)$ that consist of those initial values in $\mathbb{C}^N$ that lead to solutions of (2.9),

$$\frac{\mathrm{d}}{\mathrm{d}x}u=A(x;\lambda)u,\tag{5.1}$$

which are bounded on $[x,\infty)$ and $(-\infty,x]$, respectively; see [15]. All aforementioned spaces can be continued analytically in λ for λ close λ_*: in particular, we have the generalized eigenspaces $E^{\text{ss}}_0(\lambda)$, $E^{\text{c}}_0(\lambda)$ and $E^{\text{uu}}_0(\lambda)$ of $A_0(\lambda)$, as well as the x-dependent spaces $E^{\text{cs}}_+(x;\lambda)$ and $E^{\text{cu}}_-(x;\lambda)$ that consist of all initial conditions which lead to solutions to (5.1) that are of the order $\mathrm{O}(\mathrm{e}^{\eta|x|})$ for $x>0$ and $x<0$, respectively, for some small fixed $\eta>0$. For λ close to λ_*, we denote by $\mathrm{i}\omega(\lambda)$ the unique eigenvalue of $A_0(\lambda)$ that is close to $\mathrm{i}\omega(\lambda_*)$.

We begin by investigating the intersection $E^{\text{ss}}_+(0;\lambda)\cap E^{\text{cu}}_-(0;\lambda)$ for $\lambda\in\mathcal{S}_{\text{per}}$ close to λ_*. We claim that this intersection is trivial except possibly for finitely many elements λ near λ_*. To prove this claim, we argue by contradiction: if our claim is wrong, then Remark 4.2 implies that the intersection $E_+^{\mathrm{ss}}(0;\lambda) \cap E_-^{\mathrm{cu}}(0;\lambda)$ has non-zero dimension for all λ in a small open neighborhood U_* of λ_*. Next, recall that the set $\mathcal{S}_{\mathrm{per}}$ near λ_* is the curve that consists of precisely those values of λ for which $\omega(\lambda)$ is real. In particular, $\mathcal{S}_{\mathrm{per}}$ divides U_* into two open sets B_1 and B_2, say, so that U_* is the disjoint union of B_1, B_2 and $U_* \cap \mathcal{S}_{\mathrm{per}}$. Since $(\mathrm{d}\omega/\mathrm{d}\lambda)|_{\lambda_*} \neq 0$ by Hypothesis 6, we have that $\mathrm{Re}\,\mathrm{i}\omega(\lambda) > 0$ for all λ in either B_1 or B_2; suppose that $\mathrm{Re}\,\mathrm{i}\omega(\lambda) > 0$ for $\lambda \in B_1$, say. Therefore, we conclude that $E_-^{\mathrm{cu}}(0;\lambda)$ consists of all initial conditions that, for $\lambda \in B_1$, lead to solutions to (5.1) that decay exponentially as $x \to -\infty$ since $\mathrm{i}\omega(\lambda)$ is then an additional unstable eigenvalue of $A_0(\lambda)$. We are now in a position to reach the desired contradiction: we assumed that the intersection $E_+^{\mathrm{ss}}(0;\lambda) \cap E_-^{\mathrm{cu}}(0;\lambda)$ has non-zero dimension for all λ in an entire neighborhood of λ_*. For $\lambda \in B_1$, any solution of (5.1) associated with an initial condition in this intersection decays exponentially as $|x| \to \infty$; thus, any such λ is an eigenvalue in a region where $\mathcal{T}(\lambda)$ is Fredholm with index zero. This contradicts Hypothesis 3.

In summary, we conclude that the intersection $E_+^{\mathrm{ss}}(0;\lambda) \cap E_-^{\mathrm{cu}}(0;\lambda)$ and, by the same argument, the intersection $E_+^{\mathrm{cs}}(0;\lambda) \cap E_-^{\mathrm{uu}}(0;\lambda)$ are trivial for $\lambda \in \mathcal{S}_{\mathrm{per}}$ close to λ_* except possibly for finitely many elements λ. After removing these exceptional elements from the set $\mathcal{S}_{\mathrm{per}}$, the resulting set is still dense in Σ_{ess}. We can therefore assume that the aforementioned intersections are trivial at λ_*, and therefore also in a open neighborhood of λ_* in $\mathbb{C}$.

As a consequence, the intersection

$$E_+^{\mathrm{cs}}(0;\lambda) \cap E_-^{\mathrm{cu}}(0;\lambda) = \mathrm{span}\{u_*(0;\lambda)\}$$

is one-dimensional for every λ near λ_* and

$$u_*(0;\lambda) \notin E_-^{\mathrm{uu}}(0;\lambda) \cap E_+^{\mathrm{ss}}(0;\lambda).$$

It follows then from [35] or [25, Lemma 2.2] that there is a small $\delta > 0$, certain constants $\vartheta_\pm(\lambda) \in \mathbb{C}$, and vectors $a_0(\lambda) \in E_0^{\mathrm{c}}(\lambda)$ with $a_0(\lambda) \neq 0$ so that the solution $u_*(x;\lambda)$ to (5.1) can be expressed as

$$u_*(x;\lambda) = a_0(\lambda)\exp(\mathrm{i}(\omega(\lambda)x + \vartheta_\pm(\lambda))) + \mathrm{O}(\mathrm{e}^{-\delta|x|}) \tag{5.2}$$

for $x \in \mathbb{R}$. In particular, we have

$$E_+^{\mathrm{ss}}(x;\lambda) \oplus E_-^{\mathrm{uu}}(x;\lambda) \oplus \mathrm{span}\{u_*(x;\lambda)\} = \mathbb{C}^N \tag{5.3}$$

for all λ close to λ_*.

Next, we seek solutions $u(x)$ of (5.1) that satisfy $u(-L) = u(L)$. It is a consequence of Remark 2.1 and (5.3) that any solution $u(x)$ to (2.9) can be written in the form

$$u(x) = \varphi^{\mathrm{ss}}(x,-L;\lambda)a_- + \varphi^{\mathrm{uu}}(x,L;\lambda)a_+ + u_*(x;\lambda)b,$$

where $a_- \in E_0^{\mathrm{ss}}(\lambda)$, $a_+ \in E_0^{\mathrm{uu}}(\lambda)$ and $b \in \mathbb{C}$ are arbitrary. Here, the evolution operators $\varphi^{\mathrm{ss}}(x,-L;\lambda)$ and $\varphi^{\mathrm{uu}}(x,L;\lambda)$ satisfy

$$|\varphi^{\mathrm{ss}}(x,-L;\lambda)| \leq K\exp(-\delta|x+L|), \qquad |\varphi^{\mathrm{uu}}(x,L;\lambda)| \leq K\exp(-\delta|x-L|) \tag{5.4}$$

for $|x| \leq L$, where $\delta > 0$ is a small positive constant. Thus, it suffices to find $(a_\pm, b)$ and λ so that

$$P^{\mathrm{ss}}(-L;\lambda)a_- + \varphi^{\mathrm{uu}}(-L,L;\lambda)a_+ + u_*(-L;\lambda)b = \varphi^{\mathrm{ss}}(L,-L;\lambda)a_- + P^{\mathrm{uu}}(L;\lambda)a_+ + u_*(L;\lambda)b, \tag{5.5}$$

where

$$P^{\mathrm{ss}}(-L;\lambda) = \varphi^{\mathrm{ss}}(-L,-L;\lambda), \qquad P^{\mathrm{uu}}(L;\lambda) = \varphi^{\mathrm{uu}}(L,L;\lambda)$$

are $O(e^{-\delta L})$-close, for some $\delta > 0$ that is independent of L, to the spectral projections $P^{ss}(\lambda)$ and $P^{uu}(\lambda)$, respectively, of $A_0(\lambda)$; see again Remark 2.1. Exploiting this fact together with the estimates (5.4), we see that (5.5) is equivalent to

$$(P^{ss}(\lambda) + O(e^{-\delta L}))a_- + O(e^{-\delta L})a_+ + u_*(-L;\lambda)b = O(e^{-\delta L})a_- + (P^{uu}(\lambda) + O(e^{-\delta L}))a_+ + u_*(L;\lambda)b,$$

where we replaced δ by $\min\{\delta, \theta\}$. Substituting (5.2) and using the definition of a_+ and a_-, we obtain

$$\begin{aligned}&(\mathrm{id} + O(e^{-\delta L}))a_- + O(e^{-\delta L})a_+ + a_0(\lambda)(\exp(-i(\omega(\lambda)L + \vartheta_-(\lambda))) + O(e^{-\delta L}))b\\ &\quad = O(e^{-\delta L})a_- + (\mathrm{id} + O(e^{-\delta L}))a_+ + a_0(\lambda)(\exp(i(\omega(\lambda)L + \vartheta_+(\lambda))) + O(e^{-\delta L}))b.\end{aligned}$$

