Indirect Inference and Calibration of Dynamic Stochastic General Equilibrium Models[1]

Ramdan Dridi

Alain Guay[2]

Eric Renault[3]

This version: April 22, 2003

[1] We thank Lars Peter Hansen and Frank Schorfheide for helpful comments. All remaining errors are our own.

[2] Université du Québec à Montréal, CIRPÉE. e-mail: guay.alain@uqam.ca.

[3] Université de Montréal, CIRANO and CIREQ. e-mail: Eric.Renault@UMontreal.CA .

Abstract

We advocate in this paper the use of a Sequential Partial Indirect Inference (SPII) approach, in order to account for calibration practice where dynamic stochastic general equilibrium models (DGSE) are studied only through their ability to reproduce some well-chosen moments. We stress that, despite a lack of statistical formalization, the controversial calibration methodology addresses a genuine issue on the consequences of misspecification in highly nonlinear and dynamic structural macro-models. Such likely misspecification is even more detrimental than for direct inference, since the misspecified model is used for building simulated paths. The only way to get robust estimators, but also to assess the model despite misspecification consists in examining the structural model through a convenient and parsimonious instrumental model, which basically does not capture what goes wrong in the simulated paths. We argue that a well-driven SPII strategy might be seen as a rigorous calibrationnist approach, that captures both the advantages of this approach (accounting for structural "a-statistical" ideas) and of the inferential approach (precise appraisal of loss functions and conditions of validity). This methodology should be useful for the empirical assessment of structural models such as those stemming from the Real Business Cycle theory or the asset pricing literature.

Keywords: Calibration, Indirect Inference, Structural Models, Real Business Cycle, Asset Pricing.

1 Introduction

There is a fairly general agreement about two main goals of Econometrics, as defined by Christ (1996): "the production of quantitative economic statements that either explain the behavior of variables that we have already seen, or forecast (i.e. predict) behavior that we have not yet seen, or both". In any case, this activity relies not only upon empirical facts but also upon a theory, which produces explanation or forecasting.

But, as far as a "unification of theoretical and factual studies in economics" (Frisch (1933) in his editorial statement introducing the first issue of Econometrica) is concerned, the best way to reach these goals is still a matter for controversy. Actually, in his excellent account of the problem of macroeconometrics, Hoover (1995a) points out that, besides the two main strands of econometric thinking that both refer to standard statistical methodology, another strand has begun to be investigated by Frisch (1933) "on the eve of the birth of modern macroeconomics". The aforementioned two main strands characterized the history of econometrics, as summarized by Morgan (1990), but persist to this day and differ about their way to use statistical methodology. On the one hand, statistical methods may be applied to look for some regularities in economic time series through an atheoretical approach that does not refer explicitly to any economic theory. On the other hand, as pointed out by Morgan (1990) and Hoover (1995a) for the typical example of estimation of demand curves, a second strand of Econometrics takes economic theory (e.g. of the downward-sloping demand curve) as given. The statistics aimed only at measuring the relevant elasticity or other parameters of interest. Typically, this second strand refers to the structural approach as developed by the Cowles Commission program. The Keynesian model affords a unified framework where the econometrics of the business cycle can be associated with the structural econometrics of demand measurement (Hoover (1995a)). More recently, the famous Lucas Critique has even more emphasized the necessity of a structural approach, that is to say the reference to parameters termed structural in the Hurwicz (1962) sense because they are invariant with the respect to the considered policy interventions (despite the agents expectations as stressed by Lucas (1972), (1976)). However, while the Cowles Commission program, providing structural statistical inference through the simultaneous-equation model (SEM), has "obtained widespread acceptance among academics and policy makers during the 1960s and early 1970s" (Ingram (1995)), the new classical macroeconomics, as developed around Lucas work since the 1970s has provided new arguments to those who consider that "the Cowles Commission program applied to macroeconomics is a mistake" (Hoover (1995a)). More precisely, according to Hoover (1995a), "the proponents of the so-called Calibration approach believe that the Lucas Critique, properly interpreted, undercuts the case for structural estimation at the macroeconomics level altogether". But, far from concluding that the structural approach should be abandoned, calibrators trace their methodology back to the early work of Frisch (1933), precisely mentioned by Hoover (1995a) as an alternative to the two previously described mainstream strands of Econometrics which are tied down to orthodox statistical methodology. Frisch (1933) assigned reasonable values to the parameters of a simple theoretical model of the business cycle in order to examine its simulated behavior and to compare that behavior to the actual economy. Such a methodology is clearly very close to the calibration approach, as described in further details below. Let us just stress at this stage that, faced with some empirical failure of the Cowles program, this strand of modern macroeconomics has chosen to reject, at least partially, the orthodox statistical methodology to remain true to the structural approach. It is worth noticing that this choice is something like the exact opposite of the Sims (1980, 1996) program of Vector Autoregressions (VAR) which is another form of answer to the Lucas Critique. In the latter approach, a VAR model is specified for the variables of interest on purely statistical (i.e. atheoretical) grounds and structural properties like causality and exogeneity are tested inside this framework.[1] Sims does not ignore the Lucas Critique but considers that changes in regime due to policy interventions do not invalidate the VAR framework as long as the stationarity paradigm may be maintained.

But, to some extent, one might argue that the atheoretical approaches like Sims and LSE methodologies justify a contrario the Calibration approach since they prove that, after some disappointment about the empirical performance of the Cowles program, econometricians should choose between orthodox statistic methodology and a more symbiotic relationship with Economic Theory. As far as one wants to remain true to some paradigms of modern macroeconomics, typically the bedrock of the macro general equilibrium model synthesized by the intertemporal optimization program of a representative agent (for a given specification of tastes and technology), one should relax some usual requirements for statistical orthodoxy.

Amazingly, this regained freedom of quantitative Economic Theory (as proposed by calibrators) with respect to statistical orthodoxy is acknowledged by both its detractors and its proponents. While the former consider that this makes questionable the credibility of calibrators "computational experiments", the latter claim this freedom by using "the mantel of Frisch" (Hoover (1995b)) to argue that econometrics is not coextensive with estimation and testing, that is with orthodox statistics. More precisely, Kydland and Prescott (1991) claims that Calibration is also econometrics by referring to Frisch (1970) review of the state of Econometrics: "In this review he discusses what he considers to be econometric analysis of the genuine kind and gives four examples of such analysis. None of these examples involves the estimation and statistical testing of some model. None involves an attempt to discover some true relationship. All use a model which is an abstraction of a complex reality to address some clear-cut question or issue".

Such an endorsement of Calibration as an alternative to estimation[2] (Hansen and Heckman (1996)) leads one to the conclusion that "the new classical macroeconomics is now divided between calibrators and estimators" (Hoover (1995b)). Actually, we share with Hansen and Heckman (1996) the opinion that the construction of such artificial distinctions is counterproductive and the main goal of this paper is to try to

[1] The "general-to-specific" approach was also enhanced by the LSE school. Rougly speaking, we are going to argue hereafter that the calibration approach is the exact opposite, since it can be viewed as "specific-to-general".

[2] And the related endorsement of verification as an alternative to statistical tests.

go further in the research program advocated by Hansen and Heckman (1996): "We then argue that the model calibration and verification can be fruitfully posed as econometric estimation and testing problems. In particular, we delineate the gains from using an explicit econometric framework." This does not mean in our opinion that econometricians have nothing to learn from calibrators (or that calibration is not econometrics). The now well established methodology of statistical inference is able to incorporate and to take advantage of some practices that calibrators are right to point out as relevant for empirical economics. Indeed, not only do we consider as Hansen and Heckman (1996) that properly used and qualified simulation methods can be an important source of information and an important stimulus to high-quality empirical economic research; but also that calibrators give to statisticians a useful insight about the good way to perform these simulations in the framework of general equilibrium theory. Moreover, we aim at delineating a close methodology which could be able to gather both the advantages of the inferential approach (estimation and specification testing) and also the advantages of Calibration approach, that correspond, in our opinion, to consistent estimation of some structural parameters of interest and robust predictions despite misspecifications in the structural model used as a simulator. In other words, we acknowledge with calibrators that, in order to address "genuine" econometric issues, one often needs an alternative to the "quest for the Holy Grail" (Monfort (1996)), that is the hopeless search for a well-specified parametric model that is more often than not impossible to deduce from the Economic Theory. In this respect, it is true that we should not be obsessed by estimation and statistical testing of some model, viewed as an attempt to discover some true relationship but we consider that the modern calibrationnist practice can be fruitfully posed as econometric estimation and testing problems of something different from a "true unknown model" to be discovered.

In order to be more precise on this somewhat artificial distinction drawn between calibration and estimation, it is perhaps necessary to briefly recall in what context calibration is the most repeatedly advocated in modern macroeconomics, that is the empirical dynamic stochastic general equilibrium model (DSGE)[3]. In this context, the two methodologies (calibration and estimation) appear at first glance (as well explained by Canova (1994)) to share the same strategy in terms of model specification and solution. Namely, the first step entails specifying a dynamic equilibrium model while selecting convenient functional forms for preferences and technology processes. In the second step, the modeler derives, possibly through simulation, a solution for the endogenous variables in terms of the exogenous and predetermined variables and the parameters.

But it is when it comes to choosing the parameters to be used in the simulations and in assessing the performance of the model that several differences emerge. The estimation procedure attempts to find the parameters that lead to the best statistical fit either by Maximum Likelihood or Generalized Method of Moments (GMM hereafter) when a direct approach is feasible or, otherwise, by Simulated Method of

[3] Nowadays, the calibration approach is so tightly identified with the so-called Real Business Cycles (RBC) approach to analyzing economic fluctuations that in his texbook on "Advanced Macroeconomics", Romer (1996) raises this "empirical philosophy" as one of the four most important objections to the RBC theory.

Moments, Indirect Inference or Efficient Method of Moments (EMM hereafter). The performance of the model is examined through a battery of specification and goodness of fit tests.

The second approach calibrates parameters using a set of alternative rules which includes the matching of long run averages and chosen stylized facts such as moments of interest, the use of previous estimates or a priori selection. On top of that the fit of the model (verification step) is assessed through a rather informal distance criterion based on personal expertise. It is clear that this methodology has raised a huge amount of criticism among statisticians, First, the current use of the so-called "calibrators common knowledge", that is specific parameters values deduced from previous empirical studies, is at odds with any orthodox statistical estimation theory. One may wonder why the modeler needs to refer to such a common knowledge. Second, in order to minimize the number of "evaluated" and "calibrated" parameters, the calibration methodology only aims to reproduce some stylized facts. As stressed by Hansen and Heckman (1996) this runs the danger of making many models with very different welfare implications compatible with the evidence. In this respect, to what extend can we trust such calibrated models and how should we use them for evaluating the effects of policy interventions?

These criticisms are relevant as long as one considers and acts as if the structural model was well-specified for all the salient features of the data. However, when faced with the likely misspecification of highly nonlinear structural macro-models, one can find a rationale for the calibration approach in the following sequence of arguments.

First, as already noticed by Hoover (1995b), the alleged lack of discipline of the calibration methodology is to some extent balanced by another kind of discipline: "for Lucas (1980, p. 288) and Prescott (1983, p.11), the discipline of the calibration method comes from the paucity of free parameters". Since theory places only loose restrictions on the values of key parameters and they are often deduced from econometric estimation at the microeconomic level or accounting considerations, Hoover (1995b) stresses that the calibration method actually appears to be a kind of "indirect estimation".

