TRIANGULATED SUBCATEGORIES OF EXTENSIONS AND TRIANGLES OF RECOLLEMENTS

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Abstract. Let T be a triangulated category with triangulated subcategories X and Y. We show that the subcategory of extensions $\mathsf{X} * \mathsf{Y}$ is triangulated if and only if $\mathsf{Y} * \mathsf{X} \subseteq \mathsf{X} * \mathsf{Y}$.

In this situation, we show that there is a stable t-structure $\left(\frac{\mathsf{X}}{\mathsf{X}\cap\mathsf{Y}}, \frac{\mathsf{Y}}{\mathsf{X}\cap\mathsf{Y}}\right)$ in $\frac{\mathsf{X}*\mathsf{Y}}{\mathsf{X}\cap\mathsf{Y}}$. We use this to give a recipe for constructing triangles of recollements and recover some triangles of recollements from the literature.

0. Introduction

Let T be a triangulated category. If X and Y are full subcategories of T, then the *subcategory of extensions* $\mathsf{X} * \mathsf{Y}$ is the full subcategory of objects e for which there is a distinguished triangle $x \to e \to y$ with $x \in \mathsf{X}$, $y \in \mathsf{Y}$. Subcategories of extensions have recently been of interest to a number of authors, see [2], [5], [7], [14].

We show a criterion for such a subcategory to be triangulated.

Theorem A. *Let X, Y be triangulated subcategories of T. Then $\mathsf{X} * \mathsf{Y}$ is a triangulated subcategory of T $\Leftrightarrow$ $\mathsf{Y} * \mathsf{X} \subseteq \mathsf{X} * \mathsf{Y}$.*

This is a triangulated analogue of [14, thm. 3.2]. We use it to show a result on stable t-structures in subquotient categories. Recall that a pair of triangulated subcategories (U, V) of T is called a stable t-structure if $\mathsf{U} * \mathsf{V} = \mathsf{T}$ and $\mathrm{Hom}_{\mathsf{T}}(\mathsf{U}, \mathsf{V}) = 0$, see [8, def. 9.14].

Theorem B. *Let X, Y be thick subcategories of T and assume that $\mathsf{X} * \mathsf{Y}$ is a triangulated subcategory of T. Then $\left(\frac{\mathsf{X}}{\mathsf{X}\cap\mathsf{Y}}, \frac{\mathsf{Y}}{\mathsf{X}\cap\mathsf{Y}}\right)$ is a stable t-structure in $\frac{\mathsf{X}*\mathsf{Y}}{\mathsf{X}\cap\mathsf{Y}}$.*

The stable t-structure induces an equivalence of triangulated categories $\frac{(\mathsf{X}*\mathsf{Y})/(\mathsf{X}\cap\mathsf{Y})}{\mathsf{X}/(\mathsf{X}\cap\mathsf{Y})} \simeq \frac{\mathsf{Y}}{\mathsf{X}\cap\mathsf{Y}}$ by [8, prop. 9.16 and rmk. 9.17] which implies

$$\frac{\mathsf{X} * \mathsf{Y}}{\mathsf{X}} \simeq \frac{\mathsf{Y}}{\mathsf{X} \cap \mathsf{Y}}$$

by [13, prop. 2.3.1(c)]. This recovers [12, lem. 1] which can be thought of as the Second Isomorphism Theorem for triangulated categories. That theorem is hence augmented by Theorem B.

In Theorem B, we made no comment about the important set theoretical question of whether the quotient categories have small Hom sets. However, permitting large Hom

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sets is not in itself a problem, cf. Neeman's treatment in [10, sec. 2.2], and if we do so, then the theorem is correct as stated. See also Remark 1.1.

Finally, under stronger assumptions, we show that a pair of stable t-structures induces a so-called triangle of recollements in a quotient category. A triangle of recollements is a triple of stable t-structures (U,V), (V,W), (W,U). Triangles of recollements were introduced in [4, def. 0.3] and have a very high degree of symmetry; for instance, $\mathsf{U} \simeq \mathsf{V} \simeq \mathsf{W} \simeq \mathsf{T}/\mathsf{U} \simeq \mathsf{T}/\mathsf{V} \simeq \mathsf{T}/\mathsf{W}$. They have applications to the construction of triangle equivalences, see [4, prop. 1.16].

