CYCLIC MODULES OF FINITE GORENSTEIN INJECTIVE DIMENSION AND GORENSTEIN RINGS

HANS–BJØRN FOXBY AND ANDERS J. FRANKILD

Abstract. The main result asserts that a local commutative noetherian ring is Gorenstein if it possesses a non-zero cyclic module of finite Gorenstein injective dimension. From this follows a classical result by Peskine and Szpiro: A local ring is Gorenstein if it admits a non-zero cyclic module of finite (classical) injective dimension.

The main result applies to local homomorphisms of local rings: If the target has finite Gorenstein injective dimension over the source, and the source is a homomorphic image of a Gorenstein ring, then the source is a Gorenstein ring. This, in turn, applies to the Frobenius endomorphism when the local ring has prime characteristic: If the ring is a homomorphic image of a Gorenstein ring, then it is Gorenstein precisely when some (equivalently, all) proper iteration of the Frobenius endomorphism turns the ring into a module of finite Gorenstein injective dimension over the ring.

Dedicated to the achievements of Phil Griffith

1. Introduction

In 1956 Auslander, Buchsbaum, and Serre showed that regular local rings are exactly the local rings for which every finitely generated module has finite projective dimension. Their characterization works equally well if projective dimension is replaced by injective dimension, and it is not necessary to consider finitely generated modules only. The Auslander–Buchsbaum Equality [5] states the following: If M is a non-zero finitely generated module over a local ring R and the projective dimension $\operatorname{pd}_R M$ finite, then

$$\operatorname{pd}_R M = \operatorname{depth} R - \operatorname{depth}_R M\,.$$

Moreover, for every integer n in the range from 0 to $\operatorname{depth} R$, it is easy to construct a finitely generated module M of projective dimension n. For the injective dimension of finitely generated modules, the story is somewhat different. Bass' paper [12] on the ubiquity of Gorenstein rings (1963) determines the injective dimension of a non-zero finitely generated R–module N of finite injective dimension as the value

$$\operatorname{id}_R N = \operatorname{depth} R.$$

In the same paper Bass writes [12, rmk. p. 14] that *it is conceivable that the existence of a finitely generated module of finite injective dimension over a commutative local noetherian ring implies that it is Cohen–Macaulay.* This remark is often referred to as Bass' question; it was answered—in the affirmative—over the time span 1972–86 by Peskine and Szpiro [36], and Roberts [37] for local rings of prime

Date: 23.01.2007.

1991 *Mathematics Subject Classification.* 13D05, 18E30, 18G10.

Key words and phrases. G–dimension, Gorenstein injective dimension, cyclic modules, Auslander category, Bass category, Frobenius endomorphism.

The first author was supported by FNU, the Danish Research Council.

The second author is a Steno stipendiat supported by FNU, the Danish Research Council.

characteristic, by Hochster [29] for rings containing a field, and finally for mixed characteristic by Roberts [38].

Roughly ten years after Auslander, Buchsbaum, and Serre characterized regular local rings, Auslander [3], and Auslander and Bridger [4] developed the theory of G–dimensions for finitely generated modules; it's likely that the "G" stands for Gorenstein, because the G–dimension detects Gorenstein rings exactly as projective dimension detects regular rings.

However, this G–dimension is only defined for finitely generated modules. To extend the theory of Auslander and Bridger to all modules, Enochs and Jenda [19] developed a notion of Gorenstein projective and injective modules, and Enochs, Jenda, and Torrecillas [20] developed a notion of Gorenstein flat modules; consult (3.1)–(3.3) for the definitions. By taking resolutions with these modules one obtains Gorenstein projective, Gorenstein injective, and Gorenstein flat dimensions. The Gorenstein projective dimension is a refinement for the projective dimension in the sense that the former is always less or equal to the latter, and if the latter is finite then it equals the former. Likewise, the Gorenstein flat and Gorenstein injective dimensions are refinements of the flat and injective dimensions, respectively. Moreover, for finitely generated modules the Gorenstein projective dimension agrees with the G–dimension; this follows by a result by Avramov, Buchweitz, Martsinkovsky, and Reiten (see [14]).

It is a notable feature of, say, Gorenstein projective dimension, that the underlying local ring R is Gorenstein precisely when the Gorenstein projective dimension of any module is finite. The other two homological dimensions have the same property.

Returning to the paper by Peskine and Szpiro, they obtain the following corollary to their answer to Bass' question for rings of prime characteristic: If the finitely generated module of finite injective dimension is cyclic, then the underlying ring is actually Gorenstein; see [36, cor. (5.3)]. Moreover, Peskine and Szpiro [36, thm. (5.5)] also show that this corollary holds for every local ring.

Theorem (Peskine–Szpiro). *If there exists a non-zero cyclic R–module N with the injective dimension finite, then R is Gorenstein.*

The next theorem—which is the main result of this paper—is a Gorenstein version of previous one; it is repeated as Theorem (4.7) and is proved there.

Theorem A. *If there exists a non-zero cyclic R–module N with the Gorenstein injective dimension finite, then R is Gorenstein.*

The theorem has been proved in special cases by Takahashi; see [41, thm. 3.5]. As an immediate consequence of our theorem it follows that R is Gorenstein if and only if R has finite Gorenstein injective dimension as an R–module; this is the local commutative version of the main result in Holm [30]. Another immediate consequence is that R is Gorenstein if and only if the residue field k has finite Gorenstein injective dimension as an R–module.

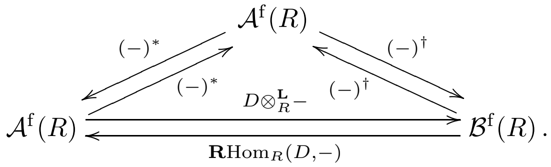
The proof of Theorem A uses Bass series $\mathrm{I}_R^M(t)$ and Poincaré series $\mathrm{P}_M^R(t)$ for finitely generated R–modules M. These series are formal power series, and their coefficients are the Bass numbers and the Betti numbers, respectively. The series and their basic properties are recalled in section 4, and Proposition (4.3) contains the next result which holds over homomorphic images of local Gorenstein rings: If N is a finitely generated R–module of finite Gorenstein injective dimension, then there exists a finitely generated R–module K of finite Gorenstein projective dimension such that there is an equality of formal power series

$$\mathrm{P}_N^R(t)t^{\operatorname{depth} R} = \mathrm{P}_K^R(t)\,\mathrm{I}_R(t)\,.$$

The proof of Proposition (4.3) requires, in turn, auxiliary techniques, and these involve two categories of bounded complexes of R–modules, namely the Auslander category $\mathcal{A}(R)$ and the Bass category $\mathcal{B}(R)$; these are defined whenever the ring R admits a dualizing complex D; see (2.5) and (2.6). In this introduction we focus on the two full subcategories of $\mathcal{A}(R)$ and $\mathcal{B}(R)$ consisting of complexes with *degreewise finitely generated* homology; these are denoted $\mathcal{A}^{\mathrm{f}}(R)$ and $\mathcal{B}^{\mathrm{f}}(R)$. The connection to Gorenstein injective dimension is provided by the main result in [16] by Christensen, Frankild, and Holm; it implies, in particular, that a finitely generated R–module is of finite Gorenstein injective dimension if and only if it belongs to $\mathcal{B}^{\mathrm{f}}(R)$. The full statement is in (3.5). Important for the proof of Theorem A are the next four functors

$$\begin{array}{ll} (-)^* = \mathbf{R}\mathrm{Hom}_R(-,R) & D \otimes^{\mathbf{L}}_R - \\ (-)^\dagger = \mathbf{R}\mathrm{Hom}_R(-,D) & \mathbf{R}\mathrm{Hom}_R(D,-)\,. \end{array}$$

The two functors in the last column are discussed in further detail in paragraph (2.6) where it is proved that these functors fit into the next diagram



Here the inner triangle and the outer triangle are commutative, each of the two pair of parallel tilted arrows provides a duality of categories, and the pair of horizontal arrows provides an equivalence of categories.

In Avramov, Iyengar and Miller [11, thm. 13.2] it is stated that: If $\varphi\colon R \longrightarrow S$ is a local homomorphism such that the target S has finite injective dimension over the source R, then R is Gorenstein and S has finite flat dimension over R. The next is a Gorenstein version of this result, modulo the fact that we require R to be a homomorphic image of a Gorenstein ring; see Theorem (5.4) and its proof.

Theorem B. *Assume that $\varphi\colon R \longrightarrow S$ is local homomorphism such that the source R is a homomorphic image of a Gorenstein local ring. If the target S has finite Gorenstein injective over the source R, then R is Gorenstein and S has has finite Gorenstein flat dimension over R.*

When the local ring R is of prime characteristic p, Theorem C below applies, in particular, to the Frobenius endomorphism given by $r \longmapsto r^p$. For every local endomorphism $\varphi\colon R \longrightarrow R$ and every R–module M, we let ${}^{\varphi^n}\!M$ denote M viewed as an R–module via φ^n, that is, the abelian group M equipped with the multiplication $(r,m) \longmapsto \varphi^n(r)m$. Recall, that a local ring is regular precisely when the projective dimension, or equivalently the flat dimension, or equivalently the injective dimension of its residue field is finite. Replacing the previous dimensions with their Gorenstein counterparts, we get a similar characterization of Gorenstein rings. When R is of prime characteristic, we may replace the residue field k with ${}^{\varphi^n}\!R$ and obtain the same conclusion. The classical results by Kunz [34, (2.1)] and

Rodicio [39] show that the ring is regular precisely when ${}^{\varphi^n}\!R$ has finite flat dimension (or, equivalently finite projective dimension). On the other hand, Avramov, Iyengar and Miller [11, thm. 13.3] showed that this is equivalent to the finiteness of the injective dimension of ${}^{\varphi^n}\!R$. Moreover, the result [31, thm. 6.5] by Iyengar and Sather-Wagstaff implies that the ring is Gorenstein exactly when the Gorenstein flat dimension of ${}^{\varphi^n}\!R$ is finite.

The Frobenius endomorphism is a particular instance of a local endomorphism $\varphi\colon (R,\mathfrak{m}) \longrightarrow (R,\mathfrak{m})$ such that $\varphi^i(\mathfrak{m}) \subseteq \mathfrak{m}^2$ for some integer $i \geqslant 1$; this condition is equivalent to the condition that for every element x from $\mathfrak{m}$ the sequence $(\varphi^i(x))_{i\geqslant 1}$ converges to zero in the $\mathfrak{m}$–adic topology. Such an endomorphism is called a *contraction*.

The next result is part of Theorem (5.8) and concerns endomorphisms that are not necessarily contractions.

Theorem C. *Let $\varphi\colon (R,\mathfrak{m}) \longrightarrow (R,\mathfrak{m})$ be a local endomorphism and assume that R is a homomorphic image of a Gorenstein local ring. The following conditions are then equivalent.*

(*i*) *R is Gorenstein.*

(*ii*) $\mathrm{Gid}_R\, {}^{\varphi^n}\!R$ *is finite for some integer $n \geqslant 1$.*

(*iii*) $\mathrm{Gid}_R\, {}^{\varphi^n}\!R$ *is finite for all integers $n \geqslant 1$.*

If one of the above conditions is met, then $\mathrm{Gid}_R\, {}^{\varphi^n}\!R = \operatorname{depth} R = \dim R$.

Theorem C is a Gorenstein version of the next theorem which is [11, thm. 13.3].

Theorem (Avramov–Iyengar–Miller). *Let $\varphi\colon (R,\mathfrak{m}) \longrightarrow (R,\mathfrak{m})$ be a contraction. If* $\mathrm{id}_R\, {}^{\varphi^n}\!R$ *is finite for some integer $n \geqslant 1$, then R is regular.*

On the other hand, the last theorem follows—in view of Avramov, Iyengar, and Miller [11, thm. 13.2]—from our Theorem C and the classical results by Kunz and Rodicio.

