# Approximation capability of TP model forms*

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## Abstract

The tensor product (TP) based models have been applied widely in approximation theory, and approximation techniques. Recently, a controller design framework working on dynamic systems has also been established based on TP model transformation combined with Linear Matrix Inequalities (LMI) within Parallel Distributed Compensation (PDC) framework. The effectiveness of the control design framework strongly depends on the approximation property of the TP model used. Therefore, the primary aim of this paper is to investigate the approximation capabilities of dynamic TP model. It is shown that the set of functions that can be approximated arbitrarily well by TP forms with bounded number of components lies no-where dense, i.e. "almost discretely" in the set of continuous functions. Consequently, this paper points out that in a class of control problems this drawback necessitates the application of trade-off techniques between accuracy and complexity of TP form. Such requirements are very difficult to consider in the analytical framework, but TP model transformation offers an easy way to deal with them.

## 1 Introduction

The demand for the decomposition of multivariate functions to univariate ones goes back to the very end of the 19th century. In 1900, in his memorable lecture at the Second International Congress of Mathematicians in Paris, D. Hilbert, the famous German mathematician, listed 23 conjectures, hypotheses concerning unsolved problems which he considered would be the most important ones to solve by the mathematicians of the 20th century [12, 9, 13]. In the 13th, he addressed the problem of multivariate continuous function decomposition to finite superposition of continuous functions of fewer variables. The motivation is straightforward: one dimensional functions are much easier to calculate with, handle and visualize.

Hilbert presumed that this problem cannot be solved in general, i.e. there exist multivariate continuous function that cannot be decomposed to univariate continuous functions. This was disproved by Kolmogorov in 1957 [15] in his general representation theorem, when he provided a constructive proof. Unfortunately, univariate function components were often highly non-smooth, so this construction could not be directly applied in practice.

Another family of univariate function based methods use tensor products of continuous univariate basis functions. This technique has been applied widely and are used in, e.g. the following fields. Non-Uniform Rational B-Spline (NURBS) technique has become industry standard throughout CAD/CAM/CAE systems for the representation, design, and data exchange of geometric information processing by computers. These systems use data-base assigned, with in a tensor product, to piecewise polynomial or rational curves and surfaces to represent complex geometry. Decomposition to primitive functions and data-base also has important role in signal processing when wavelet functions are concerned. Many of recent signal processing approaches are based on the evaluation of parameter array assigned by tensor product to wavelet or other basis.

TP based approximation has reached modelling approaches of non-linear dynamic systems, and furthermore, there are now controller design frameworks based on TP model [1, 5]. Generally, *TP model*, in broad sense, is an approximation technique where the approximating functions are in tensor product form, whereas a *TP model form* is a particular approximating function in a TP model. In this paper we consider TP model in a narrower sense, when a TP model is applied to dynamic system control (see details in Section 2.1); as a consequence we prefer the TP model denomination

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rather than TP approximation since the former reflects better our modelling point of view.

A large variety of LMI based control design techniques have been developed in the last decade [6, 7, 17]. Powerful commercialized softwares have also been developed [8] for solving LMIs and related control problems. Recently, a number of LMI based controller designs have been carried out for TP models (also termed as polytop or TS models in fuzzy theory) under PDC [20]. Further, a *TP model transformation* has been developed to transfer non-linear dynamic models to TP model whereupon PDC design frameworks can readily be executed [1, 5, 10, 11]. One can find a case study of TP model transformation in the control design of a prototypical aeroelastic wing section [2] that exhibits various control phenomena such as limit cycle oscillation and chaotic vibration [14].

A crucial point of these control design frameworks is the modelling accuracy. If TP model does not appropriately describe the real system the resulting control may not ensure the required control performance. This paper is devoted to analyze the approximation capabilities of TP model forms when applied to dynamic system control.

The paper is organized as follows: Section 2 introduces the fundamentals of TP modelling. Section 3 proves the main result of the paper. Section 4 introduces an example of state-space model that can be exactly described in TP form, and such a one that only can be approximated. Section 5 derives some conclusions.

# 2 Preliminaries

## 2.1 TP model

Consider parametrically varying state-space dynamic model:

$$\begin{aligned} s\,\mathbf{x}(t) &= \mathbf{A}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{p}(t))\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{D}(\mathbf{p}(t))\mathbf{u}(t), \end{aligned} \tag{1}$$

The system matrix

$$\mathbf{S}(\mathbf{p}(t)) = \begin{pmatrix} \mathbf{A}(\mathbf{p}(t)) & \mathbf{B}(\mathbf{p}(t)) \\ \mathbf{C}(\mathbf{p}(t)) & \mathbf{D}(\mathbf{p}(t)) \end{pmatrix} \in \mathbb{R}^{O\times I} \tag{2}$$

is a parametrically varying object, where $\mathbf{p}(t) \in \Omega$ is time varying $N$-dimensional parameter vector, where $\Omega \subset \mathbb{R}^N$ is a closed hypercube. $\mathbf{p}(t)$ can also include some elements of $\mathbf{x}(t)$. Further, for a continuous-time system $s\mathbf{x}(t) = \dot{\mathbf{x}}(t)$; or for a discrete-time system $s\mathbf{x}(t) = \mathbf{x}(t+1)$ holds.

The *TP model*, to be studied in this paper, applies univariate basis functions, which is described as:

$$\begin{pmatrix} s\mathbf{x}(t) \\ \mathbf{y}(t) \end{pmatrix} = \left( \sum_{i_1=1}^{I_1} \cdots \sum_{i_N=1}^{I_N} \prod_{n=1}^{N} w_{n,i_n}(p_n(t))\mathbf{S}_{i_1,\dots,i_N} \right) \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{pmatrix} \tag{3}$$

Function $w_{n,j}(p_n(t))$ is the $j$-th univariate basis function defined on the $n$-th dimension of $\Omega$, and $p_n(t)$ is the $n$-th element of vector $\mathbf{p}(t)$. $I_n$ $(n = 1, \dots, N)$ is the number of univariate basis functions used in the $n$-th dimension of the parameter vector. The multi-index $(i_1, i_2, \dots, i_N)$ refers to the coefficient vertex system corresponding to the $i_n$-th basis function in the $n$-th dimension $(n = 1, \dots, N)$. Hence, the number of vertex systems $\mathbf{S}_{i_1,i_2,\dots,i_N}$ is obviously $\prod_{n=1}^{N} I_n$.