Theorem C. *Let* (X,Y) *and* (Y,Z) *be stable t-structures in* T *with* $\mathsf{Z} * \mathsf{X} = \mathsf{T}$*, and let* $Q : \mathsf{T} \to \mathsf{T}/(\mathsf{X} \cap \mathsf{Z})$ *be the quotient functor.*

Then $\mathsf{T}/(\mathsf{X} \cap \mathsf{Z})$ *has small* Hom *sets and has the stable t-structures*

$$\big(Q(\mathsf{X}),Q(\mathsf{Y})\big)\ ,\ \big(Q(\mathsf{Y}),Q(\mathsf{Z})\big)\ ,\ \big(Q(\mathsf{Z}),Q(\mathsf{X})\big).$$

The paper is organized as follows: Section 1 proves Theorems A, B, and C. Section 2 shows two corollaries and some related results. Section 3 uses Theorem C to recover two triangles of recollements in homotopy categories known from [4, thm. 5.8] and [7, thm. 2.10].

Throughout, T is a triangulated category with small Hom sets. All subcategories are full and closed under isomorphisms.

1. Proofs of Theorems A, B, and C

Proof of Theorem A.

$\Rightarrow$ is clear.

$\Leftarrow$: It is clear that $\mathsf{X} * \mathsf{Y}$ is closed under (de)suspension, so it is enough to show that it is closed under extensions.

Indeed, $*$ is associative by [2, sec. 2.2] so the inclusion $\mathsf{Y}*\mathsf{X} \subseteq \mathsf{X}*\mathsf{Y}$ implies $(\mathsf{X}*\mathsf{Y})*(\mathsf{X}*\mathsf{Y}) = (\mathsf{X} * (\mathsf{Y} * \mathsf{X})) * \mathsf{Y} \subseteq (\mathsf{X} * (\mathsf{X} * \mathsf{Y})) * \mathsf{Y}) = (\mathsf{X} * \mathsf{X}) * (\mathsf{Y} * \mathsf{Y}) = \mathsf{X} * \mathsf{Y}$. □

Remark 1.1. If U is a triangulated subcategory of T, then we can consider the (Verdier) quotient category T/U. There is a quotient functor $Q : \mathsf{T} \to \mathsf{T}/\mathsf{U}$. We sometimes write $Q(\mathsf{T})$ instead of T/U. In particular, the homomorphism functor in the quotient category is denoted by $\mathrm{Hom}_{Q(\mathsf{T})}(-,-)$.

Note that T/U may not have small Hom sets, but this is not in itself a problem, see [10, sec. 2.2]. In particular, we can permit large Hom sets in the situation of the following lemma and of Theorem B.

On the other hand, in the situation of Theorem C, we will show that the quotient $\mathsf{T}/(\mathsf{X}\cap\mathsf{Z})$ does have small Hom sets, as formulated in the theorem. This is specific to Theorem C and fails for general quotients of the form $\mathsf{T}/(\mathsf{X} \cap \mathsf{Z})$, for instance if $\mathsf{X} = \mathsf{T}$ and the quotient T/Z fails to have small Hom sets.

Lemma 1.2. *Let* X*,* Y *be thick subcategories of* T *and assume that* $\mathsf{X} * \mathsf{Y}$ *is a triangulated subcategory of* T*. Let* $Q : \mathsf{T} \to \mathsf{T}/(\mathsf{X} \cap \mathsf{Y})$ *be the quotient functor.*

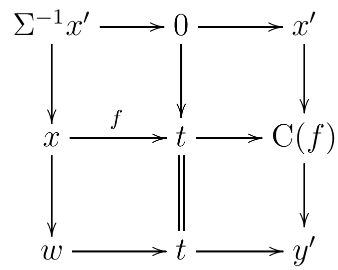
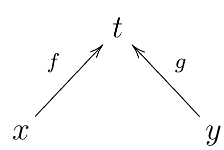


