Mirror Symmetry is T-Duality

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Abstract

It is argued that every Calabi-Yau manifold X with a mirror Y admits a family of supersymmetric toroidal 3-cycles. Moreover the moduli space of such cycles together with their flat connections is precisely the space Y. The mirror transformation is equivalent to T-duality on the 3-cycles. The geometry of moduli space is addressed in a general framework. Several examples are discussed.

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1. Introduction

The discovery of mirror symmetry in string theory [1] has led to a number of mathematical surprises. Most investigations have focused on the implications of mirror symmetry of the geometry of Calabi-Yau moduli spaces. In this paper we shall consider the implications of mirror symmetry of the spectrum of BPS soliton states, which are associated to minimal cycles in the Calabi-Yau. New surprises will be found.

The basic idea we will investigate is briefly as follows. Consider IIA string theory compactified on a large Calabi-Yau space X. In four dimensions there are BPS states arising from the reduction of the ten-dimensional 0-brane. The moduli space of this 0-brane is X itself. In four dimensions the 0-brane can be described by a supersymmetric worldbrane sigma model with target X. The BPS states are the cohomology classes on X, and interactions will involve other invariants associated to X.

Quantum mirror symmetry[1] implies that an identical theory is obtained by compactifying the IIB theory on the mirror Y of X. In this formulation of the theory, all BPS states arise from supersymmetric 3-branes wrapping 3-cycles in Y. (The condition for supersymmetry is [4] that the 3-cycle is a special lagrangian submanifold [6] and the $U(1)$ connection on the 3-cycle is flat.) Hence there must be such a 3-brane in Y whose moduli space is X.[2] The 3-brane moduli space arises both from deformations of the 3-cycle within Y as well as the flat $U(1)$ connection on the 3-cycle.[3] Both of these are generated by harmonic 1-forms on the 3-cycle [7], and the real dimension of the moduli space is accordingly $2b_1$. Since this moduli space is X this 3-brane must have $b_1 = 3$ in order for the dimensions to match.[4] Now the moduli space of the flat connections on a 3-cycle with $b_1 = 3$ at a

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[1] This goes beyond the usual assertion that the conformal field theories are equivalent, and requires that the full quantum string theories are equivalent [2][3][4][5].

[2] Equality of the BPS spectrum only implies that the cohomologies are the same, but equality of the full theory, as well as their perturbation expansions, with all interactions will require that the spaces are actually the same. For example if X is large we can localize the 0-brane near a point in X at a small cost in energy. For very large X the space of such configurations approaches X. Such strong relations are not possible *e.g.* in the context of heterotic-IIA duality because the perturbation expansions are not directly related.

[3] As discussed in section two, a careful definition of the geometry involves open string instantons.

[4] It should also have no self-intersections because massless hypermultiplets at the intersection point have no apparent analog on the 0-brane side.

fixed location in in Y is a three-torus which, as we shall see in the following, is itself a supersymmetric 3-cycle in X. Hence this construction describes X as a three-parameter family of supersymmetric three-tori. We will refer to this as a supersymmetric T^3 fibration of X, which in general has singular fibers (whose nature is not yet well-understood, but is constrained by supersymmetry). Consideration of IIB rather than IIA on X similarly yields a description of Y as a family of supersymmetric three-tori. Hence the 3-cycles with $b_1 = 3$ must be three-tori.

To summarize so far, mirror symmetry of the BPS states implies that every Calabi-Yau X which has a mirror has a supersymmetric T^3 fibration.[5] The moduli space of the supersymmetric 3-tori together with their flat connections is then the mirror space Y. Note that this is an intrinsic formulation of the mirror space.

Now consider the action of T-duality on the supersymmetric T^3 fibers. A 0-brane sitting on the fiber will turn into a 3-brane, while a 3-brane will turn into a 0-brane. T-duality does not change the moduli space of the D-brane so this is the same 0-brane - 3-brane pair discussed above. We conclude that mirror symmetry is nothing but T-duality on the T^3 fibers!

We will provide a check on this last statement in a certain limit in section two by an explicit local computation of both the T-dual and moduli space geometries. The equality of these two geometries will be seen to follow from the fact that supersymmetric 3-cycles are volume-minimizing. We also discuss the interesting role of open-string disc instantons in correcting the moduli space geometry.

The preceding arguments *assume* quantum mirror symmetry. It is important to understand how much of this structure can be directly derived without this assumption. As a step in this direction in section three we show directly that the moduli space is Kahler and that it is endowed with a holomorphic b_1 form, in agreement with conclusions following from mirror symmetry. Section four contains some examples. Concluding remarks, including a prescription for constructing M-theory duals for a large class of $N = 1$ heterotic string compactifications, are in section five.

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[5] This is reminiscent of the situation found for other types of dualities. For example it is suspected that a type II compactification has a dual heterotic representation if and only if the Calabi-Yau admits a $K3$ fibration [8][9].

2. Mirror Symmetry and T-Duality

Consider a family of toroidal, supersymmetric 3-branes L in a Calabi-Yau space X endowed with metric g_{MN} and 2-form B_{MN}. These are defined by the map

$$X^M(\sigma^i, t^a) \tag{2.1}$$

where σ^i, $i = 1, 2, 3$ is a periodic spatial coordinate on L and t^a, $a = 1, 2, 3$ is a coordinate on the moduli space of supersymmetric maps. There is also a $U(1)$ connection A on the 3-brane:

$$A = a_i d\sigma^i \tag{2.2}$$

Supersymmetry requires [4] that (i) the connection A is flat (ii) the pullback of the Kahler form on to the 3-cycle vanishes (iii) the pullback of the holomorphic 3-form on to the 3-cycle is a constant times the volume element. These last two conditions imply that the 3-cycle is a special Lagrangian submanifold [6](cf. section four also).

We are interested in the full moduli space $\mathcal{M}$, parametrized by (a, t), of supersymmetric 3-branes with flat connections. We will find it convenient to choose local coordinates

$$X^i = \sigma^i, \quad X^{a+3} = t^a, \quad a = 1, 2, 3. \tag{2.3}$$

The line element on X is then

$$ds_X^2 = g_{ab}dt^a dt^b + 2g_{ai}dt^a d\sigma^i + g_{ij}d\sigma^i d\sigma^j. \tag{2.4}$$

(2.4) locally describes a T^3 fibration of X. The two form is

$$B = \frac{1}{2}B_{ab}dt^a \wedge dt^b + B_{ai}dt^a \wedge d\sigma^i + \frac{1}{2}B_{ij}d\sigma^i \wedge d\sigma^j. \tag{2.5}$$

In general there will of course be singular points at which the fibers degenerate.

