Derivatives of Motors and Screws

Roy Featherstone
Department of Systems Engineering
Australian National University
Canberra, ACT 0200, Australia

Submitted to *Mechanism and Machine Theory*, June 24, 2003

Abstract

There are two competing definitions for the derivative of a twist, wrench, motor, or similar 6-D vector quantity: one obtained by treating it as a 6-D vector, and the other by treating it as a pair of 3-D vectors. This paper shows that the former is the absolute derivative, while the latter is an apparent derivative with respect to a moving reference point. The proof makes use of a mapping, based on a set of Plücker basis vectors, that defines the exact relationship between a motor and the 3-D vectors used to represent it. Both derivatives can be interpreted as motors. Differentiation in the dual space is also considered.

Keywords: screw theory, motor algebra, acceleration, Plücker coordinates.

1 Introduction

This paper considers the question of how to differentiate a twist, wrench, motor, or other similar 6-D vector quantity, with respect to a scalar variable. However, for the sake of clarity, a detailed argument is presented only for the time derivative of a motor. The result generalizes trivially to twists and wrenches, and to derivatives with respect to variables other than time. The only detail that is lost in this treatment is the (disputed) issue of duality, which is considered in Section 4.

There is much disagreement on the question of how to differentiate a twist, wrench or motor. This is because there are two opposing views on how to define the derivative, leading to two different expressions for the differential operator. One is obtained by treating the quantity as a 6-D vector, and the other by treating it as a pair of 3-D vectors. I shall call these operators D_6 and D_3, respectively. They are defined by the equations

$$D_6(\underline{\hat{\mathbf{m}}}_A) = \begin{bmatrix} \dot{\mathbf{m}} \\ \dot{\mathbf{m}}_A + \mathbf{v}_A \times \mathbf{m} \end{bmatrix} \tag{1}$$

and

$$D_3(\underline{\hat{\mathbf{m}}}_A) = \begin{bmatrix} \dot{\mathbf{m}} \\ \dot{\mathbf{m}}_A \end{bmatrix}, \tag{2}$$

where: $\underline{\hat{\mathbf{m}}}_A$ is a representation of an abstract motor, $\hat{\mathbf{m}}$, at a reference point A, in terms of a pair of 3-D vectors $\mathbf{m}$ and $\mathbf{m}_A$; $\dot{\mathbf{m}}$ and $\dot{\mathbf{m}}_A$ are the time-derivatives of $\mathbf{m}$ and $\mathbf{m}_A$; and $\mathbf{v}_A$ is the velocity of A.[1]

Motors are unusual in that there are two distinct representations in common use, which I shall call component motors and coordinate motors. A component motor is a pair of 3-D vectors, and a coordinate motor is a sextuple of real numbers. The quantity they represent is called an abstract motor. $\underline{\hat{\mathbf{m}}}_A$ in Eqs. 1 and 2 is a component motor.

This paper shows that Eq. 1 is the correct expression for the absolute differentiation operator, which I shall call D_{abs}, and that Eq. 2 must therefore be interpreted an an apparent or componentwise derivative. It also shows that the RHS of Eq. 2 can be interpreted as a motor if one takes into account the velocity of the reference point.

Briefly, the argument proceeds as follows. If $\underline{\hat{\mathbf{m}}}_A$ is a representation of $\hat{\mathbf{m}}$ then there exists a mapping, L, such that $\hat{\mathbf{m}} = L\,\underline{\hat{\mathbf{m}}}_A$. The absolute derivative of $\underline{\hat{\mathbf{m}}}_A$ is the motor that represents $\dot{\hat{\mathbf{m}}}$, so

$$\begin{aligned} D_{abs}\,\underline{\hat{\mathbf{m}}}_A &= L^{-1}\,\dot{\hat{\mathbf{m}}} \\ &= L^{-1}\,\frac{d}{dt}\,(L\,\underline{\hat{\mathbf{m}}}_A) \\ &= L^{-1}\,(\dot{L}\,\underline{\hat{\mathbf{m}}}_A + L\,\dot{\underline{\hat{\mathbf{m}}}}_A) \\ &= \dot{\underline{\hat{\mathbf{m}}}}_A + L^{-1}\,\dot{L}\,\underline{\hat{\mathbf{m}}}_A\,. \end{aligned}$$

This paper proves that $D_{abs} = D_6$ by obtaining an explicit expression for L, substituting it into the above equation, and showing that the result is identical to Eq. 1.

D_6 is the operator used in [4], and is apparently the operator used by Mises [16]. It is also the real-number equivalent of the motor derivative in Brand [3]. D_6 is compatible with the way that general vectors are differentiated (e.g. in analytical mechanics); and the D_6 derivative of a motor is itself a motor, which makes quantities like acceleration easier to use [7, 16].

D_3 is the operator implied by the classical treatment of rigid-body dynamics via 3-D vectors, and its appeal lies in its compatibility with 3-D methods and concepts. For example, if D_3 is applied to the velocity motor of a rigid body, then the result is a quantity composed of the 3-D angular acceleration of the rigid body and the 3-D linear acceleration of a particular body-fixed point. These are exactly the quantities that would be used in the traditional 3-D vector approach. In contrast, if D_6 is applied to a rigid-body velocity then the linear component of the result is the rate of change in the velocity with which successive body-fixed points pass through a fixed point in space [4, 7]. This quantity does not appear in the classical treatment of rigid-body acceleration. Indeed, this discrepancy has prompted some authors to say that the D_6 derivative should not be called simply 'the acceleration', but should be given some other name [15, 16].

The crucial difference between D_3 and D_6 lies in the treatment of the reference point, A. The argument that leads to D_6 says that A is part of the mapping from $\hat{\mathbf{m}}_A$ to $\hat{\mathbf{m}}$, so if A is moving then we have something analogous to a moving coordinate system. Therefore the absolute derivative must be the sum of the componentwise derivative of

[1] In this paper, symbols denoting vectors appear in a bold typeface, and motors are distinguished from other vectors by a caret. Vectors with some internal structure, such as coordinate vectors and component vectors, are distinguished from abstract vectors by an underline.

$\hat{\mathbf{m}}_A$ and a term that accounts for the time-varying nature of the mapping from $\hat{\mathbf{m}}_A$ to $\hat{\mathbf{m}}$. In contrast, the argument that leads to D_3 starts from a premise that the absolute derivative of $\hat{\mathbf{m}}_A$ is the concatenation of the absolute derivatives of its two component vectors, regardless of the velocity of A. Proponents of D_6 see D_3 as an apparent derivative in a moving coordinate system; whereas proponents of D_3 see D_6 as a partial derivative, since it does not account for the motion of the reference point.