We can rewrite (3) in the concise tensor product form as:

$$\begin{pmatrix} s\mathbf{x}(t) \\ \mathbf{y}(t) \end{pmatrix} = \mathcal{S} \bigotimes_{n=1}^{N} \mathbf{w}_n(p_n(t)) \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{pmatrix}. \tag{4}$$

Here, row vector $\mathbf{w}_n(p_n) \in \mathbb{R}^{I_n}$ contains the basis functions $w_{n,i_n}(p_n)$, the $N+2$-dimensional coefficient tensor $\mathcal{S} \in \mathbb{R}^{I_1\times I_2\times\cdots\times I_N\times O\times I}$ is constructed from the linear constant system matrices $\mathbf{S}_{i_1,i_2,\dots,i_N} \in \mathbb{R}^{O\times I}$. (Calligraphic typesetting is used for tensors.) The first $N$ dimensions of $\mathcal{S}$ are assigned to the dimensions of $\Omega$. The convexity of the basis functions is ensured by conditions:

$$\begin{aligned} &\forall n, i, p_n(t) : w_{n,i}(p_n(t)) \in [0,1]; \\ &\forall n, p_n(t) : \sum_{i=1}^{I_n} w_{n,i}(p_n(t)) = 1. \end{aligned} \tag{5}$$

The TP model transformation, described in details in [1, 5, 4], that transforms a state-space model into TP forms has the following syntax

$$(\mathbf{w}_{n=1,\dots,N}(p_n(t)); \mathcal{S}) = \text{TP-transf}(\mathbf{S}(\mathbf{p}(t)); \Omega; \text{options}); \tag{6}$$

where $\mathbf{S}(\mathbf{p}(t)) \in \mathbb{R}^{O\times I}$ from (2), and $\Omega$ is the bounded parameters space the transformation is performed over, and "option" is to define some characteristics of the resulting basis function system $\mathbf{w}_{n=1,\dots,N}(p_n(t)$: the transformation can generate "minimal", "convex" and "close-to-localized" basis. For further details see the referred papers [1, 5, 4].

Our goal in this paper is to characterize the approximation behavior of the transformation of the dynamic state-space model form to TP form with respect to accuracy and feasibility. More precisely, we intend to show that there exists functions that cannot be approximated arbitrarily well by a bounded number of basis functions. As a consequence, it will be shown that the set of functions that TP forms of bounded order can approximate arbitrarily well is no-where dense in the set of continuous functions. More precisely, there are two possible cases: 1. there exists an *exact* TP model of the dynamic system, thus this model is ready to use in practice, and the stability of the controller can be easily obtained through LMI techniques. 2. When exact model does not exist, we need to approximate the systems behavior by TP forms with acceptable accuracy for the given problem. The price of the accuracy is paid by the complexity of the TP forms, i.e. the increasing number of basis functions in each dimension. This necessitates the use of trade-off techniques, which is an amenable way of solving practical problems when TP model transformation is applied.

## 2.2 Approximation theory framework

For the sake of simplicity, we restrict to bivariate control surfaces and models with only two input variables $(N = 2)$. All definitions and statements can be carried over trivially to the multivariate case. Therefore, all results presented in the section are equally valid for any finite $N \in \mathbb{N}$ value. Further, without the loss of generality, we restrict the range of input to the unit interval, because any finite ranged interval can be mapped into another.

**Definition 1** *We refer to the matrix function of the following form as TP form of order* $(I_1, I_2)$

$$TP(z_1, z_2) = \mathbf{S}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \tag{7}$$
$$= \left(\sum_{i_1=1}^{I_1}\sum_{i_2=1}^{I_2} w_{1,i_1}(z_1)w_{2,i_2}(z_2)\mathbf{S}_{i_1,i_2}\right)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

*where the number of the component is bounded by* $I_1$ *in the first and by* $I_2$ *in the second variable. Matrices* $\mathbf{S}_{i_1,i_2}$ *contain the coefficients as defined in* (2). $w_{1,i_1}(z_1)$ *and* $w_{2,i_2}(z_2)$ *are the normalized basis functions that satisfy condition* (5).

As we noted above, all definitions and proofs can be generalized to multivariate case. Let us investigate now the correspondence between the inputs $z_i$ and the vectors $\mathbf{p}$ and $\mathbf{x}$ in the general, multivariate case. The dimension of vector $\mathbf{z}$ can be considered as the union the dimensions of $\mathbf{x}$ and $\mathbf{p}$. Since $\mathbf{p}$ can contain some elements of $\mathbf{x}$, $\mathbf{z}$ is an extension of $\mathbf{x}$ with the elements of $\mathbf{p}$ that are not in $\mathbf{x}$. Formally, if $\mathbf{x} = (x_1, \ldots, x_M)^T$ and $\mathbf{p} = (p_1, \ldots, p_N)^T$ such that $p_i = x_i$ for $i = 1, \ldots, K$, $K \leq M$, then $\mathbf{z} = (x_1, \ldots, x_M, p_{K+1}, \ldots, p_N)^T \in \mathbb{R}^{M+N-K}$, assuming that the first $K$ elements of $\mathbf{x}$ and $\mathbf{p}$ are ordered correspondingly. If $K < M$, then for $z_i$, $K+1 \leq i \leq M$ there is a sole basis function $w_i$ that is identically 1 on the whole range of $z_i$, i.e. these elements has no effect on the output of the TP model. If $N = K$ then the size of $\mathbf{z}$ is $M$.

Let us denote the set of TP forms (7) defined in Definition 1 by

$$\mathbf{TP}_{(I_1,I_2)}\left([0,1]^2\right) \tag{8}$$

where the pair $(I_1, I_2)$ is an upper bound for the number of components in the input space $[0,1]^2$. For $p \in [1,\infty]$ we introduce the set

$$\mathbf{TP}^{(p)}_{(n_1,n_2)}\left([0,1]^2\right) \tag{9}$$

which is the subset of (8) and $L^p\left([0,1]^2\right)$, as well. Therefore, the set (9) is equipped with the $L^p$-norm $\|\cdot\|_p$. An elements of sets (9) and (8) is a *TP model forms of order* $(I_1, I_2)$. In general, TP model forms with finite bounds in each dimension are called *TP model forms of bounded order*.

# 3 Main results

Moser has proven the no-where denseness of Sugeno controllers [18] in two steps [16]. Our proof to be presented here is an extension of Moser's result and its deduction technique is analogous of Moser's one. First, we prove that a special function cannot be approximated arbitrary well by TP model forms of bounded order regardless of what $L^p$-norm is chosen (cf. Lemma 1). Then it is presented (see Theorem 1) that for every TP model form, $t$, and for arbitrary $\varepsilon > 0$ there exists a function $\omega$ in the $\varepsilon$-environment of $s$, which cannot be approximated with arbitrary accuracy by TP model forms of bounded order. This immediately implies that TP model forms of bounded order are no-where dense in the space of continuous functions. The construction of $\omega$ in Theorem 1 is based on the special function of the lemma.

Now, we point out that if there is one element of a matrix function $TP$ that cannot be approximated arbitrarily well by the corresponding elements of bounded TP model forms, then this property can be carried over for the whole matrix function. Therefore in the next lemma, we investigate only one element of matrix function $TP$, $t(z_1, z_2) = TP(z_1, z_2)_{[i,j]}$. That element is a linear function of $(z_1, z_2)$ according to (1), where the coefficients depend on the $[i,j]$ elements of $\mathbf{S}_{i_1,i_2}$ matrices. For brevity, we refer to $t$ as TP model form in the next, keeping in mind that it is an arbitrary element of the matrix function $TP$.