Second, such an indirect estimation, which can be traced back to the early works of Klein and others ("Indirect Least Squares" for SEM) is now endowed with a close framework termed Indirect Inference by Gouriéroux, Monfort and Renault (1993). The core of this methodology is a family of instrumental parameters, possibly defined from some auxiliary model, for which consistent estimators are available. Then, the structural parameters are indirectly identified through a binding function which relates them to instrumental parameters. When, due to the complexity of nonlinear rational expectation models, the binding function is not available in closed form, it can be estimated from simulated paths drawn from the structural model. Then, the likely mispecification of some features of the structural model is even more detrimental than for direct inference since the vector of structural parameters is only identified as a whole, through the binding function.

This will more often than not produce a contamination of the estimation of the structural parameters of interest by the ill estimation of some nuisance parameters in the structural model. This contamination is even more striking when one realizes that the binding function is estimated from simulated paths produced by a misspecified structural model used as a simulator. In this respect, the Partial Indirect Inference (PII hereafter), as proposed by Dridi and Renault (1998) (DR hereafter), addresses the issue of consistent estimation and testing of some structural parameters of interest, when a potential misspecification of the fully parametric structural model is acknowledged. The crucial problem is that, on the one hand, a fully parametric structural model is needed to be able to draw simulated paths conformable to it (or equivalently to characterize a binding function) but, on the other hand, this complete parametric specification is likely to be misspecified and thus to provide a wrong simulator. In this context, it is shown that the only way to protect oneself against likely misspecifications of the structural model, while it is used for building simulated paths, is to examine it through a convenient instrumental model which does not capture what goes wrong in the simulated paths.

The starting point of this paper is that the methodology of Partial Indirect Inference provides some statistical foundations for the calibration methodology: since we know that the structural model is misspecified but we really need it for the interpretation of some structural parameters, we try to estimate it only through well-chosen characteristics which are conformable to the main purpose of the model. The underlying philosophy is that some elements of truth involved in the model should be caught by matching only some "well-chosen moments" and not a too large set of moments prompted by an automatic statistical process. Otherwise, we might get an inconsistent estimator of the parameters of interest as well as unreliable predictions, due to a contamination in dimensions where the model may do miserably. This is nothing but the aforementioned "discipline of the calibration method" which "comes from the paucity of free parameters".

The verification step can then be performed by using consistent PII estimators. Such strategy disentangles the calibration and verification steps and reconciles them with their econometric counterparts of estimation and testing. The criterion used for the verification step corresponds to the economic phenomena that the model is addressed to reproduce. This criterion may then be different from the one used to obtain consistent estimators in the first step. For instance, a common practice is to assess the goodness of fit of the model through its ability to reproduce second moments of aggregate time series characterizing U.S. business cycle. In fact, the choice of the criteria is tightly related to the "clear-cut question" addressed by the model. But, by contrast with the R.B.C. calibrationnist approach, the proposed verification strategy is not informal but based on well-defined statistical tests. However, we follow the calibrationnists approach by considering that the specification tests should only be focused on the reproduction of stylized facts the structural model is aimed to capture. Indeed, one can always find a dimension of the data for which the model is rejected since the model is for sure misspecified. The structural model must not reproduce all empirically aspects of the data but only the well-chosen moments corresponding to the question of interest.

The issue of statistical formalization of the calibration methodology has already been addressed in particular by Gregory and Smith (1990), Watson (1993) for a classical approach and by Canova (1994), Dejong, Ingram and Whiteman (1996), Geweke (1999) and Schorfheide (2000) for a Bayesian one. In both cases, the emphasis is laid on the ability of the structural model to reproduce some features of interest. In this paper, we focus not only on the ability of the structural model to reproduce selected moments of interest but we also address the issue of consistent estimation of some parameters of interest. Because the proposed procedure is a two step one, we call it Sequential Partial Indirect Indirect (SPII). In our opinion, this two-step statistical methodology remains exactly true to the calibrationnist point of view: reproducing some dimensions of interest under the constraint that some parameters of interest are consistently estimated.

The paper is organized as follows. In section 2, the issues of interest and the general framework to address them are defined through some template examples of the calibration literature. The statistical theory of Sequential Partial Indirect Inference is stated in section 3. Section 4 summarizes the contributions of SPII to afford a close framework to calibration.

2 Calibration as econometrics of misspecified models

The statistical assessment of economic models raises a specific issue: as already pointed out by Canova (1994) the probability structure is, to a large extent, completed in an arbitrary way (in comparison with what the structural model really specifies) and the "economic model is seen, at best, as an approximation of the true DGP which need not be either accurate or realistic and, as such, should not be regarded as a null hypothesis to be statistically tested" so that "the degree of confidence in the results depends on both the degree of confidence in the theory and in the underlying measurement of parameters". These observations pave the way for a rehabilitation of some common calibrators' practices while statisticians like Pagan (1995) use to bring against them the accusation to be "very close to blaming the data if the calibrator's model fails to fit".

Actually, Canova (1994) pleads guilty concerning this accusation when he acknowledges that "the degree of confidence in the results depends on both the degree of confidence in the theory and the underlying measurement of parameters" but this practice is not, in our opinion, open per se to criticism. This proves that the old debate "measurement" versus "theory" as popularized by Koopmans (1947) is still a matter of controversy. How could we then explain the calibration approach in comparison with a more traditional statistical methodology?

We share Gregory and Smith (1990) opinion that it is not fortuitous if Calibration and GMM (Hansen (1982)) were introduced to macroeconomics at the same time and in the same journal, "Econometrica 1982". If one reduces these two approaches to their statistical apparatus, they look very similar at first sight:

- They both focus on structural parameters (as taste parameters) and neglect to a large extent other parameters such as the technology parameters.
- Both approaches are based on "matching moments".
- Both can lead to simulation-based versions since moments of interest to be matched are often cumbersome either computationally or analytically.

But we agree with Canova (1994) to argue that the differences between the two Schools of Thought "are tightly linked to the questions the two approaches ask". Roughly speaking, the estimation approach asks the question "given that the model is true, how false it is?" In other words, considering that the true unknown DGP belongs to the class of p.d.f. delineated by the structural model, how should the econometrician efficiently provide confidence intervals, specification tests as well as optimal forecasts.

By contrast, the calibration approach asks: "given that the model is false, how true is it?" That is to say, acknowledging that any structural model is misspecified, how should the econometrician rely on this model to perform robust estimation of structural parameters of interest as well as robust predictions.

A recent illustration of this debate is the divergence between two apparently similar methodologies proposed by Gallant and Tauchen (1996) on the one hand and by Gouriéroux, Monfort and Renault (1993) on the other hand with respective names "Efficient Method of Moments" (EMM) and "Indirect Inference". While the EMM method asks the question "given that the model is true, how false is it", or, according to the conclusion of Bansal, Gallant, Hussey and Tauchen (1995) "if a structural model is to be implemented and evaluated on statistical criteria i.e., one wants to take seriously statistical test and inference, then the structural model has to face all empirically relevant aspects of the data", the Indirect Inference is rather based on the idea that "it is possible that a model structure that does a good job in matching some chosen moments may do miserably in other dimensions" (Bansal, Gallant, Hussey and Tauchen (1995)). In some sense, "given that the model is false", some elements of truth involved in the model (for instance some taste parameters) should be caught by matching only "some chosen moments" and not a too large set of moments prompted by an automatic statistical procedure. Otherwise, we might get an inconsistent estimator of parameters of interest, due to a contamination in dimensions where the model may do miserably. The problem is that, even though one acknowledges some empirical weaknesses of any theoretical model as for instance the fact that any equilibrium model is too smooth to produce realistic nonlinearity (Bansal, Gallant, Hussey and Tauchen (1995)), nobody suggests to abandon the equilibrium model. One of the main goals of this paper is precisely to provide some sensible guidance to the economist's confusion as stressed by Bansal, Gallant, Hussey and Tauchen (1995): "the findings about an equilibrium model being too smooth left the reader alone in front of the central question of the usefulness of the structural model, if one excludes the possibility of isolating a few selected dimensions along which it does well and along which it could be used".

In other words, the present paper is conformable to the Pagan (1994) research agenda: "there is now extensive material on how to perform comparison between misspecified models (see Smith (1993); Gouriéroux and Monfort (1995)), although much of the theory assumes that θ has been estimated by maximum likelihood rather than GMM estimator that is most popular among calibrators.[4] Extension of this theory to GMM estimators should make it possible to effect comparisons between models".

According to Kydland and Prescott (1991) the so-called calibration methodology was first introduced in economics by Frisch (1933) in his pioneering work "Propagation Problems and Impulse Response Problems in Dynamic Economics", already addressing some business cycle issues. It is nowadays popular, not only in the Real Business Cycles literature (following Kydland and Prescott (1982)) but also for understanding asset pricing puzzles, starting from Mehra and Prescott (1985) on the Equity Premium puzzle. It has more recently been developed and applied not only to the Real Business Cycles (strand of the literature initiated by Kydland and Prescott (1982)) but also to the Equity Premium Puzzle.

We focus in the sequel on these two strands of the literature that we consider as representative illustrations of the calibrationnist practices.

2.1 The Equity Premium Puzzle

In their presentation of the calibration approach, Kydland and Prescott (1991) lays the emphasis on the crucial role of the research question which must be clearly defined [5]. Mehra and Prescott (1985) addresses the question whether the large differential between the average return on equity and average risk free interest rate can be accounted for by models neglecting any frictions in the Arrow and Debreu set up. The simple statement of this question defines on the one hand the structural parameters of interest and on the other hand the instrumental parameters through which the empirical evidence is summarized.

In order to statistically formalize the calibration concepts, we introduce in this section general notations that are consistently maintained herein.

First, the structural parameters of interest for Mehra and Prescott's question are two taste parameters of a representative agent: $\theta_1 = (\gamma, \alpha)'$ in Lucas (1978) type consumption based CAPM. The representative agent preferences over random consumption paths are described by a time-separable expected power utility function

$$E_0 \sum_{t=0}^{\infty} \gamma^t U(c_t)$$

where

$$U(c_t) = \frac{c_t^{1-\alpha} - 1}{1 - \alpha}$$

[4] see Guay and Renault (2003) for comparison between misspecified model in GMM and SPII contexts.

[5] Actually, the sole word question is used for a section title.

and c_t denotes the consumption at time t. Of course, this way of economically defining the structural parameters of interest is tightly linked to the economic setting the modeler has in mind and might be reducing since, while γ represents the subjective discount factor, α represents both relative risk aversion and inverse of the elasticity of intertemporal substitution. This implicitly assumes that this reduction has no incidence on the answer to the aforementioned question of interest. Anyway, we stress here that the structural parameters of interest θ_1 are intrinsically defined through economic paradigms rather than through falsifiable statistical relations.

Second, in this approach the structural model is empirically assessed through its ability to reproduce some stylized facts of interest like here the high value of the equity premium. In our statistical framework, these stylized facts are referred to as the set of instrumental parameters denoted β. The empirical relevance of the structural model is assessed precisely through the matching between the observed instrumental characteristics and their theoretical counterparts consistent with the structural model.

Perhaps one of the most difficult issue for a close statement of the calibration methodology is that the reality check relies on additional assumptions which are not part of the economic theory of interest. These additional assumptions may require the specification of additional parameters θ_2 possibly of infinite dimension. In Mehra and Prescott (1985), these parameters θ_2 define the technology, that is the Markov chain assumed to govern the gross rate of dividend payments. More precisely, this gross rate x_t is described by a two states Markov chain:

$$Pr\{x_{t+1} = \lambda_j \,|x_t = \lambda_i\} = \phi_{ij}, \;\; i, j = \{1, 2\},$$

where

$$\lambda_1 = 1 + \mu + \delta, \qquad \lambda_2 = 1 + \mu - \delta,$$

and

$$\phi_{11} = \phi_{22} = \phi, \qquad \phi_{12} = \phi_{21} = 1 - \phi.$$

In other words $\theta_2 = (\mu, \delta, \phi)$. More generally, the vector θ of structural parameters is split into two parts θ_1 and θ_2 where θ_1 gathers the characteristics of interest while θ_2 corresponds to nuisance parameters which are needed for the statistical assessment. The most usual case is the one where θ_1 is related to preference specifications (taste parameters) and θ_2 describes environmental characteristics (technology parameters). However it may be the case that, as it is for the question above, one is not interested in a complete description of preferences. Then the specification of θ_1 focuses only on a subset of taste parameters (discount factor, risk aversion coefficient) while θ_2 may include other behavioral characteristics (e.g. elasticity of intertemporal substitution).