Another way to approach the question is to invoke the equivalence between motors and helicoidal vector fields [7, 9, 14, 15, 16].[2] The derivative of a screw can then be defined via the derivative of a vector field. Unfortunately, this approach also leads to two possible answers: D_6 and D_3. The argument that leads to D_6 starts from the premise that the derivative of a vector field is the rate of change of the field itself. The argument that leads to D_3 is best understood with reference to the velocity field of a moving rigid body. Since this field maps points in space to the velocities of the body-fixed points currently passing through them, it is possible to argue that its derivative should be the field that maps points in space to the accelerations of the body-fixed points passing through them.

Good expositions of the D_3 argument can be found in [9, 15], and of the D_6 argument in [3, 4, 7]. A more equivocal treatment can be found in [16], where Wohlhart argues that the Mises derivative (D_6) is the right one to use, but Sections 5 and 6 still contain traces of the D_3 argument. A derivation of D_3 via the vector field argument can be found in [14]. (By the way, this paper does not prove the assertion in its abstract that the derivatives of twists and wrenches are screws, but only a weaker statement that they can be screws under certain special circumstances.) Other contributions to the subject in recent years include [2, 10, 11, 12]; and a survey of who uses which derivative in the robotics literature can be found in [7].

Most of the results in this paper have been published before (e.g. [7]); so the novel contribution of this paper is not the results themselves, but the argument used to obtain them. In particular, this paper considers the mapping from a representation to the quantity it represents, and presents explicit expressions for the mappings from coordinate motors and component motors to the abstract motors they represent. These expressions involve the basis vectors that define systems of Plücker ray and axis coordinates on twist, wrench and motor spaces. Given their role, it seems reasonable to call them *Plücker bases*. Although it is obvious that any coordinate system in a vector space must have an associated basis, and Ball himself used explicit bases for his screw coordinates [1], the only reference I can find that makes explicit use of Plücker bases is [4, §2.4], where they are called standard bases. Plücker bases provide the key to the correct interpretation of Eqs. 1 and 2, but they appear to have been largely ignored up to now.

The significance of the argument presented here is that it forces a change in the way we conceive of 6-D quantities: rigid-body acceleration really is a vector, and it really is a screw; whereas the classical description of rigid-body acceleration, if translated into a 6-D vector, really is not the absolute acceleration of a rigid body, but merely its apparent acceleration in a moving coordinate system. It also forces a change in the way we think of coordinate systems: the basis vectors of a coordinate motor are not the basis vectors for 3-D coordinate vectors. The former are a set of six Plücker basis vectors, all of them motors; while the latter are a set of three 3-D Euclidean vectors, each having properties

[2] My references to [9] are based on the reviewed version of this paper, since the final version did not yet exist at the time of writing.

of magnitude and direction, but no definite location in space. The fact that the latter do not depend on the position of the reference point is a red herring: the Plücker basis vectors do, so if the reference point moves then the Plücker basis moves with it.

The rest of this paper proceeds as follow. First, the concepts of representation and differentiation are explained in the setting of general n-dimensional vectors, so they can be understood free of any distractions caused by motors. These concepts are then applied directly to the problem of defining the exact relationship between a motor and its representations; which in turn leads to a proof that D_6 is the absolute differentiation operator for a component motor referred to a moving point A. The correct interpretation of D_3 is then discussed; and finally, the (very minor) impact of duality on the differentiation of screws and motors is considered.

2 Derivatives of General Vectors

Suppose we have two n-dimensional vector spaces, V_A and V_R, over the field of real numbers; and suppose we are really interested in the elements of V_A, but wish to use the elements of V_R as a means of representing them. This is possible if we define a $1:1$ mapping between V_A and V_R so that each element of V_A can be associated uniquely with an element of V_R, and vice versa. Typically, the elements of V_A are described as abstract vectors, on the grounds that no mathematical properties have been specified beyond the fact that they are elements of a particular vector space; whereas the elements of V_R usually do have some internal structure. For example, V_R might be a space of n-dimensional coordinate vectors, in which case the elements of V_R will be n-tuples of real numbers—quantities with a specific internal structure.

Let $L : V_R \mapsto V_A$ be a $1:1$ mapping such that, for any $\underline{\mathbf{u}} \in V_R$, the abstract vector $\mathbf{u} = L\,\underline{\mathbf{u}}$ is the quantity represented by $\underline{\mathbf{u}}$; and let us also require that L be a linear mapping, which means that it must satisfy $\alpha\, L\,\underline{\mathbf{u}}_1 + \beta\, L\,\underline{\mathbf{u}}_2 = L(\alpha\,\underline{\mathbf{u}}_1 + \beta\,\underline{\mathbf{u}}_2)$ for all $\underline{\mathbf{u}}_1, \underline{\mathbf{u}}_2 \in V_R$ and all real numbers α, β. Technically, this makes L an isomorphism—a $1:1$ mapping that preserves the mathematical structure of a vector space.

Now suppose we wish to study the derivative of a vector $\mathbf{u} \in V_A$ by means of elements and operations in V_R. Differentiation is defined in any vector space by the general formula

$$\dot{\mathbf{u}} = \frac{d}{dt}\,\mathbf{u}(t) = \lim_{\delta t \to 0} \frac{\mathbf{u}(t+\delta t) - \mathbf{u}(t)}{\delta t}\,. \tag{3}$$

However, differentiation in V_A is not necessarily the same operation as differentiation in V_R, because L might itself depend on t. This is where the notions of absolute and apparent differentiation make an entrance. Both are defined in the representation space, V_R, but refer to the differential operator in V_A. If $\mathbf{u} = L\,\underline{\mathbf{u}}$ then the absolute derivative of $\underline{\mathbf{u}}$ is the vector that represents $\dot{\mathbf{u}}$; and the apparent derivative is what the absolute derivative appears to be to an observer that is unaware of the time dependence of L (i.e., an observer that thinks $\dot{L} = 0$).