We wish to compute the metric on $\mathcal{M}$. We first consider the tree-level contribution. This is derived from the Born-Infeld action for the 3-brane [10,11]

$$S_3 = -\int d^4\sigma \sqrt{-det(E - F)}. \tag{2.6}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$
$$E_{\mu\nu} = E_{MN}\partial_\mu X^M \partial_\nu X^N,$$
$$E_{MN} = g_{MN} + B_{MN},$$
$$\mu, \nu, \alpha, \beta = 0, 1, 2, 3, \tag{2.7}$$

and we have fixed the string coupling.

To describe slowly time-varying configurations we insert the ansatze

$$
\begin{aligned}
\dot{X}^{a+3} &= \dot{t}^a, \\
\dot{X}^i &= -E^{ji}E_{aj}\dot{t}^a, \\
F_{0i} &= \dot{a}_i,
\end{aligned}
\tag{2.8}
$$

where the dot denotes differentiation with respect to time σ^0, and $E^{ij}E_{jk} = \delta^i{}_k$.[6] The $\dot{X}^i$ term is arranged so that the motion of the 3-brane is normal to itself, with a B-dependent rotation in L. Expanding (2.6) to quadratic order in time derivatives, the leading kinetic part of the action is then

$$
\begin{aligned}
S_3 = \int d^4\sigma\sqrt{-E}[&(E_{ab} - E_{ai}E^{ij}E_{jb})\dot{t}^a\dot{t}^b \\
&+ (E^{ij}E_{ja} - E_{aj}E^{ji})\dot{a}_i\dot{t}^a + E^{ij}\dot{a}_i\dot{a}_j)] + ...
\end{aligned}
\tag{2.9}
$$

Hence the tree-level line element on $\mathcal{M}$ is

$$
ds^2_{\mathcal{M}} = (E_{ab} - E_{ai}E^{ij}E_{jb})dt^a dt^b + (E^{ij}E_{ja} - E_{aj}E^{ji})da_i dt^a + E^{ij}da_i da_j. \tag{2.10}
$$

Mirror symmetry implies that the full metric on $\mathcal{M}$ equals the metric on the mirror Y of X. The leading result (2.10) can not be the full story in general. The metric (2.10) has a $U(1)^3$ isometry generated by shifts in a_i. This is certainly not present for a generic Calabi-Yau, so (2.10) can not in general be the exact metric on the mirror. However there are instanton corrections which can resolve this apparent discrepancy. The motion of the 3-brane is generated by open strings with Dirichlet boundary conditions on the 3-brane. The metric on $\mathcal{M}$ is defined by a two point function on a disc whose boundary is the 3-brane. This receives instanton corrections from minimal area discs whose boundaries wrap a nontrivial 1-cycle in L. (If Y is simply connected all such cycles bound discs.) Generically such corrections become exponentially small at large radius. However even at large radius the corrections are non-negligible near the singularities of the fibration where the T^3 degenerates. After including these corrections mirror symmetry predicts that (2.10) will agree with the metric on Y. This is quite analogous to the usual mirror story where

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[6] Note that E^{ij} is the inverse of the tensor E_{ij} on L. So, for example, E^{ia} makes no sense, and $E^{ij}E_{ja} \neq 0$ in general.

the flat metric on the complexified Kahler cone agrees with the metric on the space of complex structures only after including world sheet instanton corrections.

Instanton corrections to the moduli space geometry of 3-cycles in X are suppressed in the limit of large radius away from the singular fibers. The moduli space geometry is then the large complex structure limit of Y. In this limit the metric takes the symmetric form (2.10) and hence admits a local $U(1)^3$ action away from the singular fibers. We refer to this as a "semi-flat" Calabi-Yau metric. A lower-dimensional example of such a metric is found in equation 4.1 of [12], where stringy cosmic strings are described as a flat T^2 fibered over R^2 with a singularity at the origin.[7] A semi-flat $K3$ metric can be obtained by patching together 24 such singular cosmic strings. A 6-dimensional example can be obtained by simply tensoring this $K3$ with T^2. The semi-flat metric on the quintic will be discussed in section four.

Because there is locally a $U(1)^3$ symmetry, an equivalent string theory can be explicitly constructed locally (away from the singularities) by T-duality on the flat T^3 fibers. This duality interchanges IIA and IIB, and maps IIA 0-branes and supersymmetric IIB 3-branes tangent to the fibers into one another. Hence this T-duality is the same as mirror symmetry.

It is illuminating to check in detail by a local calculation that the T-dual and moduli space geometries are indeed the same. The string sigma model action on a flat worldsheet is

$$S_\sigma = \frac{1}{2\pi} \int (E_{ab}\partial t^a \bar\partial t^b + E_{ia}\partial \sigma^i \bar\partial t^a + E_{ai}\partial t^a \bar\partial \sigma^i + E_{ij}\partial \sigma^i \bar\partial \sigma^j). \qquad (2.11)$$

The three symmetries of shifts of the σ^i imply that the three currents

$$j_i = E_{ji}\partial \sigma^j + E_{ai}\partial t^a \qquad (2.12)$$

are conserved. We may therefore define three real worldsheet scalars a_i by

$$j_i = i\partial a_i. \qquad (2.13)$$

Eliminating the scalars σ^i in favor of a_i one finds an equivalent T-dual sigma model. The T-dual metric is precisely (2.10), see for example [13] where various subtleties in the transformation are also discussed.

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[7] The resolution described in [12] of those singularities can be understood as arising from disc instanton corrections.

In general the T-dual of a Ricci-flat metric obeys the low energy string equations of motion. However it is not necessarily itself Ricci-flat because T-duality generically generates an axion field strength $H = dB$ and a dilaton field Φ with a nonvanishing gradient, both of which act as a source for the Ricci tensor. For simplicity consider in the following the special case in which B is zero before T-dualizing. The new Φ is then

$$\Phi = \ln \det g_{ij}. \tag{2.14}$$

The new H is

$$H^i{}_{ab} = g^{ij}{}_{,a} g_{jb} + g^{ij} g_{jb,a} - g^{ij}{}_{,b} g_{ja} - g^{ij} g_{ja,b}. \tag{2.15}$$

The metric g on X in the coordinates (2.4) obeys various differential identities by virtue of the fact that all three-tori at constant t^a minimize the volume [4]. These constraints on g imply that H vanishes and Φ is constant after T-duality, as required.