In any case, the main role of these nuisance parameters θ_2 consists in indexing a binding function between the structural parameters of interest θ_1 and the instrumental parameters β:

$$\beta = \tilde{\beta}(\theta_1, \theta_2). \tag{2.1}$$

Of course, the value β of the instrumental parameters defined by (2.1) is the theoretical one and does not coincide in general with the (population) value of the observed one ; this is precisely the question addressed by the calibration exercise. For sake of illustration, let us go into further details in the presentation of the Mehra and Prescott (1985) model. They show that the period return for the equity if the current state is i (with a level c_t of consumption) and the next period state is j is given by:

$$r_{ij}^e = \frac{\lambda_j (w_j + 1)}{w_i} - 1, \tag{2.2}$$

where w_1 and w_2 are computed from the Euler equation through the linear system of two equations:

$$w_i = \gamma \sum_{j=1}^{2} \phi_{ij} \lambda^{1-\alpha} (w_j + 1), \quad i = 1, 2. \tag{2.3}$$

In other words the expected return on the equity is:

$$R^e = \sum_{i,j=1}^{2} \pi_i \phi_{ij} r_{ij}^e, \tag{2.4}$$

where $\pi = (\pi_1, \pi_2)'$ corresponds to the vector of stationary probabilities of the Markov chain. The same type of characterization is available for the risk free return R^f and omitted here.

Therefore the above formulas (2.2)-(2.4) provide the already announced binding function between $\theta_1 = (\gamma, \alpha)'$ and $\beta^g = g(R^f, R^e) = \left(R^f, R^e - R^f\right)'$ where the vector $g(\cdot)$ contains the moments of interest.[6] Of course, this function is indexed by the additional parameters $\theta_2 = (\mu, \delta, \phi)'$ which characterize the Markov chain through.

The specific feature of the calibration methodology with respect to more standard statistical inference appears precisely at this stage: since our goal is to ask whether, given the technology, there exist taste parameters capable of matching the returns data, this, according the Cechetti, Lam and Mark (1993) "dictates that we proceed in two steps, first estimating the parameters of the endowment process, and then computing a confidence bound for the taste parameters γ and α".

With respect to more orthodox econometrics, this two steps procedure may arouse, at least, two types of criticism: First, even though the only parameters of interest are the taste parameters θ_1, one get in general more accurate estimators by a joint, possibly efficient, estimation of $\theta = (\theta_1', \theta_2')'$. Second, even when ignoring the efficiency issue, it is somewhat questionable with regard to consistent estimation to focus on taste parameters while the technology corresponds obviously to a caricature of the reality. Nobody may believe that the endowment process is conformable with a two states Markov chain and this misspecification presumably contaminates the estimation of the parameters of interest.

[6] The notation β^g is introduced to specified that the binding function is defined relative to the vector of moments g.

2.2 Encompassing assessment of the computational experiment

In our opinion, a garbled answer to the above criticisms would consist in claiming that this procedure should not be regarded as an econometric one attempting to consistently estimate the parameters of interest. In this respect, we share Hansen and Heckman (1996) point of view that the distinction drawn between calibrating and estimating the parameters of a model is artificial at best.

Actually, the core principle of the calibration approach as illustrated in Mehra and Prescott paper's consists in concluding that the structural model is rejected on grounds of "computational experiments" leading to unlikely values of the parameters of interest. Namely, in Mehra and Prescott (1985) it is argued that computed values of the discount factor and the relative risk aversion parameter outside their commonly acknowledged range $(0 < \gamma < 1, 0 \leq \alpha \leq 10)$ proves the misspecification of the structural model. How could they maintain such an argument if they did not think that these computed values are consistent estimators of something which makes sense?

Consequently, we think that calibration should also be interpreted in terms of consistent estimation of the parameters of interest, even though this issue is addressed in a non standard way in several respects:

- First, as explained above, it is often addressed in a negative way. The model is rejected because the estimators of its alleged parameters are obviously inconsistent.

- Second, consistency is the only focus of interest. Efficiency is irrelevant in this setting since the calibration exercises gather a huge amount of historical information such as series of asset returns over the whole last century in such way that the efficient use of the information is not an issue at all.

- Third, calibrators are fully aware that consistency might fail, precisely due to the misspecification of the technology or more generally of the additional assumptions about the nuisance parameters θ_2. Indeed, fully cautious about that, they advocate calibration as a search for sensible values of θ_2.

The main goal of this paper is to statistically analyze into further details the latter point. To the extent that the aforementioned consistency requirement is maintained, the crucial concern is the following: When one uses the binding function $\tilde{\beta}^g(\cdot, \bar{\theta}_2)$ indexed by a hypothetical value $\bar{\theta}_2$ of θ_2 to recover an estimate $\hat{\theta}_1$ of the parameters of interest θ_1 from an empirical measurement $\hat{\beta}^g$ of the instrumental parameters β^g by solving [7]:

$$\hat{\beta}^g = \tilde{\beta}^g(\hat{\theta}_1, \bar{\theta}_2), \tag{2.5}$$

is there any hope that $\hat{\theta}_1$ consistently estimates the true unknown value θ_1^0 of the structural parameters of interest? Before answering this question, three preliminary remarks are in order:

[7] We do not mention the issue on overidentification which might prevent one from finding an exact solution to (2.5). See section 3 for more details.

1. On the one hand, the sole idea of a true unknown value θ_1^0 of the structural parameters relies on the maintained hypothesis that the DGP is conformable to our structural ideas. This does not prevent from accounting for the calibrationnist approach which considers the estimation issue in a negative way as already explained.

2. On the other hand, we do not question here the consistency of the instrumental estimator $\hat{\beta}^g$ since the instrumental parameters $\beta^{g,0}$ are essentially defined as the population value of $\hat{\beta}^g$.

3. Finally, we consider for the moment that the binding function $\tilde{\beta}^g(\cdot, \bar{\theta}_2)$, for any reasonable value $\bar{\theta}_2$, is well defined and known as is the case of the Mehra and Prescott (1985) framework. However, to capture complicated features of richer models, simulations at different levels of the forcing processes and parameters may be useful when analytical computation is intractable. This is perhaps the reason why calibrators have extensively used simulations.

The hope for getting a consistent estimator $\hat{\theta}_1$ of θ_1° by solving (2.5) can then be supported by two alternative arguments according to our degree of optimism: Either, one adopts an optimistic approach wishing that history has provided sufficiently rich empirical evidence to determine without ambiguity a value $\bar{\theta}_2$ of the nuisance parameters. This is typically what is referred to as the calibration step. However, one should keep in mind that the technology is crudely misspecified (see the two states Markov chain above) in such a way that the estimator $\hat{\theta}_1$ can be consistent only by chance whatever the choice of $\bar{\theta}_2$. Or, to be more cautious, one tries different values of $\bar{\theta}_2$ to check whether the outcome of the computational experiments is drastically changed. This is what is called the robustness of results in Mehra and Prescott (1985) and more generally the sensitivity analysis in the calibration literature.

Of course, an ingenuous comment about this debate would be: one should jointly statistically estimate $(\beta^g, \theta_1, \theta_2)$ under the constraint (2.5). But this proposal is irrelevant in the calibration framework since the modeler knows a priori and before any statistical inference that the nuisance parameters θ_2 do not make sense on their own. Moreover, one of the main recommendations of this paper is to be suspicious in front of sophisticated strategies of model choice and fit about the technology characteristics. For instance, following Bonomo and Garcia (1994) it is true that by contrast with Cecchetti, Lam and Mark (1990) “a well-fitted equilibrium asset pricing model” may account for some stylized facts but one cannot be sure that the improvement in the technology specification is really relevant for the question of interest since misspecification is always guaranteed. For the same reason, the modern literature on EMM through fitting a nonlinear semi-nonparametric score (see Bansal, Gallant, Hussey and Tauchen (1995), Gallant, Hsieh and Tauchen (1997), Tauchen, Zhang and Liu (1997)) suffers from the same drawback since it often forgets that the crucial point is the so-called encompassing condition that is precisely defined in section 3.

Roughly speaking, we shall say that, endowed with the pseudo true value $(\theta_1^\circ, \bar{\theta}_2)$, the structural model encompasses the instrumental one when the following consistency condition is guaranteed:

$$\beta^{g,0} = \tilde{\beta}^{g}\left(\theta_1^0, \bar{\theta}_2\right).$$

We want to stress here that this consistency condition is what really matters to validate the calibration exercises. This has almost nothing to do with the accuracy of the proxy of the technology provided by the nuisance parameters to the extent that the structural models is always "an abstraction of a complex reality" (Kydland and Prescott (1991)).

The calibration strategy adopted by Cechetti, Lam and Mark (1993) reflects the concern for a parsimonious choice of the instrumental model given the technology process. These authors also investigate the equity premium through the first and second moments of the risk-free rate and the return to equity. As in Mehra and Prescott, the utility function is time-separable with a constant relative risk aversion. While Mehra and Prescott consider consumption and dividend as equal and then calibrate on an univariate Markov process, the model developed by Cechetti, Lam and Mark (1993) explicitly disentangles consumption from dividends and the endowment process is defined by a bivariate consumption-dividends Markov-Switching model.

Cechetti, Lam and Mark (1993) are clearly aware of the problem of choosing a too large set of moments to estimate both structural parameters of interest and the endowment. The explicitly argue that it would not be well-suited to estimate the parameters of interest and the endowment process jointly by maximum likelihood procedure. Such an estimation strategy forces the model to match all the aspects of the data and it is unlikely that a simple model could reproduce adequately all those aspects.

Cechetti, Lam and Mark (1993) proceed in two steps: first, they estimate the parameters of endowment process through a subset of moments chosen to match the maximum likelihood estimates of a bivariate consumption-dividends Markov-Switching model. In the second step, they compute a confidence interval bound for the taste parameters through first and second moments of returns data for a given endowment process. In our notation, this defines two subvectors of instrumental parameters namely;

$$\begin{aligned} \beta_1^g &= \beta_1^g\left(\theta_1(\bar{\theta}_2)\right) \\ \beta_2^g &= \beta_2^g(\theta_2). \end{aligned}$$

where $\theta_1 = (\gamma, \alpha)'$ and θ_2 gathers the parameters for the endowment process. The subvector $\beta_2^g(\cdot)$ corresponds to the subset of moments chosen to match the maximum likelihood estimates of the bivariate consumption-dividends Markov-Switching model. The subvector $\beta_1^g(\cdot)$ contains the first and second moments of return data used to estimate the structural parameters θ_1 given the technology characterized by $\bar{\theta}_2$. The notation $\bar{\theta}_2$ stresses the fact that the concept of true unknown value does not make sense for the nuisance parameters. As in Mehra and Prescott, the model evaluation relies on the plausibility of the confidence interval bound for the discount factor parameter (γ) and the relative risk aversion parameter (α). The calibrator does not want to match the entire set of instrumental parameters β^g to avoid contamination by the misspecification of the endowment process. The adequacy of the model is judged only through the plausibility of the structural parameters given the fit of a parsimonious subset of moments corresponding to β_1^g.