FIGURE 1. The octahedral axiom applied to the composition $t \to \mathrm{C}(f) \to y'$

(i) $\mathrm{Hom}_{Q(\mathsf{T})}(Q(\mathsf{X}), Q(\mathsf{Y})) = 0$.
(ii) $Q(\mathsf{X}) * Q(\mathsf{Y}) = Q(\mathsf{X} * \mathsf{Y})$.
(iii) $Q(\mathsf{X}) * Q(\mathsf{Y}) = Q(\mathsf{T}) \Leftrightarrow \mathsf{X} * \mathsf{Y} = \mathsf{T}$.

Proof. (i) Given $x \in \mathsf{X}$ and $y \in \mathsf{Y}$, let $Qx \to Qy$ be a morphism in $Q(\mathsf{T})$. It is represented by a "roof"

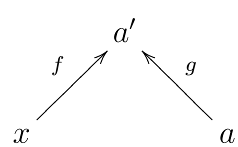


with the mapping cone $\mathrm{C}(g)$ in $\mathsf{X} \cap \mathsf{Y}$. Since $y \in \mathsf{Y}$, we have $t \in \mathsf{Y}$.

Note that $Qx \to Qy$ is the same as $Qx \overset{Qf}{\to} Qt$ up to isomorphism, so it is enough to show $Qf = 0$. Since $\mathsf{X} * \mathsf{Y}$ is triangulated and contains x and t, the cone $\mathrm{C}(f)$ is also in $\mathsf{X} * \mathsf{Y}$, so there is a distinguished triangle $x' \to \mathrm{C}(f) \to y'$ in T with $x' \in \mathsf{X}$, $y' \in \mathsf{Y}$. The octahedral axiom gives the diagram in Figure 1. The first column shows $w \in \mathsf{X}$ and the last row shows $w \in \mathsf{Y}$, so we have factored f through $w \in \mathsf{X} \cap \mathsf{Y}$. This shows $Qf = 0$.

(ii) "$\supseteq$" is clear. "$\subseteq$": The objects of $Q(\mathsf{T}) = \mathsf{T}/(\mathsf{X} \cap \mathsf{Y})$ have the form Qt for $t \in \mathsf{T}$. The condition $Qt \in Q(\mathsf{X}) * Q(\mathsf{Y})$ means that there is a distinguished triangle $Qx \to Qt \to Qy$ in $Q(\mathsf{T})$ with $x \in \mathsf{X}$, $y \in \mathsf{Y}$.

Such a triangle is isomorphic to a triangle $Qa \to Qb \to Qc$ obtained by applying Q to a distinguished triangle $a \to b \to c$ from T; see [8, def. 8.4]. There is a "roof" representing the isomorphism $Qx \cong Qa$.



Since the roof represents an isomorphism, we have $\mathrm{C}(f), \mathrm{C}(g) \in \mathsf{X} \cap \mathsf{Y}$. Since $x, \mathrm{C}(f) \in \mathsf{X}$, we have $a' \in \mathsf{X}$. Since $a', \mathrm{C}(g) \in \mathsf{X}$, we have $a \in \mathsf{X}$. A similar argument shows $c \in \mathsf{Y}$. But then $b \in \mathsf{X} * \mathsf{Y}$ whence $Qt \cong Qb \in Q(\mathsf{X} * \mathsf{Y})$.

(iii) "$\Leftarrow$" is clear. "$\Rightarrow$": By part (ii), we need to see that $Q(\mathsf{X} * \mathsf{Y}) = Q(\mathsf{T})$ implies $\mathsf{X} * \mathsf{Y} = \mathsf{T}$. But it is easy to show, using arguments similar to the ones above, that if U is any triangulated subcategory with $\mathsf{X} \cap \mathsf{Y} \subseteq \mathsf{U} \subseteq \mathsf{T}$ and $Q(\mathsf{U}) = Q(\mathsf{T})$, then $\mathsf{U} = \mathsf{T}$. □

Proof of Theorem B. This is immediate from Lemma 1.2(i+ii). □

Remark 1.3. Yoshizawa gives the following example in [14, cor. 3.3]: If R is a commutative noetherian ring and S is a Serre subcategory of $\mathsf{Mod}\,R$, then $(\mathsf{mod}\,R) * \mathsf{S}$ is a Serre subcategory of $\mathsf{Mod}\,R$. Here $\mathsf{Mod}\,R$ is the category of R-modules and $\mathsf{mod}\,R$ is the full subcategory of finitely generated R-modules.