To see this consider a small normal perturbation of the three-torus at $t = 0$ described by

$$X^i = \sigma^i - g^{ij}(0) g_{ja}(0) \Lambda^a(\sigma),$$
$$X^a = \Lambda^a(\sigma). \tag{2.16}$$

Through first order in Λ and first order in $\partial \Lambda$ the induced metric h is given by, in the semi-flat case where the metric is independent of σ,

$$h_{ij} = \big[g_{ij} + g_{ij,a} \Lambda^a + g_{ia,b} \Lambda^b \partial_j \Lambda^a + g_{ja,b} \Lambda^b \partial_i \Lambda^a$$
$$- g_{il,b} g^{lk} g_{ka} \Lambda^b \partial_j \Lambda^a - g_{jl,b} g^{lk} g_{ka} \Lambda^b \partial_i \Lambda^a \big]_{t=0}. \tag{2.17}$$

The change in the volume may be written, using (2.14) and (2.15) and integrating by parts

$$\delta V_3 = \int d^3\sigma \delta \sqrt{h}$$
$$= \frac{1}{2} \int d^3\sigma \sqrt{g} \big(\partial_a \Phi \Lambda^a + H^i{}_{ab} \Lambda^a \partial_i \Lambda^b \big). \tag{2.18}$$

The fact that the area does not change at first order in Λ implies that the dilaton is constant. The $\Lambda \partial \Lambda$ variation will be non-negative for all Λ if and only if $H = 0$. Hence H vanishes and the T-dual metric must be Ricci-flat.

At this point it is easy to see that, as mentioned in the introduction, the three-tori defined by $t = constant$ in the moduli space metric (2.10) are themselves supersymmetric 3-cycles in the mirror Y of X. One way of stating the condition for supersymmetry is that

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the T-dual geometry should have vanishing H and constant Φ. This is obviously the case since T-dualizing just returns us back to X.

Now consider perturbing the Calabi-Yau moduli so that the metric is no longer semi-flat. In that case the action of T-duality on the fibers still exists but is complicated because there are no isometries. The construction of the moduli space $\mathcal{M}$ of supersymmetric 3-branes is complicated by instantons. Nevertheless since they are both described as perturbations of the same theory the equivalence between T^3 T-duality and mirror symmetry should be valid for all Calabi-Yau geometries in a neighborhood of the semi-flat geometries. Furthermore since the supersymmetric 3-cycles correspond to minimally-charged BPS states we expect that they survive sufficiently small perturbations. In conclusion the relation between fiberwise T-duality and mirror symmetry is quite general and not restricted to the semi-flat limit.

3. D-brane moduli space

We now turn to some mathematical generalities of the above discussion. We will not be able to prove all the statements of the previous section regarding D-brane moduli space, but we hope to provide a framework for future work on these matters. We will show that with a certain natural metric, the D-brane moduli space has a Kähler structure. In addition, there are several natural forms on this moduli space which suggest that its interpretation as a Calabi-Yau for toroidal D-branes makes good sense. In fact, a "dual" geometry emerges which corresponds to the mirror symmetry interpretation of the previous section.

Recall that we are interested in a special Lagrangian submanifold, L, of a Calabi-Yau manifold, M. Let $f : L \to M$ be the map of the imbedding (generally, f will only need to be an immersion). The special Lagrangian condition is equivalent ([14], Corollary 7.39) to the statement that

$$f^*\omega = 0 \qquad \text{and} \qquad f^*\kappa = 0,$$

where ω is the Kähler form and κ is the imaginary part of the Calabi-Yau form on M. The asterisk denotes the pull-back operation. We will study the space of special Lagrangian immersions f (which arise as deformations of a fixed immersion) and call this space $\mathcal{M}_{sl}$.

We use the following conventions:

- $\alpha, \beta, \gamma, ...$ are indices for M.

- $y^\alpha, y^\beta, ...$ are coordinates on M.[8]
- $i, j, k, ...$ are indices for L.
- $x^i, x^j, ...$ are coordinates on L.
- $\mathcal{M}_{sl}$ is the moduli space of special Lagrangian submanifolds.
- $a, b, c, ...$ are indices on $\mathcal{M}_{sl}$
- $t_a, t_b, ...$ are coordinates on $\mathcal{M}_{sl}$.
- $g_{\alpha\beta}, g_{ij}, g_{ab}$ are metrics on $M, L, \mathcal{M}_{sl}$.[9] Without indices, we write $\overline{g}, g, g_{SL}$.
- $J^\alpha{}_\beta$ is the complex structure on M. $J^\alpha{}_\gamma J^\gamma{}_\beta = -\delta^\alpha{}_\beta$.
- $\omega_{\alpha\beta} = g_{\alpha\gamma} J^\gamma{}_\beta$.
- $f^\alpha{}_i = \frac{\partial f^\alpha}{\partial x^i}$, etc.
- All indices are raised or lowered by the appropriate metric.
- For a Lagrangian submanifold, a tangent vector $w^i e_i$ (where $e_i = \frac{\partial}{\partial x^i}$) determines

a normal vector $\underline{w} = w^i (J \cdot f_* e_i)$, and its associated second fundamental form $(h^{\underline{w}})_{jk}$ is given by $w^i h_{ijk}$, with h a symmetric tensor defined by $h_{ijk} = -\langle \overline{\nabla}_{f_* e_i} f_* e_j, J \cdot f_* e_k \rangle$ [15].

As yet, we have not defined the coordinates on moduli space, or even shown that it exists as a manifold.[10] We do so now, defining coordinates analogous to Riemann normal coordinates, which give a coordinatization of a Riemannian space by the tangent space of a point. Our point will be the initial special Lagrangian submanifold, L, with the map f.

The tangent space to $\mathcal{M}_{sl}$ at L can be found by considering an arbitrary one-parameter deformation $f(t)$ of $f = f(0)$. In order to remove ambiguity, we require throughout that $\dot{f}$, a vector field on $f(L)$ in M, be a normal vector field. This can be achieved through a t-dependent diffeomorphism of L. Now $\dot{f} = \frac{df^\alpha}{dt} \frac{\partial}{\partial y^\alpha}$ is orthogonal to $f(L)$. Therefore, $J \cdot \dot{f}$ is a tangent vector (by the Lagrangian condition), which we can convert to a 1-form by the metric $\overline{g}$ on M. This combination gives us a 1-form $\frac{df^\beta}{dt} \omega_{\beta\alpha} dy^\alpha$. We define θ to be the pull-back of this 1-form to L; $\theta_i = \dot{f}^\beta \omega_{\beta\alpha} f^\alpha{}_i$.[11]

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[8] We will not use complex coordinates for M; we will use real coordinates adapted to the complex structure, which will then be constant in these coordinates.

[9] This notational redundancy begs the question, "What is g_{12}?" We will be careful never to use numbers where our indices stand, and operate under the assumption that an additional identifier would be employed (e.g., $\overline{g}_{12}$ on M).