Let us introduce the symbols D_{abs} and D_{app} to denote the absolute and apparent differential operators. It turns out that the apparent derivative of a vector is the same as its ordinary derivative; i.e., $D_{app}\,\underline{\mathbf{u}} = d\underline{\mathbf{u}}/dt = \dot{\underline{\mathbf{u}}}$. However, the absolute derivative is given by the general formula

$$D_{abs}\,\underline{\mathbf{u}} \;=\; L^{-1}\,\dot{\mathbf{u}}$$

$$
\begin{aligned}
&= L^{-1}\frac{d}{dt}(L\,\underline{\mathbf{u}}) \\
&= L^{-1}(\dot{L}\,\underline{\mathbf{u}} + L\,\dot{\underline{\mathbf{u}}}) \\
&= \dot{\underline{\mathbf{u}}} + L^{-1}\,\dot{L}\,\underline{\mathbf{u}} \\
&= D_{app}\,\underline{\mathbf{u}} + L^{-1}\,\dot{L}\,\underline{\mathbf{u}}\,. \qquad (4)
\end{aligned}
$$

2.1 Coordinate Vectors

To apply the preceding theory to a particular case, we need to specify the internal structure of V_R; so let us now suppose that V_R is a space of n-dimensional coordinate vectors. The relationship between a coordinate vector and the quantity it represents is determined by a basis, which is a set of n linearly-independent vectors in the abstract vector space. Let $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \subset V_A$ be such a basis, then the relationship between $\underline{\mathbf{u}}$ and the quantity it represents is given by

$$
\mathbf{u} = L_E\,\underline{\mathbf{u}} = L_E \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \sum_{i=1}^{n} u_i\,\mathbf{e}_i\,, \qquad (5)
$$

where L_E is the representation map defined by E. L_E can be written in the form of a $1 \times n$ matrix with the basis vectors as its elements:

$$
L_E = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}. \qquad (6)
$$

With this notation, the action of L_E on $\underline{\mathbf{u}}$ is by implied matrix multiplication; i.e.,

$$
L_E\,\underline{\mathbf{u}} = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \sum_{i=1}^{n} \mathbf{e}_i\,u_i\,.
$$

To get an expression for L_E^{-1}, we must first introduce a quantity called the dual basis of E, which is given by $E^* = \{\mathbf{e}_1^*, \ldots, \mathbf{e}_n^*\}$. The vectors $\mathbf{e}_i^*$ satisfy

$$
\mathbf{e}_i^* \cdot \mathbf{e}_j = \delta_{ij}\,; \qquad (7)
$$

and we can express L_E^{-1} in terms of these vectors as follows:

$$
L_E^{-1} = \begin{bmatrix} \mathbf{e}_1^* \cdot \\ \vdots \\ \mathbf{e}_n^* \cdot \end{bmatrix}, \qquad (8)
$$

where the expression '$\mathbf{e}_i^* \cdot$' denotes an operator that maps a vector $\mathbf{v} \in V_A$ to the scalar $\mathbf{e}_i^* \cdot \mathbf{v}$. Again, the action of this operator on its operand is by implied matrix multiplication. Observe that

$$
L_E^{-1}\,L_E = \begin{bmatrix} \mathbf{e}_1^* \cdot \\ \vdots \\ \mathbf{e}_n^* \cdot \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix},
$$

which is the identity operator in V_R, and

$$L_E\, L_E^{-1} = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix} \begin{bmatrix} \mathbf{e}_1^* \cdot \\ \vdots \\ \mathbf{e}_n^* \cdot \end{bmatrix} = \sum_{i=1}^{n} \mathbf{e}_i\, \mathbf{e}_i^* \cdot \,,$$

which is the identity operator in V_A. The expression '$\mathbf{e}_i^*\, \mathbf{e}_i \cdot$' denotes an operator that maps a vector $\mathbf{v} \in V_A$ to the vector $\mathbf{e}_i\, (\mathbf{e}_i^* \cdot \mathbf{v})$.

In general, the vectors $\mathbf{e}_i^*$ must be elements of the dual space of V_A, which is denoted V_A^*; but there are some special cases where this isn't necessary, and $\mathbf{e}_i^*$ can instead be elements of V_A. Motors and Euclidean vectors are two such cases.

2.2 Component Vectors

Now let us suppose that V_R is the Cartesian product of m vector spaces; i.e., $V_R = V_1 \times V_2 \times \cdots \times V_m$, where $\dim(V_i) = n_i$ and $\sum_{i=1}^{m} n_i = n$. The elements of V_R are therefore m-tuples of smaller vectors. Specifically, if $\underline{\mathbf{u}} \in V_R$ then $\underline{\mathbf{u}} = [\mathbf{u}_1, \ldots, \mathbf{u}_m]^T$, where $\mathbf{u}_i \in V_i$. I shall call $\underline{\mathbf{u}}$ a component vector, and $\mathbf{u}_i$ is the i^{th} component of $\underline{\mathbf{u}}$. V_i are assumed to be abstract vector spaces.

The relationship between a component vector and the quantity it represents can be expressed in the form

$$\mathbf{u} = L\, \underline{\mathbf{u}} = L \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_m \end{bmatrix} = \sum_{i=1}^{m} L_i\, \mathbf{u}_i \,, \tag{9}$$

where L_i is a $1:1$ mapping from V_i to the n_i-dimensional subspace $S_i = \mathrm{range}(L_i) \subset V_A$. We can express L in terms of L_i as follows:

$$L = \begin{bmatrix} L_1 & \cdots & L_m \end{bmatrix}. \tag{10}$$

Equations 9 and 10 are the component-vector equivalents of Eqs. 5 and 6 for coordinate vectors. The mappings L_i play the same role here as that played previously by the basis vectors $\mathbf{e}_i$.

We can obtain a detailed expression for L_i by the following procedure. First, choose an arbitrary basis on V_i. Let $C_i = \{\mathbf{c}_{i1}, \ldots, \mathbf{c}_{in_i}\}$ be such a basis. Given C_i, we can define a basis $E_i = \{\mathbf{e}_{i1}, \ldots, \mathbf{e}_{in_i}\}$ on S_i such that $\mathbf{e}_{ij} = L_i\, \mathbf{c}_{ij}$; thus E_i is the image of C_i under L_i. If L_{Ci} and L_{Ei} are the linear mappings associated with C_i and E_i then

$$L_i = L_{Ei}\, L_{Ci}^{-1} = \sum_{j=1}^{n_i} \mathbf{e}_{ij}\, \mathbf{c}_{ij}^* \cdot \,. \tag{11}$$

Note that both L_{Ci} and L_{Ei} depend on the choice of C_i, but L_i is invariant.

And finally, the derivative of a component vector is given by

$$\dot{\underline{\mathbf{u}}} = \frac{d}{dt} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_m \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}\mathbf{u}_1 \\ \vdots \\ \frac{d}{dt}\mathbf{u}_m \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{u}}_1 \\ \vdots \\ \dot{\mathbf{u}}_m \end{bmatrix}. \tag{12}$$

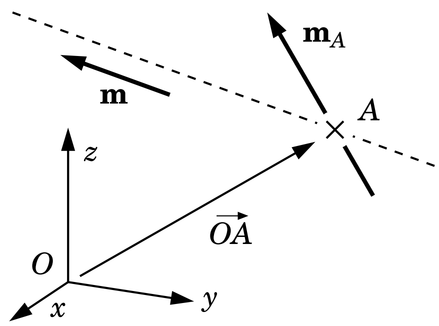

Figure 1: The description of a motor by two 3-D vectors.

