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Matrix Sum-of-Squares Relaxations for Robust Semi-Definite Programs

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Abstract. We consider robust semi-definite programs which depend polynomially or rationally on some uncertain parameter that is only known to be contained in a set with a polynomial matrix inequality description. On the basis of matrix sum-of-squares decompositions, we suggest a systematic procedure to construct a family of linear matrix inequality relaxations for computing upper bounds on the optimal value of the corresponding robust counterpart. With a novel matrix-version of Putinar's sum-of-squares representation for positive polynomials on compact semi-algebraic sets, we prove asymptotic exactness of the relaxation family under a suitable constraint qualification. If the uncertainty region is a compact polytope, we provide a new duality proof for the validity of Putinar's constraint qualification with an a priori degree bound on the polynomial certificates. Finally, we point out the consequences of our results for constructing relaxations based on the so-called full-block S-procedure, which allows to apply recently developed tests in order to computationally verify the exactness of possibly small-sized relaxations.

1. Introduction

It is well-established that a whole variety of analysis and synthesis problems in control can be reduced to scalar polynomially constrained polynomial programs.

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Only rather recently, it has been suggested how to construct semi-definite programming (SDP) relaxations of such non-convex optimization problems based on the sum-of-squares (SOS) decomposition of multivariable polynomials [23, 5, 21, 7, 6, 12, 11]. In particular in control engineering, many problems actually involve semi-definite constraints on symmetric-valued polynomial matrices, such as the spectral factorization of multidimensional transfer functions to asses dissipativity of linear shift-invariant distributed systems [25], or the synthesis of $\mathcal{H}_\infty$-optimal output feedback controllers with a constraint on the controller structure, such as an a priori bound on its McMillan degree [13].

Control systems are typically affected by uncertainty which captures the mismatch between the employed model and the real plant under consideration for analysis or synthesis. For different important classes, such as parametric or dynamic, time-invariant or time-varying deterministic uncertainties, it is well-understood how to reduce robust stability and performance analysis or robust state-feedback and estimator synthesis problems to so-called robust semi-definite programs [10, 1, 33]. Although dynamic model-mismatch in feedback interconnections leads to complex uncertainties which enter in a rational fashion [37], it is not difficult to reduce to real uncertainty and polynomial dependence [18].

This leads us to the subject of this paper, the following robust polynomial semi-definite program with optimal value v_{opt}:

$$\begin{array}{ll} \text{infimize} & c^T y \\ \text{subject to} & F(x,y) \succ 0 \quad \text{for all} \quad x \in \mathbb{R}^m \quad \text{with} \quad G(x) \preccurlyeq 0. \end{array} \tag{1}$$

Here $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathcal{S}^p$ and $G : \mathbb{R}^m \to \mathcal{S}^q$ are symmetric-valued functions which depend polynomially on the uncertainty parameter $x \in \mathbb{R}^m$, while F depends affinely on the design parameter $y \in \mathbb{R}^n$. Therefore $F(x, y) \succ 0$ is a standard linear matrix inequality (LMI) in y for fixed x, while the robust counterpart requires to satisfy the LMI for all x in the uncertainty set

$$\mathcal{G} = \{x \in \mathbb{R}^m : \ G(x) \preccurlyeq 0\}, \tag{2}$$

which itself admits a very general description in terms a polynomial semi-definite constraint. Recall that multiple polynomial SDP-constraints can be easily collected into one inequality by diagonal augmentation. We stress that, in many interesting practical cases, $\mathcal{G}$ turns out to admit an LMI representation (G is affine) or is even just a compact polytope (G is diagonal and affine). Note also that polynomial semi-definite programs $\sup\{f(x) : \ x \in \mathbb{R}^m, \ G(x) \preccurlyeq 0\}$ as considered in [19,13] are recovered from (1) with $F(x, y) = y - f(x)$ and $c = 1$. If, in addition, $G(x) = \mathrm{diag}(-g_1(x), -g_2(x), \ldots, -g_q(x))$ is scalar-diagonal, we arrive at the problem class considered in [21,24,36].

If F depends also affinely on the uncertainty x and if $\mathcal{G}$ is the convex hull of a moderate number of explicitly given generators, it is clear that (1) amounts to solving a standard LMI problem. If the number of extreme points to describe $\mathcal{G}$ is large, it is often possible to construct efficiently computable relaxation with beautiful a priori guarantees on the relaxation error [2,3]. The situation drastically differs if the uncertainties enter nonlinearly, since then such a priori guarantees are out of reach. Still, however, various relaxation schemes in robust control (such as multiplier relaxation in structured singular value theory [9,22, 15,16,30]) have been applied to construct efficiently computable relaxations.

In general, it cannot be expected that these relaxations are exact, and the only known techniques to systematically reduce the relaxation gap with guaranteed convergence is restricted to boxes [4] or to finitely generated polytopes with known generators [31,33]. As the main goal of this paper, we show how such asymptotically exact relaxation families can be constructed on the basis of matrix SOS decompositions for the much larger class of uncertainty sets $\mathcal{G}$, based directly on the implicit polynomial matrix inequality description (2) if $G(x)$ satisfies a suitable constraint qualification. In contrast to approaches based on scalarization and a subsequent application of existing relaxation techniques [21, 24,36], we will be able to show that the size of the constructed LMI relaxations grow at most bi-quadratically in the dimension p of $F(x,y)$ and q of $G(x)$ respectively. Moreover, we will reveal how the techniques in [33] can be applied in order to verify whether a given finite relaxation does not involve any conservatism.

The paper is structured as follows. In Section 2 we introduce the concept of SOS matrices and discuss how it can be verified whether a polynomial matrix is SOS by solving a linear SDP. This section comprises the exact reformulation of (1) in terms of SOS matrices as formulated in our main Theorem 1. Moreover, we point out a direct consequence, a novel generalization of Putinar's representation theorem [28] to polynomial matrices which are positive definite on $\mathcal{G}$. We also briefly discuss why existing scalar relaxation techniques fail to guarantee the claimed growth of the relaxation size in the dimension of $F(x,y)$ and $G(x)$.

The proof of our main result is provided in Section 4. As a preparation, the main purpose of Section 3 is to prove a particular version of Putinar's representation theorem [28] for polynomial matrices which are positive definite on a set described by scalar polynomial inequalities. Although the validity of the required constraint qualification for compact polytopes has been established in [17] with techniques from real algebraic geometry, we will give a new proof based on Lagrange duality, which allows to extract explicit degree bounds.

In Section 5 we describe how to construct finite-dimensional LMI relaxations with full flexibility in the choice of the underlying monomial basis, which leads to an explicit estimate of the relaxation size. In order to numerically verify whether any of these finite-dimensional relaxation is exact, we reveal, in Section 6, how to subsume (1) to the general framework in [33]. Finally, in Section 7 we provide a numerical illustration for a well-known academic example in robust control.

2. Construction of an exact SOS reformulation

2.1. Sum-of-squares decomposition of polynomial matrices

A $p \times p$-polynomial matrix $S(x)$ in $x \in \mathbb{R}^m$ is said to be a sum-of-squares (SOS) if there exists a (not necessarily square and typically tall) polynomial matrix $T(x)$ such that

$$S(x) = T(x)^T T(x).$$

If $p = 1$ and if $T_j(x)$ denote the components of the column vector $T(x)$ of length r, we infer $S(x) = \sum_{j=1}^r T_j(x)^2$ which motivates our terminology. Clearly any

SOS polynomial matrix is globally positive semi-definite, but as for $p = 1$, the converse is in general not true. The reader is referred to the nice survey [29] for a detailed discussion and a large collection of references concerning scalar SOS decompositions, and to [19,6] for related extensions to polynomial matrices.

A computational procedure for verifying whether $S(x)$ is SOS proceeds as follows. Choose pairwise different monomials $u_1(x), \ldots, u_{n_u}(x)$ and search for the coefficient matrix Y in the representation

$$T(x) = Y(u(x) \otimes I_p) \;\; \text{with} \;\; Y = \begin{pmatrix} Y_1 & \cdots & Y_{n_u} \end{pmatrix}, \;\; u(x) = \begin{pmatrix} u_1(x) \\ \vdots \\ u_{n_u}(x) \end{pmatrix}.$$

$S(x)$ is said to be SOS with respect to $u(x)$ if there exists some Y satisfying $S(x) = T(x)^T T(x) = (u(x) \otimes I_p)^T (Y^T Y)(u(x) \otimes I_p)$. This motivates the variable substitution $Z = Y^T Y$ to arrive at the following result.

Lemma 1. *The polynomial matrix $S(x)$ of dimension p is SOS with respect to the monomial basis $u(x)$ iff there exists a symmetric Z with*

$$S(x) = (u(x) \otimes I_p)^T Z (u(x) \otimes I_p) \;\; \textit{and} \;\; Z \succcurlyeq 0. \tag{3}$$

Proof. We have proved 'only if'. For the proof of 'if' suppose Z satisfies (3). If we factorize Z as $Y^T Y$, we infer that $T(x) = Y(u(x) \otimes I_p)$ is as desired. ■

Since the first relation (3) just amounts to an affine equation constraint on Z, it is a standard LMI problem to verify the existence of some Z with (3). In summary, one can check whether $S(x)$ is SOS with respect to some monomial basis by simple solving a linear SDP.