[10] This was shown on general grounds in [7], though without the introduction of coordinates, which we shall need for our geometric analysis.

[11] Note that the definition of θ is independent of the requirement that $\dot{f}$ be normal, for any tangent vector added to $\dot{f}$ adds nothing to θ, by the Lagrangian condition.

Proposition: $\frac{d}{dt}f^*\omega = d\theta$.

Proof: Commuting derivatives and using the closure of ω, one computes

$$
\begin{aligned}
\frac{d}{dt}f(t)^*\omega_{ij}(x) &= \frac{d}{dt}\left(f^\alpha{}_i f^\beta{}_j \omega_{\alpha\beta}(f(x))\right) \\
&= \partial_i(\dot f^\alpha)f^\beta{}_j\omega_{\alpha\beta} + f^\alpha{}_i\partial_j(\dot f^\beta)\omega_{\alpha\beta} + f^\alpha{}_i f^\beta{}_j \dot f^\gamma \partial_\gamma \omega_{\alpha\beta} \\
&= \partial_i(\dot f^\alpha)f^\beta{}_j\omega_{\alpha\beta} + f^\alpha{}_i\partial_j(\dot f^\beta)\omega_{\alpha\beta} + f^\alpha{}_i f^\beta{}_j \dot f^\gamma(-\partial_\alpha\omega_{\beta\gamma} - \partial_\beta\omega_{\gamma\alpha}) \\
&= \partial_i(\dot f^\alpha)f^\beta{}_j\omega_{\alpha\beta} + f^\alpha{}_i\partial_j(\dot f^\beta)\omega_{\alpha\beta} + \left(f^\beta{}_j\dot f^\gamma\partial_i\omega_{\gamma\beta} - f^\alpha{}_i\dot f^\gamma\partial_j\omega_{\gamma\alpha}\right) \\
&= \partial_i\left(\dot f^\alpha \omega_{\alpha\beta} f^\beta{}_j\right) - \partial_j\left(\dot f^\beta \omega_{\beta\alpha} f^\alpha{}_j\right) \\
&= (d\theta)_{ij}.
\end{aligned}
$$

Therefore, to preserve the Lagrangian condition (to first order), the normal vector from the deformation $f(t)$ must give rise to a 1-form which pulls back to a closed 1-form θ on L.

Likewise, we have the following proposition.

Proposition: $\frac{d}{dt}f^*\kappa = 0 \Rightarrow d^\dagger\theta = 0$.

The proof, similar to (though not quite as straightforward as) the above, may be found in [7]. Therefore, θ must be closed and co-closed, i.e. harmonic.

The tangent space to deformations of isomorphism classes of flat $U(1)$ bundles is simply described. If A represents a flat $U(1)$ connection on a bundle $E \to L$, then a deformation of the connection has the form $A + \theta$. Requiring this to be flat means that $d(A + \theta) = 0$ and hence $d\theta = 0$, since A is flat. Further, A (equivalently θ) is only defined up to gauge transformations $A \sim A + d\chi$. Therefore, the tangent space to the space of flat bundles is closed 1-forms modulo exact 1-forms, i.e. $H^1(L)$. A Riemannian metric on L – in our case the pull-back metric – provides an isomorphism $H(1) \cong \mathcal{H}^1(L)$. In fact the full space of flat $U(1)$ bundles is described by the set of monodromies, $\exp[2\pi ic_j]$, $j = 1...b_1(L)$, where b_1 is the first Betti number. This space is the torus T^{b_1}. The moduli space $\mathcal{M}$ is a fibration over the moduli space of special Lagrangian manifolds (continuously connected to $f(0)(L)$), with fiber T^{b_1}.

Therefore, the tangent space to $\mathcal{M}$ at L is $\mathcal{H}^1(L) \oplus \mathcal{H}^1(L)$, and there is a natural metric on $\mathcal{H}^1$, which defines for us a metric on $\mathcal{M}_{sl}$ and a block diagonal metric on $\mathcal{M}$. Let θ^a and θ^b be two harmonic 1-forms on L, with its induced metric g_{ij}. Then

$$
g_\mathcal{M} = \begin{pmatrix} g_{SL} & 0 \\ 0 & g_{SL} \end{pmatrix},
$$

where

$$(g_{SL})_{ab} = \int_L \theta_i^a \theta_j^b g^{ij} \sqrt{g} dx.$$

There is also a natural almost complex structure, which takes the form

$$\mathcal{J} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}.$$

The coefficients of this matrix might not be constant in a coordinate frame for $\mathcal{M}$, so we have not yet shown that it is a complex structure, or that the metric is Kähler with respect to it. We shall do this presently, after defining a coordinate frame.

Knowing that the tangent space to $\mathcal{M}$ at L is $\mathcal{H}^1 \oplus \mathcal{H}^1$ is not enough for us to be able to set up coordinates for $\mathcal{M}$. To do this, we need something analogous to the exponential map for Riemann normal coordinates. In this case, one needs, given a tangent direction, a uniquely defined one-parameter family of flows of L in M which are special Lagrangian, i.e. for which $f(t)^*\omega = f(t)^*\kappa = 0$ for all t. Consider a harmonic 1-form θ on L, with w its corresponding vector, $w^i = g^{ij}\theta_j$. w obeys $\nabla_i w^i = 0$. By continually pushing forward w and rotating it by the complex structure, we can attempt to define a flow through the equation

$$\dot{f}(t) = J \cdot f(t)_* w.$$

However, there is no guarantee that the manifolds $f(t)(L)$ will continue to be special Lagrangian manifolds at all times. The problem is that $g = f^* \overline{g}$ evolves in t, and so θ may not remain harmonic. To remedy this, we *require* θ to be harmonic by including a time-dependence explicitly. We do so in such a way that $[\theta]$, the cohomology class defined by θ, does not have a t-dependence. Therefore, we define

$$\widetilde{\theta}(t) = \theta - d\phi(t);$$

then the requirement that $\widetilde{\theta}$ be harmonic becomes $\nabla_i(g^{ij}\partial_j\phi) = \nabla_i w^i$ or

$$\triangle \phi = \nabla_i w^i = d^\dagger \theta.$$

Do not forget the implicit t-dependence of the operators! Note that at a given time t, the last equation determines ϕ up to an additive constant, which is irrelevant to the definition of $\widetilde{\theta}(t)$.

We can now try to define a one-parameter flow starting from L with initial normal vector $J \cdot f(0)_* w$ as the solution to the coupled equations

$$\frac{df^\alpha}{dt} = J^\alpha{}_\beta \frac{\partial f^\beta}{\partial x^i} g^{ij}(\theta_j - \partial_j \phi)$$
$$\triangle \phi = d^\dagger \theta, \tag{3.1}$$

where θ has no t-dependence. This is analogous to geodesic flow on Riemannian manifolds (though we don't claim that this flow is geodesic with respect to the metric on moduli space).