**Lemma 1** *Let* $\omega : [0,1] \to \mathbb{R}$ *be a function of the form* $\omega(z_1, z_2) = \alpha$ *if* $z_2 \geq z_1$, $\omega(z_1, z_2) \neq \alpha$ *else, where* $\alpha \in \mathbb{R}$. *Then for each* $p \in [1,\infty]$ *and* $I_1, I_2 \in \mathbb{N}$ *there holds*

$$\inf\left\{\|\omega - t\|_p \,\middle|\, t = TP_{[i,j]}TP \in \mathbf{TP}^{(p)}_{(I_1,I_2)}\right\} > 0. \tag{10}$$

*for arbitrary* $(i,j)$ *pair.*

**Proof.** The proof proceeds indirectly. First, we suppose that opposing the assumption (10) there exist a sequence $\hat{t}_n$, $n \in \mathbb{N}$ of TP model forms, $t \in \left(\mathbf{TP}^{(p)}_{(I_1,I_2)}\right)_{[i,j]}$, which ensures norm-convergence to the function $\omega$:

$$\lim_n \|\hat{t}_n - \omega\|_p = 0 \tag{11}$$

where $\hat{t}_n$ being TP model forms of order $(I_1, I_2)$ have the form

$$\hat{t}_n(z_1, z_2) = \sum_{i_1=1}^{I_1}\sum_{i_2=1}^{I_2} w^{(n)}_{1,i_1}(z_1)w^{(n)}_{2,i_2}(z_2)\left(\widehat{\mathbf{S}}^{(n)}_{i_1,i_2}\right)_{[i,j]}$$

where $w^{(n)}_{1,i_1}(z_1), w^{(n)}_{2,i_2}(z_2)$ are basis functions and $\left(\widehat{\mathbf{S}}^{(n)}_{i_1,i_2}\right)_{[i,j]}$ are real numbers. If we modify properly TP model forms $\hat{t}_n$ then we obtain TP model forms $t_n$, according to the assumption (11) that approximate arbitrarily well functions having value 0 for $y \geq x$ and that differs from 0 for $y < x$ w.r.t. $\|\cdot\|_p$. Therefore we introduce a sequence of TP model forms $(t_n)$ $(n \in \mathbb{N})$ as

$$t_n(z_1, z_2) = \sum_{i_1=1}^{I_1}\sum_{i_2=1}^{I_2} w^{(n)}_{1,i_1}(z_1)w^{(n)}_{2,i_2}(z_2)\left(\mathbf{S}^{(n)}_{i_1,i_2}\right)_{[i,j]} \tag{12}$$

where $\mathbf{S}^{(n)}_{[i,j]} = \widehat{\mathbf{S}}^{(n)}_{[i,j]} - \alpha$. Because norm-convergence implies pointwise convergence at almost everywhere from (11) and (12) we have

$$\lim_n \hat{t}_n = \omega - \alpha \tag{13}$$

Let $\Omega'$ be the set where pointwise convergence holds.

Moser showed that there are points of the open four-dimensional unit hypercube $(0,1)^4$

$$\left(z^*_{1,1}, z^*_{2,1}\right), \left(z^*_{1,1}, z^*_{2,2}\right) \text{ and } \left(z^*_{1,2}, z^*_{2,2}\right) \tag{14}$$

which satisfy the following inequalities (see also figure 1)

$$z^*_{2,1} < z^*_{1,1} \leq z^*_{2,2} < z^*_{1,2} \tag{15}$$

as well as lie in the set $\Omega$ and do not fall only in the set $[0,1]^2/\Omega$ that has Lesbegue measure zero. These points are essential in the last step of the proof.

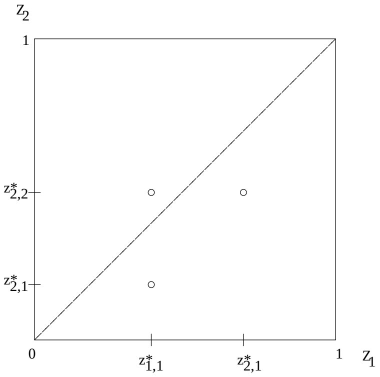


Figure 1: The points $(z^*_{1,1}, z^*_{2,1}), (z^*_{1,1}, z^*_{2,2})$ and $(z^*_{1,2}, z^*_{2,2})$

Now, we apply diagonal construction due to Cantor on the basis of $\Omega^*$, which is countable subset of $\Omega'$ lying dense in $[0,1]^2$, as well. Therefore, we can write that $\{(z_{1,q}, z_{2,q}) | i \in \mathbb{N}\} = \Omega^* \subset \Omega$. Note that $\Omega^*$ also contains points in (14). For simplicity, let us denote $\Omega^*_{z_1} = z^*_{1,1}, z^*_{1,2}, \ldots$ and $\Omega^*_{z_2} = z^*_{2,1}, z^*_{2,2}, \ldots$.

Now, we require that for the sequence of appropriate elements of $\mathbf{S}$

$$\left(\mathbf{S}^{(q,n)}_{i_1,i_2}\right)_{[i,j]} \tag{16}$$

converge to a real number as $n$ tends to infinity. By the upper index $(q,n)$ we denote the subsequence of the sequence denoted by upper index $(n)$, furthermore, we demand the sequence with upper index $(q+1,n)$ to be the subsequence of the sequence with upper index $(q,n)$. Finally, from the above we may suppose that sequences $\left(w^{(n,n)}_{1,i_1}\right)_n$, $\left(w^{(n,n)}_{2,i_2}\right)_n$, and similarly $\left(\left(\mathbf{S}^{(n,n)}_{i_1,i_2}\right)_{[i,j]}\right)_n$ are convergent for all $(z_{1,q}, z_{2,q}) \in \Omega^*$ and all indices $1 \leq i_1 \leq I_1$, $1 \leq i_2 \leq I_2$.

It follows that the input-output functions of TP model forms $t_{n,n}$ converge to the input-output function of TP model form

$$\tilde{t}(z_1, z_2) = \sum_{i,j} \tilde{w}_{1,i_1}(z_1) \tilde{w}_{2,i_2}(z_2) \left(\tilde{\mathbf{S}}_{i_1,i_2}\right)_{[i,j]} \tag{17}$$

where

$$\lim_n w^{(n,n)}_{1,i_1} = \tilde{w}_{1,i_1}, \ \lim_n w^{(n,n)}_{2,i_2} = \tilde{w}_{2,i_2},$$
$$\lim_n \left(\mathbf{S}^{(n,n)}_{i_1,i_2}\right)_{[i,j]} = \left(\tilde{\mathbf{S}}_{i_1,i_2}\right)_{[i,j]}.$$

As the matter of fact, (17) is a tensor product of univariate functions with bounded order $J = I_1 \cdot I_2$, hence we can introduce induction similarly as in [16].