More generally, we consider that, endowed with the pseudo true value $\left(\theta_1^0, \bar{\theta}_{21}\right)$, the structural model partially encompasses the instrumental one when the following consistency condition is guaranteed:

$$\beta_1^{g,0} = \tilde{\beta}_1^g \left(\theta_1^0, \bar{\theta}_{21}\right),$$

and the vector of nuisance parameters is divided as: $\theta_2 = (\theta_{21}', \theta_{22}')'$. As far as one is mainly concerned with the estimation of the structural parameters θ_1, the crucial issue of partial encompassing is the existence of subvector β_1^g. In the case of Cechetti, Lam and Mark (1993) the vector θ_{21} is empty such that the entire vector of nuisance parameters is estimated through the subvector β_2^g.

2.3 General equilibrium approach to business cycles: an illustration

Kydland and Prescott (1982) introduced a neoclassical one-sector growth model driven by technology shocks to reproduce cyclical properties of U.S. economy. The model includes a standard neoclassical production, standard preferences to describe agent's willingness to substitute intratemporally and intertemporally between consumption and leisure and a driven exogeneous process given by the technology process. The Kydland and Prescott's model and the subsequent macro dynamic equilibrium models based only on real shocks with no role for monetary shocks are called Real Business Cycle (RBC) models.[8].

The clear-cut question addresses by Kydland and Prescott (1982) is the following: How much would the U.S. economy have fluctuated if technology shocks had been the only source of fluctuations? Obviously the model is misspecified. In particular, it implies some unrealistic stochastic singularity for the vector of endogenous variables.[9]

This question addressed by Kydland and Prescott defines the moments (instrumental parameters) through which the empirical fit of the model has to be assessed. The instrumental parameters correspond to second moments describing the cyclical properties of U.S. postwar economy. While these moments can be easily estimated from the data, simulations are often required to compute their theoretical counterpart. In the strategy advocated by Kydland and Prescott (1982) the answer to the question of interest is then given by an informal distance between empirical instrumental parameters and the instrumental parameters under the structural model. The values of the structural parameters are previously deduced from applied micro-studies or by matching long run properties of U.S. economy.

[8] For extensions of this model see e.g. Hansen (1985), Beaudry and Guay (1996) and Burnside and Eichenbaum (1996).

[9] Some empirical applications bypass this misspecification problem by augmenting the theoretical solution of the model with a measurement error for each endogenous variables. The augmented model is then estimated by maximum likelihood (see Hansen and Sargent (1979) and Christiano (1988)). See Watson (1993) and Ruge Murcia (2003) for a discussion.

For sake of illustration, we consider here a benchmark RBC model (King, Plosser and Rebelo (1988a), (1988b)). The social planner of this economy maximizes

$$E_0 \sum_{t=0}^{\infty} \gamma^t \left[\ln(C_t) + \phi \ln(L_t)\right]$$

where C_t is per capita consumption, L_t is leisure, γ is the discount factor and ϕ is the weight of leisure in the utility function. The intertemporal maximization problem is subject to the following budget constraint:

$$C_t + K_{t+1} - (1-\delta)K_t \leq K_t^{1-\alpha} \left(Z_t N_t\right)^{\alpha}$$

where K_t is the capital stock, N_t are the hours worked, Z_t is the labor augmenting technology process, α is the labor share in the Cobb-Douglas production function and δ the depreciation rate of the capital stock. As mentioned by Kydland and Prescott (1996), the law of motion of the exogeneous process Z_t in the model is not provided by any economic theory. Additional assumptions which are neither given by economic theory nor by any statistical procedure are then required. Following King, Plosser and Rebelo (1988b), we consider here that the law of motion for Z_t is characterized by the following random walk with drift:

$$\ln Z_t = \mu + \ln Z_{t-1} + \varepsilon_t$$

where μ is the growth rate of the economy and is ε_t i.i.d. Normal $(0, \sigma_\varepsilon)$. Obviously, this law of motion of the technology process is a caricature of the true unknown process. Consequently, this misspecification could presumably contaminate the estimation of the structural parameters of interest. However, with such a driven process, the log-linear solution of the model is compatible with a unit root process for output, consumption, investment and real wages (see King, Plosser and Rebelo (1988b) and King, Plosser, Stock and Watson (1991)) and cointegration relationships between these variables which are consistent with U.S. data.

We consider here that there are four deep structural parameters in this model and three auxiliary parameters. In our notation, $\theta_1 = (\gamma, \delta, \alpha, \mu)'$ gathers the interest parameters and $\theta_2 = (\phi, \mu, \sigma_\varepsilon)'$ the nuisance parameters needed for statistical implementation, that is to index the binding function. We will explain later why ϕ is considered as nuisance parameter.

While Mehra and Prescott ask the question: Is there exist a set of parameters of interest with reasonable values able to reproduce some characteristics of the data?, the RBC modeler asks the question: "Given a set of parameters of interest calibrated by micro-evidence or long run averages, what is the ability of the model to reproduce some well documented "stylized facts"?

As explained above, Mehra and Prescott (and Cechetti, Lam and Mark (1993)) considers estimation issue in a negative way: they search for values of structural parameters (θ_1 in our notation) reproducing as well as possible the observed instrumental parameters β^g (or subset of the instrumental parameters (β_1^g) for Cechetti, Lam and Mark). The goodness of fit of the model is judged by the range for these values. Kydland

and Prescott (1982) evaluate the performance of the model by its ability to reproduce well defined "stylized facts" which are computed by simulations at given values of the structural parameters (θ_1). The assigned value of the parameter vector θ_1 comes from other applied studies or by matching long run average values for the economy. In contrast to Mehra and Prescott strategy, the instrumental parameters used to assess the model differs from the ones used to obtain an estimator of the structural parameters. More precisely, the strategy advanced by Kydland and Prescott (1982) consists in two steps:

- First, structural parameters are calibrated to values used in applied studies and to match long run average values.

- Second, the verification is implemented by judging the adequacy of the model to reproduce well chosen "stylized facts". When they could not find reliable estimations of a subset of parameters in economic literature or by matching long run properties, these parameters are treated as free parameters. Their values are then chosen to minimize the distance between the well chosen "stylized facts" of the U.S. economy and the corresponding ones of the model.

The first step corresponding to calibration is the most controversial one. Indeed, several authors have shown that parameters obtained from micro-applied studies can be plugged to a representative agent model to produce empirically concordant aggregate model only under very special circumstances (see Hansen and Heckman (1996) for a discussion on this point). However, matching long run properties is more conformable to the estimation step in classical econometrics. In fact, this practice consists in matching a just-identified set of moments where the corresponding instrumental parameters are the long-run averages. For instance, Kydland and Prescott (1982) calibrate the deterministic version of their model so that consumption/investment shares, factor/income shares, capital/output ratio, leisure/market-time shares and depreciation shares match the average values of U.S. economy. However, they fit the values of those parameters without a formal estimation procedure.[10] Consequently, uncertainty inherent to those values is not taking into account in the results.

The matching of long run properties of the economy corresponds in our setting to obtain an estimator of $\theta = (\theta_1', \theta_2')'$ by

$$\beta^g = \beta^g(\theta_1, \theta_2).$$

where β^g captures these long run average properties.

The verification step (second step) performed by the calibrator is based on a quite informal distance criterion for selected "stylized facts". This evaluation process can be formalized in our setting by a choice of instrumental model parameters corresponding to the "stylized facts" to reproduce. In fact, we try to judge

[10] see Christiano and Eichenbaum (1982) and Burnside and Eichenbaum (1996) for the estimation of structural parameters by a just-identified GMM.

if we can reject with a certain metric the following null hypothesis:

$$\beta^k = \beta^k(\theta_1^0, \bar{\theta}_2)$$

evaluated at the pseudo-true value obtained $(\theta_1^0, \bar{\theta}_2)$ with the instrumental parameters β^g. The index k affected to β^k distinguishes the instrumental parameters corresponding to the "stylized facts" to the instrument parameters used to estimate the parameters of interest θ.

In presence of what Kydland and Prescott (1982) called free parameters, their strategy can be formalized by obtaining an estimator of those free parameters by minimizing the distance between the "stylized facts" from the economy and the corresponding ones for the model. Suppose that the vector of nuisance parameters $\bar{\theta}_2$ is divided as $\bar{\theta}_2 = \left(\bar{\theta}_{21}', \bar{\theta}_{22}'\right)$ and the subvector $\bar{\theta}_{22}$ contains these free parameters for which direct estimator can not be performed. The value $\bar{\theta}_{22}$ of the nuisance parameters is obtained as the solution to the following minimization program:

$$\bar{\theta}_{22} = \arg\min_{\theta_{22}\in\Theta_{22}} \left(\beta^{k,0} - \tilde{\beta}^k(\theta_1^0, \bar{\theta}_{21}, \theta_{22})\right)' \Omega^k \left(\beta^{k,0} - \tilde{\beta}^k(\theta_1^0, \bar{\theta}_{21}, \theta_{22})\right) \tag{2.6}$$

where Ω^k is a positive matrix on R^{q_k} and $q_k = \dim \beta_k$. For the benchmark RBC model, the free parameter ϕ corresponding to the weight of leisure in the utility function may be difficult to estimate at the first step. In such a situation, an estimator can then be obtained by (2.6). In a more complicated model, Kydland and Prescott fix seven parameters by minimizing the distance between the model and data for twenty-three moments describing U.S. business cycle. Those parameters are the substitutability of inventories and capital, two parameters determining intertemporal substitutability of leisure, the risk aversion parameter and three parameters for the technology process.

3 A Sequential Partial Indirect Inference approach to calibration

We present in this section the Indirect Inference principles as extended in DR (1998) as well as the available results of the Partial Indirect Inference useful for validating the calibration methodology. This formulation aims to encompass the one proposed in Gouriéroux, Monfort and Renault (1993) as well as the calibration methodology.

The main goal if this section is to give a precise content to the calibrationnist-type interpretation of Indirect Inference, as put forward in section 2, that is "given that the model is false", some elements of truth involved in the model (for instance some taste parameters) should be caught by matching some well-chosen moments. The rigorous meaning of "elements of truth" lies in the semi-parametric modelling widely adapted in modern econometrics as an alternative to the "quest for the Holy Grail" (see Monfort (1996)), that is the hopeless search for a well-specified parametric model that is more often than not impossible to deduce from economic theory. On the opposite, the partial approach to Indirect Inference specifies only some parameters of interest raised out by the underlying economic theory.

We first present the theoretical results (consistency, asymptotic probability distribution) available for Partial Indirect Inference. For sake of expositional simplicity, detailed proofs and technical assumptions are not provided. The interested reader can refer either to the companion paper DR (98) or to any standard treatment of asymptotic theory of minimum distance estimators (see e.g. Newey and McFadden (1994)).

3.1 The general framework

As in DR (1998), the data consist in the observation of s stochastic process $\{(y_t, x_t), t \in Z\}$ are dates $t = 1, \ldots, T$. We denote by P_0 the true unknown p.d.f. of $\{(y_t, x_t), t \in Z\}$.

Assumption (A1):

i) P_0 belongs to a family $\mathcal{P}$ of p.d.f. on $(\mathcal{X} \times \mathcal{Y})^Z$.

ii) $\widetilde{\theta}_1$ is an application from $\mathcal{P}$ onto a part $\Theta_1 = \widetilde{\theta}_1(\mathcal{P})$ of R^{p_1}.

iii) $\widetilde{\theta}_1(\mathcal{P}_0) = \theta_1^0$, the true unknown value of the parameters of interest, belongs to the interior $\overset{\circ}{\Theta}_1$ of Θ_1.