One might suspect a triangulated analogue to say that if T is compactly generated and U is a triangulated subcategory of T, then so is $\mathsf{T}^{\mathrm{c}} * \mathsf{U}$ where T^{c} denotes the triangulated subcategory of compact objects. See [9, defs. 1.6 and 1.7]. However, this is false:

Set $\mathsf{T} = \mathsf{D}(\mathbb{Z})$ and $\mathsf{U} = \mathsf{D}(\mathbb{Q})$. Then T is compactly generated by $\{ \Sigma^i \mathbb{Z} \,|\, i \in \mathbb{Z} \}$. There is a homological epimorphism of rings $\mathbb{Z} \to \mathbb{Q}$ which induces an embedding of triangulated categories $\mathsf{U} \hookrightarrow \mathsf{T}$, see [3, def. 4.5] and [11, thm. 2.4]. Since $\mathbb{Q}$ is a field, each object of U has homology modules of the form $\coprod \mathbb{Q}$. This means that viewed in T, the only object of U which has finitely generated homology modules is 0. Hence 0 is the only object of U which is compact in T, see [9, cor. 2.3]. That is, $\mathsf{T}^{\mathrm{c}} \cap \mathsf{U} = 0$.

If $\mathsf{T}^{\mathrm{c}} * \mathsf{U}$ were a triangulated subcategory of T, then Theorem B would give that $(\mathsf{T}^{\mathrm{c}}, \mathsf{U})$ was a stable t-structure in $\mathsf{T}^{\mathrm{c}} * \mathsf{U}$, but this is false since the canonical map $\mathbb{Z} \to \mathbb{Q}$ is a non-zero morphism from an object of T^{c} to an object of U.

Proof of Theorem C. We have that $\big((\mathsf{X} \cap \mathsf{Z}) \cap \mathsf{X}, (\mathsf{X} \cap \mathsf{Z}) \cap \mathsf{Y}\big) = (\mathsf{X} \cap \mathsf{Z}, 0)$ is (trivially) a stable t-structure in $\mathsf{X} \cap \mathsf{Z}$. Hence $\big(Q(\mathsf{X}), Q(\mathsf{Y})\big)$ is a stable t-structure in $\mathsf{T}/(\mathsf{X} \cap \mathsf{Z})$ by [4, prop. 1.5]. The second stable t-structure in the theorem is obtained analogously and the third one comes from Theorem B.

To show that $\mathsf{T}/(\mathsf{X} \cap \mathsf{Z})$ has small Hom sets, let $t, t' \in \mathsf{T}$ be given. There are distinguished triangles $y \to t \to z$ and $x' \to t' \to y'$ in T with $x' \in \mathsf{X}$, $y, y' \in \mathsf{Y}$, $z \in \mathsf{Z}$. The first one gives an exact sequence

$$\mathrm{Hom}_{Q(\mathsf{T})}\big(Qz, Qt'\big) \to \mathrm{Hom}_{Q(\mathsf{T})}\big(Qt, Qt'\big) \to \mathrm{Hom}_{Q(\mathsf{T})}\big(Qy, Qt'\big)$$

so it is enough to check that the two outer Homs are small sets. The second distinguished triangle gives similar exact sequences which show that it is enough to check that the following Homs are small sets.

$$\begin{aligned} &\mathrm{Hom}_{Q(\mathsf{T})}\big(Qz, Qx'\big) \;,\; \mathrm{Hom}_{Q(\mathsf{T})}\big(Qz, Qy'\big), \\ &\mathrm{Hom}_{Q(\mathsf{T})}\big(Qy, Qx'\big) \;,\; \mathrm{Hom}_{Q(\mathsf{T})}\big(Qy, Qy'\big). \end{aligned}$$

The first of these is 0 since $\big(Q(\mathsf{Z}), Q(\mathsf{X})\big)$ is a stable t-structure. Inspection of the other three shows that it is enough to check that Homs out of and into objects of $Q(\mathsf{Y})$ are small sets.