This equation follows directly from Eq. 3 and the basic properties of vectors that are m-tuples. It therefore also applies to coordinate vectors (with trivial modification). As the quantity on the RHS consists of the derivatives of the components of $\underline{\mathbf{u}}$, this operation is sometimes called the componentwise derivative of $\underline{\mathbf{u}}$; but the name 'componentwise derivative' is really just a longer, more descriptive name for the standard derivative of a vector, as in Eq. 3, when that vector just happens to be an m-tuple.

3 Derivatives of Motors

This section constructs explicit expressions for the mappings between abstract motors, coordinate motors and component motors, and uses them to derive expressions for the absolute derivatives of coordinate and component motors. It is shown that Eq. 1 is the correct expression for the absolute derivative, while Eq. 2 is the apparent derivative as viewed by an observer translating with the reference point. It is also shown that the latter behaves like a motor under a change of reference point, provided the new reference point has the same velocity as the old.

3.1 Coordinate Motors and Component Motors

Referring to Figure 1, the usual method for defining a motor proceeds as follows. First, a reference point must be chosen. This is point A in Figure 1. The motor itself can then be described by two 3-D vectors. The first vector, $\mathbf{m}$, does not depend on the choice of A. It is sometimes called the resultant vector, and it describes the magnitude and direction of the line-vector component (or screw axis) of the motor. The second vector, $\mathbf{m}_A$, does depend on the choice of A. It is sometimes called the moment vector, and it describes the 'moment' of the motor about A. Together, these vectors form a componentwise representation of an abstract motor, $\hat{\mathbf{m}}$, referred to the point A:

$$\underline{\hat{\mathbf{m}}}_A = \begin{bmatrix} \mathbf{m} \\ \mathbf{m}_A \end{bmatrix}. \tag{13}$$

The subscripts on $\underline{\hat{\mathbf{m}}}_A$ and $\mathbf{m}_A$ indicate that these quantities depend on the location of A. The abstract motor represented by $\underline{\hat{\mathbf{m}}}_A$ is, of course, independent of A.

One of the defining properties of a motor is that if we choose a different reference point, say B, then the moment motor at B is related to the moment motor at A by

$$\mathbf{m}_B = \mathbf{m}_A + \overrightarrow{BA} \times \mathbf{m}\,, \tag{14}$$

which is known as the shifting formula.

Eq. 13 describes a motor by means of a component vector comprising two 3-D abstract Euclidean vectors. If we introduce a Cartesian coordinate frame ($Oxyz$ in Figure 1) then we can obtain the Cartesian coordinates of $\mathbf{m}$ and $\mathbf{m}_A$, which can be assembled into a coordinate motor, $\underline{\hat{\mathbf{m}}}_{Axyz}$, as follows:

$$\underline{\hat{\mathbf{m}}}_{Axyz} = \begin{bmatrix} \underline{\mathbf{m}} \\ \underline{\mathbf{m}}_A \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} \\ \begin{bmatrix} m_{Ax} \\ m_{Ay} \\ m_{Az} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} m_x \\ m_y \\ m_z \\ m_{Ax} \\ m_{Ay} \\ m_{Az} \end{bmatrix}. \tag{15}$$

In contrast, $\hat{\mathbf{m}}_A$ is given by

$$\hat{\mathbf{m}}_A = \begin{bmatrix} \mathbf{m} \\ \mathbf{m}_A \end{bmatrix} = \begin{bmatrix} m_x\,\mathbf{i} + m_y\,\mathbf{j} + m_z\,\mathbf{k} \\ m_{Ax}\,\mathbf{i} + m_{Ay}\,\mathbf{j} + m_{Az}\,\mathbf{k} \end{bmatrix}, \tag{16}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the basis vectors of the coordinate system in Euclidean 3-space defined by $Oxyz$. $\underline{\hat{\mathbf{m}}}_{Axyz}$ and $\hat{\mathbf{m}}_A$ are therefore quite different quantities.

Equation 15 shows the most common coordinate representation of a motor, and this particular coordinate system is called Plücker ray coordinates. The subscript $Axyz$ is unusual. In standard motor notation, the subscript would be A. I am using this longer subscript to indicate that the coordinate system, and hence also the mapping from the coordinate motor to the abstract motor, depends on both the position of A and the orientation of $Oxyz$. In fact, the coordinate system is defined by a single coordinate frame, $Axyz$, with its origin at A and its axes parallel to $Oxyz$.

From Eqs. 15 and 16, the exact relationship between $\hat{\mathbf{m}}_A$ and $\underline{\hat{\mathbf{m}}}_{Axyz}$ is given by

$$\hat{\mathbf{m}}_A = L_{2xyz}\,\underline{\hat{\mathbf{m}}}_{Axyz}\,, \tag{17}$$

where

$$L_{2xyz} = \begin{bmatrix} L_{xyz} & 0 \\ 0 & L_{xyz} \end{bmatrix} \tag{18}$$

and

$$L_{xyz} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \end{bmatrix}. \tag{19}$$

L_{xyz} is just the mapping from 3-D coordinate vectors to abstract vectors, and L_{2xyz} performs this mapping on pairs of vectors.

Having established the mapping between $\hat{\mathbf{m}}_A$ and $\underline{\hat{\mathbf{m}}}_{Axyz}$, all that remains is to find a mapping between $\hat{\mathbf{m}}$ and either $\hat{\mathbf{m}}_A$ or $\underline{\hat{\mathbf{m}}}_{Axyz}$. The latter is easier, since all we have to do is identify the basis vectors. Consider the six motors whose Plücker ray coordinates are $[1,0,0,0,0,0]^T$, $[0,1,0,0,0,0]^T$, and so on. The first three are unit line vectors with lines of action passing through A in the x, y and z directions. Let us call them $\hat{\mathbf{e}}_{Ax}$,

$\hat{\mathbf{e}}_{Ay}$ and $\hat{\mathbf{e}}_{Az}$, respectively. The remainder are three unit free vectors in the x, y and z directions, which we can call $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$. These six motors constitute a Plücker basis on the coordinate frame $Axyz$.

We can now say that the relationship between the abstract motor, $\hat{\mathbf{m}}$, and the coordinate motor that represents it, $\underline{\hat{\mathbf{m}}}_{Axyz}$, is

$$\hat{\mathbf{m}} = m_x\,\hat{\mathbf{e}}_{Ax} + m_y\,\hat{\mathbf{e}}_{Ay} + m_z\,\hat{\mathbf{e}}_{Az} + m_{Ax}\,\hat{\mathbf{e}}_x + m_{Ay}\,\hat{\mathbf{e}}_y + m_{Az}\,\hat{\mathbf{e}}_z\,. \tag{20}$$

Notice the pattern of subscripts: the coordinates that don't change when you move A are multiplied by the basis vectors that do change, and vice versa.