$\widetilde{\theta}_1(\mathcal{P}) = \theta_1$ is the vector unknown parameters of interest. Typically, in the case of a stationary process $\{(y_t, x_t), t \in Z\}$, it may be defined through a set h of identifying moment restrictions:

$$E_P h\left(y_t, x_t, y_{t-1}, x_{t-1}, \ldots, y_{t-K}, x_{t-K}, \theta_1\right) = 0 \quad \Longrightarrow \quad \theta_1 = \widetilde{\theta}_1(P).$$

In such a semi-parametric model, not only the Maximum Likelihood estimator is no longer available, but even more robust M-estimators or Minimum distance estimators may be intractable due to a complicated dynamic structure of P. This is the reason why we refer to indirect inference associated with a given pair of "structural" model (used as simulator) and "auxiliary" (or "instrumental") criterion.

In order to get a simulator useful for partial indirect inference on θ_1, we plug the semi-parametric model defined by $(A1)$ into a structural model that is fully parametric and misspecified in general since it introduces additional assumptions on the law of motion of (y, x) which are not suggested by any economic theory. These additional assumptions require a vector θ_2 of additional parameters. The vector θ of "structural parameters" is thus given by $\theta = (\theta_1', \theta_2')'$. We then formulate a nominal assumption (B1) to specify a structural model conformable to the previous section, even though we know that (B1) is likely to be inconsistent with the true DGP. [11]

Nominal assumptions (B1): $\{(y_t, x_t), t \in Z\}$ is a stationary process conformable to the following nonlinear simultaneous equations model:

[11] We denote by B the nominal assumptions, that is assumptions that are used for a quasi-indirect inference (by extension of the Quasi Maximum Likelihood).

•

$$\begin{cases} r\left(y_t, y_{t-1}, x_t, u_t, \theta\right) = 0, \\ \varphi\left(u_t, u_{t-1}, \varepsilon_t, \theta\right) = 0 \end{cases}$$

$\theta = (\theta_1', \theta_2')' \in (\Theta_1 \times \Theta_2) = \Theta \subset$ a compact subset of $R^{p_1+p_2}$

- the exogenous process $\{x_t, t \in Z\}$ is independent of $\{\varepsilon_t, t \in Z\}$,
- $\{\varepsilon_t, t \in Z\}$ is a white noise with a known distribution G_*.

We denote by P_* the probability distribution of the process $\{y_t, x_t, \varepsilon_t, t \in Z\}$.

We focus here on indirect inference about the true value θ_1° of the parameters of interest θ_1. The Indirect Inference principle is still defined from the two basic components: a "structural" model (B1) and the instrumental criterion $\mathcal{N}_g$:

$$Q_T\left(\underline{y}_T, \underline{x}_T, \beta^g\right) = \frac{1}{2}\left(\frac{1}{T}\sum_{t=1}^{T} g(w_t) - \beta^g\right)'\left(\frac{1}{T}\sum_{t=1}^{T} g(w_t) - \beta^g\right),$$

where $w_t = (y_t, y_{t-1}, x_{t-1}, \ldots, y_{t-K}, x_{t-K})$ for a fixed number of K lags. Note that all the interpretations that are done in the sequel are still valid when one is interested in partial indirect inference through general extremum instrumental model as defined in DR (1998).

We introduce the estimators $\hat{\beta}_T^g$ and $\tilde{\beta}_{TS}^g(\theta_1, \theta_2)$ associated with the instrumental model:

$$\hat{\beta}_T^g = \frac{1}{T}\sum_{t=1}^{T} g(w_t)$$

$$\tilde{\beta}_{TS}^g(\theta_1, \theta_2) = \frac{1}{TS}\sum_{s=1}^{S}\sum_{t=1}^{T} g(\tilde{w}_t^s(\theta))$$

where $(\tilde{w}_t^s(\theta)) = \left\{\tilde{y}_t^s(\theta), \tilde{y}_{t-1}^s(\theta), x_{t-1}, \ldots, \tilde{y}_{t-K}^s(\theta), x_{t-K}\right\}, t = 1, \ldots, T$ denote S simulated paths $s = 1, 2 \cdots S$ associated to a given value $\theta = (\theta_1', \theta_2')'$ of the structural parameters. Under usual regularity conditions, these estimators converge uniformly in (θ_1, θ_2) to:

$$P_0 \lim_{T\to\infty} \hat{\beta}_T^g = \beta^{g,0} = E_0 g(w_t)$$

$$P_* \lim_{T\to\infty} \tilde{\beta}_{TS}^g = \tilde{\beta}^g(\theta_1, \theta_2) = E_* g(\tilde{w}_t(\theta)).$$

We refer to $P_0 \lim_{T\to+\infty}$ and $P_* \lim_{T\to+\infty}$ as the limit with respect to the P_0 and the P_* probabilities when T goes to infinity.

We assume (A2) that $\tilde{\beta}^g(\cdot, \cdot)$ is one-to-one. According to Gouriéroux and Monfort (1995) terminology, $\beta^{g,0}$ is the true value of instrumental parameters and $\tilde{\beta}^g(\cdot, \cdot)$ is the binding function from the structural model to the instrumental one. The instrumental parameters β^g correspond precisely to the moments of interest $E_0 g(w_t)$ for calibration exercises.

A partial indirect inference estimators $\hat{\theta}_{1,TS}$ is then defined as follows:

$$\hat{\theta}_{TS} = \left(\hat{\theta}'_{1,TS}, \hat{\theta}'_{2,TS}\right)' = \arg\min_{(\theta_1,\theta_2)\in\Theta_1\times\Theta_2} \left[\hat{\beta}^g_T - \tilde{\beta}^g_{TS}(\theta_1,\theta_2)\right]' \hat{\Omega}^g_T \left[\hat{\beta}^g_T - \tilde{\beta}^g_{TS}(\theta_1,\theta_2)\right],$$

where $\underset{T\to+\infty}{P_* \lim} \hat{\Omega}^g_T = \Omega^g$ is positive definite matrix on R^q.

In order to derive a necessary and sufficient condition for the consistency of the PII estimator $\hat{\theta}_{1,TS}$ to θ^0_1, we define the so-called "generalized inverse" $\tilde{\beta}^{g-}$ of $\tilde{\beta}^g$ by:

$$\tilde{\beta}^{g-}(\beta^g) = \arg\min_{(\theta_1,\theta_2)\in\Theta_1\times\Theta_2} \|\beta^g - \tilde{\beta}^g(\theta_1,\theta_2)\|_{\Omega_g}.$$

In our semi-parametric setting, we are only interested in the projection of $\tilde{\beta}^{g-}[\beta^g(P)]$ on the set Θ_1 of the parameters of interest. Let us denote by Q_1 the projection operator:

$$\begin{aligned} Q_1 \quad &: \quad R^{p_1}\times R^{p_2} \to R^{p_1} \\ &\quad\ (\theta'_1,\theta'_2)' \to \theta_1. \end{aligned}$$

From DR (1998), we have the following consistency criterion:

Proposition 3.1 Under assumptions (A1)-(A2), $\hat{\theta}_{1,TS}$ is a consistent estimator of the parameters of interest θ°_1 if and only if, for any P in the family $\mathcal{P}$ of p.d.f. delineated by the model (A1):

$$Q_1\left[\tilde{\beta}^{g-}\circ\beta^g(P)\right] = \widetilde{\theta}_1(P).$$

In order to test the consistency property, we focus on a sufficient encompassing condition. We say that (B1) endowed with the pseudo-true value $(\theta^{0\prime}_1, \bar{\theta}'_2)'$ fully encompasses $(\mathcal{N}_g)$ if:

$$\beta^{g,0} = \tilde{\beta}^g(\theta^0_1, \bar{\theta}_2).$$

In this framework, we are able to prove the following sufficient condition for the consistency of the PII estimator $\hat{\theta}_{1,TS}$:

Proposition 3.2 Under assumptions (A1)-(A2), if there exists $\bar{\theta}_2 \in \Theta_2$ such that (B1) endowed with the pseudo-true value $(\theta^{0\prime}_1, \bar{\theta}'_2)'$ fully encompasses $(\mathcal{N}_g)$, then $\hat{\theta}_{1,TS}$ is a consistent estimator of the parameters of interest θ^0_1.

When the structural misspecified model (B1) endowed with the pseudo-true value $(\theta^{\circ\prime}_1, \bar{\theta}'_2)'$ for $\bar{\theta}_2 \in \Theta_2$ does not fully encompasses the instrumental model $\mathcal{N}_g$, we know from DR (1998) that we can extend the encompassing concept to a property of partial encompassing defined through a subvector $\beta^{g,0}_1$ of q_1 instrumental parameters ($q_1 \leq q$). The corresponding subvector function $\tilde{\beta}^g_1(\cdot,\cdot)$ of the binding function is defined from $\Theta_1 \times \Theta_{21}$ onto R^{q_1}:

$$\tilde{\beta}^g_1 \quad : \quad \Theta_1 \times \Theta_{21} \to R^{q_1} \tag{3.7}$$

$$(\theta_1, \theta_{21}) \to \tilde{\beta}^g_1(\theta_1, \theta_{21}), \tag{3.8}$$

where θ_{21} corresponds to the subvector of the nuisance parameters $\theta_2 = (\theta_{21}', \theta_{22}')'$ which does play a role in the first q_1 components of the binding function β^g. θ_{21} belongs to Θ_{21}, subset of $R^{p_{21}}$ with the assumed factorization of the nuisance parameters set of $\Theta_2 = \Theta_{21} \times \Theta_{22}$. We say that (B1) endowed with the pseudo-true value $(\theta_1^{0\prime}, \bar{\theta}_2')'$ partially encompasses $\mathcal{N}_g$ if the following conditions are fulfilled:

i) $\tilde{\beta}_1^g(\cdot,\cdot)$ is one-to-one,

ii) $\beta_1^{g,0} = \tilde{\beta}_1^g(\theta_1^0, \bar{\theta}_{21})$.

We introduce the following estimators $\hat{\beta}_{1,T}^g$, and $\tilde{\beta}_{1,T}^{g,s}(\theta_1, \theta_2)$ respectively defined as the subvectors of size q_1 of the estimators $\hat{\beta}_T^g$ and $\tilde{\beta}_{TS}^g(\theta_1, \theta_2)$. These estimators converge uniformly in θ_1, θ_2 to:

$$\begin{aligned} P_0 \lim_{T\to\infty} \hat{\beta}_{1,T}^g &= \beta_1^{g,0} = E_0 g_1(\omega_t), \\ P_* \lim_{T\to\infty} \tilde{\beta}_{1,TS}^g(\theta_1, \theta_2) &= \tilde{\beta}_1^g(\theta_1, \theta_{21}) = E_* g_1(\tilde{\omega}_t(\theta_1, \theta_2, z_0)), \end{aligned}$$

where $g_1(\cdot)$ is naturally defined as the components of $g = (g_1', g_2')'$ corresponding respectively to $\beta = (\beta_1', \beta_2')'$.

In this context, since the PII estimator $\hat{\theta}_{1,TS}$ is possibly not consistent for θ_1^0, we propose to focus on another class of partial indirect estimator $\hat{\theta}_{1,TS}(\bar{\theta}_{22})$ based on a subvector β_1 of the instrumental parameters and defined by:

$$\begin{aligned} \hat{\theta}_{TS}^1(\bar{\theta}_{22}) &= \left(\hat{\theta}_{1,TS}^{1\prime}(\bar{\theta}_{22}), \hat{\theta}_{21,TS}^{1\prime}(\bar{\theta}_{22})\right)' = \\ & \arg\min_{(\theta_1,\theta_{21})\in\Theta_1\times\Theta_{21}} \left[\hat{\beta}_{1,T}^g - \tilde{\beta}_{1,TS}^g(\theta_1, \theta_{21}, \bar{\theta}_{22})\right]' \hat{\Omega}_{1,T}^g \left[\hat{\beta}_{1,T}^g - \tilde{\beta}_{1,TS}^g(\theta_1, \theta_{21}, \bar{\theta}_{22})\right], \end{aligned}$$

where $P_* \lim_{T\leftarrow+\infty} \hat{\Omega}_{1,T}^g = \Omega_1^g$ is a positive definite matrix. We denote by $\bar{\theta}_{22}$ the value assigned to the nuisance parameters θ_{22} in order to perform the simulations. In this framework, we are able to prove the following sufficient condition for the consistency of the Partial II estimator $\hat{\theta}_{1,TS}^1(\bar{\theta}_{22})$:

Proposition 3.3 Under assumptions (A1)-(A2), and if there exists $\bar{\theta}_2 \in \Theta_2$ such that (B1) endowed with the pseudo-true value $(\theta_1^{0\prime}, \bar{\theta}_2')'$ partially encompasses $\mathcal{N}_g$, then $\hat{\theta}_{1,TS}^1(\bar{\theta}_{22})$ is a consistent estimator of the parameters of interest θ_1^0.