Let us check the first of these statements, as the second has a similar proof. So let $\widetilde{y} \in \mathsf{Y}$ and $t'' \in \mathsf{T}$ be given. There is a distinguished triangle $y'' \to t'' \to z''$ with $y'' \in \mathsf{Y}$, $z'' \in \mathsf{Z}$. It induces a distinguished triangle $Qy'' \to Qt'' \to Qz''$ in $\mathsf{T}/(\mathsf{X} \cap \mathsf{Z})$. We have shown that $\big(Q(\mathsf{Y}), Q(\mathsf{Z})\big)$ is a stable t-structure, so $\mathrm{Hom}_{Q(\mathsf{T})}(Q\widetilde{y}, \Sigma^* Qz'') = 0$. So the distinguished triangle induces a bijection

$$\mathrm{Hom}_{Q(\mathsf{T})}(Q\widetilde{y}, Qy'') \to \mathrm{Hom}_{Q(\mathsf{T})}(Q\widetilde{y}, Qt'').$$

It follows from [4, prop. 1.5] that $\mathrm{Hom}_{Q(\mathsf{T})}(Q\widetilde{y}, Qy'') \cong \mathrm{Hom}_{\mathsf{T}}(\widetilde{y}, y'')$. But $\mathrm{Hom}_{\mathsf{T}}(\widetilde{y}, y'')$ is a small set by assumption, so it follows that $\mathrm{Hom}_{Q(\mathsf{T})}(Q\widetilde{y}, Qt'')$ is a small set. □

2. Corollaries and related results

Let us show two corollaries of Theorem A. The first one is well known but we have been unable to locate a reference. We believe that the second one is new.

Corollary 2.1. *Let* X*,* Y *be triangulated subcategories of* T*. If* $\operatorname{Hom}_{\mathsf{T}}(\mathsf{X},\mathsf{Y}) = 0$ *then* $\mathsf{X} * \mathsf{Y}$ *is a triangulated subcategory of* T*.*

Proof. For $e \in \mathsf{Y} * \mathsf{X}$, there is a distinguished triangle $y \to e \to x \to \Sigma y$ with $x \in \mathsf{X}$, $y \in \mathsf{Y}$. When $\operatorname{Hom}_{\mathsf{T}}(\mathsf{X},\mathsf{Y}) = 0$, the last morphism is 0 so $e \cong y \oplus x$. In particular, $e \in \mathsf{X} * \mathsf{Y}$ and Theorem A gives that $\mathsf{X} * \mathsf{Y}$ is triangulated. □

Corollary 2.2. *Let* (U,V) *be a stable t-structure in* T *and let* X *be a thick subcategory of* T*. The following conditions are equivalent.*

(i) $\mathsf{U} * \mathsf{X}$ *is a triangulated subcategory of* T*.*
(ii) $\mathsf{X} * \mathsf{V}$ *is a triangulated subcategory of* T*.*
(iii) $(\mathsf{X} \cap \mathsf{U}, \mathsf{X} \cap \mathsf{V})$ *is a stable t-structure in* X*.*

When the conditions hold, we have $\mathsf{U} * \mathsf{X} = \mathsf{U} * (\mathsf{X} \cap \mathsf{V})$ *and* $\mathsf{X} * \mathsf{V} = (\mathsf{X} \cap \mathsf{U}) * \mathsf{V}$*.*

Proof. (i) ⇒ (iii): For $x \in \mathsf{X}$, there is a distinguished triangle

$$u \to x \to v \to \Sigma u \tag{1}$$

with $u \in \mathsf{U}$, $v \in \mathsf{V}$. This shows $v \in \mathsf{X} * \mathsf{U}$ and when (i) holds then $v \in \mathsf{U} * \mathsf{X}$ by Theorem A. So there is a distinguished triangle $u' \to v \to x' \to \Sigma u'$ with $u' \in \mathsf{U}$, $x' \in \mathsf{X}$. But $\operatorname{Hom}_{\mathsf{T}}(u', v) = 0$ so it follows that v is a direct summand of x'. Since X is thick we have $v \in \mathsf{X}$, and the distinguished triangle (1) then implies $u \in \mathsf{X}$. Hence (1) establishes (iii).