Organization of the paper. The main results, Theorems A, B, and C, belong to *classical homological algebra.* Their proofs, however, take—of necessity—place in the derived category $\mathsf{D}(R)$ of the category of R–modules. For now there is no suitable description of the applications of this *hyperhomological algebra* in commutative ring theory. Thus, necessary background material is scattered throughout sections 2–4,

2. Dualities and Equivalences

(2.1) **Derived category.** Throughout the paper, we will work within the derived category $\mathsf{D}(R)$ of the module category over a local commutative noetherian ring $(R,\mathfrak{m},k)$; here $\mathfrak{m}$ denotes the unique maximal ideal, and k is the residue field $R/\mathfrak{m}$.

The objects in $\mathsf{D}(R)$ are complexes of R–modules. A complex M is a sequence of R–modules $(M)_{n\in\mathbb{Z}}$ equipped with R–linear differentials $\partial^M_n\colon M_n \longrightarrow M_{n-1}$. If m is an integer, the symbol $\Sigma^m M$ denotes the complex M shifted (or translated, or suspended) m degrees to the left, that is, against the direction of the differential; its modules are given by $(\Sigma^m M)_i = M_{i-m}$, and its differential is $\partial^{\Sigma^m M}_i = (-1)^m \partial^M_{i-m}$. The symbol $\simeq$ will denote isomorphisms in the derived category.

The full subcategory of $\mathsf{D}(R)$ consisting of complexes with bounded homology is denoted $\mathsf{D}_\square(R)$, while $\mathsf{D}^{\mathsf{f}}_\square(R)$ denotes the full subcategory of $\mathsf{D}_\square(R)$ consisting of complexes with each homology module finitely generated; the complexes in $\mathsf{D}^{\mathsf{f}}_\square(R)$

will be referred to as *finite* complexes. Each R–module M may be viewed, in a canonical way, as a complex concentrated in degree zero. Moreover, each complex M of R–modules with homology concentrated in degree zero is isomorphic in $\mathsf{D}(R)$ to the module $\mathrm{H}_0(M)$. Thus we identify R–modules with complexes homologically concentrated in degree zero.

For details on the derived category and derived functors, the reader should consult the original texts, Verdier's thesis [42] and Hartshorne's notes [28]. For a modern account, the reader is referred to Gelfand and Manin [27].

To capture the homological size of a complex M we consider its *homological infimum*, its *homological supremum* and its *amplitude*

$$\inf M = \inf\{\, \ell \mid \mathrm{H}_\ell(M) \neq 0\,\}, \quad \sup M = \sup\{\, \ell \mid \mathrm{H}_\ell(M) \neq 0\}, \quad \text{and}$$

$$\operatorname{amp} M = \sup M - \inf M.$$

We always operate under the conventions that $\inf \varnothing = \infty$ and $\sup \varnothing = -\infty$. Thus, M belongs to $\mathsf{D}_\square(R)$ if and only if $\sup M < \infty$ and $\inf M > -\infty$. Moreover, $\operatorname{amp} M = 0$ if and only if M is isomorphic (in the derived category) to $\Sigma^n K$ for some non-zero R–module K and some integer n; in this case, K is isomorphic to $\mathrm{H}_n(M)$.

If M is a homologically bounded complex, then M is said to be of *finite projective dimension*, *finite injective dimension*, or *finite flat dimension*, when M is isomorphic (in $\mathsf{D}(R)$) to a bounded complex consisting of, respectively, projective modules, injective modules, or flat modules, in which case we write, respectively, $\operatorname{pd}_R M < \infty$, $\operatorname{id}_R M < \infty$, or $\operatorname{fd}_R M < \infty$. For details the reader is referred to [6].

(2.2) **Derived functors.** The left derived tensor product functor $- \otimes^{\mathbf{L}}_R -$ applied to a pair (M,N) of complexes of R–modules is defined, up to isomorphism in $\mathsf{D}(R)$, as follows

$$P \otimes_R N \simeq M \otimes^{\mathbf{L}}_R N \simeq M \otimes_R Q$$

whenever $P \xrightarrow{\simeq} M$ is a semi-projective resolution of M or $Q \xrightarrow{\simeq} N$ is a semi-projective resolution of N. An R–complex P is said to be semi-projective, if the functor $\mathbf{R}\mathrm{Hom}_R(P,-)$ preserves surjective quasi-isomorphisms. It turns out that an R–complex is semi-projective, if it is bounded to the right and consists of projective modules. A semi-projective resolution of M is a quasi-isomorphism $\pi \colon P \longrightarrow M$ with semi-projective source. For existence of semi-projective resolutions consult [9].

Dually, an R–complex I is said to be semi-injective, if the functor $\mathbf{R}\mathrm{Hom}_R(-,I)$ carries injective quasi-isomorphisms into surjective quasi-isomorphisms; and a semi-injective resolution of N is a quasi-isomorphism $\iota \colon N \longrightarrow I$ with semi-injective target. An R–complex is semi-injective, if it is bounded to the left and consists of injective modules. For existence of semi-projective resolutions consult [9].

The right derived homomorphism functor $\mathbf{R}\mathrm{Hom}_R(-,-)$ applied to a pair (M,N) of complexes of R–modules is defined, up to isomorphism in $\mathsf{D}(R)$, as follows

$$\mathrm{Hom}_R(P,N) \simeq \mathbf{R}\mathrm{Hom}_R(M,N) \simeq \mathrm{Hom}_R(M,I)$$

whenever and $P \xrightarrow{\simeq} M$ is a semi-projective resolution of M or $N \xrightarrow{\simeq} I$ is a semi-injective resolution of N.

If M and N are R–modules, then there are isomorphisms

$$\mathrm{H}_\ell(M \otimes^{\mathbf{L}}_R N) \cong \mathrm{Tor}^R_\ell(M,N) \quad \text{and} \quad \mathrm{H}_\ell(\mathbf{R}\mathrm{Hom}_R(M,N)) \cong \mathrm{Ext}^{-\ell}_R(M,N)$$

for all integers ℓ.

The next standard isomorphisms are used throughout the paper. To facilitate the description here, also the other ring S is supposed to be commutative, and not all the boundedness conditions imposed on the complexes are strictly necessary.

(2.3) **Functorial Isomorphisms.** Let K, L, and M belong to $\mathsf{D}(R)$, let P belong to $\mathsf{D}(S)$, and let N belong to the derived category $\mathsf{D}(R,S)$ of the category of R–S–bimodules. There are then the next functorial isomorphisms in $\mathsf{D}(R,S)$.

(Comm) $$L \otimes^{\mathbf{L}}_R M \xrightarrow{\simeq} M \otimes^{\mathbf{L}}_R L\,.$$

(Assoc) $$(M \otimes^{\mathbf{L}}_R N) \otimes^{\mathbf{L}}_S P \xrightarrow{\simeq} M \otimes^{\mathbf{L}}_R (N \otimes^{\mathbf{L}}_S P)\,.$$

(Adjoint) $$\mathbf{R}\mathrm{Hom}_S(M \otimes^{\mathbf{L}}_R N, P) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_S(N,P))\,.$$

(Swap) $$\mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_S(P,N)) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_S(P, \mathbf{R}\mathrm{Hom}_R(M,N))\,.$$

Moreover, there are the following evaluation morphisms.

(Tensor–eval) $$\alpha_{KNP}\colon \mathbf{R}\mathrm{Hom}_R(K,N) \otimes^{\mathbf{L}}_S P \longrightarrow \mathbf{R}\mathrm{Hom}_R(K, N \otimes^{\mathbf{L}}_S P) \quad \text{and}$$

(Hom–eval) $$\beta_{PNM}\colon P \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_R(N,M) \longrightarrow \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_S(P,N), M)\,.$$

The next two facts are useful

- The morphism α_{KNP} is an isomorphism, if K is finite, $\mathrm{H}(N)$ is bounded, and either $\mathrm{fd}_S\, P$ or $\mathrm{pd}_R\, K$ is finite.
- The morphism β_{PNM} is an isomorphism, if P is finite, $\mathrm{H}(N)$ is bounded, and either $\mathrm{pd}_S\, P$ or $\mathrm{id}_R\, M$ is finite.

For details the reader is referred to [14, A.4] and the references therein.

(2.4) **Dimension and Depth.** For a finite complex M we define its *depth* and *dimension* as follows

$$\dim_R M = \sup\{\, \dim_R \mathrm{H}_\ell(M) - \ell \mid \ell \in \mathbb{Z} \,\} \quad \text{and}$$
$$\mathrm{depth}_R M = \inf\{\, \ell \mid \mathrm{H}_{-\ell}(\mathbf{R}\mathrm{Hom}_R(k,M)) \neq 0 \,\}\,.$$

Here $\dim_R \mathrm{H}_\ell(M)$ denotes the (Krull) dimension of the module $\mathrm{H}_\ell(M)$. If M is a finitely generated module, then these invariants yield the classical depth and dimension of M. As shown by Foxby, see [21, (16.3)], the dimension of a complex M may be computed as

$$\dim_R M = \sup\{\, \dim R/\mathfrak{p} - \inf M_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Spec}\, R \,\}.$$

The ring R is a Cohen–Macaulay ring if its depth equals its dimension, that is, $\mathrm{depth}\, R = \dim R$. Finally, the width of a complex M is defined to be

$$\mathrm{width}_R M = \inf\{\, \ell \mid \mathrm{H}_\ell(k \otimes^{\mathbf{L}}_R M) \neq 0 \,\}\,.$$

In general there is the next inequality

$$\mathrm{width}_R M \geqslant \inf M\,.$$

and equality holds (by Nakayama's lemma), if the homology modules of M are finitely generated.

(2.5) **Dagger Duality.** In this paragraph we assume that R admits a *dualizing complex* D, that is, D is a finite R–complex, its injective dimension $\mathrm{id}_R\, D$ is finite, and the canonical morphism $\mu_D\colon R \longrightarrow \mathbf{R}\mathrm{Hom}_R(D,D)$ is an isomorphism. If D and E are dualizing complexes for R, then there exists an integer n such that $E \simeq \Sigma^n D$. A dualizing complex is *normalized* when $k \simeq \mathbf{R}\mathrm{Hom}_R(k,D)$; in this situation it follows that

$$\sup D = \dim R \quad \text{and} \quad \inf D = \mathrm{depth}\, R.$$

When M is a finite complex, we consider the *dagger dual* $M^\dagger = \mathbf{R}\mathrm{Hom}_R(M,D)$, and the next hold

$$\sup M^\dagger = \dim_R M \quad \text{and} \quad \inf M^\dagger = \operatorname{depth}_R M,$$

for details see [21, (16.20)]. In particular, if $\mathrm{H}(M) \neq 0$ then $\operatorname{depth}_R M \leqslant \dim_R M$, and the Cohen–Macaulay defect of M is defined to be the non-negative integer $\operatorname{cmd}_R M = \dim_R M - \operatorname{depth}_R M$.

If R is a homomorphic image of a local Gorenstein ring Q, then $\mathbf{R}\mathrm{Hom}_Q(R,Q)$ is a dualizing complex over R, and the shifted complex $\Sigma^n\,\mathbf{R}\mathrm{Hom}_Q(R,Q)$ is normalized for $n = \dim Q - \dim R$. Consequently, every local ring which is complete in the topology induced by the maximal ideal admits a dualizing complex. On the other hand, if a local ring admits a dualizing complex, then it is a homomorphic image of a Gorenstein ring by Kawasaki's proof of Sharp's conjecture; for detail see [33].