3.2 Asymptotic probability distribution of partial indirect inference estimators

In this section we recall the main asymptotic results derived in DR(98) in two cases. The first one maintain the full-encompassing assumption while the second relies only on partial encompassing.

3.2.1 Full-encompassing partial indirect inference estimator

We focus here on the asymptotic properties if the indirect inference estimator $\hat{\theta}_{TS}$ under the full-encompassing hypothesis: there exists $\bar{\theta}_2 \in \Theta_2$ such that (B1) endowed with the pseudo-true value $(\theta_1^{\circ\prime}, \bar{\theta}_2')'$ fully encompasses $\mathcal{N}_g$, and we maintain the following assumptions:

$$(A3) \qquad \frac{1}{\sqrt{T}}\sum_{t=1}^{T}(g(w_t)-\beta^{g,0}),$$

is asymptotically normally distributed with mean zero and with an asymptotic covariance matrix $I^g_\circ$.

$$(A4) \qquad \lim_{T\to+\infty}\mathrm{Cov}_*\left\{\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(g(w_t)),\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(g(\tilde{w}^s_T(\theta^0_1,\bar{\theta}_2,z^s_0))\right\}=K^g_0,$$

independent of the initial values $z^s_0, s=1,\dots,S$.

$$(A5) \qquad \frac{1}{\sqrt{T}}\sum_{t=1}^{T}(g(\tilde{w}^s_T(\theta^0_1,\bar{\theta}_2,z^s_0))-\beta^{g,0}),$$

is asymptotically normally distributed with mean zero and with an asymptotic covariance matrix $I^{g,*}_0$ and independent of the initial values $z^s_0, s=1,\dots,S$.

$$(A6) \qquad \lim_{T\to+\infty}\mathrm{Cov}_*\left\{\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(g(\tilde{w}^s_T(\theta^0_1,\bar{\theta}_2,z^s_0)),\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(g(\tilde{w}^l_T(\theta^0_1,\bar{\theta}_2,z^l_0))\right\}=K^{g,*}_0,$$

independent of the initial values z^s_0 and z^l_0, for $s\neq\ell$.

$$(A7) \qquad P_*\lim_{T\to+\infty}\frac{\partial\tilde{\beta}^{g,s}_T}{\partial\theta'}(\theta^0_1,\bar{\theta}_2)=\frac{\partial\tilde{\beta}^g}{\partial\theta'}(\theta^0_1,\bar{\theta}_2),$$

is full-column rank (p) . We are then able to prove the following result:

Proposition 3.4 Under the null hypothesis of full encompassing and assumptions (A1)-(A7), the optimal indirect inference estimator $\hat{\theta}^*_{TS}$ is obtained with the weighting matrix Ω^{g*} defined below. It is asymptotically normal, when S is fixed and T goes at infinity:

$$\sqrt{T}\begin{pmatrix}\hat{\theta}^*_{1,TS}-\theta^0_1\\ \hat{\theta}^*_{2,TS}-\bar{\theta}_2\end{pmatrix}\xrightarrow{D}\mathcal{N}(0,W^g(S,\Omega^{g,*})),$$

with:

$$\begin{aligned}W^{g,*}_S &= W^g\left(S,\Omega^{g,*}\right)=\left\{\frac{\partial(\tilde{\beta}^g)'}{\partial\theta}(\theta^0_1,\bar{\theta}_2)\left(\Phi^{g,*}_0(S)\right)^{-1}\frac{\partial\tilde{\beta}^g}{\partial\theta'}(\theta^0_1,\bar{\theta}_2)\right\}^{-1}, \qquad (3.9)\\ \Omega^{g,*} &= \Phi^{g,*}_0(S)^{-1},\\ \Phi^{g,*}_0(S) &= I^g_0+\frac{1}{S}I^{g,*}_0+\left(1-\frac{1}{S}\right)K^{g,*}_0-K^g_0-K^{g\prime}_0.\end{aligned}$$

3.2.2 Partial-encompassing partial indirect inference estimator

We now focus on the asymptotic properties of the indirect inference estimator $\hat{\theta}^1_{TS}(\bar{\theta}_{22})$ under the partial encompassing hypothesis $H^1_0(\bar{\theta}_{22})$. We first maintain assumption (A3) and we denote $\tilde{\beta}^{g,0}(\bar{\theta}_{22})=\tilde{\beta}^g(\theta^0_1,\bar{\theta}_2)$ for the given value $\bar{\theta}_{22}$ of the nuisance parameters. We made the following assumptions for a given value $\bar{\theta}_{22}$:

$$(A8) \qquad \lim_{T\to+\infty}\mathrm{Cov}_*\left\{\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(g(w_t)),\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(g(\tilde{w}^s_t(\theta^0_1,\bar{\theta}_2,z^s_0))\right\}=K^g_0(\bar{\theta}_{22}),$$

independent of the initial values $z_0^s, s = 1, \ldots, S$.

$$(A9) \qquad \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left(g(\tilde{w}_t^s(\theta_1^0, \bar{\theta}_2, z_0^s)) - \beta^{g,0}(\bar{\theta}_{22}) \right),$$

is asymptotically normally distributed with mean zero and with an asymptotic covariance matrix $I_0^{g,*}(\bar{\theta}_{22})$ and independent of the initial values $z_0^s, s = 1, \ldots, S$.

$$(A10) \qquad \lim_{T \to +\infty} \mathrm{Cov}_* \left\{ \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (g(\tilde{w}_t^s(\theta_1^0, \bar{\theta}_2, z_0^s))), \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (g(\tilde{w}_t^l(\theta_1^0, \bar{\theta}_2, z_0^l))) \right\} = K_0^{g,*}(\bar{\theta}_{22}),$$

independent of the initial values z_0^s and z_0^l, for $s \neq \ell$.

$$(A11) \qquad P_* \lim_{T \to +\infty} \frac{\partial \tilde{\beta}_{1,T}^{g,s}}{\partial \binom{\theta_1}{\theta_{21}}'}(\theta_1^0, \bar{\theta}_2) = \frac{\partial \tilde{\beta}_1^g}{\partial \binom{\theta_1}{\theta_{21}}'}(\theta_1^0, \bar{\theta}_{21}),$$

is full-column rank $(p_1 + p_{21})$. We are then able to prove the following result:

Proposition 3.5 Under the null hypothesis $H_0^1(\bar{\theta}_{22})$, assumptions (A1)-(A3), (A8)-(A11), the optimal indirect inference estimator $\hat{\theta}_{TS}^{1*}(\bar{\theta}_{22})$ is obtained with the weighting matrix $\Omega_1^{*g}(\bar{\theta}_{22})$ defined below. It is asymptotically normal, when S is fixed and T goes to infinity:

$$\sqrt{T} \begin{pmatrix} \hat{\theta}_{1,TS}^{1,*}(\bar{\theta}_{22}) - \theta_1^0 \\ \hat{\theta}_{21,TS}^{1,*}(\bar{\theta}_{22}) - \bar{\theta}_{21} \end{pmatrix} \xrightarrow{D} \mathcal{N}(0, W_1^g(S, \Omega_1^{g,*}, \bar{\theta}_{22})),$$

with:

$$W_{1,S}^{g,*}(\bar{\theta}_{22}) = W_1^g(S, \Omega_1^{g,*}(\bar{\theta}_{22})) = \left[\frac{\partial \tilde{\beta}_1^{g'}}{\partial \binom{\theta_1}{\theta_{21}}}(\theta_1, \bar{\theta}_{21}) \left(\Phi_{0,1}^{g,*}(S, \bar{\theta}_{22}) \right)^{-1} \frac{\partial \tilde{\beta}_1^g}{\partial \binom{\theta_1}{\theta_{21}}'}(\theta_1^0, \bar{\theta}_{21}) \right]^{-1},$$

$$\Omega_1^{g,*}(\bar{\theta}_{22}) = \Phi_{0,1}^{g,*}(S, \bar{\theta}_{22})^{-1},$$

$$\Phi_0^{g,*}(S, \bar{\theta}_{22}) = I_0^g + \frac{1}{S} I_0^{g,*}(\bar{\theta}_{22}) + \left(1 - \frac{1}{S}\right) K_0^{g,*}(\bar{\theta}_{22}) - K_0^g(\bar{\theta}_{22}) - K_0^{g\prime}(\bar{\theta}_{22}),$$

and $\Phi_{0,1}^*(S, \bar{\theta}_{22})$ is the $(q_1 \times q_1)$ left-upper bloc diagonal submatrix of the $(q \times q)$ matrix $\Phi_0^{g,*}(S, \bar{\theta}_{22})$.

DR (1998) have shown that we can replace the value $\bar{\theta}_{22}$ of the nuisance parameters θ_{22} by a consistent estimator $\hat{\theta}_{22,TS}$ such that $\sqrt{T} \left(\hat{\theta}_{22,TS} - \bar{\theta}_{22} \right) = O_{P*}(1)$ without modifying the asymptotic probability distribution of the PII estimator.

3.3 Identifying the Moments to Match

We follow in this subsection the testing procedure as proposed in DR (1998) and Guay and Renault (2003). This procedure starts with a set of moments to match which are suggested by economic theory or any other features of the data the econometrician wishes to reproduce. Then it seeks to identify which projection of these instrumental characteristics should be selected in order to build a consistent partial indirect estimator as well as reliable predictions under hypothetical policy interventions.

Proposition 3.6 Under assumptions (A1)-(A7) and the null hypothesis $H_\circ$ of full-encompassing of $\mathcal{N}_g$ by (B1),

$$\xi_{T,S} = T \min_{\theta\in\Theta} \left[\frac{1}{T}\sum_{t=1}^{T} g(w_t) - \frac{1}{TS}\sum_{t=1}^{T}\sum_{s=1}^{S} g(\tilde{w}_t^s(\theta))\right]]'\hat{\Omega}_T^{g,*} \left[\frac{1}{T}\sum_{t=1}^{T} g(w_t) - \frac{1}{TS}\sum_{t=1}^{T}\sum_{s=1}^{S} g(\tilde{w}_t^s(\theta))\right]],$$

where $\hat{\Omega}_T^{g,*}$ is a consistent estimator of the optimal metric $\Omega^{g,*} = \Phi_0^{g,*}(S)^{-1}$ defined in Proposition 3.4, is asymptotically distributed as a chi-square with $(q-p)$ degrees of freedom where q=dimg and p=dim θ.

The proof is omitted here since it is a simple extension of standard indirect inference theory. The associated specification test of asymptotic level α is defined by the critical region:

$$\mathcal{W}_a = \left\{\xi_{T,S} > \chi^2_{1-\alpha}(q-p)\right\}.$$

In case of rejection, we may look for a reduction through an appropriate projection of the set of moments. This is based on the following partial encompassing test.