(iii) ⇒ (i): When (iii) holds then $\mathsf{U} * \mathsf{X} = \mathsf{U} * ((\mathsf{X} \cap \mathsf{U}) * (\mathsf{X} \cap \mathsf{V})) = (\mathsf{U} * (\mathsf{X} \cap \mathsf{U})) * (\mathsf{X} \cap \mathsf{V}) = \mathsf{U} * (\mathsf{X} \cap \mathsf{V})$. The last of these categories is triangulated by Corollary 2.1 so (i) holds, and the computation also shows one of the equations in the last line of the corollary.

The other equation in the last line and (ii) ⇔ (iii) are proved analogously. □

The following proposition provides pairs of stable t-structures to which Theorem C applies.

Proposition 2.3. *Let* (X,Y) *and* (Y,Z) *be stable t-structures in* T *with* $\mathsf{Z} * \mathsf{X}$ *triangulated. Then*

$$\big(\mathsf{X}\,,\ \mathsf{Y} \cap (\mathsf{Z} * \mathsf{X})\big)\ ,\ \ \big(\mathsf{Y} \cap (\mathsf{Z} * \mathsf{X})\ ,\ \mathsf{Z}\big)$$

are stable t-structures in $\mathsf{Z} * \mathsf{X}$ *(so Theorem C applies if we replace* T *with* $\mathsf{Z} * \mathsf{X}$*).*

Proof. Let $m \in \mathsf{Z} * \mathsf{X}$ be given. There is a distinguished triangle $z \to m \to x$ with $z \in \mathsf{Z}$, $x \in \mathsf{X}$. Let $x' \to m \to y'$ be a distinguished triangle with $x' \in \mathsf{X}$, $y' \in \mathsf{Y}$. The octahedral axiom gives Figure 2. Since $x', x \in \mathsf{X}$ we have $x'' \in \mathsf{X}$ so the last column shows $y' \in \mathsf{Z} * \mathsf{X}$.

The middle row hence shows that $\big(\mathsf{X}, \mathsf{Y} \cap (\mathsf{Z} * \mathsf{X})\big)$ is a stable t-structure in $\mathsf{Z} * \mathsf{X}$ and a similar argument handles $\big(\mathsf{Y} \cap (\mathsf{Z} * \mathsf{X}), \mathsf{Z}\big)$. □

$$
\begin{array}{ccccc}
0 & \longrightarrow & z & = & z \\
\downarrow & & \downarrow & & \downarrow \\
x' & \longrightarrow & m & \longrightarrow & y' \\
\| & & \downarrow & & \downarrow \\
x' & \longrightarrow & x & \longrightarrow & x''
\end{array}
$$

FIGURE 2. The octahedral axiom applied to the composition $x' \to m \to x$

The following proposition is another way of producing pairs of stable t-structures. If X is a subcategory of T, then we write

$$\mathsf{X}^{\perp} = \{\, t \in \mathsf{T} \mid \operatorname{Hom}_{\mathsf{T}}(\mathsf{X}, t) = 0 \,\} \ \ , \ \ {}^{\perp}\mathsf{X} = \{\, t \in \mathsf{T} \mid \operatorname{Hom}_{\mathsf{T}}(t, \mathsf{X}) = 0 \,\}.$$

Proposition 2.4. *Let X be a triangulated subcategory of T. Set*

$$\mathsf{F} = \mathsf{X} * \mathsf{X}^{\perp} \ , \ \mathsf{G} = {}^{\perp}\mathsf{X} * \mathsf{X} \ , \ \mathsf{U} = \mathsf{F} \cap \mathsf{G}.$$

Then U is a triangulated subcategory of T and

$$({}^{\perp}\mathsf{X} \cap \mathsf{F} \ , \ \mathsf{X}) \ \ , \ \ (\mathsf{X} \ , \ \mathsf{X}^{\perp} \cap \mathsf{G})$$

are stable t-structures in U.