Let us denote by $\mathbf{Ten}_J(D, Z_1 \times Z_2)$ those functions $f : Z_1 \times Z_2 \mapsto \mathbb{R}$ can be, whose restrictions $f|_D$ can be represented as tensor product of maximal $J \in \mathbb{N}$ basis functions for each dimension, i.e. for an $f \in \mathbf{Ten}_n(D, Z_1 \times Z_2)$ iff $f(z_1, z_2) = \sum_{i=1}^{J} \phi_i(z_1)\psi_i(z_2)$ holds for all $(z_1, z_2) \in D$, where $\phi_1, \ldots, \phi_J : Z_1 \mapsto \mathbb{R}$ and $\psi_1, \ldots, \psi_J : Z_2 \mapsto \mathbb{R}$.

In order to deduce a contradiction from the assumption that $\omega$ can be approximated arbitrarily well by TP model forms we proceed to show that in fact the limit input-output function of TP model forms (17) is element of $\mathbf{Ten}_1(\Omega^* \cap [\gamma, 1]^2, [0,1]^2)$, for some $0 < \gamma < z^*_{2,1}$. Then from the fact that the function $\tilde{t}(z_1, z_2)$ equals by assumption the function $\omega - \alpha$ on the domain $\Omega^*$ and the construction that the points (14) are elements of $\Omega^* \cap [\gamma, 1]^2$ a contradiction can immediately be derived analogously as in [16].

Since $\tilde{t} \equiv \omega - \alpha$ on $\Omega^*$, there must be an index $i_0 \in \{1, \ldots, J\}$ for which $\phi_{i_0} \not\equiv 0$ on $\Omega^*_{z_2} \cap (0, z^*_{2,1}/4)$.

Without loss of generality let $i_0 = J$. Thus, there is an element

$$z_{1,0} \in \Omega^*_{z_1} \cap (0, z^*_{2,1}/4), \tag{18}$$

such that $\phi_J(z_{1,0}) \neq 0$. Consequently, for $z_2 \in \Omega^*_{z_2} \cap [z_{1,0}, 1]$ we obtain

$$\psi_J(z_2) = -\sum_{i=1}^{J-1} \psi_i(z_2) \frac{\phi_i(z_{1,0})}{\phi_J(z_{1,0})},$$

as the function $\tilde{t}$ vanishes on $\{(z_1, z_2) \in \Omega^* | \, z_2 \geq z_1\}$. Hence, for all pairs $(z_1, z_2) \in \Omega^*$ with $z_1, z_2 \geq z^*_{2,1}/4$ we get

$$\tilde{t}(z_1, z_2) = \sum_{i=1}^{J-1} \left(\phi_i(z_1) - \phi_J(z_1) \cdot \frac{\phi_i(z_{1,0})}{\phi_J(z_{1,0})}\right) \psi_i(z_2)$$

showing that

$$\tilde{t}(z_1, z_2) \in \mathbf{Ten}_{J-1}\left(\Omega^* \cap [z^*_{2,1}/4, 1]^2, [0,1]^2\right). \tag{19}$$

In order to apply induction we set

$$\gamma_k = (1 - (3/4)^{J-k}) z^*_{2,1}$$

for $1 \leq k \leq J$. Analogously as we have deduced (19) from (18),

$$\tilde{t}(z_1, z_2) \in \mathbf{Ten}_{k-1}\left(\Omega^* \cap [\gamma_{k-1}, 1]^2, [0,1]^2\right)$$

can be deduced from

$$\tilde{t}(z_1, z_2) \in \mathbf{Ten}_{k}\left(\Omega^* \cap [\gamma_{k}, 1]^2, [0,1]^2\right)$$

for $(2 \leq k \leq J)$. Applying induction we obtain

$$\tilde{t}(z_1, z_2) \in \mathbf{Ten}_1\left(\Omega^* \cap [\gamma_1, 1]^2, [0,1]^2\right). \tag{20}$$

From formula (20) we conclude that there are functions $\phi$ and $\psi : [0,1] \mapsto \mathbb{R}$ such that

$$\tilde{t}(z_1, z_2) = \omega(z_1, z_2) - \alpha = \phi(z_1)\psi(z_2) \tag{21}$$

for $(z_1, z_2) \in \Omega^* \cap (\gamma_1, 1]^2$.

As $\gamma_k < z^*_{2,1}$ for all $1 \leq k \leq J$, the points (14) are elements of the set $\Omega^* \cap (\gamma_1, 1]^2$.

This and equation (21) leads to a contradiction as on the one hand we have $\phi(z^*_{1,1})\psi(z^*_{2,1}) > 0$ and $\phi(z^*_{1,2})\psi(z^*_{2,2}) > 0$, hence

$$\phi(z^*_{1,1}) \neq 0 \quad \text{and} \quad \psi(z^*_{2,2}) \neq 0,$$

while on the other hand, by the definition of $\omega$, we obtain $\phi(z^*_{1,1})\psi(z^*_{2,2}) = 0$. $\square$

Lemma 1 exemplifies a function that cannot be approximated arbitrarily well by TP model forms of order $(I_1, I_2)$ w.r.t. the $L_p$ norm. The $\omega$ that is approximated above, thus not en element of cl $\left(\mathbf{TP}^{(p)}_{(I_1,I_2)}\left([0,1]^2\right)\right)$, that is the closure (w.r.t the $L_p$ norm) of the subset $\mathbf{TP}^{(p)}_{(I_1,I_2)}\left([0,1]^2\right)$ of $L_p\left([0,1]^2\right)$

While Lemma 1 only states the existence of such a function the next theorem states that $L_p\left([0,1]^2\right)\backslash\text{cl}\left(\mathbf{TP}^{(p)}_{(I_1,I_2)}\left([0,1]^2\right)\right)$ lies dense in $L_p\left([0,1]^2\right)$. Its idea is to show that in each neighborhood of an arbitrary function one can find a function which cannot be approximated arbitrarily well by TP forms of bounded order. These neighboring functions are constructed by means of Lemma 1. The proof basically exploits the special form of $\omega$, and it can be trivially obtained from the proof of the no-where denseness of Sugeno controllers [16], thus here it is omitted.

**Theorem 1** *To each* $p \in [1,\infty]$, $\varepsilon > 0$ *and* $t = TP_{[i,j]}$, $TP \in \mathbf{TP}^{(p)}_{(I_1,I_2)}\left([0,1]^2\right)$, $I_1, I_2 \in \mathbb{N}$ *there is a continuous function*

$$\omega \in L^p([0,1]^2)\backslash cl\left(\mathbf{TP}^{(p)}_{(I_1,I_2)}\left([0,1]^2\right)\right)$$

*fulfilling* $\|\omega - t\|_p < \varepsilon$.