Proposition: This flow exists, at least infinitesimally, and is unique.

Proof: A proper demonstration of the above uses results we will derive in this section. Basically, one can recursively solve for every derivative of f with respect to t. At first order, the bottom equation specifies $\phi = \frac{1}{\triangle} d^\dagger \phi + constant$, which gives a unique expression for the top equation (the constant falls out). To find $\frac{d^2 f}{dt^2}$ one can differentiate the bottom equation to get $\triangle \dot\phi = d^\dagger \psi$, for some ψ (specifically, $\psi_j = 2(h\underline{w})_{jk}g^{kl}(\partial_l \phi - \theta_l)$), and so on for the higher derivatives. We have not determined the radius of convergence (possibly zero) for this expansion in t. For the remainder of this paper, we assume that this flow exists for finite time.

Let θ^a, $a = 1...b_1$ be a basis for $\mathcal{H}^\infty(\mathcal{L})$, w_a the corresponding vectors, and t_a the corresponding coordinates in moduli space. We call L_t the submanifold $f(1)(L)$, where the flow is defined by the 1-form $\theta = t_a \theta^a$. We can coordinatize the full moduli space $\mathcal{M}$ with t_a and s_a, $a = 1...b_1$, where the s_a simply tell us to use the connection $A + s_a \theta^a$ on L_t. Now while we know that θ does not change in cohomology during the flow it defines, other 1-forms might do so. As a result, the almost complex structure, $\mathcal{J}$, could pick up a coordinate dependence.[12] The calculation in the appendix will show that, to first order, this does not occur, and as a result the Nijenhuis tensor vanishes. This shows that $\mathcal{J}$ is indeed a complex structure on moduli space.

We turn now to the geometry of the flow. Let $f(0)$ be an imbedding and $f(t)$ a family of special Lagrangian immersions inducing Then we have the following result.

Proposition: $\frac{d}{dt_a} g_{ij} = 2h_{ijk}w_a^k$, which is twice $(h\underline{w})_{ij}$, the second fundamental form defined by the normal vector $J \cdot w$.

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[12] If $[\theta^b(t)]$ is not constant, write $[\theta^b(t)] = y^b{}_c(t)\theta^c$. Then $\mathcal{J} = \begin{pmatrix} 0 & y^{-1} \\ -y & 0 \end{pmatrix}$.

Here

$$h_{ijk} = -\langle \overline{\nabla}_{f_* e_i} f_* e_j, J \cdot f_* e_k \rangle,$$

and $\overline{\nabla}$ represents the covariant derivative on M. It is not hard to show that by the Lagrangian property h is a symmetric three-tensor on L. These and other properties of Lagrangian flow are investigated in [15]. The proof of the proposition is straightforward – one simply puts $g_{ij} = f^\alpha{}_i f^\beta{}_j g_{\alpha\beta}$ and differentiates, substituting (3.1) for time derivatives (note that $\frac{d}{dt_a} g_{\alpha\beta} = \dot{f}^\gamma \partial_\gamma g_{\alpha\beta}$).

As with Riemann normal coordinates, we have to express the vector field defined by θ^b as a function of the coordinate t (analogous to push-forward by the exponential map). As a result, the forms pick up a t-dependence. We hope to find that our flow gives the θ^a a t-dependence such that the cohomology class does not change as a function of time, but note that the form itself must change, since it must remain harmonic with respect to the changing metric. That is, we want that $\frac{d}{dt_a}[\theta^b] = 0$, where the brackets represent cohomology (the flows are already defined so that $\frac{d}{dt_a}[\theta^a] = 0$, no sum) – i.e., that $\frac{d}{dt_a}\theta^b = d\psi^{ab}$ for some function $\psi^{ab}(t; x)$. We cannot solve the flows for finite time, but we can compute $\frac{d}{dt_a}\theta^b|_{t=0}$ as follows. Consider a one-parameter family of flows $f_r(t)$ defined with initial vector $\theta^a + r\theta^b$. These define a two-dimensional cone in moduli space, and we can take the derivative in the b direction at time t_a by computing $\frac{1}{t_a}\frac{d}{dr}|_{r=0}$.[13] This gives a normal vector (it may not be normal, but the pull-back to a 1-form will be insensitive to tangent directions), which then defines a 1-form representing $\theta^b(t_a)$. Specifically,

$$\theta^b(t_a)_i = -\frac{1}{t_a} J^\alpha{}_\beta \left(\frac{df_r^\beta}{dr}\Big|_{r=0} \right) g_{\alpha\gamma} \frac{\partial f^\gamma}{\partial x^i}, \tag{3.2}$$

where the family of flows is defined by the equations

$$\frac{df^\alpha}{dt} = J^\alpha{}_\beta \frac{\partial f^\beta}{\partial x^i} g^{ij}(\theta_j^a + r\theta_j^b - \partial_j \phi)$$
$$\triangle \phi = d^\dagger(\theta^a + r\theta^b).$$

Do not forget the r and t dependence hidden in the second equation. Note, too, that θ^a and θ^b have no t- or r-dependence.

We can see that $\lim_{t_a \to 0}\theta^b(t_a) = \theta^b(0)$, as it must, by L'Hôpital's rule and some algebra, noting that $f(0)$ is independent of r.

---

[13] If we write t_a, we mean to set $\vec{t} = (0, ..., t_a, ...0)$, with lone nonvanishing ath component.

It remains to compute $\lim_{t_a \to 0} \frac{d\theta^b(t)}{dt_a}$. This computation, performed in the appendix, depends on using L'Hôpital's rule and applying the definition of the flow for each time derivative. After some algebra, one finds

$$\frac{d\theta^b(t_a)}{dt_a}\Big|_{t=0} = -\frac{1}{2}d\left(\frac{d^2\phi}{drdt}\right)_{r=t=0} \equiv d\psi^{ab}. \tag{3.3}$$

This shows, too, that

$$\begin{aligned}
\partial_a g_{bc} &= \frac{d}{dt_a}\int \theta^b(t)_i\theta^c(t)_j g^{ij}\sqrt{g}dx \\
&= \int\langle d\psi^{ab}, \theta^c\rangle + \int\langle\theta^b, d\psi^{ac}\rangle - 2\int h_{ijk}w_a^i w_b^j w_c^k \sqrt{g}dx \\
&= \partial_b g_{ac},
\end{aligned} \tag{3.4}$$

where we have used harmonicity and the symmetry of h_{ijk}, as well as the fact that the derivative of $\sqrt{g}$ is proportional to the mean curvature, which vanishes for minimal manifolds (h is traceless). The result (3.3) also shows that the derivatives of $\mathcal{J}$ are zero, which immediately implies that it is a complex structure.[14] In addition, (3.4) tells us that the Kähler form on moduli space, $\omega_{ab}^{\mathcal{M}} = g_{ac}\mathcal{J}^c_b$, is closed at L, which was an arbitrary point on $\mathcal{M}$.