Proof. It is immediate from Corollary 2.1 that F and G are triangulated subcategories of T, so the same follows for U.

To see that $({}^{\perp}\mathsf{X} \cap \mathsf{F}, \mathsf{X})$ is a stable t-structure in U, observe that $\operatorname{Hom}_{\mathsf{U}}({}^{\perp}\mathsf{X} \cap \mathsf{F}, \mathsf{X}) = 0$ is clear.

We must also show $({}^{\perp}\mathsf{X} \cap \mathsf{F}) * \mathsf{X} = \mathsf{U}$. The inclusion $\subseteq$ follows because ${}^{\perp}\mathsf{X} \cap \mathsf{F}, \mathsf{X} \subseteq \mathsf{U}$. For the inclusion $\supseteq$, let $u \in \mathsf{U}$ be given. Then $u \in \mathsf{G}$ so there is a distinguished triangle

$$p \to u \to x \tag{2}$$

with $p \in {}^{\perp}\mathsf{X}$, $x \in \mathsf{X}$. Since $u, x \in \mathsf{F}$ we have $p \in \mathsf{F}$ whence $p \in {}^{\perp}\mathsf{X} \cap \mathsf{F}$, and (2) shows $u \in ({}^{\perp}\mathsf{X} \cap \mathsf{F}) * \mathsf{X}$.

A similar argument shows that $(\mathsf{X}, \mathsf{X}^{\perp} \cap \mathsf{G})$ is a stable t-structure in U. □

3. EXAMPLES

3.a. **The homotopy category of projective modules.** Let R be an Iwanaga-Gorenstein ring, that is, a noetherian ring which has finite injective dimension from either side as a module over itself. Let $\mathsf{T} = \mathsf{K}_{(\mathrm{b})}(\operatorname{Prj} R)$ be the homotopy category of complexes of projective right-R-modules with bounded homology. Define subcategories of T by

$$\mathsf{X} = \mathsf{K}^{-}_{(\mathrm{b})}(\operatorname{Prj} R) \ \ , \ \ \mathsf{Y} = \mathsf{K}_{\mathrm{ac}}(\operatorname{Prj} R) \ \ , \ \ \mathsf{Z} = \mathsf{K}^{+}_{(\mathrm{b})}(\operatorname{Prj} R)$$

where $\mathsf{K}^{-}_{(\mathrm{b})}(\operatorname{Prj} R)$ is the isomorphism closure of the class of complexes P with $P^i = 0$ for $i \gg 0$ and $\mathsf{K}^{+}_{(\mathrm{b})}(\operatorname{Prj} R)$ is defined analogously, while $\mathsf{K}_{\mathrm{ac}}(\operatorname{Prj} R)$ is the subcategory of acyclic (that is, exact) complexes.

Note that Y is equal to $\mathsf{K}_{\mathrm{tac}}(\mathrm{Prj}\ R)$, the subcategory of totally acyclic complexes, that is, acyclic complexes which stay acyclic under the functor $\mathrm{Hom}_R(-,Q)$ when Q is projective, see [6, cor. 5.5 and par. 5.12].

By [4, prop. 2.3(1), lem. 5.6(1), and rmk. 5.14] there are stable t-structures (X,Y), (Y,Z) in T.

If $P \in \mathsf{T}$ is given, then there is a distinguished triangle $P^{\geq 0} \to P \to P^{<0}$ where $P^{\geq 0}$ and $P^{<0}$ are hard truncations. Since $P^{\geq 0} \in \mathsf{Z}$ and $P^{<0} \in \mathsf{X}$, we have $\mathsf{T} = \mathsf{Z} * \mathsf{X}$.