Proposition 3.7 Under assumptions (A1)-(A3), (A8)-(A11) and the null hypothesis $H_\circ(\bar{\theta}_{22})$ of partial encompassing of $\mathcal{N}_g$ by (B1)

$$\begin{aligned}\xi^1_{T,S}(\bar{\theta}_{22}) \quad &= \quad T \min_{\theta_1,\theta_{21}\in\Theta_1\times\Theta_{21}} \left[\frac{1}{T}\sum_{t=1}^{T} g_1(w_t) - \frac{1}{TS}\sum_{t=1}^{T}\sum_{s=1}^{S} g_1(\tilde{w}_t^s(\theta_1,\theta_{21},\bar{\theta}_{22}))\right]]'\hat{\Omega}_{1,T}^{g,*} \\ &\left[\frac{1}{T}\sum_{t=1}^{T} g_1(w_t) - \frac{1}{TS}\sum_{t=1}^{T}\sum_{s=1}^{S} g_1(\tilde{w}_t^s(\theta_1,\theta_{21},\bar{\theta}_{22}))\right]],\end{aligned}$$

where $\Omega_{1,T}^{g,*}$ is a consistent estimator of the optimal metric $\Omega_1^{g,*}(\bar{\theta}_{22}) = \Phi_{0,1}^{g,*}(S,\bar{\theta}_{22})^{-1}$ defined in Proposition 3.5, is asymptotically distributed as a chi-square with $(q_1-p_1-p_{21})$ degrees if freedom where $q_1 = \dim g_1, p_1 = \dim\theta, p_{21} = \dim\theta_{21}$.

The associated specification test of asymptotic level α is defined by the following critical region:

$$\mathcal{W}^1_a = \left\{\xi^1_{T,S}(\bar{\theta}_{22}) > \chi^2_{1-\alpha}(q_1-p_1-p_{21})\right\}.$$

The previous result is not modified if θ_{22} is replaced by a consistent estimator $\hat{\theta}_{22,TS}$ such that $\sqrt{T}\left(\hat{\theta}_{22,TS}-\bar{\theta}_{22}\right) = O_{P*}(1)$. In case of rejection of any trial run of partial encompassing, the pair (structure model, instrumental model) is inadequate and has to be changed. However, it may also be the case that several pairs lead to acceptation.

3.4 Sequential Partial Indirect Inference

The previous sections show how a well-driven Partial Indirect Inference estimation strategy yields a consistent estimator for the structural parameters of interest θ_1 given θ_2. With this estimator in hand, one can now evaluate the model through dimensions of interest.

We consider an instrumental model N_k with $\beta^{k,0}$ and $\hat{\beta}^k(\theta_1, \theta_2)$ the moments of interest associated with N_k, that is:

$$\begin{aligned} \beta^{k,0} \quad &= E_0 k(w_t) \\ \tilde{\beta}^k(\theta_1, \theta_2) \quad &= E_* k(\tilde{w}_t(\theta)) \end{aligned}$$

which, under usual regularity conditions, can be consistently estimated by the following estimators:

$$\begin{aligned} \hat{\beta}^k_T \quad &= \tfrac{1}{T} \textstyle\sum_{t=1}^T k(w_t), \\ \tilde{\beta}^k_{TS}(\theta_1, \theta_2) \quad &= \tfrac{1}{TS} \textstyle\sum_{s=1}^S \sum_{t=1}^T k(\tilde{w}^s_t(\theta)) \end{aligned}$$

where $(\tilde{w}^s_t(\theta)) = \left\{\tilde{y}^s_t(\theta), \tilde{y}^s_{t-1}(\theta), x_{t-1}, \dots, \tilde{y}^s_{t-K}(\theta), x_{t-K}\right\}, s = 1, \dots, S, t = 1, \dots, T$, correspond to simulated paths of the endogenous variables for a given value $\theta = (\theta_1', \theta_2')'$ of the structural parameters.

An evaluation of the structural model can be performed by measuring a distance between the empirical instrumental parameters $\hat{\beta}^k$ and the theoretical one $\tilde{\beta}_{k,TS}(\theta_1, \theta_2)$.

As discussed above with the RBC illustration, in the case of partial encompassing, an estimator of the nuisance parameter vector θ_{22} can be obtained through the instrumental model of interest $\mathcal{N}_k$. We then define the estimator $\hat{\theta}_{22,TS}$ as follows:

$$\hat{\theta}_{22,TS} = \arg \min_{\theta_{22} \in \Theta_{22}} \left(\hat{\beta}^k_T - \tilde{\beta}^k_{TS}(\hat{\theta}^1_{TS}(\bar{\theta}_{22}), \theta_{22})\right)' \hat{\Omega}^k \left(\hat{\beta}^k_T - \tilde{\beta}^k_{TS}(\hat{\theta}^1_{TS}(\bar{\theta}_{22}), \theta_{22})\right),$$

for a given initial value $\bar{\theta}_{22}$.

Following Newey (1984), we can show the following proposition with assumptions (A.12)-(A.15) stated in the appendix:

Proposition 3.8 Under the null hypothesis of the instrumental model $\mathcal{N}_k$ and assumptions (A1)-(A3),(A8)-(A15), the optimal indirect inference estimator $\hat{\theta}^*_{22,TS}$ is obtained with the weighting matrix defined below. It is asymptotically normal, when S is fixed and T goes at infinity:

$$\sqrt{T}\left(\hat{\theta}^*_{22,TS} - \theta^*_{22}\right) \xrightarrow{D} \mathcal{N}(0, W^k(S, \Omega^{k,*})), \theta$$

where:

$$\begin{aligned} \theta^*_{22} \quad &= \quad \underset{T\to\infty}{P_0} \lim \hat{\theta}^*_{22,TS} \\ W^{k,*}_S \quad &= \quad W^k\left(S, \Omega^{k,*}\right) = \left\{ \frac{\partial(\tilde{\beta}^k)'}{\partial\theta_{22}}(\theta^0_1, \bar{\theta}_{21}, \theta^*_{22}) \left(\Phi^{k,*}_0(S)\right)^{-1} \frac{\partial\tilde{\beta}^k}{\partial\theta'_{22}}(\theta^0_1, \bar{\theta}_{21}, \theta^*_{22}) \right\}^{-1}, \end{aligned}$$

$$\Omega^{k,*} = \Phi^{k,*}_0(S)^{-1}$$

$$\Phi^{k,*}_0(S) = [A, I]\, \Phi^*_0\, [A, I]'$$

$$A = \left[-\frac{\partial \tilde{\beta}^k}{\partial \binom{\theta_1}{\theta_{21}}'} (\theta_1^0, \bar{\theta}_{21}, \theta_{22}^*) \left(W_{1,S}^{g,*}(\bar{\theta}_{22}) \right)^{-1} \frac{\partial \tilde{\beta}_1^{g\prime}}{\partial \binom{\theta_1}{\theta_{21}}} (\theta_1^0, \bar{\theta}_{21}, \theta_{22}^*) \, \Omega_1^{g,*}(\bar{\theta}_{22}) \right]$$

and Φ_0^* is defined in the Appendix.

It should be emphasized that the asymptotic distribution given by Proposition 3.8 holds only for the same simulated values ε_t^s, $t = 1, \dots, T$, $s = 1, \dots, S$ for both instrumental models $\mathcal{N}_g$ and $\mathcal{N}_k$.

Our proposed approach is then a two step procedure. For this reason, we call this procedure as Sequential Partial Inference Indirect. Consider the partial encompassing case. At the first step, the estimators of $\theta_1^0(\bar{\theta}_{22})$ and $\bar{\theta}_{21}(\bar{\theta}_{22})$ are given by minimizing the following objective function:

$$J_{1,TS}(\theta_1(\bar{\theta}_{22}), \theta_{21}(\bar{\theta}_{22})) = \left[\hat{\beta}_{1,T}^g - \tilde{\beta}_{1,TS}^g(\theta_1, \theta_{21}, \bar{\theta}_{22}) \right]' \hat{\Omega}_{1,T}^g \left[\hat{\beta}_{1,T}^g - \tilde{\beta}_{1,TS}^g(\theta_1, \theta_{21}, \bar{\theta}_{22}) \right],$$

for a given $\bar{\theta}_{22}$.

At the second step, the estimator of the nuisance parameters θ_{22} is given by minimizing the following objective function:

$$J_{2,TS}(\theta_{22}) = \left(\hat{\beta}_T^k - \tilde{\beta}_{TS}^k(\hat{\theta}_{TS}^1(\bar{\theta}_{22}), \theta_{22}) \right)' \hat{\Omega}^k \left(\hat{\beta}_T^k - \tilde{\beta}_{TS}^k(\hat{\theta}_{TS}^1(\bar{\theta}_{22}), \theta_{22}) \right),$$

An evaluation of the structural model can then be performed by measuring a distance between the empirical instrumental parameters $\hat{\beta}^k$ and the theoretical one $\tilde{\beta}_{k,TS}(\theta_1, \theta_2)$. In the case of partial-encompassing, the test corresponds to an overidentifying restrictions test. The test statistic is given by:

$$T J_{2,TS}(\hat{\theta}_{22,TS}).$$

This statistic is asymptotically distributed as a chi-square with $(dim(\beta_k) - dim(\theta_{22}))$ degrees of freedom. In the case of full-encompassing, this corresponds to a Wald test and the statistic test is given by:

$$T \left(\hat{\beta}_T^k - \tilde{\beta}_{TS}^k(\hat{\theta}_{TS}^1(\bar{\theta}_{22}), \bar{\theta}_{22}) \right)' \hat{\Omega}_T^k \left(\hat{\beta}_T^k - \tilde{\beta}_{TS}^k(\hat{\theta}_{TS}^1(\theta_{22}), \bar{\theta}_{22}) \right),$$

where $\hat{\Omega}_T^k$ is a estimator of $\Omega^{k,*}$ and

$$\Omega^{k_*} = \Phi_0^{k,*}(S)^{-1},$$

$$\Phi_0^{k,*}(S) = [A, I] \, \Phi_0^*(S) \, [A, I]'$$

$$A = \left[-\frac{\partial \tilde{\beta}^k}{\partial \theta'} \left(\theta_1^0, \bar{\theta}_2 \right)' (W^g(S, \Omega^{g,*}))^{-1} \frac{\partial \tilde{\beta}^{g\prime}}{\partial \theta} \left(\theta_1^0, \bar{\theta}_2 \right) \Omega^{g,*} \right].$$

This statistic is asymptotically distributed as a chi-square with $dim(\beta_k)$ degrees of freedom.

4 Concluding Remarks

The SPII methodology proposed in this paper aims at reconciling the calibration and verification steps proposed by the calibrationnist approach with their econometric counterparts, that is, estimation and testing procedures. We propose a general framework of multistep estimation and testing:

- First, for a given (calibrated) value $\bar{\theta}_{22}$ of some nuisance parameters, a consistent asymptotically normal estimator $\hat{\theta}^1_{1,TS}\left(\bar{\theta}_{22}\right)$ of the vector θ_1 of parameters of interest is obtained by partial indirect inference. A pseudo-true value $\bar{\theta}_{21}$ of some other nuisance parameters may also be consistently estimated by the same token.

- Second, the overidentification of the vector (θ_1, θ_{21}) of structural parameters by the selected instrumental moments β^g_1 provides a specification test of the pair (structural model, instrumental model).

- Finally, the verification step, including a statistical assessment of the calibrated value $\bar{\theta}_{22}$, can be performed through another instrumental model $\mathcal{N}_k$.

The proposed formalization enables us to answer most of the common statistical blames on the calibration methodology, insofar as one succeeds to split the model in some true identifying moment conditions and some nominal assumptions. The main message is twofold: First, acknowledging that any structural model is misspecified while aiming at producing consistent estimators of the true unknown value of some parameters of interest as well as robust predictions, one should rely, as informally advocated in calibration exercises, on parsimonious and well chosen dimensions of interest. Second, in so doing, it may be the case that simultaneous joint estimation of the true unknown value of the parameters of interest as well as of the pseudo-true value of the nuisance parameters is impossible. In this context, one should resort to a two step procedure that we call Sequential Partial Indirect Inference (SPII). This basically introduces a general loss function. This again corresponds to a statistical formalization of the common practice in calibration exercises using previous estimates and a priori selection.