2.2. The main result

In order to systematically construct relaxations, let us introduce the bilinear mapping

$$(.,.)_p : \mathbb{R}^{pq\times pq} \times \mathbb{R}^{pq\times pq} \to \mathbb{R}^{p\times p}, \quad (A,B)_p = \mathrm{tr}_p(A^T(I_p \otimes B))$$

with

$$\mathrm{tr}_p(C) := \begin{pmatrix} \mathrm{tr}(C_{11}) & \cdots & \mathrm{tr}(C_{1p}) \\ \vdots & \ddots & \vdots \\ \mathrm{tr}(C_{p1}) & \cdots & \mathrm{tr}(C_{pp}) \end{pmatrix} \quad \text{for} \quad C \in \mathbb{R}^{pq\times pq}, \quad C_{jk} \in \mathbb{R}^{q\times q}.$$

Clearly, $(A,B)_1$ just equals the standard inner product $\langle A,B\rangle = \mathrm{Trace}(A^TB)$. For later reference let us note that

$$\mathrm{tr}_p(A^T(I_p \otimes B)) = \mathrm{tr}_p((I_p \otimes B)A^T). \tag{4}$$

This allows to prove the following essential property:

$$(A,B)_p \succcurlyeq 0 \;\text{ if }\; A \succcurlyeq 0 \;\text{ and }\; B \succcurlyeq 0. \tag{5}$$

Indeed, if we decompose $B = DD^T$, we infer $(I_p \otimes D)^T A (I_p \otimes D) \succcurlyeq 0$. Since the trace operator as a mapping from $\mathbb{R}^{p\times p}$ into $\mathbb{R}$ is completely positive [8], we obtain $\mathrm{tr}_p((I_p \otimes D)^T A(I_p \otimes D)) \succcurlyeq 0$. If exploiting (4), we can conclude that $\mathrm{tr}_p(A(I_p \otimes D)(I_p \otimes D^T)) \succcurlyeq 0$ and hence $\mathrm{tr}_p(A(I_p \otimes DD^T)) = (A,B)_p \succcurlyeq 0$.

With the help of this bilinear mapping, let us now define u_{opt} as the value of the optimization problem

$$\begin{array}{ll} \text{infimize} & c^Ty \\ \text{subject to} & \epsilon > 0,\; S(x) \text{ and } F(x,y) + (S(x),G(x))_p - \epsilon I_p \text{ are SOS in } x \end{array} \tag{6}$$

in the decision variables y, ϵ, and the polynomial matrix $S(x)$ of dimension pq. As the main result of this paper, we will prove that the values of (1) and (6) are equal, $u_{\rm opt} = v_{\rm opt}$, under a rather mild constraint qualification on $G(x)$. Due to Lemma 1, computing (6) amounts to solving an infinite dimensional but linear semi-definite program, which can be easily relaxed to a standard LMI problem by restricting $S(x)$ to finite-dimensional subspaces. These insights form the basis for constructing a sequence of LMI relaxations, whose values define upper bounds of $v_{\rm opt}$ which converge to $v_{\rm opt}$, as discussed in Section 5.

Without any hypothesis it is easy to see that $u_{\rm opt} \geq v_{\rm opt}$. Indeed, suppose y, ϵ and $S(x)$ are feasible for (6). Choose an arbitrary $x_0 \in \mathbb{R}^m$ with $G(x_0) \preccurlyeq 0$; since $S(x)$ is SOS we infer $S(x_0) \succcurlyeq 0$; since $F(x,y) + (S(x), G(x))_p - \epsilon I_p$ is SOS, we similarly infer $F(x_0, y) + (S(x_0), G(x_0))_p \succ 0$; it finally remains to exploit (5) to conclude $F(x_0, y) \succ 0$; since $x_0 \in \mathcal{G}$ was arbitrary, we have indeed shown that y is also feasible for (1).

Theorem 1. *Suppose the following constraint qualification holds true: There exist some $r \in \mathbb{R}$ and some SOS matrix $\Psi(x)$ of dimension q such that*

$$r^2 - \|x\|^2 + \langle \Psi(x), G(x) \rangle \quad \text{is SOS.} \tag{7}$$

If v_{opt}, u_{opt} denote the optimal values of (1), (6) respectively then $v_{opt} = u_{opt}$.

Remark 1. The constraint qualification is a natural generalization of that used by Schweighofer [36] for multiple scalar polynomial constraints. It implies that $\mathcal{G}$ is contained in $\{x \in \mathbb{R}^m : \|x\| \leq r\}$, and hence compact. Conversely, if $\mathcal{G}$ is known to be contained in the ball around zero with radius r, we can replace $G(x) \preccurlyeq 0$

by $\tilde{G}(x) := \text{diag}(G(x), \|x\|^2 - r^2) \preccurlyeq 0$ in (1) without modifying the optimization problem, and the constraint qualification is satisfied for $\tilde{G}(x)$. Similarly as in the scalar case, we note that the constraint qualification can be alternatively formulated as follows: There exist SOS polynomial matrices $\Psi(x)$ and $\psi(x)$ of dimension q and 1 such that $\{x \in \mathbb{R}^m : \ \langle \Psi(x), G(x) \rangle - \psi(x) \geq 0\}$ is compact.

To the best of our knowledge, this exactness result for robust polynomial matrix inequalities ($p > 1$) with matrix-valued polynomial constraints on the uncertainties ($q > 1$) is new. Theorem 1 hence combines the results on polynomial semi-definite programming [19,13] (with scalar-valued F) and robust LMI problems with polytopic uncertainty regions (with diagonal and affine G) [31, 34] to a very general formulation with a wide range of applications, in particular in control.

As an immediate corollary, we can extract the following new representation result for positive definite polynomial matrices on sets that are described by semi-definite polynomial inequalities.

Corollary 1. *Suppose $G(x)$ satisfies the constraint qualification in Theorem 1. If $H(x)$ is a symmetric-valued polynomial matrix of dimension p which is positive definite on $\mathcal{G}$, there exists some SOS matrix $S(x)$ of dimension pq such that*

$$H(x) + (S(x), G(x))_p \quad \textit{is SOS.}$$

Proof. Just choose $c = 0$ and $F(x, y) = H(x)$ such that $v_{\text{opt}} = 0$. By Theorem 1, we infer $u_{\text{opt}} = 0$ which implies that (6) is feasible. Hence there exists $\epsilon > 0$ and some SOS matrix $S(x)$ such that $H(x) + (S(x), G(x))_p - \epsilon I_p$ and hence also $H(x) + (S(x), G(x))_p$ are SOS. ■

If $H(x)$ is a scalar polynomial ($p = 1$), this result has been obtained independently in [19,13], while generalizations to constraints defined by polynomial inequalities in Jordan algebras can be found in [20]. If further specializing to

$$G(x) = -\operatorname{diag}(g_1(x), \ldots, g_q(x)) \text{ with scalar polynomials } g_1(x), \ldots, g_q(x), \quad (8)$$

we arrive at Putinar's fundamental representation theorem [28] for positive polynomials on semi-algebraic sets.

Remark 2. Let us explicitly formulate the specialization of Theorem 1 for (8). The constraint qualification then requires the existence of r and scalar SOS polynomials $\psi_0(x), \psi_1(x), \ldots, \psi_q(x)$ such that

$$r^2 - \|x\|^2 = \psi_0(x) + \sum_{j=1}^{q} \psi_j(x) g_j(x). \quad (9)$$

Due to Theorem 1, the value $v_{\rm opt}$ then equals the infimal $c^T y$ such that there exist $\epsilon > 0$ and SOS polynomial matrices $S_0(x), S_1(x), \ldots, S_q(x)$ with

$$F(x, y) = S_0(x) + \sum_{j=1}^{q} S_j(x) g_j(x) + \epsilon I_p. \quad (10)$$

2.3. Scalarization

To scalarize the particular problem as described in Remark 2, one introduces the new variables $v \in \mathbb{R}^p$ and defines the polynomials

$$f(v, x, y) := v^T F(x, y) v, \quad g_{q+1}(v, x) = 2 - v^T v, \quad g_{q+2}(v, x) = v^T v - 1.$$

Then $v_{\rm opt}$ equals the infimal $c^T y$ such that

$$f(v,x,y) > 0 \quad \text{for all} \quad (x,v) \quad \text{with} \quad g_j(v,x) \geq 0, \quad j = 1,\ldots,q+2. \tag{11}$$

The SOS counterpart with optimal value $u_{\rm opt}$ amounts to infimizing $c^T y$ over $\epsilon > 0$ and SOS polynomials $s_j(v,x)$, $j = 0,\ldots,q+2$, such that

$$f(v,x,y) = s_0(v,x) + \sum_{j=1}^{q+2} s_j(v,x) g_j(v,x) + \epsilon. \tag{12}$$

If $g_j(v,x)$, $j = 1,\ldots,q+2$, satisfy Putinar's constraint qualification then $v_{\rm opt} = u_{\rm opt}$ [28,21]. However, although the scalar polynomials $f(v,x,y)$ and $g_j(v,x)$ are quadratic in v, no available result allows to draw the conclusion $v_{\rm opt} = u_{\rm opt}$ after imposing any constraints on the degrees of the SOS polynomials $s_j(v,x)$ with respect to v. The particular version of Theorem 1 in Remark 2 implies that one can indeed confine the search to $s_{q+1}(v,x) = 0$, $s_{q+2}(v,x) = 0$ and to $s_j(v,x) = v^T S_j(x) v$, $j = 0,1,\ldots,q$, which are homogenously quadratic in v, without violating $v_{\rm opt} = u_{\rm opt}$. In this sense, Theorem 1 can be interpreted as providing a generic degree bound in the SOS reformulation of the scalarized problem, which is in turn the key for the bi-quadratic growth of the relaxation size in the dimension of $F(x,y)$ and $G(x)$ as discussed in Section 5.