We can write this equation, which is linear in (a_-, a_+, b), in components according to the direct-sum decomposition

$$E_0^{ss}(\lambda) \oplus E_0^{uu}(\lambda) \oplus E_0^{c}(\lambda) = \mathbb{C}^N,$$

and solve the first two components for (a_-, a_+) as a function of b. We arrive at the equation

$$(\exp(-i(\omega(\lambda)L + \vartheta_-(\lambda))) + O(e^{-\delta L}))b = (\exp(i(\omega(\lambda)L + \vartheta_+(\lambda))) + O(e^{-\delta L}))b,$$

which, after dividing by b, is equivalent to the reduced equation

$$e^{2i\omega(\lambda)L} = \exp(i(\vartheta_-(\lambda) - \vartheta_+(\lambda))) + O(e^{-\delta L}). \tag{5.6}$$

To solve this equation, it suffices to find all solutions to

$$2\omega(\lambda)L = \vartheta_-(\lambda) - \vartheta_+(\lambda) + O(e^{-\delta L}) + 2\pi n,$$

where $n \in \mathbb{Z}$ is arbitrary. Dividing by $2L$, we get

$$\omega(\lambda) = \frac{\pi n}{L} + \frac{1}{2L}(\vartheta_-(\lambda) - \vartheta_+(\lambda)) + O(e^{-\delta L}).$$

Since $\omega(\lambda_*)$ is real, there are unique numbers $n_0(L) \in \mathbb{N}$ and $r(L) \in [0, 1)$ such that

$$n_0(L) + r(L) = \frac{\omega(\lambda_*)L}{\pi}.$$

Thus,

$$\frac{\pi n_0(L)}{L} = \omega(\lambda_*) - \frac{\pi r(L)}{L},$$

and upon setting $n = n_0(L) + m$, we obtain the equation

$$\omega(\lambda) = \omega(\lambda_*) + \frac{1}{2L}(2\pi(m - r(L)) + \vartheta_-(\lambda) - \vartheta_+(\lambda)) + O(e^{-\delta L}). \tag{5.7}$$

Since $(d\omega/d\lambda)(\lambda_*) \neq 0$, Eq. (5.7) can be solved with respect to λ for λ near λ_* for every L sufficiently large and all $m \in \mathbb{N}$ such that m/L is smaller than some constant $\epsilon > 0$. In particular, (5.7) has $O(L)$ different solutions. This proves that λ_* is indeed in the extrapolated essential spectral set Σ^{e}_{ext}. □

5.3. Separated boundary conditions

Finally, we consider separated boundary conditions. We show that $\Sigma_{\text{ext}}^{\text{e}}$ is again determined by spectral properties of $\mathcal{T}$ on $\mathbb{R}$ but in general does not coincide with Σ_{ess}. Roughly speaking, separated boundary conditions stabilize up to an optimal exponential weight. Throughout this section, we once again use the set-up and the notation introduced in Section 4.3.1.

Hypothesis 7 (*Non-degenerate boundary conditions*). There is a discrete (possibly empty) set $\mathcal{C} \subset \mathbb{C}$ with no accumulation points in $\mathbb{C}$ so that $Q_- \oplus \tilde{E}_-^{\text{s}}(\lambda) = \mathbb{C}^N$ and $Q_+ \oplus \tilde{E}_+^{\text{u}}(\lambda) = \mathbb{C}^N$ for all $\lambda \notin \Sigma_{\text{abs}} \cup \mathcal{C}$.

Recall that $\tilde{E}_\pm^{\text{s,u}}(\lambda)$ have been defined in Section 4.3.1. Note that the above hypothesis is often violated when we consider systems of decoupled equations together with boundary conditions that also decouple. An example is the operator introduced in Example 2 with either Dirichlet or Neumann conditions. It is possible to adapt the results to such cases, but we do not pursue this here.

Proposition 5. *Assume that Hypotheses* 1, 2 *and* 7 *are met. Furthermore, assume that $\mathcal{T}^\eta$ satisfies Hypothesis* 3 *for every $\eta \in \mathbb{R}^2$. Under these assumptions, we have $\Sigma_{\text{ext}}^{\text{e}} \subset \Sigma_{\text{abs}}$.*

Proof. If $\lambda \notin \Sigma_{\text{abs}}$, then $\mathcal{T}^\eta(\lambda)$ is Fredholm with index zero for an appropriately chosen weight $\eta \in \mathbb{R}^2$. Considering $\mathcal{T}^\eta(\lambda)$ on L^2, we have to replace (2.9) by the equation

$$\frac{\mathrm{d}}{\mathrm{d}x} v = (A(x;\lambda) + \eta_\pm)v. \tag{5.8}$$

The associated operator on L^2, which we again denote by $\mathcal{T}^\eta(\lambda)$, is then also Fredholm with index zero. Note that $v(\cdot)$ has to satisfy the same boundary conditions at $x = \pm L$ as $u(\cdot)$. We have therefore reduced the problem to a setting that is similar to the case of periodic boundary conditions. Isolated eigenvalues of finite multiplicity persist with their multiplicity provided the boundary conditions are transverse to the stable and unstable eigenspaces of $A_+(\lambda)$ and $A_-(\lambda)$, respectively; see Lemma 4.3. Norms on the finite interval $(-L, L)$ are equivalent, and invertibility of the v-equation therefore implies invertibility of the u-equation. If the boundary conditions are not transverse, only finitely many eigenvalues are generated, and their number, counting multiplicity, is independent of L; see Theorem 3. This completes the proof of the proposition. □

We remark that, for reversible systems, we expect that $\Sigma_{\text{abs}} = \Sigma_{\text{ess}}$. In general, however, we have $\Sigma_{\text{abs}} \neq \Sigma_{\text{ess}}$; see Section 3.2. In the remaining part of this section, we prove that $\Sigma_{\text{ext}}^{\text{e}} = \Sigma_{\text{abs}}$ under the following additional assumption.

Hypothesis 8 (*Reducible absolute spectrum*). The subset $\mathcal{S}_{\text{sep}}$, defined below, of the absolute spectrum Σ_{abs} is dense in Σ_{abs}. Here, $\lambda_* \in \mathcal{S}_{\text{sep}} \subset \Sigma_{\text{abs}}$ provided one of the following two conditions is met.

1. *Pulses* (i.e. $A_+(\lambda) = A_-(\lambda) =: A_0(\lambda)$ for all λ):

$$\operatorname{Re} \nu_{i_\infty-1}(\lambda_*) > \operatorname{Re} \nu_{i_\infty}(\lambda_*) = \operatorname{Re} \nu_{i_\infty+1}(\lambda_*) > \operatorname{Re} \nu_{i_\infty+2}(\lambda_*)$$

   with $\nu_{i_\infty}(\lambda_*) = -\eta_0 + \mathrm{i}\omega_1(\lambda_*)$ and $\nu_{i_\infty+1}(\lambda_*) = -\eta_0 + \mathrm{i}\omega_2(\lambda_*)$, where $\omega_1(\lambda_*) \neq \omega_2(\lambda_*)$ and $(\mathrm{d}(\omega_1 - \omega_2)/\mathrm{d}\lambda)|_{\lambda_*} \neq 0$.

2. *Fronts*: either

$$\operatorname{Re} \nu^+_{i_\infty-1}(\lambda_*) > \operatorname{Re} \nu^+_{i_\infty}(\lambda_*) = \operatorname{Re} \nu^+_{i_\infty+1}(\lambda_*) > \operatorname{Re} \nu^+_{i_\infty+2}(\lambda_*)$$

with $\nu^+_{i_\infty}(\lambda_*) = -\eta_+ + \mathrm{i}\omega_1(\lambda_*)$ and $\nu^+_{i_\infty+1}(\lambda_*) = -\eta_+ + \mathrm{i}\omega_2(\lambda_*)$, where $\omega_1(\lambda_*) \neq \omega_2(\lambda_*)$ and $(\mathrm{d}(\omega_1 - \omega_2)/\mathrm{d}\lambda)|_{\lambda_*} \neq 0$ while

$$\operatorname{Re} \nu^-_{i_\infty}(\lambda_*) > -\eta_- > \operatorname{Re} \nu^-_{i_\infty+1}(\lambda_*)$$

for some $\eta_\pm$, or vice versa.