On $\mathcal{M}_{sl}$, there exists a natural n-form (where $2n$ is the real dimension of M,) Θ, defined as follows [7]:

$$\Theta(\theta^{a_1}, ..., \theta^{a_n}) = \int_L \theta^{a_1} \wedge ... \wedge \theta^{a_n}.$$

Now since in our coordinates the θ^{a_i} change by exact terms, all derivatives of Θ vanish, and Θ is closed. Further, we can extend Θ to a form on $\mathcal{M}$ by defining,

$$\Theta^{\mathcal{M}} = \Theta_{a_1...a_n}dz^{a_1}...dz^{a_n},$$

where $dz^a = dt_a + ids_a$. The assertion that $d\Theta^{\mathcal{M}} = 0$ follows from the above, and fact that there is no s-dependence. We could have complex conjugated some of the dz's and obtained other forms. We don't know how many of these forms will survive the compactification procedure, but we point out that in the special case of toroidal submanifolds, $b_1 = n$ and so $\Theta^{\mathcal{M}}$ is a holomorphic b_1-form, which could be the Calabi-Yau form. This lends support, at least, to the conjectures of the previous section. Note, too, that one can also, given a

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[14] The Nijenhuis tensor, $\mathcal{N}^c_{ab} = \mathcal{J}^d_a(\partial_c\mathcal{J}^c_b - \partial_b\mathcal{J}^c_d) - (a \leftrightarrow b)$, clearly vanishes.

2-form potential $B_{\alpha\beta}$, define a 2-form on moduli space $B_{ab} = \int B^{ij} \theta_i^a \theta_j^b \sqrt{g}$, where B_{ij} are the components of f^*B, and coordinates are raised, as usual, by g^{ij}.

Of course, there is more about this moduli space to study. We need to determine its curvature. The variation of the metric means that our coordinates are not normal. A coordinate transformation mixing t and s is needed to make it so. Therefore, the fibration may not be holomorphic with respect to this metric. This makes some sense, as we don't expect, for example for K3, to find the fibers always perpendicular to the base. However, we do not yet know how this metric should get corrected, or what the proper compactification of moduli space looks like. We do not know how the Kähler class gets corrected, as a result. We hope more rigorous mathematical results will ensue.

4. Examples

4.1. K3

The only well-understood example of the preceding is K3 (or $K3 \times T^2$, if we want a three-fold). The supersymmetric cycles can be related, by a rotation of complex structures[15] to Riemann surfaces sitting in the K3. For genus one, these are tori, which are easily apparent if the K3 is written as an elliptic fibration: $\pi : K3 \to \mathbf{P}^1$. Each point p on the K3 determines a torus $\pi^{-1}(\pi(p))$, and exists as a point on that torus (which is its own Jacobian). Given a section, this uniquely determines a submanifold and a flat bundle on it. Thus K3 appears as the moduli space for submanifolds which are tori. This is appropriate, as K3 is its own mirror!

4.2. The Quintic

The example of the quintic has been analyzed in great detail [16]. It can be seen as follows that, at least for sufficiently large complex structure, there is a supersymmetric T^3 cycle in the homology class predicted by mirror symmetry. On the mirror quintic Q there is a a 0-brane, with moduli space equal to Q. There is also a 6-brane which wraps Q and has no moduli (note that there are no flat bundles since Q is simply connected).

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[15] The K3 has three complex structures, I, J, K, giving three 2-forms, $\omega_I, \omega_J, \omega_K$ which can be interchanged by SO(3) rotations. With complex structure J, we have $\omega = \omega_J$ and $\Omega_J = \omega_I + i\omega_K$, so we can relate the condition $f^*(\omega) = f^*(\text{Im}\Omega_J) = 0$ with complex structure J to $f^*(\Omega_K) = 0$, in complex structure K. These surfaces are then holomorphic submanifolds with respect to K.

Quantum mirror symmetry tells us that the corresponding 6- and 0-cycles should be mirror to two distinct 3-cycles on the quintic, $\widetilde{Q}$. The mirror transformations relating the even cohomology on Q to the odd cohomology on $\widetilde{Q}$ have been explicitly displayed in [16]. Using these transformations it was shown in [17] that the 6-cycle on Q is mirror to the 3-cycle on $\widetilde{Q}$ which degenerates at the conifold. Since the 0-cycle is Poincare dual to the 6-cycle (at large radius where instanton corrections can be ignored) it must be mirror to a 3-cycle Poincare dual to the one which degenerates at the conifold. Consider the 3-cycle defined in [16] by

$$|z_1| = a, \quad |z_2| = b, \quad |z_3| = c, \tag{4.1}$$

where a, b, c are real. Consider a patch in which we can fix $z_5 = -1$ and take z_4 to be a root of the polynomial equation for a quintic

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 = 1. \tag{4.2}$$

If a, b, c are sufficiently small the roots are non-degenerate and as the phases of z_1, z_2, z_3 vary a three-torus is swept out in $\widetilde{Q}$. Topologically (4.1) defines a T^3 "fibration" of $\widetilde{Q}$ with base parameterized by a, b, c. The fibers become singular at values of a, b, c for which the roots collide. It was shown in [16] that this 3-cycle is Poincare dual to the vanishing cycle at the conifold. Hence it is in the correct homology class to be mirror to the 0-brane on Q. To see that there is actually a supersymmetric cycle in this class consider the large complex structure limit in which a term $\lambda z_1 z_2 z_3 z_4 z_5$ is added and the coefficient λ taken to infinity. There is a branch on which we may take $z_5 = -1$ and z_4 near zero and the metric approaches (up to a constant)

$$ds^2 = d\ln z_1 d\ln \bar{z}_1 + d\ln z_2 d\ln \bar{z}_2 + d\ln z_3 d\ln \bar{z}_3 \tag{4.3}$$

for finite z_1, z_2, z_3. In this limit (4.1) obviously defines a family of supersymmetric 3-cycles. Perturbing away from the large complex structure limit produces a small effect on the Ricci-flat metric for finite z_1, z_2, z_3. Since supersymmetric 3-cycles are expected to be stable under small perturbations of the metric, such a 3-cycle should exist in a neighborhood of the large complex structure limit.