To scalarize a genuine semi-definite constraint $G(x) \preccurlyeq 0$ in the general situation of (1), one can equivalently rewrite it in terms of the principal minors as $M_i(G(x)) \leq 0$, $i = 1,\ldots,2^q$, whose number grows exponentially in the dimension q of $G(x)$ [14]. Even if $p = 1$, the size of scalar SOS relaxations [23,21,36] depend exponentially on q. In contrast, Theorem 1 allows the construction of relaxations whose size grows at most bi-quadratically in p and q respectively, as demonstrated in Section 5.

3. Matrix positivity on semi-algebraic sets

The main goal of this section is to prove the following particular version of Corollary 1 for (8) under the hypothesis that the constraint qualification as formulated in Remark 2 holds true.

Theorem 2. *If the symmetric-valued polynomial matrix $H(x)$ is positive definite on $\mathcal{G}$, there exist $\epsilon > 0$ and SOS polynomial matrices $S_0(x), S_1(x), \ldots, S_q(x)$ of dimension p such that*

$$H(x) = S_0(x) + \sum_{j=1}^{q} S_j(x) g_j(x) + \epsilon I_p. \tag{13}$$

The proof is provided by combining a matrix version of Pólya's classical theorem [27,31] with a nice penalty technique suggested by Schweighofer [36].

3.1. Reduction to matrix positivity on the standard simplex

Since $\mathcal{G}$ is compact there exist constants $\lambda_0 > 0$ and $\mu_0 > 0$ such that

$$H(x) \succcurlyeq \lambda_0 I \ \text{ for all } \ x \in \mathbb{R}^m \ \text{ satisfying } \ g_j(x) \geq -\mu_0, \ \ j = 1, \ldots, q. \tag{14}$$

W.l.o.g. we assume that r in (9) is positive. With the all-ones vector e and this r define $\Sigma := \{x \in \mathbb{R}^m : \ x \geq -re, \ e^T x \leq r\sqrt{m}\}$. Again by compactness, there exist real λ_1 and $\mu_1 > 0$ with

$$H(x) \succcurlyeq \lambda_1 I \ \text{ and } \ g_j(x) \leq \mu_1 \ \text{ for all } \ x \in \Sigma, \ \ j = 1, \ldots, q. \tag{15}$$

With fixed $\rho = 1/(m + \sqrt{m}) > 0$ let us now perform the change of coordinates

$$z = \rho(x/r + e) \iff x = r(z/\rho - e).$$

Under this transformation, Σ maps into the standard unit simplex

$$\hat{\Sigma} := \{z \in \mathbb{R}^m : \ z \geq 0, \ e^T z \leq 1\} = \rho(\Sigma/r + e).$$

Introduce

$$\hat{H}_0(z) := H(r(z/\rho - e)) \ \text{ and } \ \hat{g}_j(z) := g_j(r(z/\rho - e))/\mu_1$$

and substitute $x = r(z/\rho - e)$ in (9). With the SOS polynomials

$$\hat{\psi}_0(z) := \psi_0(r(z/\rho - e))/r, \quad \hat{\psi}_j(z) := \mu_1 \psi_j(r(z/\rho - e))/r, \quad j = 1, \ldots, q, \tag{16}$$

we infer that the constraint qualification equation reads in new coordinates as

$$1 - \|z/\rho - e\|^2 = \hat{\psi}_0(z) + \sum_{j=1}^{q} \hat{g}_j(z) \hat{\psi}_j(z). \tag{17}$$

Clearly (14) and (15) imply, with $\mu := \mu_0/\mu_1$, that

$$\hat{H}_0(z) \succcurlyeq \lambda_0 I \ \text{ for all } \ z \in \mathbb{R}^m \ \text{ with } \ \hat{g}_j(z) \geq -\mu, \quad j = 1, \ldots, q, \tag{18}$$

as well as

$$\hat{H}_0(z) \succcurlyeq \lambda_1 I \ \text{ and } \ \hat{g}_j(z) \leq 1 \ \text{ for all } \ z \in \hat{\Sigma}. \tag{19}$$

Our main goal is to prove that there are $\lambda > 0$ and SOS polynomials $\hat{s}_j(z)$ with

$$P(z) := \hat{H}_0(z) - \sum_{j=1}^{q} \hat{s}_j(z) \hat{g}_j(z) I - \lambda I \succcurlyeq \lambda I \ \text{ for all } \ z \in \hat{\Sigma}. \tag{20}$$

For this purpose, we follow (a slight variant) of the arguments in Schweighofer [36]. Fix an integer $k \geq 0$ and some $\xi > 0$ with

$$\lambda := \frac{1}{2} \min \left\{ \lambda_0 - \frac{q\xi}{2k+1},\ \lambda_1 - \frac{q\xi}{2k+1} + \xi(1+\mu)^{2k}\mu \right\} > 0. \tag{21}$$

Let us now reveal that we can just take

$$\hat{s}_j(z) := \xi(1 - \hat{g}_j(z))^{2k}, \quad j = 1, \ldots, q.$$

Since $(1-t)^{2k} t \leq 1/(2k+1)$ for all $t \in [0,1]$, we infer from (19) that

$$\sum_{\{j:\ \hat{g}_j(z) \geq 0\}} (1 - \hat{g}_j(z))^{2k} \hat{g}_j(z) \leq q/(2k+1) \quad \text{for all} \quad z \in \hat{\Sigma}.$$

This implies (after legitimately dropping the terms with $0 > \hat{g}_j(z) > -\mu$) that

$$P(z) \succcurlyeq \hat{H}_0(z) - \frac{q\xi}{2k+1} I - \xi \sum_{\{j:\, -\mu \geq \hat{g}_j(z)\}} (1 - \hat{g}_j(z))^{2k} \hat{g}_j(z) I - \lambda I.$$

If $z \in \hat{\Sigma}$ satisfies $\hat{g}_j(z) > -\mu$ for all $j = 1, \ldots, q$, then (20) follows from (18) and the definition of λ in (21). If there is at least one j_0 with $-\mu \geq \hat{g}_{j_0}(z)$ we infer $(1 - \hat{g}_{j_0}(z))^{2k} \geq (1+\mu)^{2k}$ and hence

$$-\xi \sum_{\{j:\, -\mu \geq \hat{g}_j(z)\}} (1 - \hat{g}_j(z))^{2k} \hat{g}_j(z) \geq \xi(1+\mu)^{2k} [-\hat{g}_{j_0}(z)] \geq \xi(1+\mu)^{2k}\mu.$$

Then (20) is a consequence of (19) and, again, of the definition of λ in (21).

3.2. Characterizing matrix positivity on the standard simplex

Throughout this section, we assume that the $p \times p$-dimensional polynomial matrix $P(z)$ has degree d and satisfies, for some $\lambda > 0$,

$$P(z) \succcurlyeq \lambda I \quad \text{for all} \quad z \in \hat{\Sigma}. \tag{22}$$

If $p = 1$ and if $P(z)$ is homogenous, the classical theorem of Pólya's [26] implies that $(e^T z)^N P(z)$ has only positive coefficients for sufficiently large N. Let us first provide a generalization to $p > 1$ with an explicit bound for N in terms of d and λ [27]. For this purpose, recall the standard notations $|\alpha| = \alpha_1 + \cdots + \alpha_m$, $\alpha! = \alpha_1! \cdots \alpha_m!$, $x^\alpha = z_1^{\alpha_1} \cdots z_m^{\alpha_m}$, and $D_\alpha = \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m}$ for any multi-index $\alpha \in \mathbb{N}_0^m$ and $z \in \mathbb{R}^m$. With the spectral norm $\|.\|$ we follow [27,36] and define

$$L(P) := \max_{\alpha \in \mathbb{N}_0^m} \frac{\|D_\alpha P(0)\|}{|\alpha|!}.$$

Theorem 3. *If $P(z)$ is homogenous and $N + d > d(d-1)L(P)/(2\lambda)$ then all coefficients of $(e^T z)^N P(z)$ are positive definite.*

Proof. In what follows, $\alpha \in \mathbb{N}_0^m$ and $s(z) = e^T z$. We have

$$P(z) = \sum_{|\alpha|=d} \frac{D_\alpha P(0)}{\alpha!} z^\alpha \quad \text{and} \quad s(z)^N P(z) = \sum_{|\alpha|=d+N} \frac{D_\alpha (s^N P)(0)}{\alpha!} z^\alpha.$$

Choose any $v \in \mathbb{R}^p$ of norm one. On the one hand, we infer $v^T P(z) v \geq \lambda$ for all $z \in \hat{\Sigma}$. On the other hand, since the coefficients of the z-polynomial $v^T P(z) v$ are $v^T [D_\alpha P(0)] v / \alpha!$, we have

$$\max_{\alpha \in \mathbb{N}_0^m} \left| \frac{\alpha!}{|\alpha|!} \frac{v^T [D_\alpha P(0)] v}{\alpha!} \right| \leq \max_{\alpha \in \mathbb{N}_0^m} \frac{\|D_\alpha P(0)\|}{|\alpha|!} = L(P).$$

By [27] we conclude that the z-polynomial $s(z)^N v^T P(z) v = v^T (s(z)^N P(z)) v$ has positive coefficients. This implies $v^T D_\alpha (s^N P)(0) v > 0$ for all $|\alpha| = N + d$. Since v was chosen arbitrarily of norm one and since N is independent of v, we end up with $D_\alpha (s^N P)(0) \succ 0$ for all $|\alpha| = N + d$. ∎

Similarly as in [27], even if P is not homogenous, it not difficult to derive the following generalization of Handelman's theorem for the standard simplex and with degree bounds.