We observe that the reducible absolute spectrum $\mathcal{S}_{\mathrm{sep}}$ consists of regular curve segments.

Theorem 5. *Assume that Hypotheses* 1, 2, 7 *and* 8 *are met. Furthermore, assume that Hypothesis* 3 *is satisfied for* $\mathcal{T}^\eta$ *for every* $\eta \in \mathbb{R}^2$. *We then have* $\Sigma^{\mathrm{e}}_{\mathrm{ext}} = \Sigma_{\mathrm{abs}}$.

Proof. We have to show that $\lambda_* \in \mathcal{S}_{\mathrm{sep}}$ implies $\lambda_* \in \Sigma^{\mathrm{e}}_{\mathrm{ext}}$. We again consider (5.8) using the weights $\eta_\pm$ that appear in Hypothesis 8. In contrast to the notation introduced in Section 4.3.1, we omit in this proof the tilde ($\tilde{}$) that referred to quantities computed with respect to (5.8). In other words, for the sake of simplicity, we assume that $\eta = 0$ (possibly after changing the equation appropriately). We then use the notation and conventions introduced in the proof of Theorem 4; in fact, the proofs for separated and periodic boundary conditions are quite similar. Finally, we restrict ourselves to the case of fronts; the proof for pulses proceeds in a similar fashion.

First, we claim that we can assume that $Q_- \oplus E^{\mathrm{s}}_-(\lambda_*) = \mathbb{C}^N$. Indeed, suppose that Q_- and $E^{\mathrm{s}}_-(\lambda_*)$ have a non-trivial intersection. Since $\mathcal{S}_{\mathrm{sep}}$ consists locally of regular curve segments, we can vary λ in $\mathcal{S}_{\mathrm{sep}}$ near λ_*. As a consequence, the subspaces Q_- and $E^{\mathrm{s}}_-(\lambda)$ intersect either only at the origin for any λ close to λ_* with $\lambda \neq \lambda_*$, or else they intersect non-trivially for all λ in an open neighborhood of λ_* due to analyticity of $E^{\mathrm{s}}_-(\lambda)$ in λ; the latter case, however, contradicts Hypothesis 7. Thus, the first case occurs, and we can replace λ_* by any nearby $\lambda \in \mathcal{S}_{\mathrm{sep}}$. This proves our claim.

As a consequence, if we transport the subspace Q_- using the evolution $\varphi(x, -L; \lambda)$ associated with (2.9), then, by hyperbolicity of $A_-(\lambda)$, the transported subspace $\varphi(0, -L; \lambda)Q_-$ is close to $E^{\mathrm{u}}_-(0; \lambda)$ for all large L.

Next, consider the situation at the right endpoint $x = L$ of the interval $(-L, L)$. By Hypothesis 8, we have

$$E^{\mathrm{ss}}_+(\lambda) \oplus \operatorname{span}\{a_1(\lambda), a_2(\lambda)\} \oplus E^{\mathrm{uu}}_+(\lambda) = \mathbb{C}^N$$

for all λ near λ_*, where $a_1(\lambda)$ and $a_2(\lambda)$ are non-zero eigenvectors of $A_+(\lambda)$ associated with the eigenvalues $\omega_1(\lambda)$ and $\omega_2(\lambda)$. Using the roughness theorem for exponential dichotomies [33,35], we can continue any combination of these subspaces to x-dependent invariant subspaces of (2.9). In particular, using also [25, Lemma 2.2], there are subspaces

$$E^{\mathrm{ss}}_+(x; \lambda) \oplus \operatorname{span}\{a_1(x; \lambda)\}, \qquad E^{\mathrm{ss}}_+(x; \lambda) \oplus \operatorname{span}\{a_2(x; \lambda)\} \tag{5.9}$$

that converge to the corresponding x-independent eigenspaces of $A_+(\lambda)$ as $x \to \infty$. Note that $E^{\mathrm{cs}}_+(x; \lambda)$ is $(N - i_\infty + 1)$-dimensional, while $E^{\mathrm{u}}_-(x; \lambda)$ has dimension i_∞. Therefore, these two subspaces intersect in a non-trivial fashion; in fact, we may assume that $E^{\mathrm{u}}_-(x; \lambda)$ and $E^{\mathrm{cs}}_+(x; \lambda)$ intersect transversely in a one-dimensional subspace which is not contained in either of the two spaces appearing in (5.9). Otherwise, we reach a contradiction to Hypothesis 3; see Remark 4.2 and [25].

Hence, as a consequence of the discussion in the last few paragraphs, $E^{\mathrm{cs}}_+(0; \lambda)$ and the transported subspace $\varphi(0, -L; \lambda)Q_-$ intersect in a one-dimensional subspace that is spanned by a vector $u_*(0; \lambda)$. The solution associated with the initial condition $u_*(0; \lambda)$ can be written as

$$u_*(x; \lambda) = a_1(\lambda) \exp(\mathrm{i}\omega_1(\lambda)x) + a_2(\lambda) \exp(\mathrm{i}\omega_2(\lambda)x) + \mathrm{O}(\mathrm{e}^{-\theta x}) \tag{5.10}$$

as $x \to \infty$, where θ is again the rate of convergence of $A(x;\lambda)$ as $x \to \infty$, and $a_1(\lambda)$ and $a_2(\lambda)$ are certain non-zero eigenvectors of $A_+(\lambda)$ associated with the eigenvalues $\mathrm{i}\omega_1(\lambda)$ and $\mathrm{i}\omega_2(\lambda)$. This expansion can be proved by using exponential weights and dichotomies for an appropriate variation-of-constant formula; see, for instance [33,35] or [25, Lemma 2.2]. In addition, we have

$$\varphi(L,-L;\lambda)Q_- = \mathrm{span}\{u_*(L;\lambda)\} + E_+^{\mathrm{uu}}(\lambda) + \mathrm{O}(\mathrm{e}^{-\delta L})) \tag{5.11}$$

for all L sufficiently large. Here, and in the following, δ denotes a small positive constant determined by θ and the rates of hyperbolicity of $A_\pm(\lambda)$.

In the next step, we focus on the boundary conditions at the right endpoint of the interval. Arguing as above, we can assume that

$$Q_+ = \mathrm{span}\{v_*\} \oplus \tilde{Q}_+,$$

where

$$v_* \in \mathrm{span}\{a_1(\lambda), a_2(\lambda)\} \oplus E_+^{\mathrm{uu}}(\lambda), \qquad v_* \notin \mathrm{span}\{a_j(\lambda)\} \oplus E_+^{\mathrm{uu}}(\lambda) \tag{5.12}$$

for $j = 1, 2$ and

$$\tilde{Q}_+ \oplus E_+^{\mathrm{c}}(\lambda) \oplus E_+^{\mathrm{uu}}(\lambda) = \mathbb{C}^N \tag{5.13}$$

otherwise, we reach a contradiction to Hypothesis 3. In particular, we have

$$v_* = a_1^+(\lambda) + a_2^+(\lambda) + v_*^{\mathrm{u}}(\lambda) \tag{5.14}$$

with $a_j^+(\lambda) \in \mathrm{span}\{a_j(\lambda)\}$ for $j = 1, 2$ and $v_*^{\mathrm{u}}(\lambda) \in E_+^{\mathrm{uu}}(\lambda)$. Note that $a_j^+(\lambda)$ is not equal to zero for $j = 1, 2$.