As Theorem 1 guarantees that to each $\varepsilon > 0$ and $t = TP_{[i,j]}$, $TP \in \mathbf{T}^{(p)}_{(I_1,I_2)}\left([0,1]^2\right)$ there is a function $\omega \in L^p([0,1]^2)\backslash\text{cl}\left(\mathbf{TP}^{(p)}_{(I_1,I_2)}\left([0,1]^2\right)\right)$ with $\|\omega - t\|_p < \varepsilon$, or equivalently, there is no inner point of the set $\mathbf{TP}^{(p)}_{(I_1,I_2)}\left([0,1]^2\right)$, because every $\varepsilon$-environment of an arbitrary element of the set contains functions not included in the closure of the set. Thus, we immediately obtain that $\mathbf{TP}^{(p)}_{(I_1,I_2)}$ is no-where dense in $L^p([0,1]^2)$.

# 4 Examples

In this section we present two examples. The examples are introduced by means of a mechanical system depicted on Figure 2. First, we show that the original non-linear state-space model of the system can exactly be represented by a TP model. This TP model is actually obtained by TP model transformation [1, 5, 4], but can also be derived in an analytic way [19]. Here, the number of basis functions in each dimension is 2. Second, we point out that if a non-linear term of the mechanical system is changed to a function having the same properties as $\omega$ in Lemma1 then regardless how many basis functions are used the exact TP model representation can not be achieved. This is obviously a synthetical example created in order to show that in such cases TP approximation based controller design framework has the ability to deal with trade-off consideration, that is not feasible by the analytic way. We remark that the achievable accuracy of the TP model, and further control solutions, depends on the capacity and speed of the computer used for modelling. It should be also noted that the computational complexity of the TP model transformation is strongly exponential,and further, the number of LMIs created from the TP model, and especially the computational requirement of LMI solvers explode with the complexity of the TP model, except a few extreme cases. These facts significantly affect the maximum number of basis functions, hence, the maximal achievable accuracy both of modelling and control design.

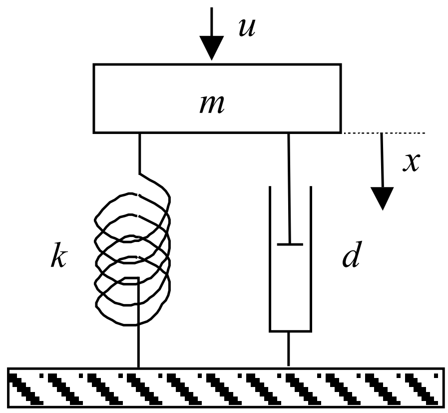

Figure 2: Mass–spring–damper system

## 4.1 The original mass–spring–damper mechanical system

The dynamical equation of the mechanical system of Figure 2 is given as:

$$m \cdot \ddot{x}(t) + g(x(t), \dot{x}(t)) + k(x(t)) = \phi(\dot{x}(t)) \cdot u(t), \quad (22)$$

where $m$ is the mass and $u(t)$ represents the force. The function $k(x)$ is the non-linear or uncertain stiffness coefficient of the spring, $g(x, \dot{x})$ is the non-linear or uncertain term damping coefficient of the damper, and $\phi(\dot{x}(t))$ is the non-linear input term. Assume that $g(x(t), \dot{x}(t)) = d(c_1x(t)+c_2\dot{x}^3(t))$, $k(x(t)) = c_3x(t) + c_4x^3(t)$, and $\phi(\dot{x}(t)) = 1 + c_5\dot{x}^3(t)$. Furthermore, assume that $x \in [-a, a]$, $\dot{x}(t) \in [-b, b]$ and $a, b > 0$. The above parameters are set as follows [22]: $m = 1$, $d = 1$, $c_1 = 0.01$, $c_2 = 0.1$, $c_3 = 0.01$, $c_4 = 0.67$, $c_5 = 0$, $a = 1.5$, and $b = 1.5$. Equation (22) then becomes:

$$\ddot{x}(t) = -0.1\dot{x}^3(t) - 0.02x(t) - 0.67x^3(t) + u(t). \quad (23)$$

The non-linear terms are $-0.1\dot{x}^3(t)$ and $-0.67x^3(t)$. The state-space model is:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(x_1(t), x_2(t))\mathbf{x}(t) + \begin{pmatrix}1\\0\end{pmatrix} u(t)$$

that is in the present case:

$$\dot{\mathbf{x}}(t) = \begin{pmatrix}-0.1x_1^2 & -0.02 - 0.67x_2^2\\ 1 & 0\end{pmatrix}\mathbf{x}(t) + \begin{pmatrix}1\\0\end{pmatrix} u(t). \quad (24)$$

When TP model transformation is applied, we obtain the following TP model form as detailed in [1, 5, 4]:

$$\dot{\mathbf{x}}(t) = \mathcal{A} \mathop{\otimes}_{n=1}^{2} \mathbf{w}_n(x_n(t))\mathbf{x}(t) + \mathbf{Bu}(t).$$

As it is noted in the referred papers, if the density of sampling grid is increased to infinity the limes of the rank of the sampled tensor remains 2 on the first two dimensions. This experimental observation is actually proved by the analytic derivation presented in [19]. Thus, (24) can exactly be represented by a TP model form as:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \sum_{i=1}^{2}\sum_{j=1}^{2} w^a_{1,i}(x_1(t))w^a_{2,j}(x_2(t))\mathbf{A}^a_{i,j}\mathbf{x}(t) + \mathbf{B}u(t)\\ &= \mathcal{A}^a \mathop{\otimes}_{n=1}^{2} \mathbf{w}^a_n(x_n(t))\mathbf{x}(t) + \mathbf{B}u(t).\end{aligned} \quad (25)$$

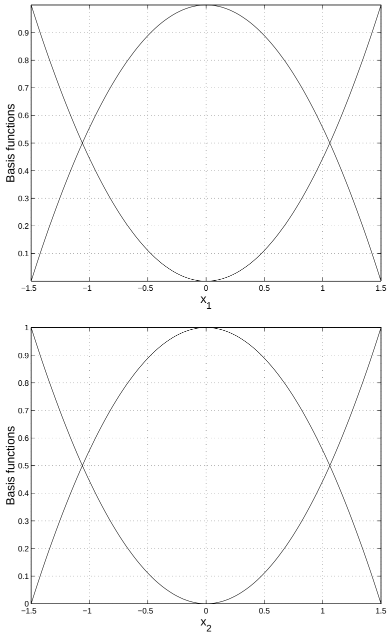


Figure 3: The basis functions obtained by TP model transformation; 2 functions in each dimension assure exact approximation

Here, superscript "$a$" denotes that this result was obtained via analytic derivations. For further details see [19].