The contravariant functor $(-)^\dagger = \mathbf{R}\mathrm{Hom}_R(-,D)$ carries the category $\mathsf{D}^{\mathsf{f}}_{\square}(R)$ into itself, and for every finite M there is the next biduality isomorphism

$$\delta^M_D \colon\; M \xrightarrow{\simeq} M^{\dagger\,\dagger} = \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M,D),D)$$

which is obtained from the next commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{\quad\delta^M_D\quad} & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M,D),D) \\
\Big\| & & \simeq\Big\uparrow \beta_{MDD} \\
M \otimes^{\mathbf{L}}_R R & \xrightarrow[M\otimes^{\mathbf{L}}_R\mu_D]{\simeq} & M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D,D)
\end{array}$$

where the rightmost vertical isomorphism is (Hom–eval) in (2.3). This induces the next duality of categories.

$$\mathsf{D}^{\mathsf{f}}_{\square}(R) \underset{(-)^\dagger}{\overset{(-)^\dagger}{\rightleftarrows}} \mathsf{D}^{\mathsf{f}}_{\square}(R)\,.$$

If R is a Cohen–Macaulay ring of dimension d possessing a normalized dualizing complex D, then $\mathrm{H}_n(D) = 0$ for $n \neq d$, and $\mathrm{H}_d(D)$ is said to be the dualizing (or canonical) module for R, [13, chap. 3].

(2.6) **Dualizing Equivalence.** Let D be a dualizing complex for R. The pair of adjoint functors

$$D \otimes^{\mathbf{L}}_R - \quad \text{and} \quad \mathbf{R}\mathrm{Hom}_R(D,-)$$

is naturally equipped with the unit morphism $\eta^R \colon 1_{\mathsf{D}(R)} \longrightarrow \mathbf{R}\mathrm{Hom}_R(D, D\otimes^{\mathbf{L}}_R -)$ and the counit morphism $\varepsilon^R \colon D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D,-) \longrightarrow 1_{\mathsf{D}(R)}$. The unit fits into the commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{\quad\eta^R_M\quad} & \mathbf{R}\mathrm{Hom}_R(D, D\otimes^{\mathbf{L}}_R M) \\
\Big\| & & \Big\uparrow \alpha_{DDM} \\
R \otimes^{\mathbf{L}}_R M & \xrightarrow[\mu_D\otimes^{\mathbf{L}}_R M]{\simeq} & \mathbf{R}\mathrm{Hom}_R(D,D) \otimes^{\mathbf{L}}_R M
\end{array}$$

where M is any R–complex. The counit fits into the commutative diagram

$$\begin{array}{ccc} N & \xleftarrow{\varepsilon_N^R} & D \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(D,N) \\ \Big\| & & \Big\downarrow \beta_{DDN} \\ \mathbf{R}\mathrm{Hom}_R(R,N) & \xleftarrow[\mathbf{R}\mathrm{Hom}_R(\mu_D,N)]{\simeq} & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(D,D),N) \end{array}$$

where N is any R–complex.

The *Auslander category* $\mathcal{A}(R)$ and the *Bass category* $\mathcal{B}(R)$ with respect to D are the full subcategories of $\mathsf{D}_\square(R)$ defined as follows.

(a) M is in $\mathcal{A}(R)$ if and only if $D \otimes_R^{\mathbf{L}} M$ is homologically bounded and the unit η_M^R is an isomorphism.

(b) N is in $\mathcal{B}(R)$ if and only if $\mathbf{R}\mathrm{Hom}_R(D,N)$ is homologically bounded and the counit ε_N^R is an isomorphism.

The definitions of the Auslander and Bass categories are designed such that we have an equivalence of categories

$$\mathcal{A}(R) \underset{\mathbf{R}\mathrm{Hom}_R(D,-)}{\overset{D\otimes_R^{\mathbf{L}}-}{\rightleftarrows}} \mathcal{B}(R).$$

This equivalence was introduced in [8]. It follows from (2.3) that every complex of finite flat dimension belongs to $\mathcal{A}(R)$, and that every complex of finite injective dimension belongs to $\mathcal{B}(R)$. By restricting to the full subcategory $\mathsf{D}_\square^{\mathrm{f}}(R)$ (consisting of complexes having degreewise finitely generated homology), we obtain the next equivalence of categories

$$\mathcal{A}^{\mathrm{f}}(R) \underset{\mathbf{R}\mathrm{Hom}_R(D,-)}{\overset{D\otimes_R^{\mathbf{L}}-}{\rightleftarrows}} \mathcal{B}^{\mathrm{f}}(R).$$

For details, the reader is referred to [8].

In the following lemma we consider the two functors $(-)^* = \mathbf{R}\mathrm{Hom}_R(-,R)$ and $(-)^\dagger = \mathbf{R}\mathrm{Hom}_R(-,D)$.

(2.7) **Lemma.** *The following hold for the next diagram.*

$$\begin{array}{ccccc} & & \mathcal{A}^{\mathrm{f}}(R) & & \\ & {\scriptstyle (-)^*}\swarrow\nearrow{\scriptstyle (-)^*} & & {\scriptstyle (-)^\dagger}\nwarrow\searrow{\scriptstyle (-)^\dagger} & \\ \mathcal{A}^{\mathrm{f}}(R) & & \underset{\mathbf{R}\mathrm{Hom}_R(D,-)}{\overset{D\otimes_R^{\mathbf{L}}-}{\rightleftarrows}} & & \mathcal{B}^{\mathrm{f}}(R)\,, \end{array}$$

(1) *The inner triangle is commutative (up to canonical isomorphism).*
(2) *The outer triangle is commutative (up to canonical isomorphism).*
(3) *The left pair of parallel tilted arrows provides a duality of categories.*
(4) *The right pair of parallel tilted arrows provides a duality of categories.*
(5) *The pair of horizontal arrows provides an equivalence of categories.*

Proof. The assertion concerning the horizontal arrows follows by dualizing equivalence (2.6). The functor $(-)^*$ provides a duality on $\mathcal{A}^{\mathrm{f}}(R)$, see also [8, (4.1.7)] or [15, thm. (4.7)]. The corresponding assertion concerning $(-)^\dagger$ follows from [14, lem. (3.2.9)].

The commutativity (up to canonical isomorphism) of the inner triangle is established as follows

$$\begin{aligned}(D \otimes^{\mathbf{L}}_R M)^{\dagger} &= \mathbf{R}\mathrm{Hom}_R(D \otimes^{\mathbf{L}}_R M, D)\\ &\xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_R(D, D))\\ &\xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(M, R) = M^* .\end{aligned}$$

Here the first isomorphism is by (Adjoint) in (2.3), and the second follows from the isomorphism $R \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(D, D)$. The commutativity (up to canonical isomorphism) of the outer triangle is established as follows

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(D, M^{\dagger}) &= \mathbf{R}\mathrm{Hom}_R(D, \mathbf{R}\mathrm{Hom}_R(M, D))\\ &\xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_R(D, D))\\ &\xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(M, R) = M^* .\end{aligned}$$

Here the first isomorphism is by (Swap) in (2.3), and the second follows from the isomorphism $R \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(D, D)$. □

3. Gorenstein Homological Dimensions

(3.1) **Gorenstein Injective Dimension.** A complex I of R–modules is said to be a *complete injective resolution*, if it is exact and consists of injective modules, and it is such that $\mathrm{Hom}_R(I', I)$ is exact for all injective R–modules I'. An R–module J is said to be *Gorenstein injective*, if it is a cokernel in a complete injective resolution. Thus, every injective module is Gorenstein injective.

The *Gorenstein injective dimension* $\mathrm{Gid}_R\, M$ of $M \in \mathsf{D}_{\square}(R)$ is defined to be the infimum of the set of integers n such that there exists a complex I consisting of Gorenstein injective modules satisfying $M \simeq I$ and $I_\ell = 0$ for $-\ell > n$. (Recall that we always use homological notation.) Thus, a complex of R–modules has finite Gorenstein injective dimension if and only if it is isomorphic in $\mathsf{D}(R)$ to a bounded complex of Gorenstein injective modules. Moreover, the Gorenstein injective dimension is a *refinement* of the (classical) injective dimension

$$\mathrm{Gid}_R\, M \leqslant \mathrm{id}_R\, M \quad \text{with equality if} \quad \mathrm{id}_R\, M \quad \text{is finite.}$$

For details consult [14, chap. 6].

(3.2) **Gorenstein Projective Dimension.** The *Gorenstein projective dimension* $\mathrm{Gpd}_R\, M$ of $M \in \mathsf{D}_{\square}(R)$ was introduced by Enochs and Jenda and is defined dually to the injective one above. It is a refinement of the (classical) projective dimension

$$\mathrm{Gpd}_R\, M \leqslant \mathrm{pd}_R\, M \quad \text{with equality if} \quad \mathrm{pd}_R\, M \quad \text{is finite.}$$

For details consult [14, chap. 4].

(3.3) **Gorenstein Flat Dimension.** The definition of the *Gorenstein flat dimension* $\mathrm{Gfd}_R\, M$ of $M \in \mathsf{D}_{\square}(R)$ is similar to the Gorenstein injective dimension above. A complex F of R–modules is said to be a *complete flat resolution*, if it is exact and consists of flat modules, and it is such that also $I' \otimes_R F$ is exact for all injective R–modules I'. An R–module G is said to be *Gorenstein flat*, if it is a cokernel in a complete flat resolution. Thus, every flat module is Gorenstein flat.

The *Gorenstein flat dimension* $\mathrm{Gfd}_R\, M$ of $M \in \mathsf{D}_{\square}(R)$ is defined to be the infimum of the set of integers n such that there exists a complex F consisting of Gorenstein flat modules satisfying $M \simeq F$ and $F_\ell = 0$ for $\ell > n$. Thus, a complex of R–modules has finite Gorenstein flat dimension if and only if it is isomorphic in $\mathsf{D}(R)$

to a bounded complex of Gorenstein flat modules. Moreover, the Gorenstein flat dimension is a refinement of the (classical) flat dimension

$$\operatorname{Gfd}_R M \leqslant \operatorname{fd}_R M \quad \text{with equality if} \quad \operatorname{fd}_R M \quad \text{is finite.}$$

For details consult [14, chap. 5].

(3.4) **Auslander's G–dimension.** If G is a *finitely generated* R–module, then it turns out, see for example, [14, (4.4.6)], that it is Gorenstein projective if and only if it satisfies the next three conditions.

(1) $\operatorname{Ext}_R^\ell(G,R)=0$ for $\ell>0$,
(2) $\operatorname{Ext}_R^\ell(\operatorname{Hom}_R(G,R),R)=0$ for $\ell>0$, and
(3) the canonical homomorphism $G \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(G,R),R)$ is an isomorphism.

Auslander's *Gorenstein dimension* $\operatorname{G–dim}_R M$ of a finite R–complex M is defined to be at most n exactly when M is isomorphic in $\mathsf{D}(R)$ to a bounded complex G such that $G_\ell = 0$ for $\ell > n$, and such that each G_ℓ is a finitely generated R–modules satisfying the above three condition, confer [14, (2.3.2)]. The next is known as the *Auslander–Bridger Equality.*

(3.4.1) $$\operatorname{G–dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M\,, \quad \text{if} \quad \operatorname{G–dim}_R M < \infty\,,$$

confer [14, (1.4.8), (2.3.13)]. Furthermore, it turns out that the next holds.

(3.4.2) $$\operatorname{G–dim}_R M = \operatorname{Gpd}_R M = \operatorname{Gfd}_R M\,,$$

when M is a finite R–complex; for details consult [16, prop. 3.8].

The next result by Foxby, see Yassemi [43, (2.7)], provides a relation between finiteness of G-dimension and the Auslander category.

(3.4.3) $$\operatorname{G–dim}_R M < \infty \quad \text{if and only if} \quad M \in \mathcal{A}^{\mathrm{f}}(R)\,,$$

for further details see [14, (3.1.11)].

This is extended by next result which is the main theorem in Christensen, Frankild and Holm [16]; it underlines the importance of the Auslander and Bass categories.

(3.5) **Finiteness of Gorenstein dimensions.** Assume that R is a homomorphic image of a Gorenstein ring. Let M be a complex of R–modules. Then following are equivalent.