One minor flaw in this notation emerges if we consider the identity of the dual basis. If $P_{Axyz} = \{\hat{\mathbf{e}}_{Ax}, \ldots, \hat{\mathbf{e}}_z\}$ is the Plücker basis defined by $Axyz$ then, logically, the dual basis is $P^*_{Axyz} = \{\hat{\mathbf{e}}^*_{Ax}, \ldots, \hat{\mathbf{e}}^*_z\}$. If we interpret $\hat{\mathbf{e}}^*_{Ax}$ as being the member of the dual basis corresponding to $\hat{\mathbf{e}}_{Ax}$ then all is fine; but if we interpret the subscript in $\hat{\mathbf{e}}^*_{Ax}$ to mean that it is a line vector passing through A then that is incorrect. From Eq. 7 and the definition of the scalar product of two motors [16], we can deduce that the dual basis is actually $P^*_{Axyz} = \{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \ldots, \hat{\mathbf{e}}_{Az}\}$, which is the basis for Plücker axis coordinates, and is just a permutation of P_{Axyz}.

Let L_{Axyz} be the mapping defined by the Plücker basis P_{Axyz}, so

$$L_{Axyz} = \left[\begin{array}{cccccc} \hat{\mathbf{e}}_{Ax} & \hat{\mathbf{e}}_{Ay} & \hat{\mathbf{e}}_{Az} & \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \end{array}\right], \tag{21}$$

then

$$\hat{\mathbf{m}} = L_{Axyz}\,\underline{\hat{\mathbf{m}}}_{Axyz} \tag{22}$$

and (from Eq. 17)

$$\hat{\mathbf{m}} = L_A\,\underline{\hat{\mathbf{m}}}_A\,, \tag{23}$$

where

$$L_A = L_{Axyz}\,L^{-1}_{2xyz}\,. \tag{24}$$

Note that L_A is independent of $Oxyz$, just as L_i in Eq. 11 is independent of C_i.

There is another useful expression relating $\hat{\mathbf{m}}$ and $\underline{\hat{\mathbf{m}}}_A$:

$$\hat{\mathbf{m}} = \mathit{linevec}_A(\mathbf{m}) + \mathit{freevec}(\mathbf{m}_A)\,, \tag{25}$$

where $\mathit{linevec}_A(\mathbf{m})$ is a line vector with magnitude and direction given by $\mathbf{m}$ and a line of action passing through A, and $\mathit{freevec}(\mathbf{m}_A)$ is a free vector having magnitude and direction given by $\mathbf{m}_A$. This equation says that $\hat{\mathbf{m}}$ is the sum of a line vector passing through A and a free vector, and that the two 3-D components of $\underline{\hat{\mathbf{m}}}_A$ describe the magnitude and direction of each of these vectors [4]. If we compare Eq. 25 with Eq. 23 then we get another expression for L_A:

$$L_A = \left[\begin{array}{cc} \mathit{linevec}_A & \mathit{freevec} \end{array}\right],$$

where

$$\mathit{linevec}_A = \hat{\mathbf{e}}_{Ax}\,\mathbf{i}\cdot + \hat{\mathbf{e}}_{Ay}\,\mathbf{j}\cdot + \hat{\mathbf{e}}_{Az}\,\mathbf{k}\cdot$$

and

$$\mathit{freevec} = \hat{\mathbf{e}}_x\,\mathbf{i}\cdot + \hat{\mathbf{e}}_y\,\mathbf{j}\cdot + \hat{\mathbf{e}}_z\,\mathbf{k}\cdot\,.$$

There is one important detail that should be mentioned here before we move on. It is sometimes said that $\mathbf{m}$ describes (or even *is*) the part of the motor that is independent of A. This can be misleading, since, as we have just seen, $\mathbf{m}$ represents a line vector passing through A. If we switch to another reference point, B, then the $\mathbf{m}$ in $\underline{\hat{\mathbf{m}}}_B$ represents a different line vector to the $\mathbf{m}$ in $\underline{\hat{\mathbf{m}}}_A$. It is precisely because the line vector changes when we move the reference point, that it becomes necessary to add a different free vector so that the sum remains unchanged. Indeed, if we set $\mathbf{m} = \mathbf{0}$ then $\mathbf{m}_A$, despite its name, does not vary with A because it represents a free vector. I believe equations like Eq. 20 give a clearer picture of the relationship between a motor and the quantities used to represent it: $\mathbf{m}$ is $m_x\,\mathbf{i} + m_y\,\mathbf{j} + m_z\,\mathbf{k}$, but $\mathbf{m}$ represents $m_x\,\hat{\mathbf{e}}_{Ax} + m_y\,\hat{\mathbf{e}}_{Ay} + m_z\,\hat{\mathbf{e}}_{Az}$.

3.2 Absolute Derivatives

From Eqs. 4, 22 and 23, we can immediately write

$$D_{abs}\,\underline{\hat{\mathbf{m}}}_{Axyz} = \dot{\underline{\hat{\mathbf{m}}}}_{Axyz} + L_{Axyz}^{-1}\,\dot{L}_{Axyz}\,\underline{\hat{\mathbf{m}}}_{Axyz} \tag{26}$$

and

$$D_{abs}\,\hat{\mathbf{m}}_A = \dot{\hat{\mathbf{m}}}_A + L_A^{-1}\,\dot{L}_A\,\hat{\mathbf{m}}_A\,. \tag{27}$$

If the frame $Axyz$ is moving with twist velocity $\underline{\hat{\mathbf{v}}}_{Axyz} = [\,\underline{\boldsymbol{\omega}}^T\ \underline{\mathbf{v}}_A^T\,]^T$ (which is the same thing as saying $Oxyz$ is rotating with angular velocity $\underline{\boldsymbol{\omega}}$ and A is moving with velocity $\underline{\mathbf{v}}_A$), then we have the following two established results:

$$L_{xyz}^{-1}\,\dot{L}_{xyz} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = \underline{\boldsymbol{\omega}}\times \tag{28}$$

and

$$L_{Axyz}^{-1}\,\dot{L}_{Axyz} = \begin{bmatrix} \underline{\boldsymbol{\omega}}\times & 0 \\ \underline{\mathbf{v}}_A\times & \underline{\boldsymbol{\omega}}\times \end{bmatrix} = \underline{\hat{\mathbf{v}}}_{Axyz}\times\,. \tag{29}$$

The latter is the motor equivalent of the former, and can be found in [3, 4, 16]. The cross notation is interpreted as follows. As $\underline{\boldsymbol{\omega}}$ is a coordinate vector, the expression '$\underline{\boldsymbol{\omega}}\times$' denotes the operator that maps a coordinate vector $\underline{\mathbf{v}}$ to $\underline{\boldsymbol{\omega}} \times \underline{\mathbf{v}}$. The equivalent abstract operator is denoted $\boldsymbol{\omega}\times$. Likewise, $\underline{\hat{\mathbf{v}}}_{Axyz}\times$ is the operator that maps a coordinate motor $\underline{\hat{\mathbf{m}}}_{Axyz}$ to $\underline{\hat{\mathbf{v}}}_{Axyz} \times \underline{\hat{\mathbf{m}}}_{Axyz}$, and the equivalent operators on component motors and abstract motors are $\hat{\mathbf{v}}_A\times$ and $\hat{\mathbf{v}}\times$.