The basis functions of the model obtained by TP model transformation are depicted on Figure 3. These functions were calculated with the "close-to-localized" options of TP model transformation (see (6)). Note that the functions coincide with the ones obtained analytically [19].

## 4.2 The mass–spring–damper system with modified non-linear term

Let us consider the dynamic system (22) when an extra term $f(x,\dot{x})\dot{x}$ is added to the non-linear term $g(x,\dot{x})$ as

$$g(x,\dot{x}) = d(c_1 x + f(x,\dot{x})\dot{x} + c_2\dot{x}^3)$$

where

$$f(x_1,x_2) = \begin{cases} 0.01 & \text{if } x_2 \geq x_1 \\ 0.01 + (x_1 - x_2) & \text{if } x_2 < x_1 \end{cases}$$

is a continuous function constructed based upon the $\omega$ in Lemma 1 (see also Figure 4).

The parameters $c_i$ $(i = 0,\dots,5)$ are set as follows: $c_1 = 0.01, c_2 = 0, c_3 = 0.01, c_4 = 0, c_5 = 0$. All other parameters remain unchanged. Then from (22) we obtain

$$\ddot{x} = -f(x,\dot{x})\dot{x} - 0.02x + u$$

so the state-space model is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -f(x_1(t),x_2(t)) & -0.02 \\ 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \quad (26)$$

Let us make now a short detour to recall the modelling characteristics of TP model transformation in terms of accuracy and complexity. TP model transformation is a numerical method that has a maximum achievable accuracy for each setting. This accuracy depends on the number of the measurement points the modelled system (or the approximated function) is sampled at. As a consequence of the internal operation of the TP model transformation (see details in [1, 5, 4, 3, 21]) the number of basis functions is limited by the number of measurement points in each variable. In such favorable cases like the previous example of Section 4.1 the number of required basis functions does not increase with the number of measurement points, or in other words, the "rank" of the system modelled (function approximated) is finite (in this case 2). However, when TP model transformation is applied to state-space model (26) the necessary number of basis functions tends to infinity with the increase of the number of measurement points, i.e. there is no limit of the "rank" of the system model (26). Figure 5 shows when $2\times2$ basis functions are used on the dimensions of the state vector. The maximum error of the model is 7.8. In order to improve the approximation accuracy we increase the number of the basis functions, see Figure 6. The maximum error is decreased to 4.0425. Along in the same line we can generate basis as depicted on Figure 7 that yields error 2.6152. Figure 8 presents the case when $50$ basis functions are used for the approximating TP model form. These basis functions were obtained without the "close-to-localized" option of the TP model transformation, because this option would explode the computational need of transformation, since the determination of a tight convex hull consisting of $N$ point of an $N$ dimensional point set is an NP hard problem.

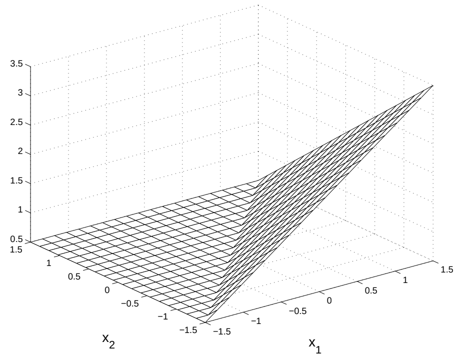


Figure 4: The graph of function $f(x_1,x_2)$ over the region $[-1.5,1.5]\times[-1.5,1.5]$

Table 1: Maximum error vs. number of basis functions in each dimension

| number of basis functions | maximum error | number of LMIs |
|---|---|---|
| $2\times2$ | 7.8 | 11 |
| $3\times2$ | 4.0425 | 22 |
| $3\times3$ | 2.6152 | 46 |
| ⋮ | ⋮ | ⋮ |
| $20\times20$ | 0.02838 | 80 201 |
| $30\times30$ | 0.01476 | 405 451 |
| $40\times40$ | 0.00895 | 1 280 801 |
| $50\times50$ | 0.00555 | 3 126 251 |

Table 1 shows the maximum error of the approximation when the function was sampled on a $100 \times 100$ equidistant grid on the parameter space. One can observe that the maximum error decreases. However, the table shows obviously, that we can apply trade-off consideration between accuracy and the number of basis. Such problems raise severe difficulties when to be solved by analytical derivations.

## 4.3 Control of the modified mass–spring–damper system

In this subsection we apply one of the simplest LMI theorem under the PDC framework to the modified mass–spring–damper system to yield a controller capable of ensuring global and asymptotic stabilization. Before dealing with LMI theorems, we introduce a simple indexing technique, in order to have direct link between TP model form and the typical LMI notations [1, 5, 4]):

(Index transformation) Let

$$\mathbf{S}_r = \begin{pmatrix} \mathbf{A}_r & \mathbf{B}_r \\ \mathbf{C}_r & \mathbf{D}_r \end{pmatrix} = \mathbf{S}_{i_1,i_2,..,i_N},$$

where $r = ordering(i_1, i_2, \ldots, i_N)$ ($r = 1, \ldots, R = \prod_n I_n$). The function "ordering" results in the linear index equivalent of an $N$ dimensional array's index $i_1, i_2, \ldots, i_N$, when the size of the array is $I_1 \times I_2 \times \cdots \times I_N$. Let the basis functions be defined according to the sequence of $r$:

$$w_r(\mathbf{p}(t)) = \prod_n w_{n,i_n}(p_n(t)).$$

The controller design can be derived from the Lyapunov stability theorems for global and asymptotic stability as shown in [17, 20]:

**Theorem 2** *(*Global and asymptotic stabilization of the convex TP model (4)) *Assume a given state-space model in TP form (4) with conditions (5).*

*Find* $\mathbf{X} > 0$ *and* $\mathbf{M}_r$ *satisfying equ.*

$$-\mathbf{X}\mathbf{A}_r^T - \mathbf{A}_r\mathbf{X} + \mathbf{M}_r^T\mathbf{B}_r^T + \mathbf{B}_r\mathbf{M}_r > 0 \qquad (27)$$

*for all* $r$ *and*

$$-\mathbf{X}\mathbf{A}_r^T - \mathbf{A}_r\mathbf{X} - \mathbf{X}\mathbf{A}_s^T - \mathbf{A}_s\mathbf{X} + \qquad (28)$$
$$+\mathbf{M}_s^T\mathbf{B}_r^T + \mathbf{B}_r\mathbf{M}_s + \mathbf{M}_r^T\mathbf{B}_s^T + \mathbf{B}_s\mathbf{M}_r \geq 0.$$

*for* $r < s \leq R$, *except the pairs* $(r, s)$ *such that* $w_r(\mathbf{p}(t))w_s(\mathbf{p}(t)) = 0, \forall \mathbf{p}(t)$.