A Appendix

We define the vector of empirical moments $f(\omega_t) = (g(\omega)', k(\omega_t)')'$ and the vector of moments from the model $f\left(\tilde{\omega}_T^s(\theta_1^0, \bar{\theta}_2)\right) = \Big(g\big(\big(\tilde{\omega}_T^s(\theta_1^0(\bar{\theta}_{22}, \bar{\theta}_{12}(\bar{\theta}_{22})\big)', k\big(\big(\tilde{\omega}_T^s(\theta_1^0, \bar{\theta}_{12}, \bar{\theta}_{22})\big)'\Big)$

We make the following assumptions:

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} (f(w_t) - \beta^0), (A13) \tag{A12}$$

is asymptotically normally distributed with mean zero and with an asymptotic covariance matrix I_0.

$$\lim_{T\to+\infty} \mathrm{Cov}_* \left\{ \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (f(w_t)), \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (f(\tilde{w}_T^s(\theta_1^0, \bar{\theta}_2, z_0^s)) \right\} = K_0, (A14) \tag{A13}$$

independent of the initial values $z_0^s, s = 1, \ldots, S$.

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} (f(\tilde{w}_T^s(\theta_1^0, \bar{\theta}_2, z_0^s)) - \beta^0), (A15) \tag{A14}$$

is asymptotically normally distributed with mean zero and with an asymptotic covariance matrix I_0^* and independent of the initial values $z_0^s, s = 1, \ldots, S$.

$$\lim_{T\to+\infty} \mathrm{Cov}_* \left\{ \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (f(\tilde{w}_T^s(\theta_1^0, \bar{\theta}_2, z_0^s)), \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (f(\tilde{w}_T^l(\theta_1^0, \bar{\theta}_2, z_0^l)) \right\} = K_0^*, (A16) \tag{A15}$$

independent of the initial values z_0^s and z_0^l, for $s \neq \ell$.

We can show following DR (1998) that

$$\Phi_0^*(S) = I_0 + \frac{1}{S} I_0^* + \left(1 - \frac{1}{S}\right) K_0^* - K_0 - K_0'.$$

References

Bansal, R., R.A. Gallant, R. Hussey and G. Tauchen (1995), "Nonparametric Estimation of Structural Models for High-Frequency Currency Market Data" Journal of Econometrics, 66, 251-287.

Beaudry, P. and A. Guay (1996), "What Do Interest Rates Reveal about the Functioning of Real Business Cycle Models ?", Journal of Economic Dynamics and Control,, 20, 1661-1682.

Bonomo, M. and R. Garcia (1994), "Can a Well-Fitted Asset-Pricing Model Produce Mean Reversion?", Journal of Applied Econometrics, 9, 19-29.

Burnside, C. and M. Eichenbaum (1996), "Factor-Hoarding and the Propagation of Business-Cycle Shocks", American Economic Review, 86, 1154-1174.

Canova, F. (1994), "Statistical Inference in Calibrated Models", Journal of Applied Econometrics, 9, 123-144.

Cechetti, S. G., P.Lam and N. C. Mark (1990), "Evaluating Empirical Tests of Asset Pricing Models: Alternative Interpretations", American Economic Review, 80, 48-51.

Cechetti, S. G., P.Lam and N. C. Mark (1993), "The Equity Premium and the Risk Free Rate: Matching the Moments", Journal of Monetary Economics, 31, 21-45.

Christ, C. F. (1996), "Econometric Models and Methods", New York: Wiley.

Christiano, L. (1988), "Why Does Inventory Investment Fluctuate So Much", Journal of Monetary Economics, 21, 247-280.

Christiano, L. and M. Eichenbaum (1992), "Current Real Business Cycle Theories and Aggregate Labor Market Fluctuations", American Economic Review, 82, 430-450.

Constantinides, G. M. (1990), "Habit Formation: A Resolution of the Equity Premium Puzzle", Journal of Political Economy, 98, 519-543.

Dejong, D. N., B. Fisher Ingram and C. H. Whiteman (1996), "A Bayesian Approach to Calibration", Journal of Business and Economic Statistics, 14, 1-9.

Dridi, R. and E. Renault (1998), "Semiparametric Indirect Inference", Mimeo Université de Toulouse.

Epstein, L. G. and S. E. Zin (1989), "Substitution, Risk Aversion, and the Temporal Behavior of Consumption Growh and Asset Returns: a Theoretical Framework", Econometrica, 57, 937-969.

Frisch, R. (1933), "Propagation Problems and Impulse Problems in Dynamic Economics", in Economic Essays in Honor of Gustav Cassel, London: Allen and Unwin, 171-205.

Frisch, R. (1970), "Econometrics in the World of Today", in W. A. Eltis, M. F. G. Scott, and J. N. Wolfe (eds.), Induction, Growth and Trade: Essays in Honour of Sir Roy Harrod, Oxford: Clarendon Press, 152-166.

Gallant, R. A., R. Hsieh and G. Tauchen (1997), "Estimating SV Models with Diagnostic", Journal of Econometrics, 81, 159-192.

Gallant and G. Tauchen (1996), "Which Moments to Match", Econometric Theory, 12, 657-681.

Geweke, J. (1999), "Computational experiments and reality", Working Paper, University of Iowa.

Gouriéroux, C. and A. Monfort (1995), "Testing, Encompassing and Simulating Dynamic Econometric Models", Econometric Theory, 11, 195-228.

Gouriéroux, C., A. Monfort and E. Renault (1993), "Indirect Inference", Journal of Applied Econometrics, 8, 85-118.

Gregory, A. W. and G. W. Smith (1990), "Calibration as Estimation", Econometric Reviews, 9, 57-89.

Guay, A. and E. Renault (2003), "Indirect Encompassing with Misspecified Models", Mimeo.

Hansen, G. (1985), "Indivisible Labor and the Business Cycle",Journal of Monetary Economics, 16, 309-328.

Hansen, L. P. (1982), "Large Sample Properties of Generalized Method of Moments Estimators",Econometrica, 50, 1029-1054.

Hansen, L. P. and J. J. Heckman (1996), "The Empirical Foundations of Calibration", Journal of Economic Perspectives, 10, 87-104.

Hansen, L. P. and T. J. Sargent (1979), "Formulating and Estimating Dynamic Linear Rational Expectations Models", Journal of Economic Dynamics and Control, 2, 7-46.

Hendry, D. F. and J-F. Richard (1982), "On the Formulation of Empirical Models in Dynamic Econometrics", Journal of Econometrics, 20, 3-33.

Hoover, K. D. (1995a), "The Problem of Macroeconometrics", in Macroeconometrics, Developments, Tensions and Prospects, Kluwer Academic Publishers Boston/Dordrecht/London, Recent Economic Thought, 1-12.

Hoover, K. D. (1995b), "Facts and Artifacts: Calibration and Empirical Assessment of Real-Business-Cycle Models", Oxford Economic Papers, 47, 24-44.

Hurwicz, L. (1962), "On the Structural Form of Interdependent Systems", in Ernest Nagel et al. (eds.), Logic and Methodology in the Social Sciences., Stanford, CA: Stanford University Press, 232-239.

Ingram, B. F. (1995), "Recent Advances in Solving and Estimating Dynamic, Stochastic Macroeconomic Models", in Kevin D. Hoover (ed.), Macroeconometrics Developments, Tensions and Prospects, Kluwer Academic Publishers Boston/ Dordrecht/ London: Recent Economic Thought Series, 15-46.

Ingram, B. F. and B. S. Lee (1991), "Estimation by Simulation of Time Series Models", Journal of Econometrics, 47, 197-207.

King, R.G., C.I. Plosser and S. Rebelo (1988a), "Production, Growth and Business Cycles: I. The

Basic Neoclassical Model", Journal of Monetary Economics, 21, 195-232.

King, R.G., C.I. Plosser and S. Rebelo (1988b), "Production, Growth and Business Cycles: II. The Basic Neoclassical Model", Journal of Monetary Economics, 21, 309-342.

King, R.G., C.I. Plosser, J.H. Stock and M. Watson (1991), "stochastic Trends and Economic Fluctuations", American Economic Review, 81, 819-840.

Koopmans, T. (1947), "Measurement without Theory", Review of Economic Statistics, 29, 161-172.

(1990), Journal of Business Economics and Statistics, 8.

Kydland, F. E. and E. C. Prescott (1982), "Time to Build and Aggregate Fluctuations", Econometrica, 50, 1345-1370.

Kydland, F. E. and E. C. Prescott (1991), "The Econometrics of the General Equilibrium Approach to Business Cycles", Scandinavian Journal of Economics, 93:2, 161-178.

Kydland, F. E. and E. C. Prescott (1996), "The Computational Experiment: An Econometric Tool", Journal of Economic Perspectives, 10, 69-86.

Lucas, R. E. (1972), "Expectations and the Neutrality of Money",Journal of Economic Theory, 4, 103-124.

Lucas, R. E. (1976), "Econometric Policy Valuation: a Critique", in The Phillips Curve and Labor Markets, K. Brunner (ed.), supplement to the Journal of Monetary Economics, 1, 19-46.

Lucas, R. E. (1978), "Asset Prices in an Exchange Economy", Econometrica, 46, 1429-1445.

Lucas, R. E. (1980), "Methods and Problems in Business Cycle Theory", in Studies in Business-Cycle Theory, Blackwell, Oxford 1981.

Mehra, R. and E. C. Prescott (1985), "The Equity Premium: a Puzzle", Journal of Monetary Economics, 15, 145-161.

Monfort, A. (1996), "A Reappraisal of Misspecified Econometric Models", Econometric Theory, 12, 597-619.

Morgan, M. (1990), "The History of Econometric Ideas", Cambridge: Cambridge University Press.

Newey, W.K. (1984), "A Method of Moments Interpretation of Sequential Estimators", Economic Letters, 14, 201-206.

Newey, W.K. and D. McFadden (1994), "Large sample estimation and hypothesis testing", Handbook of Econometrics, Vol. IV, R.F. Engle and D. McFadden eds, 2111-2245.

Pagan, A. R. (1994), "Calibration and Econometric Research: an Overview", Journal of Applied Econometrics, 9, 1-10.

Pagan, A. R. (1995), "On Calibration", Mimeo, The Australian National University.

Prescott, E. C. (1983), "Can the Cycle be Reconciled with a Consistent Theory of Expectations" or A Progress Report on Business Cycle Theory, Working Paper, n° 239, Research Department, Federal Reserve

Bank of Mineapolis.

Romer, D. (1996), "Advanced Macroeconomics", New York: McGraw-Hill Companies, McGraw-Hill Advanced Series in Economics.

Ruge-Murcia, F. (2003), "Methods to Estimate Dynamic Stochastic General Equilibrium Models", Working Paper, CIREQ, Université de Montréal.

Schorfheide, F. (2000), "Loss function-based evaluation of DSGE models", Journal of Applied Econometrics, 15, 645-670.

Sims, C. A. (1980), "Macroeconomics and Reality", Econometrica, 48, 1-48.

Sims, C. A. (1996), "Macroeconomics and Methodology", Journal of Economic Perspectives, 10, 105-120.

Smith, A. (1993), "Estimating Nonlinear Time Series Models Using Simulated Vector Autoregressions", Journal of Applied Econometrics, 8, 63-84.

Tauchen, G., H. Zang and M. Liu (1997), "Volume, Volatility and Leverage: a Dynamic Analysis", Journal of Econometrics, 74, 177-208.

Watson, M. (1993), "Measure of Fit for Calibrated Models", Journal of Political Economy, 101, 1011-1041.