(*i*) M belongs to $\mathcal{A}(R)$.
(*ii*) M has finite Gorenstein projective dimension, that is, $\operatorname{Gpd}_R M < \infty$.
(*iii*) M has finite Gorenstein flat dimension, that is, $\operatorname{Gfd}_R M < \infty$.

Furthermore, if N is a complex of R–modules, then the following are equivalent.

(*i*) N belongs to $\mathcal{B}(R)$.
(*ii*) N has finite Gorenstein injective dimension, that is, $\operatorname{Gid}_R N < \infty$.

For details the reader is referred to [16, thm. 4.1 and thm. 4.4].

The next result on completion and Gorenstein injective dimension is due to Christensen, Frankild, and Iyengar. We thank Christensen and Iyengar for allowing us to include it here. Note that it does not require R to be a homomorphic image of a Gorenstein ring.

(3.6) **Theorem.** *Let R be a local ring, and let M be a finite R–complex. If M has finite Gorenstein injective dimension over R, then $M \otimes_R \widehat{R}$ has finite Gorenstein injective dimension over $\widehat{R}$.*

Proof. Let K^R be the Koszul complex on a set of generators for the maximal ideal $\mathfrak{m}$. Because the homology modules of K^R have finite length, there is a isomorphism $K^R \simeq \widehat{R} \otimes_R K^R$ in $\mathsf{D}(R)$. By flatness of the completion map $R \longrightarrow \widehat{R}$, a minimal set of generators for $\mathfrak{m}$ extends to a minimal set of generators of $\widehat{\mathfrak{m}}$; in particular $K^{\widehat{R}} = \widehat{R} \otimes_R K^R$ is a Koszul complex on a minimal set of generators for $\widehat{\mathfrak{m}}$. Moreover, the isomorphism of R–modules $R/\mathfrak{m} \cong k \cong \widehat{R}/\widehat{\mathfrak{m}}$ together with the fact that $\mathrm{H}_\ell(K^R) \cong \mathrm{H}_\ell(K^{\widehat{R}})$ for all $\ell \in \mathbb{Z}$ shows that $K^R \xrightarrow{\simeq} K^{\widehat{R}}$ in $\mathsf{D}(R)$.

Under the present assumptions on M, the complex

$$N = M \otimes_R K^R \simeq M \otimes_R K^{\widehat{R}} \simeq (M \otimes^{\mathbf{L}}_R \widehat{R}) \otimes_{\widehat{R}} K^{\widehat{R}}$$

has finite Gorenstein injective dimension over R; see [17, (5.5)(c')]. Note that the homology modules of N have finite length since M is finite. Hence there is an isomorphism

$$N \xrightarrow{\simeq} \mathrm{Hom}_R(\mathrm{Hom}_R(N, \mathrm{E}_R(k)), \mathrm{E}_R(k))\,.$$

Here $\mathrm{E}_R(k)$ denotes the injective envelope of the R–module k. In particular,

$$\mathrm{Gid}_R \, \mathrm{Hom}_R(\mathrm{Hom}_R(N, \mathrm{E}_R(k)), \mathrm{E}_R(k)) < \infty$$

and, therefore, $\mathrm{Gfd}_R \, \mathrm{Hom}_R(N, \mathrm{E}_R(k))$ is finite by [14, (6.4.2)]. As the homology modules of $\mathrm{Hom}_R(N, \mathrm{E}_R(k))$ has finite length, the complex

$$(*) \qquad \mathrm{Hom}_R(N, \mathrm{E}_R(k)) \otimes_R \widehat{R} \simeq \mathrm{Hom}_R(N, \mathrm{E}_R(k))$$

has finite Gorenstein flat dimension over $\widehat{R}$. Thus, using $(*)$ and (Adjoint) from (2.3) we conclude

$$N \xrightarrow{\simeq} \mathrm{Hom}_{\widehat{R}}(\mathrm{Hom}_R(N, \mathrm{E}_R(k)), \mathrm{E}_R(k))$$

has finite Gorenstein injective dimension over $\widehat{R}$ by [16, prop. 5.1]; this uses the fact that the complete ring $\widehat{R}$ admits a dualizing complex D. From [16, thm. 4.4] it follows that the complex

$$\mathbf{R}\mathrm{Hom}_{\widehat{R}}(D, N) \simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}}(D, M \otimes^{\mathbf{L}}_R \widehat{R}) \otimes^{\mathbf{L}}_{\widehat{R}} K^{\widehat{R}}$$

is homologically bounded. Thus, from [24, 1.3] it follows that the complex $\mathbf{R}\mathrm{Hom}_{\widehat{R}}(D, M \otimes^{\mathbf{L}}_R \widehat{R})$ is homologically bounded as well. Consider the commutative diagram

$$\begin{array}{ccc}
(M \otimes^{\mathbf{L}}_R \widehat{R}) \otimes^{\mathbf{L}}_{\widehat{R}} K^{\widehat{R}} & \xleftarrow[\simeq]{\varepsilon^{\widehat{R}}_{(M \otimes^{\mathbf{L}}_R \widehat{R}) \otimes^{\mathbf{L}}_{\widehat{R}} K^{\widehat{R}}}} & D \otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{R}\mathrm{Hom}_{\widehat{R}}(D, (M \otimes^{\mathbf{L}}_R \widehat{R}) \otimes^{\mathbf{L}}_{\widehat{R}} K^{\widehat{R}}) \\
\| & & \downarrow \simeq \\
(M \otimes^{\mathbf{L}}_R \widehat{R}) \otimes^{\mathbf{L}}_{\widehat{R}} K^{\widehat{R}} & \xleftarrow[(\varepsilon^{\widehat{R}}_{M \otimes^{\mathbf{L}}_R \widehat{R}}) \otimes^{\mathbf{L}}_{\widehat{R}} K^{\widehat{R}}]{} & (D \otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{R}\mathrm{Hom}_{\widehat{R}}(D, M \otimes^{\mathbf{L}}_R \widehat{R})) \otimes^{\mathbf{L}}_{\widehat{R}} K^{\widehat{R}} \,,
\end{array}$$

where the rightmost vertical isomorphism is by (Tensor-eval) and (Assoc) in (2.3). Again, using [24, 1.3] and a standard mapping cone argument it follows $\varepsilon^{\widehat{R}}_{M \otimes^{\mathbf{L}}_R \widehat{R}}$ is a isomorphism. Whence,

$$M \otimes_R \widehat{R} \cong M \otimes^{\mathbf{L}}_R \widehat{R}$$

has finite Gorenstein injective dimension over $\widehat{R}$ by [16, thm. 4.4]. □

4. Bass and Poincaré Series

(4.1) **Bass and Poincaré Series.** For a finite R–complex M and an integer ℓ, the ℓ th *Bass number* and the ℓ th *Betti numbers* are, respectively, the next vector space dimensions over the residue field k

$$\mu_R^\ell(M) = \operatorname{rank}_k \mathrm{H}_{-\ell}(\mathbf{R}\mathrm{Hom}_R(k, M)) \quad \text{and} \quad \beta_\ell^R(M) = \operatorname{rank}_k \mathrm{H}_\ell(k \otimes_R^{\mathbf{L}} M)\,.$$

The ring of *formal Laurent series with integer coefficients* is denoted $\mathbb{Z}(|t|)$; its elements are have the form $\sum_{\ell\in\mathbb{Z}} a_\ell t^\ell$ with $a_\ell \in \mathbb{Z}$ and $a_\ell = 0$ for $\ell \ll 0$. For a finite R–complex M the *Bass series* $\mathrm{I}_R^M(t)$ and the *Betti series* $\mathrm{P}_M^R(t)$ are elements in $\mathbb{Z}(|t|)$ defined as follows.

$$\mathrm{I}_R^M(t) = \sum_{\ell\in\mathbb{Z}} \mu_R^\ell(M)t^\ell \quad \text{and} \quad \mathrm{P}_M^R(t) = \sum_{\ell\in\mathbb{Z}} \beta_\ell^R(M)t^\ell\,.$$

The Bass series for the ring is denoted $\mathrm{I}_R(t)$, and the ring R is Gorenstein precisely when $\mathrm{I}_R(t) = t^s$ for some non-negative integer s. Every finite R–complex D is a normalized dualizing complex, if and only if $\mathrm{I}_R^D(t) = 1$.

The next equations for Bass and Poincaré series are used throughout. The first two are proved in [8, (1.5.3)] while the last two are proved in [23, (4.3)].

(4.2) **Bass–Poincaré Equalities.** For finite complexes M and N there are the next equalities of formal Laurent series, that is, equalities in $\mathbb{Z}(|t|)$.

(PP) $$\mathrm{P}^R_{M\otimes_R^{\mathbf{L}} N}(t) = \mathrm{P}_M^R(t)\,\mathrm{P}_N^R(t)\,.$$

(PI) $$\mathrm{I}_R^{\mathbf{R}\mathrm{Hom}_R(M,N)}(t) = \mathrm{P}_M^R(t)\,\mathrm{I}_R^N(t)\,.$$

(IP) $$\mathrm{I}_R^{M\otimes_R^{\mathbf{L}} N}(t) = \mathrm{I}_R^M(t)\,\mathrm{P}_R^N(t^{-1}) \quad \text{provided} \quad \mathrm{pd}_R\, N < \infty\,.$$

(II) $$\mathrm{P}^R_{\mathbf{R}\mathrm{Hom}_R(M,N)}(t) = \mathrm{I}_R^M(t)\,\mathrm{I}_N^R(t^{-1}) \quad \text{provided} \quad \mathrm{id}_R\, N < \infty\,.$$

(4.3) **Proposition.** *Let R be a homomorphic image of a Gorenstein ring, and let N be a finite R–complex of finite Gorenstein injective dimension. There exists then a finite R–complex K of finite Gorenstein projective dimension with* $\inf K = \inf N$ *and* $\operatorname{amp} K \leqslant \operatorname{amp} N$ *such that there is the next equality of formal Laurent series.*

$$\mathrm{P}_N^R(t)t^{\operatorname{depth} R} = \mathrm{P}_K^R(t)\,\mathrm{I}_R(t)\,.$$

If N is of finite (classical) injective dimension, then K has finite (classical) projective dimension.

Proof. Throughout the proof, we let D be a normalized dualizing complex, and set

$$M = N^\dagger \quad \text{and} \quad L = \mathbf{R}\mathrm{Hom}_R(D, N) \quad \text{and} \quad s = \operatorname{depth} R.$$

Note that $\operatorname{depth}_R M = \inf N$ by (2.5). As N belongs to $\mathcal{B}^{\mathrm{f}}(R)$ by (3.5), Lemma (2.7) yields that M and L belong to $\mathcal{A}^{\mathrm{f}}(R)$, and that $L^* \simeq M$ and $M^* \simeq L$. As M belongs to $\mathcal{A}^{\mathrm{f}}(R)$, the G–dimension of M is finite by (3.5). Whence, by [14, (1.2.7) and (1.4.8)] we obtain

$$\inf L = \inf M^* = -\,\mathrm{G\text{–}dim}_R\, M = \operatorname{depth}_R M - \operatorname{depth} R = \inf N - \operatorname{depth} R.$$

Next, [14, (A.4.6)] yields the next inequality

$$\sup L = \sup \mathbf{R}\mathrm{Hom}_R(D, N) \leqslant -\inf D + \sup N = \sup N - \operatorname{depth} R.$$

The last equality is by (2.5). Hence, $\operatorname{amp} L \leqslant \operatorname{amp} N$. Set $K = \Sigma^s L$ which has finite Gorenstein projective dimension as it belongs to $\mathcal{A}^{\mathrm{f}}(R)$. It still remains to prove the equation for the Laurent series. Recall that $s = \operatorname{depth} R$. The computation

$$K^{*\dagger} \simeq ((\Sigma^s L)^*)^\dagger = (\Sigma^{-s} L^*)^\dagger \simeq (\Sigma^{-s} M)^\dagger = \Sigma^s M^\dagger \simeq \Sigma^s N.$$

yields the next formula.