5. Conclusions

Our results should lead to the construction of a large new class of dual pairs. For example, heterotic string theory on T^3 is equivalent to M-theory on $K3$. Applying this duality fiberwise to the T^3 fibers of any $N = 1$ compactification of heterotic string theory on a Calabi-Yau space with a mirror partner, one obtains an $N = 1$ compactification of M-theory on a seven-manifold of G_2 holonomy. Examples of this type are in [18][19]. By considering perturbations this duality can be extended to $(0, 2)$ heterotic compactifications.[16] Perhaps this will provide a useful way to study a phenomenologically interesting class of string compactifications. A precise understanding of this duality, as well as the mirror relation discussed herein, will probably require a better understanding of the nature of the singular fibers. Undoubtedly, constraints from supersymmetry will play a role in controlling these singularities.

In this paper we have only considered the implications of quantum mirror symmetry for the simplest case of a single 0-brane. We expect that consideration of other p-branes and their bound states will lead to further insights into the rich structure of Calabi-Yau spaces and supersymmetric string compactifications in general.

Appendix: The calculation of $\frac{d}{dt_a}\theta^b(0)$.

We calculate $\frac{d}{dt_a}\theta^b(0)$, where conventions are as in section three. Let us now use t for t_a. We have from (3.2) that $\theta^b(t)$ is of the form $\theta^b(t) = C/t$, where $C \to 0$ as $t \to 0$ (since $\frac{df_s}{ds} = 0$ at $t = 0$). So

$$
\begin{aligned}
\lim_{t\to 0}\frac{d\theta^b}{dt} &= \lim_{t\to 0}(t\dot{C} - C)/t^2 \\
&= \lim_{t\to 0}(t\ddot{C} + \dot{C} - \dot{C})/2t \qquad \text{by L'Hôpital's rule} \\
&= \lim_{t\to 0}\ddot{C}/2 = \ddot{C}(0)/2.
\end{aligned}
$$

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[16] Recent progress on this problem has been made using F-theory duals in [20].

Let's calculate. $\theta^b(t)_i = -\frac{1}{t} J^\alpha{}_\beta \frac{df^\beta}{ds} g_{\alpha\gamma} f^\gamma{}_i$.

$$\dot\theta^b(0)_i = -J^\alpha{}_\beta \frac{d}{dt}(g_{\alpha\gamma} f^\gamma{}_i) \frac{d}{ds} \frac{df^\beta}{dt} - \frac{1}{2} g_{\alpha\gamma} J^\alpha{}_\beta f^\gamma{}_i \frac{d}{ds}\left(\frac{d^2 f^\beta}{dt^2}\right)$$

$$= -J^\alpha{}_\beta (\partial_\rho g_{\alpha\gamma})(J^\rho{}_\tau f^\tau{}_j w_a^j) f^\gamma{}_i \frac{d}{ds}\left[J^\beta{}_\mu f^\mu{}_k g^{kl}(\theta_l^a + s\theta_l^b - \partial_l \phi)\right]$$

$$\quad - J^\alpha{}_\beta g_{\alpha\gamma} \partial_i (J^\gamma{}_\mu f^\mu{}_j w_a^j) \frac{d}{ds}\left[J^\beta{}_\nu f^\nu{}_k g^{kl}(\theta_l^a + s\theta_l^b - \partial_l \phi)\right]$$

$$\quad - \frac{1}{2} g_{\alpha\gamma} J^\alpha{}_\beta f^\gamma{}_i \frac{d}{ds}\left(\frac{d^2 f^\beta}{dt^2}\right)$$

$$= -J^\alpha{}_\beta (\partial_\rho g_{\alpha\gamma}) J^\rho{}_\tau f^\tau{}_j w_a^j f^\gamma{}_i J^\beta{}_\nu f^\nu{}_k g^{kl} \theta_l^b$$

$$\quad - J^\alpha{}_\beta g_{\alpha\gamma} J^\gamma{}_\mu (f^\mu{}_{ij} w_a^j + f^\mu{}_j \partial_i w_a^j) J^\beta{}_\nu f^\nu{}_k g^{kl} \theta_l^b$$

$$\quad - \frac{1}{2} g_{\alpha\gamma} J^\alpha{}_\beta f^\gamma{}_i \frac{d}{ds}\left(\frac{d^2 f^\beta}{dt^2}\right).$$

In the above and what follows we use the fact that derivatives of indices on M with respect to t can be obtained by $\frac{df^\alpha}{dt}\partial_\alpha$. Also, we will freely use that $J^\alpha{}_\gamma J^\gamma{}_\beta = -\delta^\alpha{}_\beta$ and that $g_{\alpha b}$ is hermitian: $J^\alpha{}_\mu g_{\alpha\beta} J^\beta{}_\nu = g_{\mu\nu}$. Derivatives are finally evaluated at $t = s = 0$.

$$\dot\theta^b(0)_i = (\partial_\rho g_{\alpha\gamma}) J^\rho{}_\tau f^\tau{}_j w_a^j f^\gamma{}_i f^\alpha{}_k w_b^k - g_{\beta\mu} J^\beta{}_\nu (f^\mu{}_{ij} w_a^j + f^\mu{}_j \partial_i w_a^j) f^\nu{}_k w_b^k$$

$$\quad - \frac{1}{2} g_{\alpha\gamma} J^\alpha{}_\beta f^\gamma{}_i \frac{d}{ds}\left(\frac{d^2 f^\beta}{dt^2}\right)$$

$$= (\partial_\rho g_{\alpha\gamma}) J^\rho{}_\tau f^\alpha{}_k f^\tau{}_j f^\gamma{}_i w_b^k w_a^j$$

$$\quad + \omega_{\nu\mu} f^\nu{}_k f^\mu{}_{ij} w_b^k w_a^j$$

$$\quad + \omega_{\nu\mu} f^\nu{}_k f^\mu{}_j (\partial_i w_a^j) w_b^k$$

$$\quad - \frac{1}{2} g_{\alpha\gamma} J^\alpha{}_\beta f^\gamma{}_i \frac{d}{ds}\left(\frac{d^2 f^\beta}{dt^2}\right)$$

The third line in the last equality is zero since $\omega_{\nu\mu} f^\nu{}_k f^\mu{}_j$ equals $f^* \omega_{kj}$, which vanishes.

$$\dot\theta^b(0)_i = J^\rho{}_\tau (\partial_\rho g_{\alpha\gamma}) f^\alpha{}_k f^\tau{}_j f^\gamma{}_i w_b^k w_a^j + \omega_{\nu\mu} f^\nu{}_k f^\mu{}_j (\partial_i w_a^j) w_b^k$$

$$\quad + \frac{1}{2} \omega_{\beta\gamma} f^\gamma{}_i \frac{d}{ds}\left(\frac{d^2 f^\beta}{dt^2}\right).$$