Lemma 2. *If $M > d(d^2-1)L(P)/(2\lambda)$ there exist $P_\beta \succ 0$ such that*

$$P(z) = \sum_{\beta\in\mathbb{N}_0^{m+1},\,|\beta|=M} P_\beta\, z_1^{\beta_1}\cdots z_m^{\beta_m}(1-e^Tz)^{\beta_{m+1}}. \tag{23}$$

Proof. Decompose P into $P_0+\cdots+P_d$ with homogenous $P_0,\ldots,P_d$ of degrees $0,\ldots,d$. With $t\in\mathbb{R}$, define the homogenous d-degree polynomial

$$Q(z,t) := \sum_{j=0}^{d} P_j(z)(e^Tz+t)^{d-j}$$

whose restriction to $\{(z,t):\ z\geq 0,\ t\geq 0,\ e^Tz+t=1\}$ equals $P(z)$. Therefore, the smallest eigenvalue of $Q(z,t)$ on this set is not smaller than λ. Moreover, with $s(z,t)=e^Tz+t$ we can apply Lemma 5 in the appendix to infer

$$L(Q)\leq \sum_{j=0}^{d} L(P_js^{d-j}) \leq \sum_{j=0}^{d} L(P_j)L(s^{d-j}) = \sum_{j=0}^{d} L(P_j) \leq (d+1)L(P).$$

For $N+d > d(d-1)(d+1)L(P)/(2\lambda)$ we conclude with Theorem 3 that

$$(e^Tz+t)^N Q(z,t) = \sum_{\beta\in\mathbb{N}_0^{m+1},\,|\beta|=N+d} P_\beta\, z_1^{\beta_1}\cdots z_m^{\beta_m} t^{\beta_{m+1}} \quad \text{where} \quad P_\beta\succ 0.$$

The result follows with the substitution $t=1-e^Tz$. ∎

Remark 3. With the same M as in Lemma 2 it is easy to show that

$$P(z) = \sum_{\delta\in\{0,1\}^{m+1}} S_\delta(z)\, z_1^{\delta_1}\cdots z_m^{\delta_m}(1-e^Tz)^{\delta_{m+1}}$$

for SOS polynomial matrices $S_\delta(z)$ of degree at most M. This is a matrix version of Schmüdgen's classical representation result [35] for the simplex which includes explicit bounds on the degrees of $S_\delta(z)$. ∎

The following representation result is elementary.

Lemma 3. *For nonzero $\beta \in \mathbb{N}_0^{m+1}$ there exist SOS polynomials $s_\beta(z)$ and $t_\beta(z)$ with degrees $2|\beta|$ and $2|\beta|-2$ such that*

$$z_1^{\beta_1} \cdots z_m^{\beta_m} (1 - e^T z)^{\beta_{m+1}} = s_\beta(z) + t_\beta(z)(1 - \|z/\rho - e\|^2). \tag{24}$$

Proof. With $t_0(z) := \rho\sqrt{m}/2 \geq 0$, it is easy to verify that $s_0(z) := 1 - e^T z - t_0(z)(1 - \|z/\rho - e\|^2)$ is nonnegative and hence SOS (since of degree 2). This leads to the SOS representation

$$1 - e^T z = s_0(z) + t_0(z)(1 - \|z/\rho - e\|^2)$$

with s_0, t_0 of degrees 2 and 0 respectively. Similarly we have

$$z_i = s_i(z) + t_i(z)(1 - \|z/\rho - e\|^2) \quad \text{for} \quad s_i(z) = (\rho/2)\|z/\rho - (e - e_i)\|^2,\ t_i(z) = \rho/2.$$

This finishes the proof for $|\beta| = 1$, and it is simple to recursively construct (24) from this representation for $|\beta| > 1$. ∎

If we just combine Lemma 2 with Lemma 3, we arrive at the main goal of this section, the following instrumental representation result.

Lemma 4. *If $M > d(d^2-1)L(P)/(2\lambda)$ and $\rho = 1/(m + \sqrt{m})$, there exist SOS matrices S, T of degree at most $2M$, $2M-2$ such that*

$$P(z) = S(z) + T(z)(1 - \|z/\rho - e\|^2). \tag{25}$$

Proof. With (23) and (24), we infer (25) for the SOS matrices

$$S(z) = \sum_{\beta \in \mathbb{N}_0^{m+1},\, |\beta| = M} P_\beta\, s_\beta(z) \quad \text{and} \quad T(z) = \sum_{\beta \in \mathbb{N}_0^{m+1},\, |\beta| = M} P_\beta\, t_\beta(z).$$

∎

3.3. Completion of the proof of Theorem 2

Let us combine (17), (20) and (25) to conclude

$$\hat{H}_0(z) = \hat{S}_0(z) + \sum_{j=1}^{q} \hat{S}_j(z)\hat{g}_j(z) + \lambda I_p$$

for the SOS polynomial matrices

$$\hat{S}_0(z) := S(z) + T(z)\hat{\psi}_0(z), \quad \hat{S}_j(z) := \hat{s}_j(z)I_p + T(z)\hat{\psi}_j(z), \quad j = 1, \ldots, q.$$

Then (13) just follows by substituting $z = \rho(x/r - e)$ with the SOS matrices $S_0(x) := \hat{S}_j(\rho(x/r - e))$, $S_j(x) := \hat{S}_j(\rho(x/r - e))/(r\mu_1)$, and with $\epsilon = \lambda$.

3.4. Discussion of degree bounds

Given the constants λ_0, λ_1 and μ_1, μ_2 as defined at the beginning of Section 3.1, it is simple to determine $k \geq 0$ and $\xi > 0$ which render λ in (21) positive, and to just compute P by (20). With

$$D := \deg(P)(\deg(P)^2 - 1)L(P)/\lambda - 1,$$

we can extract the following rather explicit bounds on the degree of the SOS polynomial matrices in the representation (13):

$$\deg(S_0) \leq \max\{D + 2, D + \deg(\psi_0)\}, \; \deg(S_j) \leq \max\{2k, D + \deg(\psi_j)\}.$$

As stressed in [36], this bound still depends on the degrees of the polynomials ψ_j, $j = 0, 1, \ldots, q$, which appear in the constraint qualification.

As another technical contribution of this paper, let us show that this difficulty can be overcome if the constraint functions $g_j(x)$ are all affine and define a compact polytope $\mathcal{G}$. It is a consequence of rather deep results in real algebraic geometry [17] that the constraint qualification is indeed satisfied, but no results on degree bounds for $\psi_j(x)$ are available. In Appendix B, we will provide a new proof based on semi-definite duality, which allows to show that the constraint qualification certificate polynomials can be chosen quadratic.

Theorem 4. *If the affine functions $g_1(x), \ldots, g_q(x)$ define a compact polytope $\mathcal{G} = \{x \in \mathbb{R}^m : \; g_1(x) \geq 0, \ldots, g_q(x) \geq 0\}$, then there exist $r \in \mathbb{R}$ and SOS polynomials $\psi_0(x), \psi_1(x), \ldots, \psi_q(x)$ of degree at most two with (9).*

4. Proof of Theorem 1

As a consequence of the constraint qualification, if

$$G(x) \preccurlyeq 0 \;\text{ is replaced by }\; \tilde{G}(x) := \text{diag}(G(x), \|x\|^2 - r^2) \preccurlyeq 0$$

then (1) is not modified (Remark 1). In a first step of the proof let us show that the same is true for the SOS reformulation (6).

Indeed, suppose $F(x,y) + (S(x), G(x))_p - \epsilon I_p = S_0(x)$ with SOS matrices $S_0(x)$ and $S(x)$. If we partition $S(x) = (S_{jk}(x))_{jk}$ into $q \times q$-blocks, then $\tilde{S}(x) := (\text{diag}(S_{jk}(x), 0))_{jk}$ satisfies $(\tilde{S}(x), \tilde{G}(x))_p = (S(x), G(x))_p$ and, therefore, $F(x,y) + (\tilde{S}(x), \tilde{G}(x))_p - \epsilon I_p = S_0(x)$.