(4.3.1) $$N \simeq \mathsf{\Sigma}^{-s} K^{*\dagger}.$$

Using that $\mathrm{I}_R^D(t) = 1$ by (2.5) and that $\mathrm{P}_R(t) = 1$, the formulae (II) and (PI) in (4.2) yields the next equalities

$$\mathrm{P}_N^R(t) = \mathrm{P}^R_{\mathsf{\Sigma}^{-s}K^{*\dagger}}(t) = t^{-s}\,\mathrm{P}^R_{K^{*\dagger}}(t) = t^{-s}\,\mathrm{I}_R^{K^*}(t) = t^{-s}\,\mathrm{P}_K^R(t)\,\mathrm{I}_R(t).$$

Finally, if $\mathrm{id}_R N$ is finite, it follows that $\mathrm{pd}_R N^\dagger$ is finite as well, and this implies that $\mathrm{pd}_R N^{\dagger *} < \infty$, that is, $\mathrm{pd}_R K < \infty$. □

(4.4) **Remark on depth.** For every finite R–complexes M and N there is the next equality of integers

$$\mathrm{depth}_R\, \mathbf{R}\mathrm{Hom}_R(M,N) = \inf M + \mathrm{depth}_R N\,.$$

Indeed, (PI) in (4.2) yields the identity

$$\mathrm{I}_R^{\mathbf{R}\mathrm{Hom}_R(M,N)}(t) = \mathrm{P}_M^R(t)\,\mathrm{I}_R^N(t)\,,$$

which gives the second equality below.

$$\begin{aligned}\mathrm{depth}_R\, \mathbf{R}\mathrm{Hom}_R(M,N) = \mathrm{ord}\,\mathrm{I}_R^{\mathbf{R}\mathrm{Hom}_R(M,N)}(t) &= \mathrm{ord}(\mathrm{P}_M^R(t)\,\mathrm{I}_R^N(t)) \\ = \mathrm{ord}\,\mathrm{P}_M^R(t) + \mathrm{ord}\,\mathrm{I}_R^N(t) &= \inf M + \mathrm{depth}_R N\,.\end{aligned}$$

In particular, let $M = D$ be a normalized dualizing complex and let $K = \mathsf{\Sigma}^s\, \mathbf{R}\mathrm{Hom}_R(D,N)$ be the complex considered in the proof of (4.3). From the above it thus follows that $\mathrm{depth}_R K = \mathrm{depth}_R N$.

(4.5) **Remark on dimension.** For the sake of simplicity, we assume in this paragraph that N from Proposition (4.3) is a module. Let D be a normalized dualizing complex for R. Recall that

$$K = \mathsf{\Sigma}^s L \quad \text{where} \quad L = \mathbf{R}\mathrm{Hom}_R(D,N) \quad \text{and} \quad s = \mathrm{depth}\, R\,,$$

and that K is a module. The support of D is the entire $\mathrm{Spec}\, R$, and the homology of $\mathbf{R}\mathrm{Hom}_R(D,N)$ is zero precisely when N is trivial; see e.g., [8, lem. (1.2.3)]. Thus, for a prime ideal $\mathfrak{p}$ from $\mathrm{Spec}\, R$ we have $K_\mathfrak{p} = 0$ if and only if $N_\mathfrak{p} = 0$ forcing $\mathrm{Supp}_R K = \mathrm{Supp}_R N$. In particular, using (2.4) we read off that

(4.5.1) $$\dim_R K = \dim_R N,$$

in particular, it follows from (4.4) that

$$\mathrm{cmd}_R K = \mathrm{cmd}_R N.$$

Since D is normalized, the $R_\mathfrak{p}$–complex $\mathsf{\Sigma}^{-\dim R/\mathfrak{p}} D_\mathfrak{p}$ is a normalized dualizing complex for the localized ring $R_\mathfrak{p}$; see e.g., [21, (15.17)]. Thus, we immediately conclude that

$$\begin{aligned}\inf K_\mathfrak{p} = \inf\,(\mathsf{\Sigma}^s L)_\mathfrak{p} &= \mathrm{depth}\, R + \inf \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(D_\mathfrak{p}, N_\mathfrak{p}) \\ &= \mathrm{depth}\, R - \dim R/\mathfrak{p} + \inf \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(\mathsf{\Sigma}^{-\dim R/\mathfrak{p}} D_\mathfrak{p}, N_\mathfrak{p}).\end{aligned}$$

The assumption that R admits a dualizing complex ensures that the Gorenstein injective dimension of $N_\mathfrak{p}$ is finite; see [16, prop. 5.5] and may be computed as

$$\mathrm{Gid}_{R_\mathfrak{p}} N_\mathfrak{p} = -\inf \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(\mathsf{\Sigma}^{-\dim R/\mathfrak{p}} D_\mathfrak{p}, N_\mathfrak{p}) = \mathrm{depth}\, R_\mathfrak{p},$$

for details see [16, cor. 6.7 and thm. 6.3]. Consequently, using that K is a module, and that $\mathrm{Supp}_R K = \mathrm{Supp}_R N$ we obtain

(∗) $$\mathrm{depth}\, R = \dim R/\mathfrak{p} + \mathrm{depth}\, R_\mathfrak{p},$$

for every prime $\mathfrak{p}$ in the support of N. Compare this with the fact that for any Cohen–Macaulay ring, and any prime $\mathfrak{p}$ the formula in $(*)$ hold.

(4.6) **Remark.** In this remark we use the derived local cohomology $\mathbf{R}\Gamma_{\mathfrak{m}}(-)$ and its right adjoint the derived local homology functor $\mathbf{L}\Lambda^{\mathfrak{m}}(-)$ (also referred to as the derived completion functor), confer e.g. [1]. When the finite complex N from Proposition (4.3) is a module, the finite complex K is also a module, and it has finite Gorenstein projective dimension. Let D be a normalized dualizing complex. The proof shows that

$$(*) \qquad K = \mathrm{H}_{-s}(\mathbf{R}\mathrm{Hom}_R(D, N)),$$

where $s = \operatorname{depth} R$. As D is normalized, $\mathbf{R}\Gamma_{\mathfrak{m}} D \simeq \mathrm{E}_R(k)$ by local duality, where $\mathrm{E}_R(k)$ is the injective hull of the residue field k. Therefore, we have the next string of isomorphisms

$$\mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k), N) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{m}} D, N) \simeq \mathbf{R}\mathrm{Hom}_R(D, \mathbf{L}\Lambda^{\mathfrak{m}} N),$$

for details see e.g., [26, sec. 2] and the references therein. As N is finite, we have

$$\mathbf{R}\mathrm{Hom}_R(D, \mathbf{L}\Lambda^{\mathfrak{m}} N) \simeq \mathbf{R}\mathrm{Hom}_R(D, N \otimes^{\mathbf{L}}_R \widehat{R}) \simeq \mathbf{R}\mathrm{Hom}_R(D, N) \otimes^{\mathbf{L}}_R \widehat{R}.$$

Here the first isomorphism follows from [26, prop. (2.7)] and the second from (Tensor-eval). Comparing this with $(*)$ shows that

$$\widehat{K} = K \otimes_R \widehat{R} = \mathrm{H}_{-s}(\mathbf{R}\mathrm{Hom}_R(D, N)) \otimes_R \widehat{R} \cong \mathrm{Ext}^s_R(\mathrm{E}_R(k), N).$$

The latter module is used by Peskine and Spziro in the proof of their theorem on cyclic modules of finite (classical) injective dimension; for details see [36, thm. (4.10)].

(4.7) **Theorem.** *If there exists a non-zero cyclic R–module N with the Gorenstein injective dimension $\mathrm{Gid}_R N$ finite, then R is Gorenstein.*

Proof. Note first that we may assume that R is complete and thus possesses a dualizing complex, confer by (3.6). As N is cyclic, we have that $N \cong R/\operatorname{Ann}_R N$ and that constant term in $\mathrm{P}^R_N(t)$ is then 1. Thus, (4.3) yields that the constant term in $\mathrm{P}^R_K(t)$ is also 1. In particular, the module K occurring in (4.3) is cyclic; whence $K \cong R/\operatorname{Ann}_R K$. The formula (4.3.1) gives that $\operatorname{Ann}_R N \supseteq \operatorname{Ann}_R K$. Applying the functor $(-)^{\dagger *}$ to (4.3.1) we obtain the equation $K \simeq \mathbf{\Sigma}^s N^{\dagger *}$. This yields $\operatorname{Ann}_R N \subseteq \operatorname{Ann}_R K$. Thus $\operatorname{Ann}_R N = \operatorname{Ann}_R K$, and it follows that $N \cong K$, so the equation in (4.3) implies that $\mathrm{I}_R(t) = t^s$, that is, R is Gorenstein. □

(4.8) **Remark on finite length modules.** Assume that M is an R–module of finite length and of finite Gorenstein projective dimension. As these assumptions pass to the completion, we assume that R admits a (normalized) dualizing complex D. Note that M belongs to $\mathcal{A}(R)$ by (3.5).

Since M has finite length it is easy to show that $\mathrm{H}(D \otimes^{\mathbf{L}}_R M)$ is concentrated in degree $s = \operatorname{depth} R$. By assumption it belongs to $\mathcal{B}(R)$. Applying (PP) from (4.2) we obtain that the minimal number of generators for $\mathrm{Tor}^R_s(D, M)$ equals the type of R times the minimal number of generators of M. Thus, if M is cyclic and the type of R is 1, then $\mathrm{Tor}^R_s(D, M)$ is a cyclic module of finite Gorenstein injective dimension. From (4.7) is follows that R is Gorenstein. Compare this with [40, thm. 2.3] where Takahashi shows that R is Gorenstein precisely when R has type 1 and admits a Cohen–Macaulay module of finite Gorenstein projective dimension or, if R admits a Cohen–Macaulay module of type 1 and of finite Gorenstein projective dimension.

(4.9) **Cohen–Macaulay injective dimension.** Theorem (4.10) below is an immediate consequence of (4.7), and it characterizes Cohen–Macaulay rings in terms finiteness of the Cohen–Macaulay injective dimension introduced by Holm and Jørgensen [32]. Recall from [15] that a finitely generated R–module C is *semi-dualizing* if the natural homomorphism $R \longrightarrow \mathrm{Hom}_R(C,C)$ is an isomorphism and $\mathrm{Ext}^i_R(C,C) = 0$ for all $i > 0$, that is, the homothety morphism $R \longrightarrow \mathbf{R}\mathrm{Hom}_R(C,C)$ is an isomorphism in $\mathsf{D}(R)$. The *Cohen–Macaulay injective dimension* of an R–module M is defined as

$$\mathrm{CMid}_R\, M = \inf\{\, \mathrm{Gid}_{R\ltimes C}\, M \mid C \,\text{is a semi-dualizing module over}\, R\,\}.$$

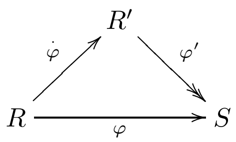
Here $R \ltimes C$ denotes the trivial extension ring; it is the R–module $R \times C$ equipped with the multiplication $(r,c)(r',c') = (rr', rc' + r'c)$. If $(R, \mathfrak{m}, k)$ is local, then so is $(R \ltimes C, \mathfrak{m} \times C, k)$. The ring homomorphism $R \ltimes C \longrightarrow R$ defined by $(r,c) \longmapsto r$ turns every R–module into an $R \ltimes C$– module; if N is cyclic over R, then it is so over $R \ltimes C$. Finally, the module C is a dualizing module precisely when the ring $R \ltimes C$ is Gorenstein; for details see Foxby [22].