It suffices to find non-trivial intersections of Q_+ and $\varphi(L,-L;\lambda)Q_-$. Exploiting (5.10)–(5.13) and using Lyapunov–Schmidt reduction as in the proof of Theorem 4 (we omit the details), we arrive at the reduced equation

$$a_1(\lambda)(\exp(\mathrm{i}\omega_1(\lambda)L) + \mathrm{O}(\mathrm{e}^{-\delta L})) + a_2(\lambda)(\exp(\mathrm{i}\omega_2(\lambda)L) + \mathrm{O}(\mathrm{e}^{-\delta L})) = r(a_1^+(\lambda) + a_2^+(\lambda)),$$

where $r \in \mathbb{C}$ is arbitrary. In other words, we shall solve

$$a_1(\lambda)(\exp(\mathrm{i}\omega_1(\lambda)L) + \mathrm{O}(\mathrm{e}^{-\delta L})) = ra_1^+(\lambda), \qquad a_2(\lambda)(\exp(\mathrm{i}\omega_2(\lambda)L) + \mathrm{O}(\mathrm{e}^{-\delta L})) = ra_2^+(\lambda). \tag{5.15}$$

Recall that $a_j(\lambda)$ and $a_j^+(\lambda)$ are not equal to zero for $j = 1, 2$. Thus, we can write

$$a_j^+(\lambda) = \frac{|a_j^+(\lambda)|}{|a_j(\lambda)|}\exp(\mathrm{i}\vartheta_j(\lambda))a_j(\lambda)$$

for certain complex numbers $\vartheta_j(\lambda)$ with $j = 1, 2$. The first equation in (5.15) can then be solved for r:

$$r = \frac{|a_1(\lambda)|}{|a_1^+(\lambda)|}\exp(-\mathrm{i}\vartheta_1(\lambda))(\exp(\mathrm{i}\omega_1(\lambda)L) + \mathrm{O}(\mathrm{e}^{-\delta L})).$$

Substituting this expression into the second equation in (5.15), we obtain

$$\exp(\mathrm{i}\omega_2(\lambda)L) + \mathrm{O}(\mathrm{e}^{-\delta L}) = \frac{|a_2^+(\lambda)|\,|a_1(\lambda)|}{|a_2(\lambda)|\,|a_1^+(\lambda)|}\exp(\mathrm{i}(\vartheta_2(\lambda) - \vartheta_1(\lambda)))(\exp(\mathrm{i}\omega_1(\lambda)L) + \mathrm{O}(\mathrm{e}^{-\delta L})),$$

which is equivalent to

$$\exp(\mathrm{i}(\omega_2(\lambda) - \omega_1(\lambda))L) = \frac{|a_2^+(\lambda)|\,|a_1(\lambda)|}{|a_2(\lambda)|\,|a_1^+(\lambda)|}\exp(\mathrm{i}(\vartheta_2(\lambda) - \vartheta_1(\lambda))) + \mathrm{O}(\mathrm{e}^{-\delta L}).$$

This equation is exactly of the type considered in the proof of Theorem 4; see (5.6) and the discussion following it. Thus, the proof of the theorem is complete. □

Remark 5.6. *In the set-up of the above theorem, we have that an eigenfunction $u(x)$ to the original Eq.* (5.1) *on the interval $(-L, L)$ with separated boundary conditions typically satisfies*

$$|u(-L)| \approx \exp(-\mathrm{Re}\,\nu^-_{i_\infty}(\lambda)L), \qquad |u(L)| \approx \exp(\mathrm{Re}\,\nu^+_{i_\infty+1}(\lambda)L).$$

The remark is a consequence of the proof of the previous theorem.

In particular, the convective properties of the absolute spectrum manifest themselves via the growth of the associated eigenmodes at $x = \pm L$ depending on the direction of transport. If the absolute eigenvalue λ is induced by unstable spatial eigenvalues $\nu^+_{i_\infty}(\lambda)$ and $\nu^+_{i_\infty+1}(\lambda)$, then the direction of transport is to the right. Note that this requires that the formerly stable spatial eigenvalue $\nu^+_{i_\infty+1}(\lambda)$ moves into the right half-plane; we would therefore need $\eta_+ < 0$ to stabilize the wave using exponential weights. Analogously, if the absolute eigenvalue λ is induced by stable spatial eigenvalues $\nu^-_{i_\infty}(\lambda)$ and $\nu^-_{i_\infty+1}(\lambda)$, then the direction of transport is to the left. In the other cases, the absolute eigenmodes transport towards $x = 0$, either from $x = L$ if $\mathrm{Re}\,\nu^+_{i_\infty}(\lambda) < 0$ or else from $x = -L$ if $\mathrm{Re}\,\nu^-_{i_\infty+1}(\lambda) > 0$; in these cases, the instability would lead to a break-up of the wave near its core, away from the asymptotic rest states.

5.4. Separated boundary conditions: the edge of the absolute spectrum

Often, the rightmost endpoint of the absolute spectrum is given by a branch point, i.e. by a double root of the dispersion relation. In that case, it is of interest how well the edge of the absolute spectrum is approximated on bounded intervals. For the sake of brevity, we only consider the case of fronts. A similar result under analogous assumptions is true for pulses.

Hypothesis 9 (*Non-degenerate double eigenvalue*). *Fronts:* We have a double eigenvalue $\nu^+_{i_\infty}(\lambda_*) = \nu^+_{i_\infty+1}(\lambda_*)$ with geometric multiplicity one so that

$$\mathrm{Re}\,\nu^+_{i_\infty-1}(\lambda_*) > \mathrm{Re}\,\nu^+_{i_\infty}(\lambda_*) > \mathrm{Re}\,\nu^+_{i_\infty+2}(\lambda_*),$$

and the Jordan block associated with $\nu^+_{i_\infty}(\lambda_*)$ is unfolded generically upon varying λ near λ_*. Let $v^+_{i_\infty}(\lambda_*)$ denote the eigenvector of $A_+(\lambda_*)$ associated with $\nu^+_{i_\infty}(\lambda_*)$. We assume that

$$\mathrm{span}\{v^+_{i_\infty}(\lambda_*)\} \oplus E^{\mathrm{uu}}_+(\lambda_*) \oplus Q_+ = \mathbb{C}^n, \tag{5.16}$$

where $E^{\mathrm{uu}}_+(\lambda_*)$ is the eigenspace of $A_+(\lambda_*)$ associated with the unstable part of the spectrum. Furthermore, we assume that $\lambda_* \notin \Sigma^-_{\mathrm{abs}}$ and that $E^{\mathrm{s}}_-(\lambda_*) \oplus Q_- = \mathbb{C}^n$. Finally, we assume that

$$E^{\mathrm{u}}_-(0;\lambda_*) \oplus \mathrm{span}\{v^+_{i_\infty}(0;\lambda_*)\} \oplus E^{\mathrm{ss}}_+(0;\lambda_*) = \mathbb{C}^n, \tag{5.17}$$

where $v^+_{i_\infty}(x;\lambda_*)$ is a solution that converges to $v^+_{i_\infty}(\lambda_*)$ as $x \to \infty$; see [15].

Lemma 5.5. *Assume that Hypotheses* 1, 2 *and* 9 *are met. In addition, suppose that λ_* is the rightmost point in the absolute spectrum. There are then constants $b_{1,2} \in \mathbb{C}$ with $b_1 \neq 0$ and $\delta > 0$ such that, if we order the eigenvalues*

$\lambda_{j,L}$ of $\mathcal{T}_L^{\mathrm{sep}}$ that are closest to λ_* by their real part, then

$$\left|\frac{1}{\sqrt{|\lambda_{j,L}-\lambda_*|}}-(b_1jL+b_2)\right| = \mathrm{O}(\mathrm{e}^{-\delta L})$$

for all L large enough (*depending on j*).

Proof. We proceed as in the proof of Theorem 5. Without loss of generality, we can assume that $\mathrm{Re}\,\nu_{i_\infty}^+ = 0$ for $\lambda=\lambda_*$. We write E_+^{ss}, E_+^{c} and E_+^{uu} for the stable, center and unstable eigenspaces of A_+, respectively, where the two-dimensional center eigenspace corresponds to the two eigenvalues near $\nu_{i_\infty}^+$.