Conversely, suppose $F(x,y) + (\tilde{S}(x), \tilde{G}(x))_p - \epsilon I_p = \tilde{S}_0(x)$ with SOS matrices $\tilde{S}_0(x)$, $\tilde{S}(x)$. Now we make explicit use of $r^2 - \|x\|^2 = \psi(x) - \langle \Psi(x), G(x)\rangle$ with

SOS matrices $\psi(x)$, $\Psi(x)$. Let us partition

$$\tilde{S}(x) = \left(\begin{pmatrix} S_{jk}(x) & * \\ * & s_{jk}(x) \end{pmatrix} \right)_{jk} \quad \text{into blocks of size} \quad (q+1) \times (q+1),$$

and define $S(x) := (S_{jk}(x) + s_{jk}(x)\Psi(x))_{jk}$, $s(x) = (s_{jk}(x))_{jk}$ of dimension pq, p respectively. It is easy to verify that both matrices are SOS and satisfy $(\tilde{S}(x), \tilde{G}(x))_p = (S(x), G(x))_p - s(x)\psi(x)$. This implies $F(x,y) + (S(x), G(x))_p - \epsilon I_p = \tilde{S}_0(x) + s(x)\psi(x)$ and it remains to observe that $\tilde{S}_0(x) + s(x)\psi(x)$ is SOS.

Therefore, from now on we can assume w.l.o.g. that

$$v_1^T G(x) v_1 = \|x\|^2 - r^2 \tag{26}$$

where $v_1 = (0, \ldots, 0, 1)^T \in \mathbb{R}^q$. It remains to show $u_{\text{opt}} \leq v_{\text{opt}}$. For this purpose, it suffices to choose an arbitrary y_0 which is feasible for (1), and to prove that y_0 is as well feasible for (6).

Let us hence assume $F(x, y_0) \succ 0$ for all $x \in \mathcal{G}$. Choose a sequence of unit vectors $v_2, v_3, \ldots$ such that v_i, $i = 1, 2, \ldots$ is dense in $\{v \in \mathbb{R}^q : \ \|v\| = 1\}$. Define $\mathcal{G}_N := \{x \in \mathbb{R}^m : v_i^T G(x) v_i \leq 0, \quad i = 1, \ldots, N\}$ to infer that $\mathcal{G}_N$ is compact (by (26)), and that $\mathcal{G}_N \supset \mathcal{G}_{N+1} \supset \mathcal{G}$ for $N = 1, 2, \ldots$. Therefore, $p_N := \min\{\lambda_{\min}(F(x, y_0)) : \ x \in \mathcal{G}_N\}$ is attained by some x_N and

$$p_N \leq p_{N+1} \quad \text{for all} \quad N = 1, 2, \ldots.$$

Let us prove that there exists some N_0 for which $p_{N_0} > 0$, which implies

$$F(x, y_0) \succ 0 \quad \text{for all} \quad x \in \mathcal{G}_{N_0}. \tag{27}$$

Indeed, otherwise $p_N \leq 0$ for all $N = 1, 2, \ldots$, and hence $\lim_{N\to\infty} p_N \leq 0$. Choose a subsequence N_ν with $x_{N_\nu} \to x_0$ to infer $0 \geq \lim_{\nu\to\infty} \lambda_{\min}(F(x_{N_\nu}, y_0)) =$

$\lambda_{\min}(F(x_0, y_0))$. This contradicts the choice of y_0 if we can show that $G(x_0) \preccurlyeq 0$. In fact, otherwise there exists a unit vector v with $\delta := v^T G(x_0)v > 0$. By convergence, there exists some K with $\|G(x_{N_\nu})\| \leq K$ for all ν. By density, there exists a sufficiently large ν such that $K\|v_i - v\|^2 + 2K\|v_i - v\| < \delta/2$ for some $i \in \{1, \ldots, N_\nu\}$. Since $v^T G(x_{N_\nu})v \to v^T G(x_0)v$, we can increase ν to even guarantee $v^T G(x_{N_\nu})v \geq \delta/2$, and we arrive at the following contradiction:

$$0 \geq v_i^T G(x_{N_\nu})v_i =$$
$$= (v_i - v)^T G(x_{N_\nu})(v_i - v) + 2v^T G(x_{N_\nu})(v_i - v) + v^T G(x_{N_\nu})v \geq$$
$$\geq -K\|v_i - v\|^2 - 2K\|v_i - v\| + \delta/2 > 0.$$

We are now in the position to apply Theorem 2 to (27) since, due to (26), the constraint qualification is trivially satisfied. Hence, there exist $\epsilon > 0$ and polynomial matrices $U_i(x)$ with p columns, $i = 1, \ldots, N_0$, such that

$$F(x, y_0) - \epsilon I + \sum_{i=1}^{N_0} [U_i(x)^T U_i(x)](v_i^T G(x)v_i) \text{ is SOS in } x. \tag{28}$$

With elementary Kronecker product manipulations and (4) we conclude

$$\begin{aligned} [U_i(x)^T U_i(x)](v_i^T G(x)v_i) &= \operatorname{tr}_p \left([U_i(x)^T U_i(x)] \otimes (v_i^T G(x)v_i)\right) \\ &= \operatorname{tr}_p \left(([U_i(x)^T U_i(x)] \otimes v_i^T)(I_p \otimes G(x))(I_p \otimes v_i)\right) \\ &= \operatorname{tr}_p \left(([U_i(x)^T U_i(x)] \otimes v_i v_i^T)(I_p \otimes G(x))\right) \\ &= ((U_i(x) \otimes v_i^T)^T (U_i(x) \otimes v_i^T), G(x))_p. \end{aligned}$$

With the SOS polynomial matrix $S(x) := \sum_{i=1}^{N_0} (U_i(x) \otimes v_i^T)^T (U_i(x) \otimes v_i^T)$, we infer that $F(x, y_0) - \epsilon I + (S(x), G(x))_p$ equals the left-hand side in (28) and is hence SOS in x. Therefore, y_0 is feasible for (6). ∎

5. Construction of LMI relaxations

Let us choose monomial vectors $u(x)$, $u_0(x)$ of length n_u, n_{u_0} and parameterize the SOS matrices $S(x)$, $S_0(x)$ with $Z \succcurlyeq 0$, $Z_0 \succcurlyeq 0$ as in Lemma 1 respectively. The infimal $c^T y$ for which there exist $\epsilon > 0$, $Z_0 \succcurlyeq 0$, $Z \succcurlyeq 0$ that satisfy

$$F(x, y) - \epsilon I_p + ((u(x) \otimes I_{pq})^T Z(u(x) \otimes I_{pq}), G(x))_p = (u_0(x) \otimes I_p)^T Z_0(u_0(x) \otimes I_p)$$

defines an upper bound of the value $u_{\rm opt}$ of (6), and hence also an upper bound of the value $v_{\rm opt}$ of (1), even without constraint qualification. Clearly, computing this bound amounts to solving a finite dimensional linear SDP. The bound does not increase, and is often improved, if adding additional monomials to the basis vectors $u(x)$, $u_0(x)$ respectively. If $u(x)$, $u_0(x)$ comprise all monomials up to a certain total degree d, the value of the computed upper bound is guaranteed to converge to $v_{\rm opt}$ for $d \to \infty$ if the constraint qualification of Theorem 1 is satisfied.

Let us finally clarify how the description of the SDP can be made explicit. For this purpose we choose pairwise different monomials $w_1(x), \ldots, w_{n_w}(x)$ such that there exist symmetric matrices P_j^0, P_j with

$$u_0(x)u_0(x)^T = \sum_{j=1}^{n_w} P_j^0 w_j(x), \quad (I_q \otimes u(x))G(x)(I_q \otimes u(x)^T) = \sum_{j=1}^{n_w} P_j w_j(x),$$

as well as symmetric-valued affine mappings $A_j(y, \epsilon)$ with

$$F(x, y) - \epsilon I_p = \sum_{j=1}^{n_w} A_j(y, \epsilon) w_j(x).$$

If Q_0 denotes the permutation for which $u_0(x) \otimes I_p = Q_0(I_p \otimes u_0(x))$, we infer

$$(u_0(x) \otimes I_p)^T Z_0 (u_0(x) \otimes I_p) = \mathrm{tr}_p (I_p \otimes u_0(x)^T) Q_0^T Z_0 Q_0 (I_p \otimes u_0(x)) =$$

$$= \mathrm{tr}_p \left(Q_0^T Z_0 Q_0 (I_p \otimes u_0(x) u_0(x)^T) \right) = \sum_{j=1}^{n_w} (Q_0^T Z_0 Q_0, P_j^0)_p \, w_j(x),$$

where we made use of (4). Similarly, with the permutation Q satisfying $u(x) \otimes I_{pq} = Q(I_{pq} \otimes u(x))$, we obtain

$$((u(x) \otimes I_{pq})^T Z (u(x) \otimes I_{pq}), G(x))_p =$$

$$= \mathrm{tr}_p \left((I_p \otimes I_q \otimes u(x)^T) Q^T Z Q (I_p \otimes I_q \otimes u(x))(I_p \otimes G(x)) \right) =$$

$$= \mathrm{tr}_p \left(Q^T Z Q [I_p \otimes (I_q \otimes u(x)) G(x) (I_q \otimes u(x)^T)] \right) =$$

$$= \sum_{j=1}^{n_w} \mathrm{tr}_p \left(Q^T Z Q [I_p \otimes P_j] \right) w_j(x) = \sum_{j=1}^{n_w} (Q^T Z Q, P_j)_p w_j(x).$$

Therefore the upper bound relaxation requires to infimize $c^T y$ over

$$\epsilon > 0, \quad Z_0 \succcurlyeq 0, \quad Z \succcurlyeq 0, \tag{29}$$

$$A_j(y, \epsilon) + (Q^T Z Q, P_j)_p = (Q_0^T Z_0 Q_0, P_j^0)_p, \; j = 1, ..., n_w. \tag{30}$$

The size of the relaxation is determined by three SDP constraints (29) in $\mathcal{S}^1 \times \mathcal{S}^{pn_{u_0}} \times \mathcal{S}^{pqn_u}$, and the n_w affine equation constraints (30) in $\mathcal{S}^p$. Moreover, it involves the unknowns y, ϵ, Z_0, Z of size n, 1, pn_{u_0}, pqn_u, which sums up to

$$n + 1 + 0.5\, pn_{u_0}(pn_{u_0} + 1) + 0.5\, pqn_u(pqn_u + 1) \tag{31}$$

scalar decision variables. Generically, this number is reduced by $0.5\, n_w p(p+1)$ through (30). This indeed reveals that the size of the relaxation grows bi-quadratically in the dimension p of $F(x, y)$ and the dimension q of $G(x)$ respectively.