(4.10) **Theorem.** *If there exists a non-zero cyclic R–module N with the Cohen–Macaulay injective dimension $\mathrm{CMid}_R\, N$ finite, then R is Cohen–Macaulay with dualizing module.* □

5. Local Homomorphisms

In this section we apply Theorem (4.7) to a homomorphisms of local rings $\varphi\colon (R, \mathfrak{m}, k) \longrightarrow (S, \mathfrak{n}, \ell)$ which is assumed to be local, that is, $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$.

(5.1) **Cohen factorizations.** We will make use of the technology know as *Cohen factorizations* of local homomorphisms due to Avramov, Foxby, and Herzog [10]. A Cohen factorization of φ is a commutative diagram of local homomorphisms



such that φ' is surjective, and $\dot{\varphi}$ is flat with the closed fiber $R'/\mathfrak{m}R'$ a regular ring and with the target R' complete. For details the reader is referred to [10].

Cohen factorizations often exists: the (so-called) *semi-completion* $\acute{\varphi}\colon R \longrightarrow S \longrightarrow \widehat{S}$ always admits a Cohen factorization; see [10, (1.1)].

(5.2) **Remark.** A result due to Avramov, Iyengar, and Miller states: Let φ be a local homomorphism. The injective dimension of S over R is finite precisely when R is Gorenstein and the flat dimension of S over R is finite. For details, the reader is referred to [11, thm. 13.2]. Theorem (5.4) below is a Gorenstein version of the this result modulo the fact that we require R to admit a dualizing complex.

(5.3) **Remark.** Over a Gorenstein ring, say R, every module has finite Gorenstein flat dimension. In particular, if $\varphi\colon R \longrightarrow S$ is a local homomorphism it follows that $\mathrm{Gfd}_R\, S$ is finite. Furthermore, when R is a homomorphic image of a Gorenstein ring, the next theorem implies that the class of local homomorphisms of finite Gorenstein injective dimension is a subclass of that of local homomorphisms of finite Gorenstein dimension.

(5.4) **Theorem.** *Let $\varphi\colon R \longrightarrow S$ be a local homomorphism, and assume that R is a homomorphic image of a Gorenstein ring. Then $\mathrm{Gid}_R\, S < \infty$ if and only if R is Gorenstein.*

Proof. If R is Gorenstein, then $\mathrm{Gid}_R\, S$ is finite. Next, assume that $\mathrm{Gid}_R\, S$ is finite. Note that (A.5) implies that $\mathrm{Gid}_R\, \widehat{S}$ is finite as well. Pick a Cohen factorization $R \longrightarrow R' \longrightarrow \widehat{S}$ of the semi-completion $\grave{\varphi}$. From (A.5) we have that the cyclic R'–module $\widehat{S}$ has finite Gorenstein injective dimension. But then (4.7) yields that R' is Gorenstein, and by flat descent [35, (23.4)], so is R. □

(5.5) **Almost finite complexes.** Let M and N be finite R–complexes such that the projective dimension of M and the injective dimension of N are both finite. The Auslander–Buchsbaum and Bass Equalities yield the equalities

$$(\star) \qquad \mathrm{fd}_R\, M = \mathrm{pd}_R\, M = \sup\,(k \otimes^{\mathbf{L}}_R M) = \mathrm{depth}\, R - \mathrm{depth}_R\, M,$$

and

$$(\star\star) \qquad \mathrm{id}_R\, N = -\inf \mathbf{R}\mathrm{Hom}_R(k, N) = \mathrm{depth}\, R - \inf N.$$

Consider next a more general setup: Let $\varphi\colon R \longrightarrow S$ be a local homomorphism, and let M and N be finite complexes of S–modules; such complexes are said to be *almost finite complexes over* R. Assume, furthermore, that $\mathrm{fd}_R\, M$ and $\mathrm{id}_R\, N$ are finite. The next results due to André [2, (2.57)] and extended to complexes by Avramov and Foxby [6, prop. 5.5] state that

$$(*) \qquad \mathrm{fd}_R\, M = \sup\,(k \otimes^{\mathbf{L}}_R M) = \mathrm{depth}\, R - \mathrm{depth}_R\, M,$$

and

$$(**) \qquad \mathrm{id}_R\, N = -\inf\,(\mathbf{R}\mathrm{Hom}_R(k, N)) = \mathrm{depth}\, R - \inf N.$$

Relaxing the homological assumption on M from finite flat dimension to finite Gorenstein flat dimension Christensen and Iyengar [18] proved

$$(***) \qquad \mathrm{Gfd}_R\, M = \sup\,(\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R M) = \mathrm{depth}\, R - \mathrm{depth}_R\, M.$$

Note that the residue field k in $(*)$ is replaced with its injective hull $\mathrm{E}_R(k)$ in $(***)$. Next, we give the Gorenstein injective version of this result modulo the fact that we require the source ring R to be a homomorphic image of a Gorenstein ring.

(5.6) **Theorem.** *Let $\varphi\colon R \longrightarrow S$ be a local homomorphism, and assume that R is a homomorphic image of a Gorenstein ring. Let N be a finite S–complex such that $\mathrm{Gid}_R\, N$ is finite. Then*

$$\mathrm{Gid}_R\, N = -\inf\,(\mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k), N)) = \mathrm{depth}\, R - \inf N.$$

Proof. First, we note that the finiteness of $\mathrm{Gid}_R\, N$ ensures that

$$-\inf\,(\mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k), N)) = \mathrm{depth}\, R - \inf N,$$

by [16, thm. 6.6]. It is an immediate consequence of e.g., [16, thm. 3.3] that

$$(*) \qquad -\inf\,(\mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k), N)) \leqslant \mathrm{Gid}_R\, N.$$

Therefore, in order to complete the argument, it suffices to show that

$$\mathrm{Gid}_R\, N \leqslant \mathrm{depth}\, R - \inf N.$$

1° First we reduce to the case where R and S are complete with respect to the topologies induced by their respective maximal ideals. As we have

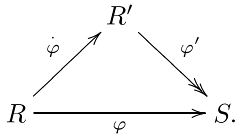
$$\mathrm{depth}\, R = \mathrm{depth}\, \widehat{R} \quad \text{and} \quad \inf N = \inf \widehat{N},$$

it immediately follows from the (in)equalities established in (A.6) in conjunction with [16, thm. 6.3] that

$$\operatorname{depth}\widehat{R} - \inf\widehat{N} = \operatorname{depth}R - \inf N \leqslant \operatorname{Gid}_R N \leqslant \operatorname{Gid}_{\widehat{R}}\widehat{N}.$$

Thus, once the statement has been proved for complete rings, it holds in general. Therefore, we may assume that R and S are complete.

2° Next, we reduce to the case where φ is flat with regular closed fiber. Since S is assumed to be complete, the local homomorphism φ admits a Cohen factorization; see (5.1). Whence φ factors according to the following diagram



where $\dot{\varphi}$ is flat, the closed fiber $R'/\mathfrak{m}R'$ is regular, and φ' is surjective. By assumption N is a finite complex consisting of S–modules. As φ' is a surjection, it follows that N is also a finite complex over the intermediate ring R'. Consequently, it suffices to consider the case where $\varphi\colon R \longrightarrow S$ is flat with regular closed fiber.

3° As φ is flat with regular closed fiber it is Gorenstein, see (A.2). Let D^S denote the normalized dualizing complex for S. Since N is a finite complex of S–modules we have that the biduality morphism

$$(\dagger)\qquad N \longrightarrow \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(N, D^S), D^S)$$

is a isomorphism in $\mathsf{D}(S)$; see (2.5). To keep notation simple, we let

$$(-)^{\dagger_S} = \mathbf{R}\mathrm{Hom}_S(-, D^S)$$

in the sequel. In particular, we may conclude that $\operatorname{Gid}_R N = \operatorname{Gid}_R(N^{\dagger_S\,\dagger_S})$. Note that the finiteness of $\operatorname{Gid}_R N$ forces $\operatorname{Gid}_S N$ to be finite as well; this is a consequence of (A.7) and uses the fact that in the present setup φ is flat and Gorenstein. Moreover, a reference to [16, thm. 3.3] yields the existence of an injective R–module, say J, such that the following equality holds

$$\operatorname{Gid}_R(N^{\dagger_S\,\dagger_S}) = -\inf \mathbf{R}\mathrm{Hom}_R(J, N^{\dagger_S\,\dagger_S}).$$

This allows us to compute as follows

$$(\ddagger)\qquad \begin{aligned} \operatorname{Gid}_R N &= -\inf \mathbf{R}\mathrm{Hom}_R(J, N^{\dagger_S\,\dagger_S}) \\ &= -\inf \mathbf{R}\mathrm{Hom}_S(N^{\dagger_S}, \mathbf{R}\mathrm{Hom}_R(J, D^S)) \\ &\leqslant \operatorname{Gpd}_S N^{\dagger_S} - \inf \mathbf{R}\mathrm{Hom}_R(J, D^S) \\ &\leqslant \operatorname{Gpd}_S N^{\dagger_S} + \operatorname{id}_R D^S. \end{aligned}$$

Here the first equality comes from (†); the second follows from (Swap) from (2.3); the first inequality from [16, thm. 3.1], as $J^{\dagger_S}$ is an S–complex of finite flat dimension it has finite projective dimension (recall, that φ is flat); the final inequality follows from [6, thm. 2.4.I]. From [16, cor. 6.4] we conclude that $\operatorname{Gpd}_S N^{\dagger_S}$ is finite. As $N^{\dagger_S}$ is finite over S, it follows from [16, prop. 3.8(b)] and [14, thm. (2.3.13)] that

$$(**)\qquad \operatorname{Gpd}_S N^{\dagger_S} = \operatorname{depth} S - \operatorname{depth}_S N^{\dagger_S}.$$

From the fact that $\varphi\colon R \longrightarrow S$ is flat we conclude from [8, prop. (4.6)] that the injective dimension of D^S is finite over R. Also, as the homology of D^S is finite over S we may employ the result by André to obtain that

$$(***)\qquad \operatorname{id}_R D^S = \operatorname{depth} R - \inf D^S = \operatorname{depth} R - \operatorname{depth} S.$$

Here the second equality stems from the fact that D^S is a normalized dualizing complex over S. Thus, inserting the equalities from $(**)$ and $(***)$ we finally get

$$\begin{aligned}\operatorname{Gid}_R N &\leqslant \operatorname{depth} S - \operatorname{depth}_S N^{\dagger_S} + \operatorname{id}_R D^S \\ &= \operatorname{depth} S - \inf N + \operatorname{depth} R - \operatorname{depth} S \\ &= \operatorname{depth} R - \inf N.\end{aligned}$$

Here the first equality again follows from the fact that D^S is normalized. □

The following result on contractions is due to Iyengar and Sather-Wagstaff, see [31, thm. 8.14 and 8.15].

(5.7) **Theorem (Iyengar–Sather-Wagstaff).** *Let $\varphi\colon (R,\mathfrak{m}) \longrightarrow (R,\mathfrak{m})$ be a contraction, that is, $\varphi^i(\mathfrak{m}) \subseteq \mathfrak{m}^2$ for some integer $i \geqslant 1$. The following conditions are equivalent.*

- (i) *R is Gorenstein.*
- (ii) *$\operatorname{Gfd}_R {}^{\varphi^n}\! R$ is finite for all integers $n \geqslant 1$.*
- (iii) *There exists a finite R–complex P with $\mathrm{H}(P) \neq 0$ and $\operatorname{pd}_R P$ finite such that $\operatorname{Gfd}_R {}^{\varphi^n}\! P$ is finite for some integer $n \geqslant 1$.*

If one of the above equivalent conditions is met, then $\operatorname{Gfd}_R {}^{\varphi^n}\! R = 0$.

What follows may be thought of as a Gorenstein injective version of the above. However, the result is not restricted to contracting endomorphisms.