We assumed that $\lambda_*\notin\Sigma_{\mathrm{abs}}^-$ and that $D_-(\lambda_*)\neq 0$. Hence, $\varphi(0,-L;\lambda)Q_-$ is exponentially close to $E_-^{\mathrm{u}}(0;\lambda)$ for all λ close to λ_*. In particular, using (5.17), we see that

$$\varphi(0,-L;\lambda)Q_-\cap E_+^{\mathrm{cs}}(0;\lambda)=\mathrm{span}\{u_*(0;\lambda)\}, \tag{5.18}$$

where $u_*(0;\lambda)$ is not equal to zero and

$$u_*(0;\lambda_*)\notin\mathrm{span}\{v_{i_\infty}^+(0;\lambda_*)\}\oplus E_+^{\mathrm{ss}}(0;\lambda_*). \tag{5.19}$$

As a consequence of (5.18), and proceeding as in the proof of Theorem 5, we have that

$$\varphi(L,-L;\lambda)Q_-=\mathrm{span}\{u_*(L;\lambda)\}\oplus(E_+^{\mathrm{uu}}(\lambda)+\mathrm{O}(\mathrm{e}^{-\delta L}))$$

for some $\delta>0$. We seek those λ close to λ_* for which $\varphi(L,-L;\lambda)Q_-$ has a non-trivial intersection with Q_+. Thus, we are interested in the space

$$[\mathrm{span}\{u_*(L;\lambda)\}\oplus(E_+^{\mathrm{uu}}(\lambda)+\mathrm{O}(\mathrm{e}^{-\delta L}))]\cap Q_+. \tag{5.20}$$

We begin by tracking $u_*(x;\lambda)$ up to $x=L$. We denote by A_+^{c} the restriction of A_+ to the center space E_+^{c}. We claim that

$$u_*(L;\lambda)=[\exp(A_+^{\mathrm{c}}(\lambda)L)+\mathrm{O}(\mathrm{e}^{-\delta L})]a_+(\lambda) \tag{5.21}$$

for some $\delta>0$ that is independent of λ and some non-zero vector $a_+(\lambda)\in E_+^{\mathrm{c}}(\lambda)$. Indeed, upon using exponential dichotomies, we can reduce the equation to an equation in $\mathbb{R}^2$. We can then use the variation-of-constant formula and exponential weights; we refer to [35] for similar arguments. See also [15, Section 3.8] for the case when the equation does not depend upon parameters. In addition, we know that $a_+(\lambda_*)\neq v_{i_\infty}^+(\lambda_*)$ due to (5.19). Next, we consider the space Q_+. Due to (5.16), we have

$$Q_+\cap(E_+^{\mathrm{c}}(\lambda)\oplus E_+^{\mathrm{uu}}(\lambda))=\mathrm{span}\{q_+(\lambda)\}$$

for some $q_+(\lambda)\neq 0$ with

$$q_+(\lambda_*)\notin\mathrm{span}\{v_{i_\infty}^+(\lambda_*)\}\oplus E_+^{\mathrm{uu}}(\lambda_*).$$

In other words, we have

$$Q_+=\mathrm{span}\{q_+(\lambda)\}\oplus\tilde{Q}_+$$

with $\tilde{Q}_+\cap E_+^{\mathrm{uu}}(\lambda)=\{0\}$. Expression (5.20) then reads

$$[\mathrm{span}\{u_*(L;\lambda)\}\oplus(E_+^{\mathrm{uu}}(\lambda)+\mathrm{O}(\mathrm{e}^{-\delta L}))]\cap[\mathrm{span}\{q_+(\lambda)\}\oplus\tilde{Q}_+].$$

Therefore, we have that

$$u = ru_*(L;\lambda) + (\mathrm{id} + \mathrm{O}(\mathrm{e}^{-\delta L}))u_+^{\mathrm{uu}}$$

is in Q_+ for appropriate choices of $r \in \mathbb{R}$ and $u_+^{\mathrm{uu}} \in E_+^{\mathrm{uu}}(\lambda)$ if, and only if,

$$u = \tilde{r}q_+(\lambda) + \tilde{q}_+,$$

where $\tilde{r} \in \mathbb{R}$ and $\tilde{q}_+ \in \tilde{Q}_+$. Using Lyapunov–Schmidt reduction, i.e. upon projecting these equations into the complementary subspaces $\tilde{Q}_+$, $E_+^{\mathrm{uu}}(\lambda)$ and $E_+^{\mathrm{c}}(\lambda)$, and solving the projected equations in the former two spaces, we finally arrive at the reduced equation

$$r[\exp(A_+^{\mathrm{c}}(\lambda)L)a_+(\lambda) + \mathrm{O}(\mathrm{e}^{-\delta L})] = \tilde{r}[q_+(\lambda) + \mathrm{O}(\mathrm{e}^{-\delta L})], \tag{5.22}$$

where we used (5.21). Note that $a_+(\lambda)$ and $q_+(\lambda)$ are smooth and that both are contained in $E_+^{\mathrm{c}}(\lambda)$. In addition, neither of these vectors is equal to $v_{i_\infty}^+(\lambda_*)$ for $\lambda = \lambda_*$. Since $A_+^{\mathrm{c}}(\lambda_*)$ is a Jordan block, we see that $\exp(A_+^{\mathrm{c}}(\lambda)L)$ corresponds to a linear second-order scalar operator, and (5.22) is the equation that appears when we seek the operator's eigenvalues. Thus, we can solve this equation by phase–plane analysis; we omit the details. □

6. Numerical computations

To illustrate and confirm the results, we compare our theoretical predictions with numerical computations. The computations are carried out for pulses in the generalized KdV equation and for fronts that arise in the Gray–Scott model. We conclude with a brief discussion on the implications that our results have for the numerical computation of spectra on large intervals.

6.1. *The generalized KdV equation*

We begin with the generalized KdV equation that is given by

$$u_t + u_{xxx} - cu_x + u^p u_x = 0, \quad x \in \mathbb{R}, \tag{6.1}$$

where c is the wave speed and p is a parameter. This equation admits a family of pulses given by

$$u(x) = [\tfrac{1}{2}c(p+1)(p+2)]^{1/p}\,\mathrm{sech}^{2/p}(\tfrac{1}{2}xp\sqrt{c}) \tag{6.2}$$

for any positive values of c and p. The linearization of (6.1) about one of these pulses is equal to

$$\mathcal{L}v = -v_{xxx} + (c - u^p)v_x + pu^{p-1}u_x v. \tag{6.3}$$

It has been shown in [31] that the pulses are marginally stable in $\mathbb{R}$ for $p < 4$ and unstable for $p > 4$. The instability is induced by a simple unstable eigenvalue that appears for $p > 4$. For any $p \neq 4$, $\lambda = 0$ is an eigenvalue with geometric multiplicity 1 and algebraic multiplicity 2; the associated eigenvectors are u_x and u_c.

To compute the essential and the absolute spectrum, we rewrite the eigenvalue problem $\mathcal{L}v = \lambda v$ about the pulses as a first-order system. The associated asymptotic matrix is given by

$$A_0(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda & c & 0 \end{pmatrix}. \tag{6.4}$$

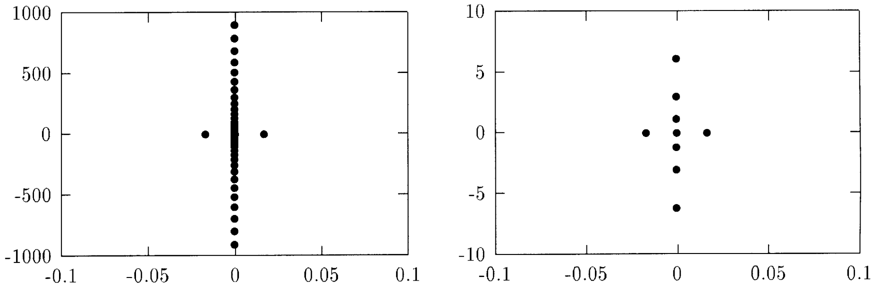

Fig. 4. The spectrum of $\mathcal{L}$ on the interval $(-7.5, 7.5)$ with periodic boundary conditions. We discretized the operator using a pseudo-spectral method with 700 Fourier modes. The resulting matrix-eigenvalue problems were always solved using the routine DGEEV from the LAPACK package [2].

Its three spatial eigenvalues $\nu_j(\lambda)$ are the roots of the dispersion relation

$$d(\lambda, \nu) = \lambda + \nu^2(\nu - c).$$

In particular, the asymptotic index i_∞ is equal to two, and we will need two boundary conditions at $x = L$ and one boundary condition at $x = -L$. Using the dispersion relation, the essential and absolute spectra of $\mathcal{L}$ can be computed

$$\Sigma_{\mathrm{ess}} = \mathrm{i}\mathbb{R}, \qquad \Sigma_{\mathrm{abs}} = (-\infty, -2(\tfrac{1}{3}c)^{3/2}], \tag{6.5}$$

see [32, Proposition 2.3]. The absolute eigenmodes induce transport towards $x = -\infty$ since, for $\lambda \in \Sigma_{\mathrm{abs}}$, the two spatial eigenvalues that have the same real part are located in the left half-plane. The discussion after Remark 5.6 then implies that the eigenmodes are exponentially growing as $x \to -\infty$. This behavior is consistent with $c > 0$.