6. Relaxations based on the S-procedure

In this section, we intend to briefly address the relation of the suggested approach to relaxations based on the so-called full-block S-procedure, as exposed with a variety of concrete control applications in [32,33] and the references therein. Indeed, even if $F(x,y)$ is rational in x without pole at $x=0$, one can construct a so-called linear fractional representation

$$F(x,y) = \Delta(x) \star \begin{pmatrix} A & B \\ C(y) & D(y) \end{pmatrix} := D(y) + C(y)\Delta(x)(I - A\Delta(x))^{-1}B \quad (32)$$

where A, B are fixed matrices and $C(y)$, $D(y)$, $\Delta(x)$ are matrix-valued affine mappings in y and x respectively. Let us stress that many robust control problems involve constraints that are naturally formulated in this fashion [9,37].

We assume the constraint qualification of Theorem 1 to hold such that $\mathcal{G}$ is compact. One can then apply the full block S-procedure [16,30] to infer that

$$\det(I - A\Delta(x)) \neq 0 \ \text{ and } \ F(x,y) \succ 0 \ \text{ for all } \ x \in \mathcal{G}$$

iff there exists a symmetric so-called multiplier matrix P such that

$$\begin{pmatrix} \Delta(x) \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta(x) \\ I \end{pmatrix} \succ 0 \quad \text{for all} \quad x \in \mathcal{G}, \quad (33)$$

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^T P \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} - \begin{pmatrix} 0 & I \\ C(y) & D(y) \end{pmatrix}^T \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ C(y) & D(y) \end{pmatrix} \prec 0. \quad (34)$$

Although the uncertainties x enter the original problem in a rational fashion, we observe that (33) is quadratic in x and affine in P. We can hence apply

Theorem 1 to infer that (33) holds iff there exist $\epsilon > 0$ and SOS matrices $S(x)$, $S_0(x)$ with

$$\begin{pmatrix} \Delta(x) \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta(x) \\ I \end{pmatrix} + (S(x), G(x))_p = S_0(x) + \epsilon I. \tag{35}$$

If constraining $S(x)$, $S_0(x)$ to be SOS with respect to fixed monomial basis vectors $u(x)$, $u_0(x)$, it is possible to turn (35) into a finite-dimensional genuine LMI constraint (Section 5). By infimizing $c^T y$ over these LMI's combined with (34), one determines an upper bound of v_{opt}, which is again guaranteed to converge to v_{opt} if $u(x)$, $u_0(x)$ comprise all monomials up to a certain degree, and if the degree grows to infinity.

Even for short monomial bases, it can often be observed for specific problem instances that the value of the relaxation equals v_{opt}, and is hence exact. Since the suggested LMI relaxation falls in the general class as discussed in [33], we can directly apply all results in this reference in order to numerically verify exactness in practice. Finally, it is suggested in [33] how to construct asymptotically exact relaxation families if $\mathcal{G}$ is a polytope with an explicit description in terms of its convex hull generators. In contrast, the above technique offers the extension to implicitly described polytopes (for affine diagonal $G(x)$), and it even includes uncertainty regions with a general description by LMI's (for affine $G(x)$) or by polynomial matrix inequalities. This flexibility goes far beyond approaches that have been proposed in the literature so far.

Relax-ation	Line in Figure 1	Monomial bases				
		$u_0(x)^T$	$u_1(x)^T$	$u_2(x)^T$	$u_3(x)^T$	$u_4(x)^T$
A	$--$	$(1, x_1, x_2)$	$(1, x_1)$	$(1, x_1)$	1	1
B	$\cdots$	$(1, x_1, x_2)$	1	1	$\begin{pmatrix} 1 \; x_1 \end{pmatrix}$	$\begin{pmatrix} 1 \; x_2 \end{pmatrix}$
C	$-$	$(1, x_1, x_2)$	$(1, x_1, x_2)$	$(1, x_1, x_2)$	$(1, x_1, x_2)$	$(1, x_1, x_2)$
D	$-$	$(1, x_1, x_2, x_1x_2)$	$(1, x_1, x_2)$	$(1, x_1, x_2)$	$(1, x_1, x_2)$	$(1, x_1, x_2)$

Table 1. Sos bases employed for relaxations.

7. Numerical example

Consider a variation of an example in [22,33]: Compute the infimal y with

$$\begin{pmatrix} y & f_a(x) \\ f_a(x) & y \end{pmatrix} \succ 0 \quad \text{for all} \quad x \in \mathcal{G} := \{x \in \mathbb{R}^2 : \; g_j(x) \geq 0, \; j = 1, \ldots, 4\}$$

for 20 equidistant values of $a \in [0.5, 1]$, where

$$f_a(x_1, x_2) := 1 - \frac{2ax_1^2 x_2}{2 - 2ax_2 + ax_1x_2 - x_1} - \frac{x_2(2ax_2 + x_1^2 + ax_1^2x_2 - 2)}{2 - 2a^2x_2^2 - x_1^2 + x_1^2a^2x_2^2},$$

$$g_1(x) := 0.8 - x_1, \; g_2(x) = 0.7 + x_1, \; g_3(x) = 0.7 - x_2, \; g_4(x) = 0.65 + x_2.$$

Note that the optimal value equals $\sup_{x \in \mathcal{G}} |f_a(x)|$. Moreover, one easily determines a linear fractional representation of f_a with $\Delta(x) = \text{diag}(x_1 I_2, x_2 I_2)$. Since the polytope $\mathcal{G}$ is compact, the constraint qualification is satisfied (Theorem 4), and we can either apply the relaxation in Section 6 (labeled by **A**, **B**, **C**) or we can use the direct approach as in Section 5 (labeled by **D**), both based on the special version of Theorem 1 as discussed in Remark 2. The corresponding upper bounds have been computed for SOS matrices $S_j(x)$, $j = 0, 1, \ldots, 4$, with respect to the monomial bases $u_j(x)$, $j = 0, 1, \ldots, 4$, as given in Table 1.

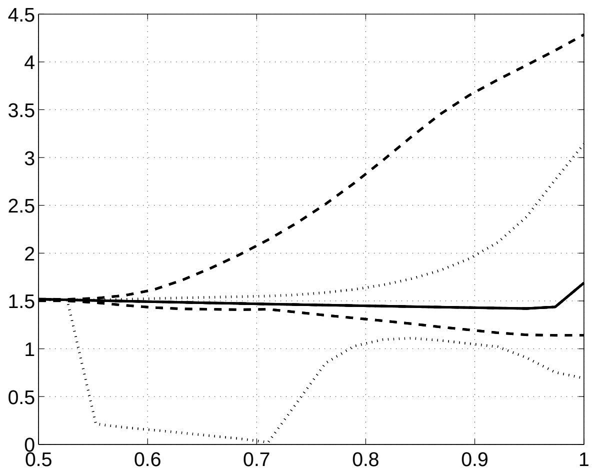

Fig. 1. Upper and lower bounds for relaxations **A** (dashed), **B** (dotted), **C** and **D** (solid).

Figure 1 depicts the computed upper bounds on v_{opt}, together with lower bounds that are obtained by constructing a worst-case uncertainty as described in [33]. Clearly, **A**, **B** suffer from a relaxation gap, while both **C** and **D** are exact as confirmed by the exactness test of [33] for the S-procedure relaxation **C**.

8. Conclusions

For a general class of robust SDP problems with polynomial or rational dependence on the uncertainties, we have shown how to approximately compute upper bounds on the optimal value. The uncertainty region can admit an implicit description in terms of a polynomial matrix inequality, with LMI regions or compact polytopes as special, yet practically important, instances. Our major technical contributions comprise an extension of Putinar's SOS representation result to matrix-valued polynomials, and the verification of the related constraint qualification for uncertainty polytopes with degree bounds on the certificate polynomials. With a suitable matrix-version of the constraint qualification, we have revealed how to construct a convergent sequence of relaxations whose size grows bi-quadratically in the dimension of the describing matrices. We have finally pointed out how one can numerically verify the exactness of even small-sized relaxations for problems with linear fractional representations.