Since the above conditions (27) and (28) are LMI's with respect to variables $\mathbf{X}$ and $\mathbf{M}_r$, we can find a positive definite matrix $\mathbf{X}$ and a matrix $\mathbf{M}_r$ or determine that no such matrices exist. This is a convex feasibility problem. Numerically, this problem can be solved very efficiently by means of the most powerful tools available in the mathematical programming literature e.g. MATLAB-LMI toolbox [8]. The feedback gains can be obtained from the solutions $\mathbf{X}$ and $\mathbf{M}_r$ as

$$\mathbf{K}_r = \mathbf{M}_r\mathbf{X}^{-1} \qquad (29)$$

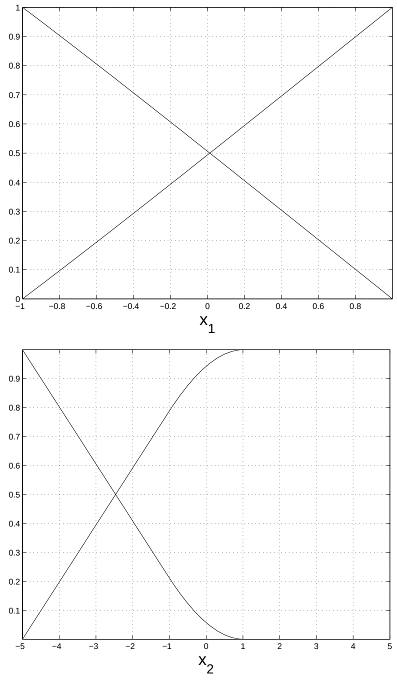


Figure 5: 2 basis functions in each dimension when (26) is approximated by TP model form

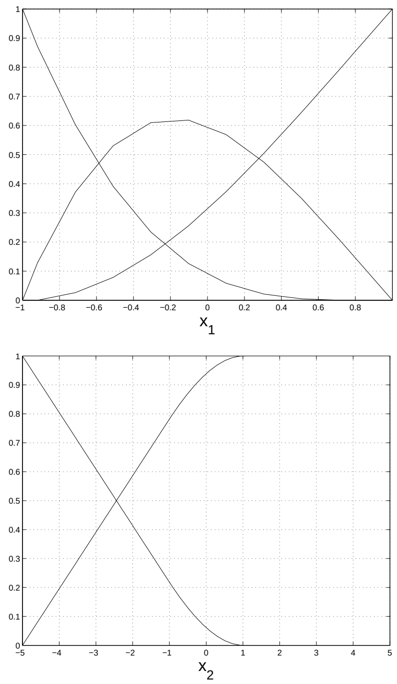


Figure 6: $3 \times 2$ basis functions when (26) is approximated by TP model form

Then, with the help of $r = ordering(i_1, i_2, \ldots, i_N)$ one can define feedbacks $\mathbf{K}_{i_1,i_2,\ldots,i_N}$ from $\mathbf{K}_r$ obtained in (29) and store into tensor $\mathcal{K}$. The control value is computed as:

$$\mathbf{u}(t) = -\left(\mathcal{K} \overset{N}{\underset{n=1}{\otimes}} \mathbf{w}_n(p_n(t))\right)\mathbf{x}(t), \qquad (30)$$

where the basis function $\mathbf{w}_n(p_n(t))$ are from (4). Figure 9 shows the control results of three controllers. Controllers 1, 2 and 3 are respectively determined by the basis functions

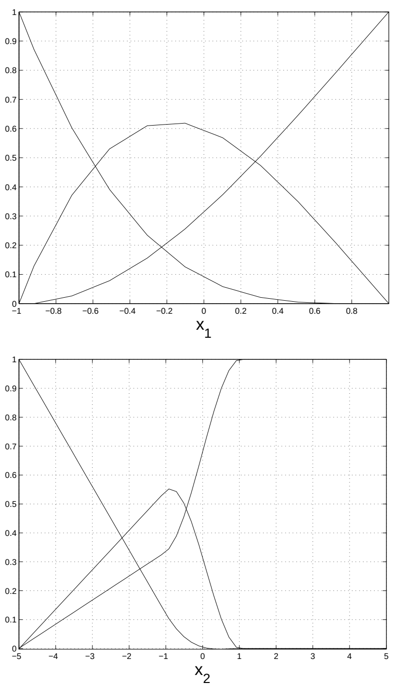


Figure 7: 2 basis functions in each dimension when (26) is approximated by TP model form

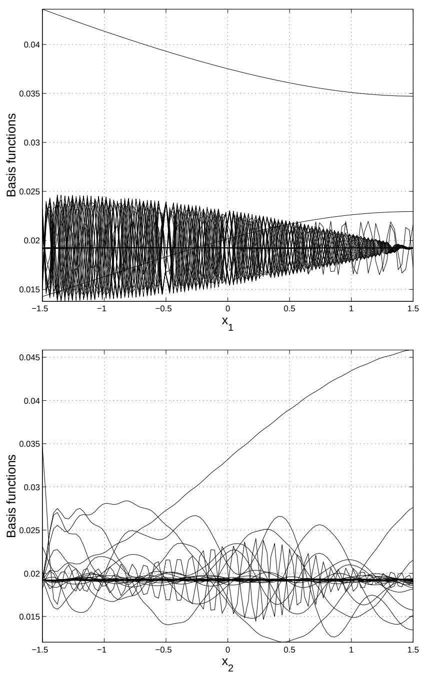


Figure 8: $50$ basis functions in each dimension when (26) is approximated by TP model form

depicted on Figures 5, 6 and 7. We can observe that the values of Controller 3 are significantly smaller than that of Controller 1. We can also observe that Controller 3 is faster than Controller 1 and 2. In this regard we can conclude that the Controller 3 exhibits better performance than Controllers 1 and 2. The reason is that Controller 3 is designed by TP model whose accuracy is significantly better than the accuracy of TP models applied in the cases of Controllers 1 and 2. One may apply the above LMI technique to the TP model

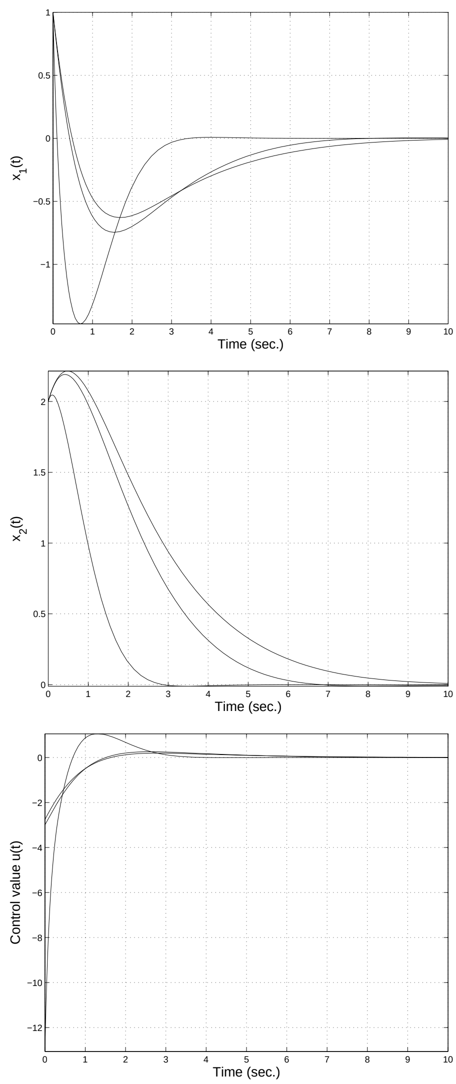


Figure 9: Control results of Controller 1, 2 and 3.