We can hence apply Theorem C. The intersection

$$\mathsf{X} \cap \mathsf{Z} \;=\; \mathsf{K}^{-}_{(\mathrm{b})}(\mathrm{Prj}\ R) \cap \mathsf{K}^{+}_{(\mathrm{b})}(\mathrm{Prj}\ R) \;=\; \mathsf{K}^{\mathrm{b}}(\mathrm{Prj}\ R)$$

is the isomorphism closure of the class of bounded complexes. If we use an obvious shorthand for quotient categories, Theorem C therefore provides a triangle of recollements

$$\big(\mathsf{K}^{-}_{(\mathrm{b})}/\mathsf{K}^{\mathrm{b}}(\mathrm{Prj}\ R)\ ,\ \mathsf{K}_{\mathrm{ac}}(\mathrm{Prj}\ R)\ ,\ \mathsf{K}^{+}_{(\mathrm{b})}/\mathsf{K}^{\mathrm{b}}(\mathrm{Prj}\ R)\big)$$

in $\mathsf{K}_{(\mathrm{b})}/\mathsf{K}^{\mathrm{b}}(\mathrm{Prj}\ R)$. Note that $\mathsf{K}_{\mathrm{ac}}(\mathrm{Prj}\ R)$ is equivalent to its projection to $\mathsf{K}_{(\mathrm{b})}/\mathsf{K}^{\mathrm{b}}(\mathrm{Prj}\ R)$ by [4, prop. 1.5], so we can write $\mathsf{K}_{\mathrm{ac}}(\mathrm{Prj}\ R)$ instead of the projection.

This example and its finite analogue were first obtained in [4, thms. 2.8 and 5.8] and motivated the definition of triangles of recollements.

3.b. **The symmetric Auslander category.** Let R be a noetherian commutative ring with a dualizing complex. In [7, def. 2.1] we introduced the symmetric Auslander category $\mathsf{A}^{\mathrm{s}}(R)$ as the isomorphism closure in $\mathsf{K}_{(\mathrm{b})}(\mathrm{Prj}\ R)$ of the class of complexes P for which there exist $T, U \in \mathsf{K}_{\mathrm{tac}}(\mathrm{Prj}\ R)$ such that $P^{\ll 0} = T^{\ll 0}$ and $P^{\gg 0} = U^{\gg 0}$.

The methods of [7, sec. 2] show that there are stable t-structures

$$\big(\mathsf{A}(R)\ ,\ \mathsf{K}_{\mathrm{tac}}(\mathrm{Prj}\ R)\big)\ \ ,\ \ \big(\mathsf{K}_{\mathrm{tac}}(\mathrm{Prj}\ R)\ ,\ S(\mathsf{B}(R))\big)$$

in $\mathsf{A}^{\mathrm{s}}(R)$ with $S(\mathsf{B}(R)) * \mathsf{A}(R) = \mathsf{A}^{\mathrm{s}}(R)$. Here $\mathsf{A}(R)$ and $\mathsf{B}(R)$ are the Auslander and Bass categories of R, see [1, sec. 3], and S is the functor introduced in [6, sec. 4.3]. It is easy to show that $\mathsf{A}(R) \cap S(\mathsf{B}(R)) = \mathsf{K}^{\mathrm{b}}(\mathrm{Prj}\ R)$ and using Theorem C recovers the triangle of recollements

$$\big(\mathsf{A}(R)/\mathsf{K}^{\mathrm{b}}(\mathrm{Prj}\ R)\ ,\ \mathsf{K}_{\mathrm{tac}}(\mathrm{Prj}\ R)\ ,\ S(\mathsf{B}(R))/\mathsf{K}^{\mathrm{b}}(\mathrm{Prj}\ R)\big)$$

in $\mathsf{A}^{\mathrm{s}}(R)/\mathsf{K}^{\mathrm{b}}(\mathrm{Prj}\ R)$ first obtained in [7, thm. 2.10]. As in the previous example, the category $\mathsf{K}_{\mathrm{tac}}(\mathrm{Prj}\ R)$ is equivalent to its projection to $\mathsf{A}^{\mathrm{s}}(R)/\mathsf{K}^{\mathrm{b}}(\mathrm{Prj}\ R)$ so we can write $\mathsf{K}_{\mathrm{tac}}(\mathrm{Prj}\ R)$ instead of the projection.

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