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A. An auxiliary result

In addition to $L(P)$ in Section 3.1, define

$$\Delta(P) := \underbrace{\max\{|\alpha| : \ D_\alpha P(0) \neq 0, \ \alpha \in \mathbb{N}_0^m\}}_{\deg_{\max}(P)} - \underbrace{\min\{|\alpha| : \ D_\alpha P(0) \neq 0, \ \alpha \in \mathbb{N}_0^m\}}_{\deg_{\min}(P)}.$$

Then, among the following properties, only (36) is not elementary.

Lemma 5. *For two matrix-valued m-variable polynomials P and Q (of compatible dimension) one has $\Delta(PQ) \leq \Delta(P) + \Delta(Q)$, $L(P+Q) \leq L(P) + L(Q)$, and $L(\xi P) \leq |\xi| L(P)$ for all $\xi \in \mathbb{R}$. Moreover*

$$L(PQ) \leq \min\{\Delta(P) + 1, \Delta(Q) + 1\} L(P) L(Q). \tag{36}$$

Proof. To prepare the proof of (36), let us apply the generalized product rule to the function $s^m = s^{\hat{m}} s^{m-\hat{m}}$ with $s(x) = x_1 + \cdots + x_m$ and with $n = |\alpha|$ for $\alpha \in \mathbb{N}_0^m$:

$$\underbrace{D_\alpha s^m(0)}_{=|\alpha|!} = \sum_{0 \leq k \leq \alpha, \, |k| = \hat{m}} \frac{\alpha!}{k!} \underbrace{D_k s^{\hat{m}}(0)}_{=|k|!} \underbrace{D_{\alpha - k} s^{(m-\hat{m})}(0)}_{=(|\alpha|-|k|)!}. \tag{37}$$

Using $\sum_\emptyset = 0$, we hence observe again with the general product rule that

$$L(PQ) = \max_{\alpha\in\mathbb{N}_0^m} \frac{1}{|\alpha|!}\|(D_\alpha PQ)(0)\| =$$

$$= \max_{\alpha\in\mathbb{N}_0^m} \frac{1}{|\alpha|!}\left\|\sum_{0\le k\le\alpha}\frac{\alpha!}{k!}D_kP(0)D_{\alpha-k}Q(0)\right\| \le$$

$$\le \max_{\alpha\in\mathbb{N}_0^m} \sum_{\substack{0\le k\le\alpha \\ \deg_{\min}(P)\le|k|\le\deg_{\max}(P) \\ \deg_{\min}(Q)\le|\alpha-k|\le\deg_{\max}(Q)}} \frac{\alpha!}{k!}\frac{|k|!(|\alpha|-|k|)!}{|\alpha|!}\frac{\|D_kP(0)\|}{|k|!}\frac{\|D_{\alpha-k}Q(0)\|}{(|\alpha|-|k|)!} \le$$

$$\le \max_{\alpha\in\mathbb{N}_0^m} \sum_{\substack{m\in[\deg_{\min}(P),\deg_{\max}(P)] \\ \hat{m}\in[|\alpha|-\deg_{\max}(Q),|\alpha|-\deg_{\min}(Q)]}} \sum_{\substack{0\le k\le\alpha \\ |k|=\hat{m}}} \frac{\alpha!}{k!}\frac{|k|!(|\alpha|-|k|)!}{|\alpha|!} L(P)L(Q) =$$

$$= L(P)L(Q) \max_{\alpha\in\mathbb{N}_0^m} \sum_{\hat{m}\in[\deg_{\min}(P),\deg_{\max}(P)]\cap[|\alpha|-\deg_{\max}(Q),|\alpha|-\deg_{\min}(Q)]} 1,$$

where the latter equation follows from (37). Since the sum consists of at most $\min\{\Delta(P)+1,\Delta(Q)+1\}$ nonzero terms, the result follows. ∎

B. Proof of Theorem 4

Let us first assume $g_i(x) = x_i$, $i = 1,\dots,m$, and $g_{m+1}(x) = 1 - e^Tx$. With $z(x) = \mathrm{col}(1,x)$, any quadratic function can be represented as $z(x)^TYz(x)$ with some symmetric matrix Y, and this function is SOS iff $Y \succcurlyeq 0$. In particular, with $r^2 - \|x\|^2 = z(x)^TRz(x)$ for $R = \mathrm{diag}(r^2,-1,\cdots,-1)$, our goal is to prove that there exist r and $Y_i \succ 0$, $i = 0,\dots,m$, with

$$z(x)^TRz(x) = r^2 - \|x\|^2 = z(x)^TY_0z(x)(1-e^Tx) + \sum_{i=1}^m z(x)^TY_iz(x)x_i =$$

$$= \langle z(x)z(x)^T, Y_0\rangle + \sum_{i=1}^m \langle x_iz(x)z(x)^T, Y_i - Y_0\rangle. \quad (38)$$

If $w_1(x) = 1$, and if $w_j(x)$, $j = 2, \ldots, k$, is the list of all pairwise different monomials in m variables of degree at least one and at most three (in any ordering), we can determine the representations

$$z(x)z(x)^T = \sum_{j=1}^{k} Q_j^0 w_j(x),\ \ x_i z(x)z(x)^T = \sum_{j=1}^{k} Q_j^i w_j(x)$$

to infer $z(x)^T R z(x) = \langle z(x)z(x)^T, R\rangle = \sum_{j=1}^{k}\langle Q_j^0, R\rangle w_j(x)$. Our problem reduces to proving the existence of positive definite $Y_0, Y_1, \ldots, Y_m$ with

$$\langle Q_j^0, R\rangle = \langle Q_j^0, Y_0\rangle + \sum_{i=1}^{m}\langle Q_j^i, Y_i - Y_0\rangle,\quad j = 1, \ldots, k. \tag{39}$$

This is true iff the infimum of all t with $Y_i + tI \succcurlyeq 0$, $i = 0, \ldots, m$, and with (39) is negative. Since this programm is strictly feasible (choose $Y_i = R$, $i = 0, \ldots, m$, and t large), we can dualize without gap. With Lagrange multipliers $\Gamma_i \succcurlyeq 0$, $i = 0, \ldots, m$, and γ_j, $j = 1, \ldots, k$, (39) has positive definite solutions iff

$$\max_{\substack{\Gamma_i \succcurlyeq 0,\ 1 - \langle \sum_{i=0}^{m} \Gamma_i, I\rangle = 0, \\ \sum_{j=1}^{k}\gamma_j Q_j^0 - \sum_{i=1}^{m}\sum_{j=1}^{k}\gamma_j Q_j^i - \Gamma_0 = 0,\ \sum_{j=1}^{k}\gamma_j Q_j^i - \Gamma_i = 0,\ i = 1, \ldots, m}} -\langle \sum_{j=1}^{k}\gamma_j Q_j^0, R\rangle < 0.$$

With $M_i^m(\gamma) := \sum_{j=1}^{k}\gamma_j Q_j^i$, $i = 0, \ldots, m$, this is equivalent to

$$M_i^m(\gamma) \succcurlyeq 0,\ \ i = 1, \ldots, m,\ \ M_0^m(\gamma) - \sum_{i=1}^{m} M_i^m(\gamma) \succcurlyeq 0,\ \ \langle M_0^m(\gamma), I\rangle = 1 \quad \Rightarrow$$

$$\Rightarrow \quad \langle M_0^m(\gamma), R\rangle > 0. \tag{40}$$

We have indicated the number of variables m in this definition since it is essential to exploit the structural relation of $M_i^{m-1}(\gamma)$ and $M_i^m(\gamma)$ that is easily identified as follows: If

$$M_m^m(\gamma) = \begin{pmatrix} c_0 & \cdots & c_{m-1} & c_m \\ & & & \\ d_0 & \cdots & d_{m-1} & d_m \end{pmatrix} \in \mathbb{R}^{(m+1)\times(m+1)}$$

for suitable $\mathbb{R}^m$-vectors c_i and scalars d_i then

$$M_i^m(\gamma) = \begin{pmatrix} M_i^{m-1}(\gamma) & c_i \\ c_i^T & d_i \end{pmatrix} \in \mathbb{R}^{(m+1)\times(m+1)} \quad \text{for} \quad i = 0, \ldots, m-1. \tag{41}$$

With the all-ones vector e and the standard unit vectors e_i let us define the length $(m+1)$-vectors

$$f_i = e - e_{i+1}, \quad g_i = e - e_1 - e_{i+1} = f_0 - e_{i+1}, \quad i = 0, \ldots, m$$

as well as the scalar

$$s^m := e^T \left(M_0^m(\gamma) - \sum_{i=1}^m M_i^m(\gamma) \right) e + 0.5 \left(\sum_{i=1}^m f_i^T M_i^m(\gamma) f_i + \sum_{i=1}^m g_i^T M_i^m(\gamma) g_i \right).$$