We first consider $p = 2$, and also fix the wave speed $c = 2$. The pulses are then transiently unstable; see [32]. On the bounded interval $(-L, L)$, we consider periodic boundary conditions as well as the separated boundary conditions

$$u_x(-L) = 0, \qquad u(L) = 0, \qquad u_x(L) = 0. \tag{6.6}$$

We begin by comparing the spectra of the operator $\mathcal{L}$ on the real line and the bounded interval $(-L, L)$ with $L = 7.5$. Fig. 4 shows that periodic boundary conditions indeed recover the essential spectrum. In addition, the two embedded eigenvalues at zero move away from the imaginary axis. For the separated boundary conditions defined in (6.6), we recover the absolute spectrum; see Fig. 5. As predicted, the two embedded eigenvalues at zero stay near the origin but split into two simple eigenvalues. It is straightforward to show that the boundary conditions are non-degenerate near $\lambda = 0$ so that no additional eigenvalues are created there.

We then compared the rate of convergence with which the embedded eigenvalues near zero approach zero as $L \to \infty$. The spatial eigenvalues $\nu_j(\lambda)$ of the asymptotic matrix $A_0(\lambda)$, see (6.4), at $\lambda = 0$ are given by

$$\nu_1(0) = 0, \qquad \nu_{2,3}(0) = \pm\sqrt{c}.$$

The spectral gap is therefore equal to $\sqrt{c}$. Since the multiplicity of the eigenvalue $\lambda = 0$ is 2, we expect that the rate of convergence is equal to $\frac{1}{2}\sqrt{c}$. We calculated the temporal eigenvalues near zero numerically using the package AUTO97, see [17], and continued in the interval length L. The results are shown in Fig. 6(a); the actual rate of convergence is $\sqrt{c}$ and not the expected $\frac{1}{2}\sqrt{c}$. The reason for the super-convergence is as follows: first, the eigenfunction of $\lambda = 0$ on the real line converges faster to zero as $x \to -\infty$ than expected; its exponential rate is $\sqrt{c}$ rather than 0. The same is true for the unique bounded solution $\psi(x)$ of the adjoint eigenvalue equation. The

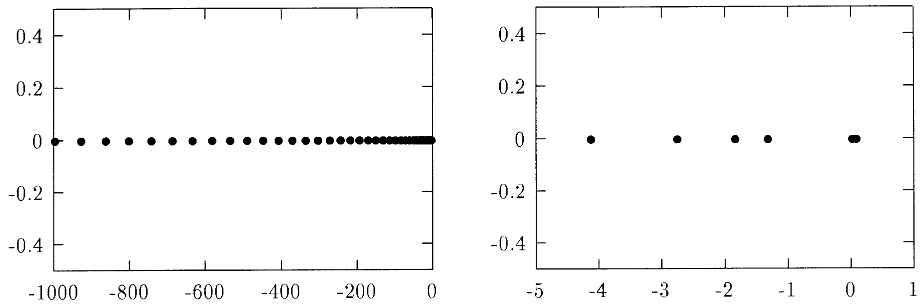

Fig. 5. The spectrum of $\mathcal{L}$ on the interval $(-7.5, 7.5)$ with the separated boundary conditions (6.6). We discretized the operator using one-sided finite differences with 1500 equidistant mesh points corresponding to a step size of $h = 0.01$.

latter is true due to the Hamiltonian nature of the KdV equation. In fact, we have $\psi(x) = \nabla H(u(x))$, where $u(x)$ is the pulse and $H(u)$ is the Hamiltonian of the KdV equation. It is then a consequence of the super-convergence results presented in [36] that the rate of convergence is $\sqrt{c}$ and not $\frac{1}{2}\sqrt{c}$.

Next, we investigate the approach of eigenvalues to the edge of the absolute spectrum located at

$$\lambda_* = -2(\tfrac{1}{3}c)^{3/2} \approx -1.089. \tag{6.7}$$

We expect that the convergence is like

$$|\lambda - \lambda_*| \approx \frac{1}{L^2}.$$

We therefore plotted $(|\lambda - \lambda_*|)^{-1/2}$ over L, and expect to see a straight line. This is confirmed in Fig. 6(b). Note that the slopes of the first three eigenvalues that we continued have a ratio of approximately $1 : \frac{1}{2} : \frac{1}{3}$ as predicted; see Lemma 5.5.

Next, we change the boundary conditions to

$$u_x(-L) = 0, \qquad u_x(L) = 0, \qquad u_{xx}(L) = 0. \tag{6.8}$$

These boundary conditions are no longer non-degenerate. It is straightforward to show that $D_+(\lambda) = \lambda + \mathrm{O}(|\lambda|^2)$. We therefore expect three eigenvalues near $\lambda = 0$. This is confirmed in Fig. 7(a).

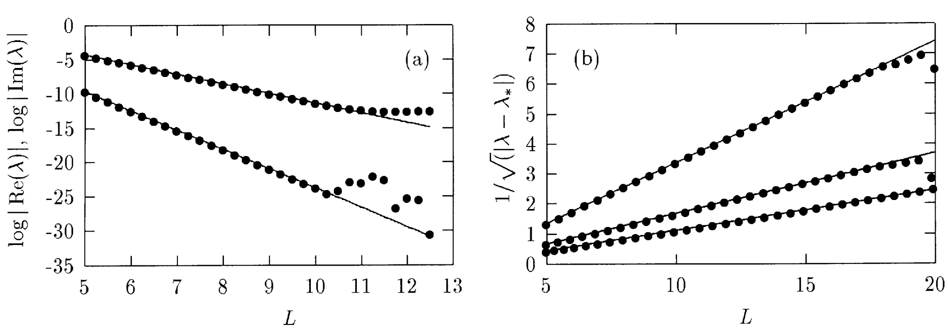

Fig. 6. (a) For the separated boundary conditions (6.6), the slopes of the curves formed by the real and imaginary parts of the eigenvalues that approach zero are given by -2.8275 and -1.4109, respectively. The overall numerically computed slope is therefore -1.4109 which is twice the expected rate of $\sqrt{2}/2$; this super-convergence phenomenon is explained in the main text. (b) The first three eigenvalues that approach the edge of the absolute spectrum are continued in L. The slopes of the scaled curves formed by these eigenvalues are given by 0.40468, 0.20170 and 0.13498, respectively.

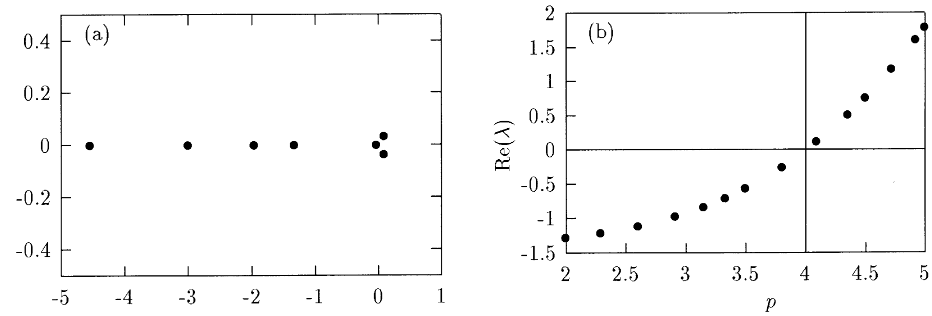

Fig. 7. (a) The spectrum of $\mathcal{L}$ near zero on the interval $(-7.5, 7.5)$ using the boundary conditions (6.8). As predicted, there are three eigenvalues near $\lambda = 0$. (b) The resonance pole is shown as a function of the parameter p. The pole crosses the imaginary axis at $p \approx 4$.

Finally, we return to the case of the transverse boundary conditions (6.6). We shall confirm that resonance poles, i.e. eigenvalues that are generated upon using exponential weights, show up on large intervals with separated boundary conditions but are not visible for periodic boundary conditions. To this end, we vary p in the interval (2, 5). It has been shown in [31] that at $p = 4$, a resonance pole crosses the imaginary axis from left to right at the origin, rendering the pulses unstable. For $p > 4$, this resonance pole is an ordinary eigenvalue that should then be picked up by periodic boundary conditions. Our numerical computations confirm that this is indeed the case; see Figs. 7(b) and 8. Recall that our theory predicts that the absolute spectrum is filled with eigenvalues as $L \to \infty$. Thus, all but finitely many eigenvalues will stay to the left of the edge λ_* of the absolute spectrum. Hence, a priori, we cannot distinguish the resonance pole from other eigenvalues until it emerges from the absolute spectrum through the edge $\lambda_* \approx -1.089$. This happens at $p = 2.551$; see Fig. 7(b).