Applying Eq. 29 to Eq. 26 gives

$$D_{abs}\,\underline{\hat{\mathbf{m}}}_{Axyz} = \dot{\underline{\hat{\mathbf{m}}}}_{Axyz} + \underline{\hat{\mathbf{v}}}_{Axyz} \times \underline{\hat{\mathbf{m}}}_{Axyz}\,, \tag{30}$$

which is the well-known formula for differentiating a coordinate motor in a moving coordinate system [3, 4, 16]. Applying Eq. 24, the identity $\dot{L}^{-1}\,L + L^{-1}\,\dot{L} = 0$, and then Eqs. 29, 18 and 28 to the expression $L_A^{-1}\,\dot{L}_A$ gives

$$\begin{aligned} L_A^{-1}\,\dot{L}_A &= L_{2xyz}\,L_{Axyz}^{-1}\,(\dot{L}_{Axyz}\,L_{2xyz}^{-1} + L_{Axyz}\,\dot{L}_{2xyz}^{-1}) \\ &= L_{2xyz}\,(L_{Axyz}^{-1}\,\dot{L}_{Axyz} + \dot{L}_{2xyz}^{-1}\,L_{2xyz})\,L_{2xyz}^{-1} \\ &= L_{2xyz}\,(L_{Axyz}^{-1}\,\dot{L}_{Axyz} - L_{2xyz}^{-1}\,\dot{L}_{2xyz})\,L_{2xyz}^{-1} \end{aligned}$$

$$
\begin{aligned}
&= L_{2xyz} \left(\begin{bmatrix} \underline{\boldsymbol{\omega}}\times & 0 \\ \underline{\mathbf{v}}_A\times & \underline{\boldsymbol{\omega}}\times \end{bmatrix} - \begin{bmatrix} \underline{\boldsymbol{\omega}}\times & 0 \\ 0 & \underline{\boldsymbol{\omega}}\times \end{bmatrix} \right) L_{2xyz}^{-1} \\
&= \begin{bmatrix} 0 & 0 \\ \mathbf{v}_A\times & 0 \end{bmatrix},
\end{aligned} \tag{31}
$$

where the final step uses the identity

$$L_{xyz}\, \underline{\mathbf{v}}_A \times L_{xyz}^{-1} = \mathbf{v}_A \times ,$$

which is the mapping from the coordinate-vector operator $\underline{\mathbf{v}}_A\times$ to its equivalent abstract operator. Substituting this result into Eq. 27 gives

$$
\begin{aligned}
D_{abs}\, \hat{\underline{\mathbf{m}}}_A &= \begin{bmatrix} \dot{\mathbf{m}} \\ \dot{\mathbf{m}}_A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mathbf{v}_A\times & 0 \end{bmatrix} \begin{bmatrix} \mathbf{m} \\ \mathbf{m}_A \end{bmatrix} \\
&= \begin{bmatrix} \dot{\mathbf{m}} \\ \dot{\mathbf{m}}_A + \mathbf{v}_A \times \mathbf{m} \end{bmatrix},
\end{aligned} \tag{32}
$$

which is in complete agreement with Eq. 1. Thus, Eq. 1 is indeed the correct definition of the absolute derivative of a component motor.

3.3 Interpreting D_3

Another way to write Eq. 32 is

$$D_{abs}\, \hat{\underline{\mathbf{m}}}_A = \dot{\hat{\underline{\mathbf{m}}}}_A + \begin{bmatrix} \mathbf{0} \\ \mathbf{v}_A \end{bmatrix} \times \hat{\underline{\mathbf{m}}}_A\,. \tag{33}$$

This equation is the component-motor equivalent of Eq. 30. It states that the absolute derivative of a component motor is the sum of its apparent derivative and a term that accounts for the velocity of the observer. In the case of Eq. 30, the observer is fixed to a moving coordinate frame with velocity $[\,\underline{\boldsymbol{\omega}}^T\ \underline{\mathbf{v}}_A\,]^T$. In the case of Eq. 33, the observer is fixed only to the point A, and therefore has a velocity of $\mathbf{v}_A$. We may therefore conclude that $D_3(\hat{\underline{\mathbf{m}}}_A)$ in Eq. 2, which equates with $\dot{\hat{\underline{\mathbf{m}}}}_A$ in Eq. 33, is the apparent derivative of a component motor as seen by an observer with the same velocity as the reference point.

The same conclusion can be reached by considering the special case of a Plücker basis with velocity $\hat{\underline{\mathbf{v}}}_{Axyz} = [\,\underline{\mathbf{0}}^T\ \underline{\mathbf{v}}_A^T\,]^T$. D_3 is defined as an operation on a component motor; but if $Oxyz$ is not rotating then the absolute and componentwise derivatives of a 3-D coordinate vector are the same, which allows us to equate D_3 with the componentwise derivative $\dot{\hat{\underline{\mathbf{m}}}}_{Axyz}$ in Eq. 30. In this case, Eq. 30 tells us that D_3 is the apparent derivative of a motor in a non-rotating Plücker basis with the same velocity as the reference point.

This observation has been made before in [7]. However, there is another related observation that should be mentioned here. It is often said that the derivative of a motor (or screw), as defined by Eq. 2, is not itself a motor (e.g. see [9, 15, 16]). This is a reasonable statement to make if you believe that D_3 is the absolute differentiation operator. However, we have just seen that D_3 is really an apparent derivative, and this fact allows us a new possibility to interpret it (correctly) as a motor.

The point about an apparent derivative is that it is calculated relative to a moving observer. Thus, D_3 yields a quantity that depends on both the position of the reference point and the velocity of the observer. Now, two observers at the same point, but with different velocities, will disagree on the value of the apparent derivative, unless an adjustment is made to account for the difference in their velocities. In other words, two apparent derivatives can only be compared if their reference points coincide and their observer velocities are the same.