As an essential ingredient of our proof, we exploit that $s^m \geq 0$ if the hypothesis in (40) is satisfied. Moreover the specific vectors in the definition of s^m are chosen to be able to obtain the following recursion. Indeed with the corresponding vectors e, e_i, f_i, g_i of length m we have, due to (41),

$$\begin{aligned} s^m = {} & e^T M_0^{m-1}(\gamma) e + 2c_0^T e + d_0 - \sum_{i=1}^{m-1} (e^T M_i^{m-1}(\gamma) e + 2c_i^T e + d_i) - \\ & - \sum_{i=0}^{m} (c_i^T e + d_i) + 0.5 \left(\sum_{i=1}^{m-1} (f_i^T M_i^{m-1}(\gamma) f_i + 2c_i^T f_i + d_i) + \right. \\ & \left. + \, (g_i^T M_i^{m-1}(\gamma) g_i + 2c_i^T g_i + d_i) + \sum_{i=0}^{m-1} c_i^T e + \sum_{i=1}^{m-1} c_i^T f_0 \right), \end{aligned}$$

which simplifies to

$$\begin{aligned} s^m = {} & s^{m-1} + 2c_0^T e + d_0 - \sum_{i=1}^{m-1} (3c_i^T e + 2d_i) - (c_0^T e + d_0) - (c_m^T e + d_m) + \\ & + \sum_{i=1}^{m-1} (c_i^T f_i + c_i^T g_i + d_i) + 0.5 \left(\sum_{i=1}^{m-1} c_i^T e + c_i^T f_0 \right) + 0.5 c_0^T e. \end{aligned}$$

Since $c_m^T e = d_0 + \cdots + d_{m-1}$ (just because $M_m^m(\gamma)$ is symmetric), we infer

$$s^m = s^{m-1} + 1.5c_0^T e - \sum_{i=1}^{m-1} 2d_i - (d_0 + d_m) + \sum_{i=1}^{m-1} c_i^T (f_i + g_i + 0.5f_0 - 2.5e).$$

With $f_i + g_i + 0.5f_0 = (e - e_{i+1}) + (e - e_1 - e_{i+1}) + 0.5(e - e_1) = 2.5e - 2e_{i+1} - 1.5e_1$ and $c_0^T e - c_1^T e_1 - \cdots - c_{m-1}^T e_1 = c_0^T e_1$ (again since $M_m^m(\gamma)$ is symmetric), we obtain

$$s^m = s^{m-1} - d_0 - d_m + 1.5c_0^T e_1 - \sum_{i=1}^{m-1} 2d_i - 2\sum_{i=1}^{m-1} c_i^T e_{i+1}.$$

Let us finally exploit $d_i = e_{m+1}^T M_i^m(\gamma) e_{m+1}$, $i = 0, \ldots, m$, and $c_i^T e_{i+1} = e_{i+1}^T M_m^m(\gamma) e_{i+1}$, $i = 1, \ldots, m-1$ as well as

$$2c_0^T e_1 = e_1^T M_0^{m-1}(\gamma) e_1 + e_{m+1}^T M_0^m(\gamma) e_{m+1} - (e_1 - e_{m+1})^T M_0^m(\gamma)(e_1 - e_{m+1})$$

to end up with

$$s^m = s^{m-1} + \frac{3}{4} e_1^T M_0^{m-1}(\gamma) e_1 - \frac{1}{4} e_{m+1}^T M_0^m(\gamma) e_{m+1} - e_{m+1}^T M_m^m(\gamma) e_{m+1} -$$
$$-\frac{3}{4}(e_1 - e_{m+1})^T M_0^m(\gamma)(e_1 - e_{m+1}) - 2\sum_{i=1}^{m-1} e_{m+1}^T M_i^m(\gamma) e_{m+1} - 2\sum_{i=1}^{m-1} e_{i+1}^T M_m^m(\gamma) e_{i+1}.$$

Due to the choice $w_1(x) = 1$, we have $e_1^T M_0^\nu(\gamma) e_1 = \gamma_1$ for all $\nu = 1, 2, \ldots, m$. Hence for $m = 1$ we conclude

$$M_1^1(\gamma) = \begin{pmatrix} c_0 & c_1 \\ d_0 & d_1 \end{pmatrix} = \begin{pmatrix} c_0 & d_0 \\ d_0 & d_1 \end{pmatrix} \quad \text{and} \quad M_0^1(\gamma) = \begin{pmatrix} \gamma_1 & c_0 \\ c_0^T & d_0 \end{pmatrix}$$

and thus

$$s^1 = (\gamma_1 - c_0) + 2(c_0 - d_0) + (d_0 - d_1) + 0.5c_0 =$$
$$= 1.75e_1^T M_0^1(\gamma) e_1 - 0.25e_2^T M_0^1(\gamma) e_2 - e_2^T M_1^1(\gamma) e_2 - 0.75(e_1 - e_2)^T M_0^1(\gamma)(e_1 - e_2).$$

Let us now assume that the hypothesis in (40) is satisfied. Just due to (41) we can conclude $M_i^\nu(\gamma) \succcurlyeq 0$ for $i = 0, \ldots, \nu$ and $\nu = m-1, m-2, \ldots, 1$ (by induction). This implies

$$s^1 \leq 1.75 e_1^T M_0^1(\gamma) e_1 - 0.25 e_2^T M_0^1(\gamma) e_2,$$
$$0 \leq s^\nu \leq s^{\nu-1} + 0.75 e_1^T M_0^{\nu-1}(\gamma) e_1 - 0.25 e_{\nu+1}^T M_0^\nu(\gamma) e_{\nu+1}, \quad \nu = 2, \ldots, m.$$

Since $e_1^T M_0^{\nu-1}(\gamma) e_1 = e_1^T M_0^m(\gamma) e_1$ and $e_{\nu+1}^T M_0^\nu(\gamma) e_{\nu+1} = e_{\nu+1}^T M_0^m(\gamma) e_{\nu+1}$ for $\nu = 2, \ldots, m$, we conclude by induction that

$$(1 + 0.75m) e_1^T M_0^m(\gamma) e_1 - 0.25 \sum_{\nu=1}^{m} e_{\nu+1}^T M_0^m(\gamma) e_{\nu+1} \geq s^m \geq 0$$

and consequently, for $r = \sqrt{5+3m}$,

$$\langle R, M_0^m(\gamma)\rangle = \langle e_1 e_1^T, M_0^m(\gamma)\rangle + \underbrace{\langle (4+3m) e_1 e_1^T - \sum_{\nu=2}^{m+1} e_{\nu+1} e_{\nu+1}^T, M_0^m(\gamma)\rangle}_{\geq 0} \geq 0.$$

If $\langle R, M_0^m(\gamma)\rangle = 0$ we infer $0 = \langle e_1 e_1^T, M_0^m(\gamma)\rangle = e_1^T M_0^m(\gamma) e_1$. This implies $e_1^T M_0^m(\gamma) e_{\nu+1} = 0$ and thus $e_1^T M_\nu^m(\gamma) e_1 = 0$ and thus $e_1^T M_\nu^m(\gamma) e_{\nu+1} = 0$ and thus $e_{\nu+1}^T M_0^m(\gamma) e_{\nu+1} = 0$ and thus $M_0^m(\gamma) e_{\nu+1} = 0$ for $\nu = 2, \ldots, m$. Therefore, $M_0^m(\gamma) = 0$ in contradiction to $\langle M_0^m(\gamma), I\rangle = 1$. This proves (40) and Theorem 4 for the particularly chosen constraint functions.

Now assume that $g_i(x)$, $i = 1, \ldots, q$, are general. If $\mathcal{G}$ is contained in the simplex $\Sigma := \{x \in \mathbb{R}^m : \ x \geq 0, \ e^T x \leq 1\}$, there exist (by LP duality) $v_{ij} \geq 0$, $i = 0, \ldots, n$, $j = 1, \ldots, q$, such that

$$1 - e^T x = \sum_{j=1}^{q} v_{0j} g_j(x), \quad x_i = \sum_{j=1}^{q} v_{ij} g_j(x), \quad i = 1, \ldots, n.$$

If we combine with (38), we conclude

$$r^2 - \|x\|^2 = \sum_{i=0}^{n} z(x)^T Y_i z(x) \sum_{j=1}^{q} v_{ij} g_j(x) = \sum_{j=1}^{q} \left[z(x)^T \left(\sum_{i=0}^{n} Y_i v_{ij} \right) z(x) \right] g_j(x)$$

which is the desired representation.

Finally, if $\mathcal{G} \not\subset \Sigma$, a suitably scaled and shifted version $(\mathcal{G} - c)/\alpha$ described as $\{z \in \mathbb{R}^m : \; g_j(\alpha z + c) \geq 0, \; j = 1, \ldots, q\}$ is contained in Σ (by compactness). Then there exists β and SOS polynomials $\hat{\psi}_j(z)$ with $\beta^2 - \|z\|^2 = \sum_{j=1}^{q} \hat{\psi}_j(z) g_j(\alpha z + c)$ which implies $\beta^2 - \|(x-c)/\alpha\|^2 = \sum_{j=1}^{q} \hat{\psi}_j((x-c)/\alpha) g_j(x)$. It remains to observe that one can easily find positive r, t such that that $r^2 - \|x\|^2 - t(\beta^2 - \|(x-c)/\alpha\|^2) =: \psi_0(x)$ is non-negative for all $x \in \mathbb{R}^m$, and hence SOS. This finally leads to

$$r^2 - \|x\|^2 = s_0(x) + t(\beta^2 - \|(x-c)/\alpha\|^2) = s_0(x) + \sum_{j=1}^{q} [t\hat{\psi}_j((x-c)/\alpha)] g_j(x).$$

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