(5.8) **Theorem.** *Let $\varphi\colon (R,\mathfrak{m}) \longrightarrow (R,\mathfrak{m})$ be a local homomorphism, and assume that R is a homomorphic image of a Gorenstein ring. The following conditions are equivalent*

- (i) *R is Gorenstein.*
- (ii) *$\operatorname{Gid}_R {}^{\varphi^n}\! R$ is finite for all integers $n \geqslant 1$.*
- (iii) *There exists a finite R–complex P with $\mathrm{H}(P) \neq 0$ and $\operatorname{pd}_R P$ finite such that $\operatorname{Gid}_R {}^{\varphi^n}\! P$ is finite for some integer $n \geqslant 1$.*

If one of the above equivalent conditions is met, we have

$$\operatorname{Gid}_R {}^{\varphi^n}\! R = \operatorname{depth} R = \dim R.$$

Proof. The equivalence of (i) and (ii) is just a reformulation of Theorem (5.4) for endomorphisms. Clearly (iii) is stronger than (ii), and it is trivial that (i) implies (iii).

By (5.6) we have $\operatorname{Gid}_R {}^{\varphi^n}\! R = \operatorname{depth} R = \dim R$ where the last equality follows from the fact that R is Gorenstein, in particular, R is Cohen–Macaulay. □

Acknowledgments

It is a pleasure to thank Lars Winther Christensen and Sean Sather-Wagstaff for carefully reading the manuscript and for numerous thoughtful comments.

Appendix: Stability results for Gorenstein injectives

(A.1) **Remark.** In this appendix we collect results needed for establishing (5.6). Although some of them may be known to experts, we have chosen to present them here to increase the readability.

When $\varphi\colon R \longrightarrow S$ is a local homomorphism and N is a complex of S–modules, we let $\widetilde{N} = N \otimes_S \widehat{S}$ where $\widehat{S}$ is the completion of $(S,\mathfrak{n})$ with respect to its $\mathfrak{n}$–adic topology.

(A.2) **Remark.** Recall, that a flat local homomorphism $\varphi\colon R \longrightarrow S$ is Gorenstein at $\mathfrak{n}$ precisely when the closed fiber $S/\mathfrak{m}S$ is a Gorenstein ring, confer e.g, [7, (4.2)].

(A.3) **Proposition.** *Let $\varphi\colon R \longrightarrow S$ be a local flat Gorenstein homomorphism. If N is a finite R–complex of such that $\operatorname{id}_S(N \otimes_R S)$ is finite, then the fiber $k(\mathfrak{p}) \otimes_R S$ is a Gorenstein ring for every contraction $\mathfrak{p} = \mathfrak{q} \cap R$ where $\mathfrak{q}$ is a prime from $\operatorname{Supp}_S(N \otimes_R S)$.*

Proof. Let N be a finite R–complex, and assume that $\operatorname{id}_S(N \otimes_R S)$ is finite. Note that [21, cor. (22.25)] forces $\operatorname{id}_R N$ to be finite as well. Pick a prime $\mathfrak{q}$ from $\operatorname{Supp}_S(N \otimes_R S)$, and let $\mathfrak{p}$ be its contraction through φ. Localizing implies that $\operatorname{id}_{S_\mathfrak{q}}(N_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q})$ is finite, and as the localized homomorphism $\varphi_\mathfrak{q}\colon R_\mathfrak{p} \longrightarrow S_\mathfrak{q}$ still is flat [21, cor. (22.25)] states that the fiber $k(\mathfrak{p}) \otimes_R S$ is a Gorenstein ring. $\square$

(A.4) **Corollary.** *Assume that R is a homomorphic image of a Gorenstein ring. Let $\varphi\colon R \longrightarrow S$ be a local flat Gorenstein homomorphism, N a finite R–complex such that $\operatorname{id}_S(N \otimes_R S)$ is finite, and assume that $\operatorname{Supp}_S(N \otimes_R S)$ equals $\operatorname{Spec} S$. Then for any prime $\mathfrak{p}$ in $\operatorname{Spec} R$ there are (in)equalities*

$$\operatorname{id}_S(\mathrm{E}_R(R/\mathfrak{p}) \otimes_R S) = \operatorname{id}(k(\mathfrak{p}) \otimes_R S) < \infty.$$

Proof. First, as φ is flat the going-down property holds for φ; consult [35, thm. 7.3(i)] for details. By assumption $\operatorname{Supp}_S(N \otimes_R S)$ equals $\operatorname{Spec} S$. Therefore, to any prime $\mathfrak{p}$ in $\operatorname{Spec} R$ we can find a prime $\mathfrak{q}$ from $\operatorname{Supp}_S(N \otimes_R S)$ contracting to $\mathfrak{p}$. For $\mathfrak{p}$ in $\operatorname{Spec} R$ the only associated prime of the injective hull $\mathrm{E}_R(R/\mathfrak{p})$ is $\mathfrak{p}$. Thus, the injective dimension of the S–module $\mathrm{E}_R(R/\mathfrak{p}) \otimes_R S$ is

$$\operatorname{id}_S(\mathrm{E}_R(R/\mathfrak{p}) \otimes_R S) = \operatorname{id}(k(\mathfrak{p}) \otimes_R S)$$

according to [25, rmk. 1]. Applying Proposition (A.3) gives the conclusion. $\square$

(A.5) **Lemma.** *Assume R is a homomorphic image of a Gorenstein ring. Let $\varphi\colon R \longrightarrow S$ be a local homomorphism, $R \longrightarrow R' \longrightarrow \widehat{S}$ a Cohen factorization of the semi-completion, and let N be a bounded complex of S–modules. Then*

$$\operatorname{Gid}_R N, \quad \operatorname{Gid}_R \widetilde{N}, \quad \textit{and} \quad \operatorname{Gid}_{R'} \widetilde{N}$$

are simultaneously finite.

Proof. Let D denote the normalized dualizing complex for R. According to [16, thm. 4.4] we are required to show the following two equivalences

$$N \in \mathcal{B}(R) \iff \widetilde{N} \in \mathcal{B}(R) \iff \widetilde{N} \in \mathcal{B}(R')\,.$$

1° The latter equivalence follows from [16, thm. 5.3].

2° We move on to establishing the first equivalence. As the completion $\widehat{S}$ is faithfully flat over S there is an isomorphism of S–complexes

$$\mathbf{R}\mathrm{Hom}_R(D,N) \otimes^{\mathbf{L}}_S \widehat{S} \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(D,\widetilde{N}).$$

This allows us to conclude that the homology of $\mathbf{R}\mathrm{Hom}_R(D,\widetilde{N})$ is bounded if and only if the homology of $\mathbf{R}\mathrm{Hom}_R(D,N)$ is. Moreover, from the commutative diagram

$$\begin{array}{ccc}
\widetilde{N} & \xleftarrow{\varepsilon^R_{\widetilde{N}}} & D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D,\widetilde{N}) \\
\Vert & & \uparrow \simeq \\
N \otimes^{\mathbf{L}}_S \widehat{S} & \xleftarrow[\varepsilon^R_N \otimes^{\mathbf{L}}_R \widehat{S}]{} & D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D,N) \otimes^{\mathbf{L}}_S \widehat{S}
\end{array}$$

it follows, using a standard mapping cone argument, that the counit $\varepsilon^R_{\widetilde{N}}$ is an isomorphism if and only if the counit ε^R_N is. □

(A.6) **Lemma.** *Assume R is a homomorphic image of a Gorenstein ring. Let $\varphi\colon R \longrightarrow S$ be a local homomorphism, and let N be a bounded complex of S–modules. Then there are (in)equalities*

$$\operatorname{Gid}_R N = \operatorname{Gid}_R \widetilde{N} \leqslant \operatorname{Gid}_{\widehat{R}} \widetilde{N}.$$

Proof. From (A.5) we know that all three numbers are simultaneously finite; we assume this in the sequel. The existence of a dualizing complex for R ensures that R has Gorenstein formal fibers. Fix a prime $\mathfrak{p}$ in $\operatorname{Spec} R$ and consider the indecomposable injective R–module $\mathrm{E}_R(R/\mathfrak{p})$. From the main result of [25, rmk. 1] we conclude that

$$\operatorname{id}_{\widehat{R}}(\mathrm{E}_R(R/\mathfrak{p}) \otimes_R \widehat{R}) = \operatorname{id}(k(\mathfrak{p}) \otimes_R \widehat{R}) \leqslant \dim \widehat{R} = \dim R.$$

In general $\mathrm{E}_R(R/\mathfrak{p}) \otimes_R \widehat{R}$ is not injective over the completion $\widehat{R}$. A straightforward application of Matlis' structure theorem shows that for any non-zero injective R–module, say J, we have that the injective dimension of $J \otimes_R \widehat{R}$ over $\widehat{R}$ is finite as well.

1° First we focus on the inequality $\operatorname{Gid}_R \widetilde{N} \leqslant \operatorname{Gid}_{\widehat{R}} \widetilde{N}$. From [16, thm. 3.3] it follows that

$$\operatorname{Gid}_{\widehat{R}} \widetilde{N} = \sup\{-\inf \mathbf{R}\mathrm{Hom}_{\widehat{R}}(J', \widetilde{N}) \mid \operatorname{id}_{\widehat{R}} J' < \infty\}.$$

and that

$$\operatorname{Gid}_R \widetilde{N} = \sup\{-\inf \mathbf{R}\mathrm{Hom}_R(J, \widetilde{N}) \mid \operatorname{id}_R J < \infty\}.$$

Let J be a non-zero R–module of finite injective dimension. By adjointness we obtain

$$-\inf(\mathbf{R}\mathrm{Hom}_{\widehat{R}}(J \otimes_R \widehat{R}, \widetilde{N})) = -\inf(\mathbf{R}\mathrm{Hom}_R(J, \widetilde{N})),$$

and it immediately follows that $\operatorname{Gid}_R N \leqslant \operatorname{Gid}_{\widehat{R}} \widetilde{N}$.

2° It remains to show $\operatorname{Gid}_R N = \operatorname{Gid}_R \widetilde{N}$. From [16, thm. 6.8] it follows that

$$\operatorname{Gid}_R N \;=\; \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

and also that

$$\operatorname{Gid}_R \widetilde{N} \;=\; \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} (\widetilde{N})_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

However, if $\mathfrak{p}$ is a prime from $\operatorname{Spec} R$ then the integer $\operatorname{width}_{R_{\mathfrak{p}}} (\widetilde{N})_{\mathfrak{p}}$ can be computed as follows

$$\begin{aligned}
\operatorname{width}_{R_{\mathfrak{p}}} (\widetilde{N})_{\mathfrak{p}} &= \inf(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} (\widetilde{N})_{\mathfrak{p}}) \\
&= \inf(k(\mathfrak{p}) \otimes^{\mathbf{L}}_R N \otimes^{\mathbf{L}}_S \widehat{S}) \\
&= \inf(k(\mathfrak{p}) \otimes^{\mathbf{L}}_R N) \\
&= \inf(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_p} N_{\mathfrak{p}}) \\
&= \operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}.
\end{aligned}$$

The third equality uses that $\widehat{S}$ is faithfully flat over S, and the others follow trivially. Reinserting this into the above expression of $\operatorname{Gid}_R \widetilde{N}$ yields the conclusion. □

(A.7) **Lemma.** *Assume that R is a homomorphic image of a Gorenstein ring. Let $\varphi\colon R \longrightarrow S$ be a flat local Gorenstein homomorphism. For any finite S–complex N the quantities $\operatorname{Gid}_R N$ and $\operatorname{Gid}_S N$ are simultaneously finite, and there is an inequality*

$$\operatorname{Gid}_R N \leqslant \operatorname{Gid}_S N\,.$$

Proof. Let D be a dualizing complex for R. As φ is assumed to be a Gorenstein (at $\mathfrak{n}$), the base–changed complex $D \otimes_R S$ is dualizing for S. From [8, cor. (7.9)] we learn that N belongs to $\mathcal{B}(R)$ precisely when N belongs to $\mathcal{B}(S)$. Hence, from [16, thm. 4.4] we conclude that $\operatorname{Gid}_R N$ is finite, if and only if $\operatorname{Gid}_S N$ is.