Suppose that $\hat{\underline{\mathbf{m}}}_A$ and $\hat{\underline{\mathbf{m}}}_{A'}$ are two component motors representing the same abstract motor, $\hat{\mathbf{m}}$; and suppose also that A and A' coincide at the current instant, but their velocities differ. It therefore follows that $\hat{\underline{\mathbf{m}}}_A = \hat{\underline{\mathbf{m}}}_{A'}$ and $D_{abs}\,\hat{\underline{\mathbf{m}}}_A = D_{abs}\,\hat{\underline{\mathbf{m}}}_{A'}$ at the current instant; so we can deduce (from Eq. 33) the following relationship between the apparent derivatives:

$$\begin{aligned} D_{app}\,\hat{\underline{\mathbf{m}}}_{A'} &= D_{abs}\,\hat{\underline{\mathbf{m}}}_A - \begin{bmatrix} \mathbf{0} \\ \mathbf{v}_{A'} \times \mathbf{m} \end{bmatrix} \\ &= D_{app}\,\hat{\underline{\mathbf{m}}}_A + \begin{bmatrix} \mathbf{0} \\ (\mathbf{v}_A - \mathbf{v}_{A'}) \times \mathbf{m} \end{bmatrix}. \end{aligned} \tag{34}$$

This is the *velocity shift* formula for a component motor. It applies only to apparent derivatives, and it accounts for the velocity difference between two coincident reference points. (Higher apparent derivatives would also need an acceleration shift formula, and so on.)

Let us now investigate the relationship between $D_3(\hat{\underline{\mathbf{m}}}_A)$ and $D_3(\hat{\underline{\mathbf{m}}}_B)$, where $\hat{\underline{\mathbf{m}}}_A$ and $\hat{\underline{\mathbf{m}}}_B$ represent the same abstract motor at two different reference points:

$$\begin{aligned} D_3(\hat{\underline{\mathbf{m}}}_B) &= D_3(\hat{\underline{\mathbf{m}}}_A) + \begin{bmatrix} \dot{\mathbf{m}} \\ \dot{\mathbf{m}}_B \end{bmatrix} - \begin{bmatrix} \dot{\mathbf{m}} \\ \dot{\mathbf{m}}_A \end{bmatrix} \\ &= D_3(\hat{\underline{\mathbf{m}}}_A) + \begin{bmatrix} \mathbf{0} \\ \frac{d}{dt}\,(\mathbf{m}_A + \overrightarrow{BA} \times \mathbf{m}) - \dot{\mathbf{m}}_A \end{bmatrix} \\ &= D_3(\hat{\underline{\mathbf{m}}}_A) + \begin{bmatrix} \mathbf{0} \\ (\frac{d}{dt}\,\overrightarrow{BA}) \times \mathbf{m} + \overrightarrow{BA} \times \dot{\mathbf{m}} \end{bmatrix} \\ &= D_3(\hat{\underline{\mathbf{m}}}_A) + \begin{bmatrix} \mathbf{0} \\ (\mathbf{v}_A - \mathbf{v}_B) \times \mathbf{m} + \overrightarrow{BA} \times \dot{\mathbf{m}} \end{bmatrix} \end{aligned} \tag{35}$$

(cf. [9, §4.3]). Now define the point B' to be instantaneously coincident with B, but travelling with velocity $\mathbf{v}_A$. According to Eq. 34, the relationship between $D_3(\hat{\underline{\mathbf{m}}}_{B'})$ and $D_3(\hat{\underline{\mathbf{m}}}_B)$ is

$$D_3(\hat{\underline{\mathbf{m}}}_{B'}) = D_3(\hat{\underline{\mathbf{m}}}_B) + \begin{bmatrix} \mathbf{0} \\ (\mathbf{v}_B - \mathbf{v}_A) \times \mathbf{m} \end{bmatrix}; \tag{36}$$

so the relationship between $D_3(\hat{\underline{\mathbf{m}}}_{B'})$ and $D_3(\hat{\underline{\mathbf{m}}}_A)$ is

$$D_3(\hat{\underline{\mathbf{m}}}_{B'}) = D_3(\hat{\underline{\mathbf{m}}}_A) + \begin{bmatrix} \mathbf{0} \\ \overrightarrow{BA} \times \dot{\mathbf{m}} \end{bmatrix}. \tag{37}$$

This is just the usual shifting formula for motors. Therefore, the apparent derivative of a motor, as given by Eq. 2, obeys the shifting formula in Eq. 14, provided the new point has the same velocity as the old. If the velocities differ then one must also use the velocity shifting formula in Eq. 34.

4 Derivatives and Duality

In analytical mechanics, a generalized velocity vector is considered to be an element of the tangent space to the configuration manifold at the current configuration; whereas a generalized force vector is considered to be an element of the cotangent space, which is the dual of the tangent space. Traditionally, neither screw theory nor motor algebra has incorporated this concept of duality, although this situation is now starting to change [6, 8, 13].

For vectorial quantities, like motors, the change from a non-dual theory to a dual theory is quite simple: replace the original structure, in which a single vector space contains all motors, with a new structure comprising two vector spaces. Twists, and other motion-like vectors, go into one space; while wrenches, and other force-like vectors, go into the other. I call these spaces M^6 and F^6, respectively. The inner product in the original motor space is replaced by a scalar product that takes one argument from each space. Thus, if $\hat{\mathbf{m}} \in \mathsf{M}^6$ and $\hat{\mathbf{f}} \in \mathsf{F}^6$ then $\hat{\mathbf{m}} \cdot \hat{\mathbf{f}}$ is the work done by a wrench $\hat{\mathbf{f}}$ on a twist $\hat{\mathbf{m}}$, but expressions like $\hat{\mathbf{m}} \cdot \hat{\mathbf{m}}$ and $\hat{\mathbf{f}} \cdot \hat{\mathbf{f}}$ are no longer permitted. For more details, see [6] and the appendix of [5].

To work in a dual-space setting, the following changes must be made to the theory in Section 3.

1. The original Plücker basis, P_{Axyz}, is placed in M^6, and its dual, P^*_{Axyz}, is placed in F^6. (They were previously both in the same vector space.) P_{Axyz} now defines a system of ray coordinates in M^6, and P^*_{Axyz} a system of axis coordinates in F^6.

2. Introduce two new mappings: $L^*_{Axyz} = [\,\hat{\mathbf{e}}^*_{Ax} \;\cdots\; \hat{\mathbf{e}}^*_{z}\,]$ and $L^*_A = L^*_{Axyz}\, L^{-1}_{2xyz}$. The differentiation of elements of M^6 now proceeds as described in Section 3, while the differentiation of elements of F^6 proceeds according to a parallel set of equations in which L^*_{Axyz} and L^*_A replace L_{Axyz} and L_A.