We used the aforementioned resonance poles, calculated for $p = 4.5$, to illustrate the difference in the convergence rates for periodic and separated boundary conditions. We computed the unstable eigenvalues for increasing values of L and compared them with the “exact” unstable eigenvalue λ^{u}. The latter was calculated using the boundary conditions (6.6) for a large value of L, namely $L_* = 40$. The spatial eigenvalues of the matrix $A_0(\lambda^{\mathrm{u}})$, see (6.4), are

$$\nu_1 = 1.15612, \qquad \nu_2 = 0.42071, \qquad \nu_3 = -1.57683.$$

Thus, from Theorem 2 and Lemma 4.3, we expect the convergence rates $2\kappa = 2\min\{\nu_2, |\nu_3|\} = 0.84143$ for periodic and $\sigma = \nu_2 - \nu_3 = 1.99754$ for separated boundary conditions. This is confirmed by numerical computations using AUTO97; see Fig. 9.

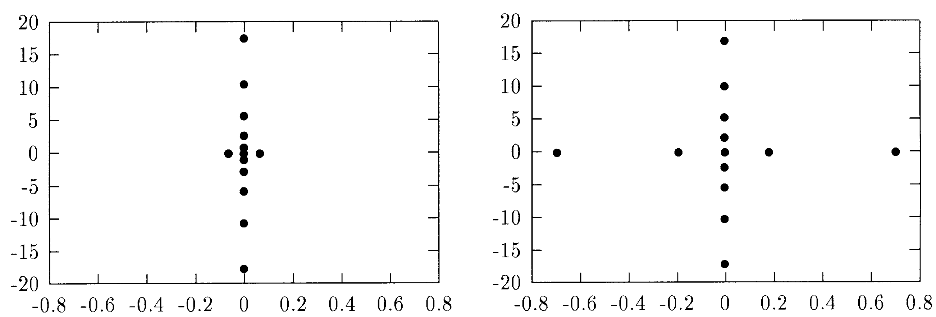

Fig. 8. The spectrum of $\mathcal{L}$ with periodic boundary conditions on the interval $(-7.5, 7.5)$ for $p = 3.5$ (left) and $p = 4.5$ (right). The resonance pole is not visible in the left plot; the unstable pair of eigenvalues for $p > 4$, however, is captured; see the plot to the right. We discretized the operator using a pseudo-spectral method with 800 Fourier modes.

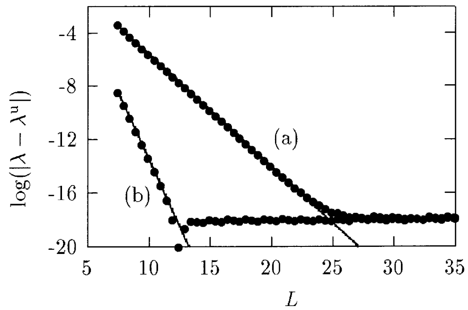

Fig. 9. The scaled error of the unstable eigenvalue, computed numerically for various values of L but for fixed $p = 4.5$, is plotted for: (a) periodic, and (b) separated boundary conditions (6.6). The eigenvalues are compared with the exact value $\lambda^u = 0.766736$ that we obtained using the boundary conditions (6.6) for large L_*. The slopes for the scaled error are -0.84154 for periodic and -2.00403 for separated boundary conditions. Our theory predicts the slopes 0.84143 and -1.99754, respectively.

6.2. The Gray–Scott model

The second equation that we investigate is the Gray–Scott model:

$$u_t = D_1 u_{xx} - cu_x - uv^2 + F(1-u), \qquad v_t = D_2 v_{xx} - cv_x + uv^2 - (F+k)v. \tag{6.9}$$

Here, c denotes again the wave speed. In the parameter regime where $\Delta = 1 - 4(F+k)^2/F$ is positive, (6.9) has three different homogeneous steady states; the two that are of concern to us are commonly referred to as the red and blue states

$$(u_r, v_r) = (1, 0), \qquad (u_b, v_b) = \left(\frac{1}{2}(1-\sqrt{\Delta}), \frac{F}{2(F+k)}(1+\sqrt{\Delta})\right). \tag{6.10}$$

If we choose the parameters according to

$$D_1 = 6.0 \times 10^{-5}, \quad D_2 = 1.0 \times 10^{-5}, \qquad c = -5.02063 \times 10^{-4}, \qquad k = 0.05, \qquad F = 0.1, \tag{6.11}$$

then numerical computations reveal that (6.9) admits a stationary front that connects the blue state at $-\infty$ with the red state at $+\infty$. The front was computed using the driver HOMCONT [10] that is built into the package AUTO97 [17]. We refer to Fig. 12(a) for a plot of the two components of the front. In fact, since $c < 0$, the front moves to the right towards the red state if considered in a non-moving coordinate frame. It can be shown that the red state is stable, while the blue state is unstable for the aforementioned choice of parameters. The linearization of (6.9) about the front is given by

$$\mathcal{L} = \begin{pmatrix} D_1\partial_{xx} - c\partial_x - v^2 - F & -2uv \\ v^2 & D_2\partial_{xx} - c\partial_x - 2uv - (F+k) \end{pmatrix}. \tag{6.12}$$

We calculate its spectrum on the interval (0, 1); note that, if we rescale the equation so that the diffusion constants are of order 1, then the length of the interval would be of the order 1×10^2. We used the boundary conditions

$$u(0) + v(0) = 0, \qquad u_x(0) - v_x(0) = 0, \qquad u(1) + v(1) = 0, \qquad u_x(1) - v_x(1) = 0. \tag{6.13}$$

Neumann boundary conditions violate Hypothesis 7 since the two components of the operator (6.12) decouple at the red state.

First, we computed the absolute and essential spectra of the asymptotic homogeneous states. This was done by continuation within AUTO97. The results are shown in Fig. 10. Note that absolute spectrum is to the left of the

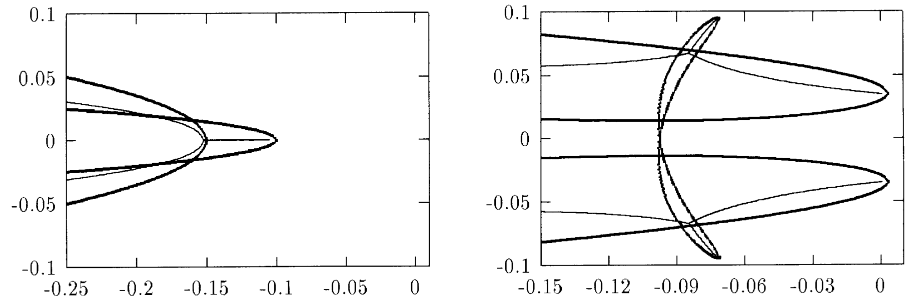

Fig. 10. The absolute and essential spectra of the red (left) and the blue (right) rest states are shown. The essential spectrum is plotted using thicker lines, while the absolute spectra are plotted with thin lines.

essential spectrum. The rightmost edge of the absolute spectrum of the blue state corresponds to a double spatial eigenvalue as does the rightmost point of the absolute spectrum of the red state.

We then computed the spectrum of the operator $\mathcal{L}$ on the interval (0, 1) with the boundary conditions (6.13); see Fig. 11. The computations confirm that the spectrum on the bounded interval asymptotes on the absolute and not on the essential spectrum. The additional eigenvalue at zero is of course due to translational invariance of (6.9).

The absolute spectrum of the blue state is caused by spatial eigenvalues that cross the imaginary axis from right to left. The exponential weight function is therefore given by $e^{\eta x}$ with $\eta > 0$, and we expect that perturbations are convected towards $-\infty$. In particular, eigenfunctions associated with eigenvalues of $\mathcal{L}$ on the bounded interval should be large at the left endpoint $x = 0$ of the domain. This is confirmed in Fig. 12(b), where the u-component of a typical eigenfunction within the absolute spectrum is plotted.