consisting of $50 \times 50$ basis functions. An $n \times m$ basis system yields $R = nm$ components in the TP model. LMIs in Theorem 2 show that employing $R$ components in the TP model results in $R + 1 + \sum_{r=1}^{R-1} r$ LMI terms. Thus, $50 \times 50$ basis functions require the solution of 3 126 251 LMIs. This is almost impossible using MATLAB on a regular PC. However, if we apply $3 \times 3$ basis functions the number of LMIs is only 46 that can be evaluated online even in real-time applications. In this case, however we should keep in mind that the controlled real system may exhibit different control performance from the controlled TP model. Table 1 shows the number of LMI terms to be evaluated in respect of the number of basis functions. Table 1 also helps in finding the trade-off between the calculation complexity of the control design and the accuracy of the controller. Note again that the computational requirement of LMI solvers exponentially increases with the increase of LMI terms.

# 5 Conclusion

This paper investigated the approximation capability of TP model forms. TP model forms can be obtained by TP model approximation from dynamic system models, and are usually used in combination with LMI methods within PDC framework for system control. We showed that the practically important subset of TP model forms that uses bounded number of components has limited approximation capability: the set of such TP model forms lie no-where dense in the space of continuous functions. As a consequence, there exist simple functions that cannot be approximated arbitrarily well regardless how many basis function is applied. Such cases necessitates the use of trade-off techniques between accuracy and complexity. The great advantage of TP model transformation over analytic transformation is that the former enables the use of trade-off methods, which is hardly possible in the analytical frameworks.

# References


[1] P. Baranyi. TP model transformation as a way to LMI based controller design. IEEE Trans. on Industrial Electronics, in press, 2004.

[2] P. Baranyi and R.J. Patton. A numerical control design method for prototypical aeroelastic wing section with structural non-linearity. In *European Control Conference (ECC'03)*, University of Cambridge, UK, 2003.

[3] P. Baranyi, D. Tikk, R. J. Patton, and P. Michelberger. Automatic TS-model representation of analytic differential equations for Tanaka's LMI based control design. In *Proc. of the Int. Conf. on Systems, Man and Cybernetics (SMC 2002)*, pages 167–173, Hammamet, Tunisia, October 6–9, 2002.

[4] P. Baranyi, D. Tikk, Y. Yam, and R. J. Patton. From differential equations to PDC controller design via numerical transformation. *Computers in Industry*, 51(3):281–297, 2003.

[5] P. Baranyi, Y. Yam, D. Tikk, and R. J. Patton. Trade-off between approximation accuracy and complexity: TS controller design via HOSVD based complexity minimization. In J. Casillas, O. Cordón, F. Herrera, and L. Magdalena, editors, *Interpretability Issues in Fuzzy Modeling*, number 128 in Studies in Fuzziness and Soft Computing, pages 249–277. Springer-Verlag, Heidelberg, 2003.

[6] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM, Philadelphia PA, 1994.

[7] M. Chilali and P. Gahinet. $H_\infty$ design with pole placement constrains: an LMI approach. *IEEE Transaction on Automatic Control*, 41(3):358–367, 1996.

[8] P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali. *The* LMI *Control Toolbox*. The Mathworks, Inc., 1995.

[9] J. Gray. The Hilbert problems 1900–2000. *Newsletter*, 36:10–13, 2000. http://www.mathematik.uni-bielefeld.de/∼kersten/hilbert/gray.html.

[10] G. Hancke and Á. Szeghegyi. Application study of the TP model transformation in the control of an inverted pendulum. In *International Conference on Computational Cybernetics*, pages 29–31, Siófok, Hungary, 2003.

[11] G. Hancke and Á. Szeghegyi. Nonlinear control via TP model transformation: the TORA system example. In *2nd Slovakian-Hungarian Joint Symposium on Applied Machine Intelligence (SAMI 2004)*, pages 333–340, Herlany, Slovakia, 2004.

[12] D. Hilbert. Mathematische Probleme. In *2nd International Congress of Mathematician*, Paris, France, 1900. http://www.mathematik.uni-bielefeld.de/∼kersten/hilbert/rede.html.

[13] I. Kaplansky. *Hilbert's Problems*. University of Chicago, Chicago, 1977.

[14] J. Ko, A. J. Kurdila, and T. W. Strganac. Nonlinear control of a ptototypical wing section with torsional nonlinearity. *J. of Guidance, Control, and Dynamics*, 20(6):1181–1189, November–December 1997.

[15] A. N. Kolmogorov. On the representation of continuous functions of many variables by superpositions of continuous functions of one variable and addition. *Dokl. Akad. SSSR*, 114:953–956, 1957. (In Russian).

[16] B. Moser. Sugeno controllers with a bounded number of rules are nowhere dense. *Fuzzy Sets and Systems*, 104(2):269–277, 1999.

[17] C. W. Scherer and S. Weiland. Linear matrix inequalities in control, 2000. DISC course lecture notes. http://www.cs.ele.tue.nl/sweiland/lmi.htm; download: http://www.cs.ele.tue.nl/SWeiland/lmid.pdf.

[18] M. Sugeno. An introductory survey of fuzzy control. *Information Science*, 36:59–83, 1985.

[19] K. Tanaka, T. Ikeda, and H. O. Wang. Robust stabilization of a class of uncertain nonlinear systems via fuzzy control: Quadratic stabilizability, $H_\infty$ control theory, and Linear Matrix Inequalities. *IEEE Trans. Fuzzy Systems*, 4(1):1–13, 1996.

[20] K. Tanaka and H. O. Wang. *Fuzzy Control Systems Design and Analysis — A Linear Matrix Inequality Approach*. John Wiley and Sons, Inc., New York — Chichester, 2001.

[21] D. Tikk, P. Baranyi, R. J. Patton, I. Rudas, and J. K. Tar. Design methodology of tensor product based control models via HOSVD and LMIs. In *Proc. of the IEEE International Conference on Industrial Technology (ICIT 2002)*, volume II, pages 1290–1295, Bangkok, Thailand, 2002.

[22] H. O. Wang, K. Tanaka, and M. F. P. Griffin. An approach to fuzzy control of non-linear systems: Stability and design issues. *IEEE Trans. Fuzzy Systems*, 4(1):14–23, 1996.