We may assume that $\operatorname{Gid}_R N$ and $\operatorname{Gid}_S N$ are finite. We will demonstrate that

$$\operatorname{Gid}_R N \leqslant \operatorname{Gid}_S N.$$

Note that by [16, cor. 3.4] the finiteness of $\operatorname{Gid}_R N$ implies the existence of a prime $\mathfrak{p}$ from $\operatorname{Spec} R$ such that

$$\operatorname{Gid}_R N = -\inf \mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(R/\mathfrak{p}), N).$$

Since N carries a S–structure compatible with its R–structure, we have $N \simeq \mathbf{R}\mathrm{Hom}_S(S, N)$, whence adjointness yields

$$\begin{aligned} \operatorname{Gid}_R N &= -\inf \mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(R/\mathfrak{p}), N) \\ &= -\inf \mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(R/\mathfrak{p}), \mathbf{R}\mathrm{Hom}_S(S, N)) \\ &= -\inf \mathbf{R}\mathrm{Hom}_S(\mathrm{E}_R(R/\mathfrak{p}) \otimes_R S, N). \end{aligned}$$

From (A.4) we know that the injective dimension of the S–module $\mathrm{E}_R(R/\mathfrak{p}) \otimes_R S$ is finite; in particular

$$\operatorname{Gid}_R N = -\inf \mathbf{R}\mathrm{Hom}_S(\mathrm{E}_R(R/\mathfrak{p}) \otimes_R S, N) \leqslant \operatorname{Gid}_S N$$

through a reference to [16, thm. 3.3]. □

(A.8) **Lemma.** *Assume R is a homomorphic image of a Gorenstein ring. Let $\varphi\colon R \longrightarrow S$ be a local homomorphism, and let P be a finite complex of S–modules with $\operatorname{pd}_S P$ finite. For any finite complex N consisting of S–modules, the two numbers*

$$\operatorname{Gid}_R N \quad \text{and} \quad \operatorname{Gid}_R(N \otimes^{\mathbf{L}}_S P)$$

are simultaneously finite.

Proof. Let D be a dualizing complex. According to [16, thm. 4.4] it is enough to show that $N \otimes^{\mathbf{L}}_S P$ belongs to $\mathcal{B}(R)$ precisely when N belongs to $\mathcal{B}(R)$. Before doing so, we need a handy observation: Under the present assumptions (Tensor–eval) from (2.3) yields an isomorphism

$$\alpha_{DNP}\colon \mathbf{R}\mathrm{Hom}_R(D, N) \otimes^{\mathbf{L}}_S P \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(D, N \otimes^{\mathbf{L}}_S P).$$

1° From e.g., [8, (1.2.2)] it follows that $\mathbf{R}\mathrm{Hom}_R(D, N)$ is homologically degree-wise finitely generated. Therefore, an application of [31, thm. 2.9] yields that $\mathbf{R}\mathrm{Hom}_R(D, N)$ and $\mathbf{R}\mathrm{Hom}_R(D, N \otimes^{\mathbf{L}}_S P)$ are bounded simultaneously.

2° Consider the commutative diagram

$$\begin{array}{ccc} N \otimes^{\mathbf{L}}_S P & \xleftarrow{\ \varepsilon_{N \otimes^{\mathbf{L}}_S P}\ } & D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, N \otimes^{\mathbf{L}}_S P) \\ \Big\| & & \simeq \Big\uparrow D \otimes^{\mathbf{L}}_R \alpha_{DNP} \\ N \otimes^{\mathbf{L}}_S P & \xleftarrow[\ \varepsilon_N \otimes^{\mathbf{L}}_S P\ ]{} & D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, N) \otimes^{\mathbf{L}}_S P, \end{array}$$

from which we read off that $\varepsilon_N \otimes^{\mathbf{L}}_S P$ is an isomorphism when and only when $\varepsilon_{N \otimes^{\mathbf{L}}_S P}$ is. Resorting to [31, prop. 2.10] we obtain that ε_N and $\varepsilon_{N \otimes^{\mathbf{L}}_S P}$ are simultaneously isomorphisms. □

References

1. Leovigildo Alonso Tarrío, Ana Jeremías López, and Joseph Lipman, *Local homology and cohomology on schemes*, Ann. Sci. École Norm. Sup. (4) **30** (1997), no. 1, 1–39. MR 98d:14028
2. Michel André, *Homologie des algèbres commutatives*, Springer-Verlag, Berlin, 1974, Die Grundlehren der mathematischen Wissenschaften, Band 206. MR MR0352220 (50 #4707)
3. Maurice Auslander, *Anneaux de Gorenstein, et torsion en algèbre commutative*, Secrétariat mathématique, Paris, 1967, Séminaire d'Algèbre Commutative dirigé par Pierre Samuel, 1966/67. Texte rédigé, d'après des exposés de Maurice Auslander, par Marquerite Mangeney, Christian Peskine et Lucien Szpiro. École Normale Supérieure de Jeunes Filles.
4. Maurice Auslander and Mark Bridger, *Stable module theory*, Memoirs of the American Mathematical Society, No. 94, American Mathematical Society, Providence, R.I., 1969. MR 42 #4580
5. Maurice Auslander and David A. Buchsbaum, *Homological dimension in Noetherian rings*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 36–38.
6. Luchezar L. Avramov and Hans-Bjørn Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), no. 2-3, 129–155. MR 93g:18017
7. ______, *Locally Gorenstein homomorphisms*, Amer. J. Math. **114** (1992), no. 5, 1007–1047.
8. ______, *Ring homomorphisms and finite Gorenstein dimension*, Proc. London Math. Soc. (3) **75** (1997), no. 2, 241–270. MR 98d:13014
9. Luchezar L. Avramov, Hans-Bjørn Foxby, and Stephen Halperin, *Differential graded homological algebra*, in preparation.
10. Luchezar L. Avramov, Hans-Bjørn Foxby, and Bernd Herzog, *Structure of local homomorphisms*, J. Algebra **164** (1994), no. 1, 124–145. MR 95f:13029
11. Luchezar L. Avramov, Srikanth Iyengar, and Claudia Miller, *Homology over local homomorphisms*, Amer. J. Math. **128** (2006), no. 1, 23–90. MR MR2197067
12. Hyman Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.
13. Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, revised ed., Cambridge University Press, Cambridge, 1998, Cambridge studies in advanced mathematics, vol. 39.
14. Lars Winther Christensen, *Gorenstein dimensions*, Lecture Notes in Mathematics, vol. 1747, Springer-Verlag, Berlin, 2000. MR 2002e:13032
15. ______, *Semi-dualizing complexes and their Auslander categories*, Trans. Amer. Math. Soc. **353** (2001), no. 5, 1839–1883 (electronic). MR 2002a:13017
16. Lars Winther Christensen, Anders Frankild, and Henrik Holm, *On Gorenstein projective, injective and flat dimensions—a functorial description with applications*, J. Algebra **302** (2006), no. 1, 231–279. MR MR2236602
17. Lars Winther Christensen and Henrik Holm, *Ascent properties of Auslander categories*, to appear in Canad. J. Math (2005), 30.
18. Lars Winther Christensen and Srikanth Iyengar, *Gorenstein dimension of modules over homomorphisms,*, J. Pure Appl. Algebra **208** (2007), 177–188.
19. Edgar E. Enochs and Overtoun M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), no. 4, 611–633. MR 97c:16011
20. Edgar E. Enochs, Overtoun M. G. Jenda, and Blas Torrecillas, *Gorenstein flat modules*, Nanjing Daxue Xuebao Shuxue Bannian Kan **10** (1993), no. 1, 1–9. MR 95a:16004
21. Hans-Bjørn Foxby, *Hyperhomological algebra & commutative rings*, notes in preparation.
22. ______, *Gorenstein modules and related modules*, Math. Scand. **31** (1972), 267–284 (1973).
23. ______, *Isomorphisms between complexes with applications to the homological theory of modules*, Math. Scand. **40** (1977), no. 1, 5–19. MR 56 #5584
24. Hans-Bjørn Foxby and Srikanth Iyengar, *Depth and amplitude for unbounded complexes*, Commutative algebra. Interaction with Algebraic Geometry (Grenoble-Lyon 2001), Amer. Math. Soc., Providence, RI, no. 331, 2003, pp. 119–137.
25. Hans-Bjørn Foxby and Anders Thorup, *Minimal injective resolutions under flat base change*, Proc. Amer. Math. Soc. **67** (1977), no. 1, 27–31.
26. Anders Frankild, *Vanishing of local homology*, Math. Z. **244** (2003), no. 3, 615–630. MR 1 992 028
27. Sergei I. Gelfand and Yuri I. Manin, *Methods of homological algebra*, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. MR 2003m:18001
28. Robin Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966. MR 36 #5145
29. Melvin Hochster, *Topics in the homological theory of modules over commutative rings*, Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, Providence, R.I., 1975, Expository lectures from the CBMS Regional Conference held

at the University of Nebraska, Lincoln, Neb., June 24–28, 1974, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 24. MR MR0371879 (51 #8096)

30. Henrik Holm, *Rings with finite Gorenstein injective dimension*, Proc. Amer. Math. Soc. **132** (2004), no. 7, 1913–1923.
31. Srikanth Iyengar and Sean Sather-Wagstaff, *G-dimension over local homomorphisms. Applications to the Frobenius endomorphism*, Illinois J. Math. **48** (2004), no. 1, 241–272. MR MR2048224 (2005c:13016)
32. Peter Jørgensen and Henrik Holm, *Cohen–Macaulay homological dimensions*, Rend. Sem. Mat. Univ. Padova (to appear).
33. Takesi Kawasaki, *On arithmetic Macaulayfication of Noetherian rings*, Trans. Amer. Math. Soc. **354** (2002), no. 1, 123–149 (electronic). MR 2002i:13001
34. Ernst Kunz, *Characterizations of regular local rings for characteristic p*, Amer. J. Math. **91** (1969), 772–784. MR MR0252389 (40 #5609)
35. Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. MR 90i:13001
36. Christian Peskine and Lucien Szpiro, *Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck*, Inst. Hautes Études Sci. Publ. Math. (1973), no. 42, 47–119.
37. Paul Roberts, *Two applications of dualizing complexes over local rings*, Ann. Sci. École Norm. Sup. (4) **9** (1976), no. 1, 103–106.
38. Paul C. Roberts, *Multiplicities and Chern classes in local algebra*, Cambridge Tracts in Mathematics, vol. 133, Cambridge University Press, Cambridge, 1998. MR MR1686450 (2001a:13029)
39. Antonio G. Rodicio, *On a result of Avramov*, Manuscripta Math. **62** (1988), no. 2, 181–185. MR MR963004 (89k:13014)
40. Ryo Takahashi, *Some characterizations of Gorenstein local rings in terms of G-dimension*, Acta Math. Hungar. **104** (2004), no. 4, 315–322. MR MR2082781 (2005f:13015)
41. ———, *The existence of finitely generated modules of finite Gorenstein injective dimension*, Proc. Amer. Math. Soc. **134** (2006), no. 11, 3115–3121 (electronic). MR MR2231892
42. Jean-Louis Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque (1996), no. 239, xii+253 pp. (1997), With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis. MR 98c:18007
43. Siamak Yassemi, *G-dimension*, Math. Scand. **77** (1995), no. 2, 161–174.

Hans–Bjørn Foxby, Afdeling for Matematiske Fag, Universitetsparken 5, DK–2100 Copenhagen Ø, Denmark

E-mail address: foxby@math.ku.dk

Anders J. Frankild, Afdeling for Matematiske Fag, Universitetsparken 5, DK–2100 Copenhagen Ø, Denmark

E-mail address: frankild@math.ku.dk

URL: http://www.math.ku.dk/~frankild/