6.3. Numerical computations of spectra on the real line

As we have seen, only periodic boundary conditions generally capture the spectrum of PDE operators on the real line. One of the exceptions is the case where the operator exhibits an additional reversibility structure so that the essential and the absolute spectrum are in fact equal.

For separated boundary conditions, the spectrum that is computed on the bounded interval is the absolute spectrum plus the set of eigenvalues and resonance poles of the original operator. Additional eigenvalues can be created

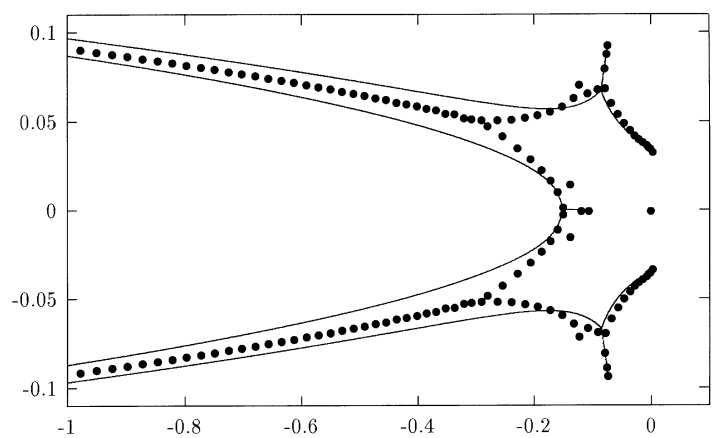

Fig. 11. The spectrum of $\mathcal{L}$ on the interval (0, 1). The thin lines indicate the location of the absolute spectrum of the operator. We used centered finite differences with 2000 equidistant mesh points corresponding to a step size of $h = 5 \times 10^{-4}$.

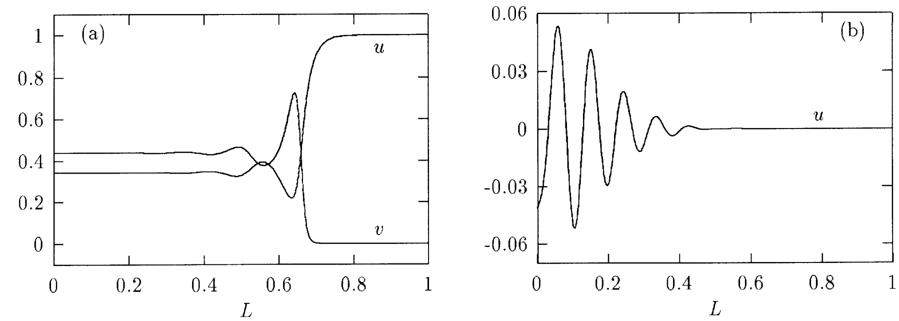

Fig. 12. (a) The left-hand side picture contains the u and v components of the front to the Gray–Scott model as a function of x. The parameters are chosen according to (6.11). (b) The u-component of the eigenfunction of $\mathcal{L}$ associated with the eigenvalue $\lambda = -3.96 \times 10^{-4} - 3.5\,\mathrm{i} \times 10^{-2}$ is plotted. This eigenvalue is close to the rightmost edge of absolute spectrum of the blue state.

through the boundary conditions. To confirm the numerical computations, one could therefore compute the absolute spectrum of the asymptotic states separately, either by using the spatial eigenvalues of the asymptotic matrices or by numerically computing the spectra of the asymptotic constant-coefficient operators. A comparison with the spectrum of the full operator then identifies the absolute spectrum. Spurious eigenvalues generated by the boundary conditions can be identified using different boundary conditions and comparing those eigenvalues that are not related to the absolute spectrum.

Finally, we emphasize that our results are true asymptotically as $L \to \infty$, but that we do not have estimates for how large L really has to be in order to resolve the absolute spectrum over a large region in the complex plane. An additional difficulty is that the operator has to be discretized so that the spectra also depend upon the step size of the discretization scheme. An example where these issues seem to play a role is the FitzHugh–Nagumo equation that has been used in [8] to illustrate domain-truncation results for isolated eigenvalues. It appears as if the computed spectrum is close to the absolute spectrum only extremely near the imaginary axis. Our calculations show that the rest of the spectrum is very sensitive to variations of the length of the interval and the choice of the number of mesh points.

7. Conclusions and discussion

Our results can be summarized as follows. As far as the original point spectrum on the real line is concerned, eigenvalues persist under truncation with their multiplicity. For separated boundary conditions, however, additional eigenvalues can be created when the boundary conditions are not transverse to certain eigenspaces. In addition, eigenvalues may appear in regions that were previously occupied by the essential spectrum; these eigenvalues are often referred to as resonance poles. The essential spectrum of the problem on the real line is recovered under domain truncation only if periodic boundary conditions are imposed. For separated boundary conditions, the spectrum on the bounded intervals asymptotes onto the absolute spectrum as the endpoints of the interval tend to $\pm\infty$.

We have taken three different viewpoints towards stability for operators on the real line: L^2-stability, convective instability, and transient instability. As far as the essential spectrum is concerned, L^2-stability implies stability on all sufficiently large intervals with periodic boundary conditions, while transient instability implies stability under separated boundary conditions. In particular, separated boundary conditions can stabilize: transiently unstable patterns may be spectrally unstable under periodic boundary conditions, while they may be stable under separated boundary conditions. Convective instability does in general not imply stability under separated boundary conditions; see Example 2.

Proving that solutions actually decay pointwise whenever the operator is convectively unstable is in general a difficult endeavor for hyperbolic or dispersive equations since it requires to show the convergence of Γ-integrals.

Uniform bounds on the resolvent usually require a scaling of the scalar product in $\mathbb{R}^N$ for large λ. For instance, the heat equation $u_{xx} = \lambda u$, when rewritten as $u_x = v$, $v_x = \lambda u$, does not admit uniform dichotomies as $\lambda \to \infty$ with $\lambda \in \mathbb{R}$ since the eigenvectors $(1, \pm\sqrt{\lambda})^{\mathrm{T}}$ are asymptotically parallel; $\mathcal{T}(\lambda)^{-1}$ is therefore not uniformly bounded in λ. The correct (space–time) scaling is $u_x = \sqrt{\lambda}v$, $v_x = \sqrt{\lambda}u$ which guarantees uniform dichotomies.

Our results are partial in the sense that they only consider the effect of the truncation on the linearization as in [8]. In general, the stationary solution of the nonlinear PDE itself is perturbed by the boundary conditions. When the essential spectrum does not contain $\lambda = 0$, these perturbations are often harmless. In many circumstances, the perturbed wave is $\mathrm{e}^{-\theta L}$-close to the original wave; see [7]. In this situation, our results are also true if the original wave is replaced by the perturbed wave. This is a consequence of the estimates for exponential dichotomies that were established in [33,35].

The approach using exponential dichotomies is suitable for problems in one-dimensional domains where dynamical-systems properties prove particularly useful. However, the results can be immediately generalized to cylindrical domains with multi-dimensional bounded cross-section and to time-periodic solutions of parabolic problems using a slightly generalized notion of dichotomies and Morse indices; we refer to [27,37,38] for related results. In particular, the absolute Morse indices considered here have to be replaced by relative Morse indices.

In general, the absolute spectrum seems to play an important role whenever boundary or, more generally, matching conditions are imposed. An interesting example is the following situation: suppose that the travelling wave ODE admits a heteroclinic cycle so that the first connection is transversely constructed while the other connection is of codimension two. This situation is often called a *T-point*. The interpretation for the PDE is then as follows. There are two homogeneous rest states so that one of them is stable while the other one is unstable. There are also two fronts that connect the first to the second and the second to the first rest state, respectively. Furthermore, these fronts have the same wave speed. It is known that, for nearby parameter values, the PDE exhibits pulses that connect the stable rest state to itself. An interesting issue is the stability of these pulses. Note that both fronts are unstable since one of their asymptotic states is unstable. Also, the pulses have a long plateau along which they are very close to the unstable rest state. Numerically, it appears as if the bifurcating pulses can be stable, see [41,45], even though in the limiting configuration, i.e. for the heteroclinic cycle, part of the essential spectrum is contained in the right half-plane. Matching or gluing the pulses from two fronts is similar to imposing a boundary condition in the middle of the domain. We therefore expect that the stability properties of the pulse are not determined by the essential spectrum of the unstable rest state but by its absolute spectrum (which can be stable even though the essential spectrum is unstable). As shown in [39], this is indeed the case.

Acknowledgements

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