At this point we'll need to calculate $\frac{d}{ds}\left(\frac{d^2 f^\beta}{dt^2}\right)|_{t=s=0}$. We have

$$\frac{df^\gamma}{dt} = J^\gamma{}_\tau f^\tau{}_l g^{lm}(\theta_m^a + s\theta_m^b - \partial_m \phi).$$

Differentiating again, we get

$$\frac{d^2 f^\gamma}{dt^2} = J^\gamma{}_\tau \frac{\partial}{\partial x^l} \left(J^\tau{}_\nu f^\nu{}_n g^{np}(\theta_p^a + s\theta_p^b - \partial_p \phi) \right) g^{lm} \left(\theta_m^a + \theta_m^b - \partial_m \phi \right)$$

$$+ J^\gamma{}_\tau f^\tau{}_l \left(-2h_a{}^{lm} - 2sh_b{}^{lm} \right) \left(\theta_m^a + s\theta_m^b - \partial_m \phi \right)$$

$$+ J^\gamma{}_\tau f^\tau{}_l g^{lm}(-\partial_m \frac{d\phi}{dt}).$$

Taking the derivative with respect to s gives (note $\phi = 0$ at t = 0, independent of s):

$$\frac{d}{ds} \left(\frac{d^2 f^\gamma}{dt^2} \right)_{t=s=0} = J^\gamma{}_\tau \partial_l \left(J^\tau{}_\nu f^\nu{}_n g^{np}\theta_p^a \right) g^{lm}\theta_m^b + J^\gamma{}_\tau \partial_l \left(J^\tau{}_\nu f^\nu{}_n g^{np}\theta_p^b \right) g^{lm}\theta_m^a$$

$$- 2J^\gamma{}_\tau f^\tau{}_l h_a{}^{lm}\theta_m^b - 2J^\gamma{}_\tau f^\tau{}_l h_b{}^{lm}\theta_m^a$$

$$- J^\gamma{}_\tau f^\tau{}_l g^{lm}\partial_m \left(\frac{d^2\phi}{dsdt} \right)$$

$$= -\partial_l(f^\gamma{}_n w_a^n)w_b^l - \partial_l(f^\gamma{}_n w_b^n)w_a^l$$

$$- 2J^\gamma{}_\tau f^\tau{}_l h_a{}^{lm}\theta_m^b - 2J^\gamma{}_\tau f^\tau{}_l h_b{}^{lm}\theta_m^a$$

$$- J^\gamma{}_\tau f^\tau{}_l g^{lm}\partial_m \left(\frac{d^2\phi}{dsdt} \right).$$

Let's now plug this result into our formula for $\dot\theta^b(0)_i$.

$$\dot\theta^b(0)_i = J^\rho{}_\tau(\partial_\rho g_{\alpha\gamma})f^\alpha{}_k f^\tau{}_j f^\gamma{}_i w_b^k w_a^j + \omega_{\nu\mu}f^\nu{}_k f^\mu{}_j(\partial_i w_a^j)w_b^k$$

$$+ \frac{1}{2}\omega_{\beta\gamma}f^\gamma{}_i \left[-\partial_l(f^\gamma{}_n w_a^n)w_b^l - \partial_l(f^\gamma{}_n w_b^n)w_a^l \right]$$

$$+ \frac{1}{2}\omega_{\beta\gamma}f^\gamma{}_i \left[-2J^\gamma{}_\tau f^\tau{}_l h_a{}^{lm}\theta_m^b - 2J^\gamma{}_\tau f^\tau{}_l h_b{}^{lm}\theta_m^a \right]$$

$$+ \frac{1}{2}\omega_{\beta\gamma}f^\gamma{}_i \left[-J^\gamma{}_\tau f^\tau{}_l g^{lm}\partial_m \left(\frac{d^2\phi}{dsdt} \right) \right]$$

$$= J^\rho{}_\tau(\partial_\rho g_{\alpha\gamma})f^\alpha{}_k f^\tau{}_j f^\gamma{}_i w_b^k w_a^j + \omega_{\nu\mu}f^\nu{}_k f^\mu{}_j(\partial_i w_a^j)w_b^k$$

$$- \frac{1}{2}\omega_{\beta\gamma}f^\gamma{}_i \left(f^\beta{}_{ln}w_a^n w_b^l + f^\beta{}_{ln}w_b^n w_a^l \right)$$

$$- \omega_{\beta\gamma}J^\beta{}_\tau f^\gamma{}_i f^\tau{}_l \left(h_a{}^{lm}\theta_m^b + h_b{}^{lm}\theta_m^a \right)$$

$$- \frac{1}{2}\omega_{\beta\gamma}f^\gamma{}_i J^\beta{}_\tau f^\tau{}_l g^{lm}\partial_m \left(\frac{d^2\phi}{dsdt} \right)$$

$$= J^\rho{}_\tau(\partial_\rho g_{\alpha\gamma})f^\alpha{}_k f^\tau{}_j f^\gamma{}_i w_b^k w_a^j + \omega_{\nu\mu}f^\nu{}_k f^\mu{}_j(\partial_i w_a^j)w_b^k$$

$$- \omega_{\beta\gamma}f^\gamma{}_i f^\beta{}_{ln}w_a^l w_b^n$$

$$- g_{li} \left(h_a{}^{lm}\theta_m^b + h_b{}^{lm}\theta_m^a \right)$$

$$- \frac{1}{2}\partial_i \left(\frac{d^2\phi}{dsdt} \right).$$

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Note that $g_{li}h_a{}^{lm}\theta_m^b = g_{li}h_b{}^{lm}\theta_m^a = h_{ijk}w_a^j w_b^k$. The rest is just algebra in combining the other terms. It is trivial to show the following quoted result in normal coordinates on the target space (which, since it is Kähler, can be chosen simul- taneously with coordinates adapted to the complex structure), in which case $h_{ijk} = \omega_{\alpha\beta}f^\alpha{}_i f^\beta{}_{jk}$. In any case, it can be shown straightforwardly, in general. We get, since the h terms cancel, the following remarkable result.

$$\frac{d\theta_i^b}{dt_a}\Big|_{t=0} = -\frac{1}{2}\frac{\partial}{\partial x^i}\left(\frac{d^2\phi}{dsdt}\right), \tag{A.1}$$

or

$$\partial_a \theta^b = d\psi \tag{A.2}$$

at $t = 0$, where $\psi = -\frac{1}{2}\frac{d^2\phi}{dsdt}\big|_{t=s=0}$.

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