Two notable differences between the the equations for F^6 and for M^6 are:

$$L^{*-1}_{Axyz}\, \dot{L}^*_{Axyz} = \begin{bmatrix} \underline{\boldsymbol{\omega}}\times & \underline{\mathbf{v}}_A\times \\ 0 & \underline{\boldsymbol{\omega}}\times \end{bmatrix} \tag{38}$$

(cf. Eq. 29) and

$$L^{*-1}_{A}\, \dot{L}^*_{A} = \begin{bmatrix} 0 & \underline{\mathbf{v}}_A\times \\ 0 & 0 \end{bmatrix} \tag{39}$$

(cf. Eq. 31). Equation 38 puts the motor cross-product notation into a bit of difficulty, since $\hat{\mathbf{v}}\times\hat{\mathbf{m}}$ is now a different operation from $\hat{\mathbf{v}}\times\hat{\mathbf{f}}$, which means that the expression $\hat{\mathbf{v}}\times$ on its own is ambiguous. Lie algebra gets over this difficulty by using two different symbols: $ad_{\hat{\mathbf{v}}}$ for the $\hat{\mathbf{v}}\times$ in $\hat{\mathbf{v}}\times\hat{\mathbf{m}}$ and $ad^*_{\hat{\mathbf{v}}}$ for the $\hat{\mathbf{v}}\times$ in $\hat{\mathbf{v}}\times\hat{\mathbf{f}}$ [13]. I have been experimenting with the notation $crm(\hat{\mathbf{v}})$ for $ad_{\hat{\mathbf{v}}}$ and $crf(\hat{\mathbf{v}})$ for $ad^*_{\hat{\mathbf{v}}}$ wherever disambiguation is required, and $\hat{\mathbf{v}}\times$ wherever it is clear from the context which operator is intended.

5 Conclusion

This paper has shown that if a twist, wrench or motor is represented by means of a pair of 3-D vectors, one of them depending on the position of a reference point, A, then the mapping from the representation to the actual motor is necessarily a function of A. If A has a nonzero velocity then we have a situation analogous to an ordinary vector represented in a time-varying coordinate system: the componentwise derivative of the motor is not its absolute derivative, but only an apparent derivative relative to an observer with the same velocity as A. The absolute derivative is the sum of the componentwise derivative and a term depending on the velocity of A, as shown in Eq. 1 in the introduction.

One immediate consequence of this result is that the absolute derivative of a motor is itself a motor. However, this paper goes on to show that the apparent derivative can also be regarded as a motor, on the grounds that it depends on both the position and the velocity of the reference point. If one changes only the position, then the apparent derivative behaves like a motor. If one changes both the position and the velocity then a separate velocity-shifting formula accounts for the change in velocity.

Most of the results in this paper have been published before, so the novelty is mainly in the argument used to get them. Explicit 1 : 1 mappings are formulated to express the relationships between motors and their representations, so as to avoid any possibility of vagueness, ambiguity, or alternative interpretations, on the critical questions of the exact interpretation and role of the two 3-D vectors and the reference point used to represent a motor. This is accomplished using Plücker basis vectors. Although the paper focuses on the time derivative of a motor, the results generalize trivially to twists and wrenches, and to differentiation with respect to variables other than time. The only detail that is not completely straightforward is the issue of duality, which is considered separately at the end of this paper.

In my opinion, the debate over the correct way to define and interpret the derivative of a twist, wrench or motor should now come to an end.

References

[1] Ball, R. S. 1900. *A Treatise on the Theory of Screws*. London: Cambridge University Press. Republished 1998.

[2] Bokelberg, E. H., Hunt, K. H., and Ridley, P. R. 1992. Spatial Motion I: Points of Inflection and the Differential Geometry of Screws. *Mechanism and Machine Theory*, vol. 27, no. 1, pp. 1–15.

[3] Brand, L. 1953. *Vector and Tensor Analysis*, 4th ed. New York/London: Wiley/Chapman and Hall.

[4] Featherstone, R. 1987. *Robot Dynamics Algorithms*. Boston: Kluwer Academic Publishers.

[5] Featherstone, R. 1999. A Divide-and-Conquer Articulated-Body Algorithm for Parallel $O(\log(n))$ Calculation of Rigid-Body Dynamics. Part 1: Basic Algorithm. *Int. J. Robotics Research*, vol. 18, no. 9, pp. 867–875.

[6] Featherstone, R. 2000. On the Limits to Invariance in the Twist/Wrench and Motor Representations of Motion and Force Vectors. Proc. Ball 2000, Cambridge, UK, July 9–12. Available: `http://www.helix.gatech.edu/ball2000/`.

[7] Featherstone, R. 2001. The Acceleration Vector of a Rigid Body. *Int. J. Robotics Research*, vol. 20, no. 11, pp. 841–846.

[8] Griffis, M., and Rico, J. M. 2003. The Nut in Screw Theory. *J. Robotic Systems*, vol. 20, no. 7, pp. ??–??.

[9] Lipkin, H. 2003. Properties of Screw Derivatives. *accepted for publication in Mechanism and Machine Theory.*

[10] Rico Martínez, J. M., and Duffy, J. 1996. An Application of Screw Algebra to the Acceleration Analysis of Serial Chains. *Mechanism & Machine Theory*, vol. 31, no. 4, pp. 445–457.

[11] Rico, J. M., Gallardo, J., and Duffy, J. 1999. Screw Theory and Higher Order Kinematic Analysis of Open Serial and Closed Chains. *Mechanism & Machine Theory*, vol. 34, no. 4, pp. 559–586.

[12] Ridley, P. R., Bokelberg, E. H., and Hunt, K. H. 1992. Spatial Motion II: Acceleration and the Differential Geometry of Screws. *Mechanism and Machine Theory*, vol. 27, no. 1, pp. 17–35.

[13] Selig, J. M. 1996. *Geometrical Methods in Robotics*. New York: Springer.

[14] Stramigioli, S., and Bruyninckx, H. 2001. Geometry of Dynamic and Higher-Order Kinematic Screws. Proc. IEEE Int. Conf. Robotics and Automation, Seoul, Korea, pp. 3344–3349.

[15] Wohlhart, K. 2001. Time Derivatives of the Velocity Motor. Proc. SYROM 2001, Bucharest, Romania, vol. 2, pp. 371–376.

[16] Wohlhart, K. 2002. Mises Derivatives of Motors and Motor Tensors. Proc. RoManSy 2002, Udine, Italy, pp. 87–98.